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Central limit theorems for law-invariant coherent risk measures

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Abstract

In this paper we study the asymptotic properties of the canonical plug-in estimates for law-invariant coherent risk measures. Under rather mild conditions not relying on the explicit representation of the risk measure under consideration, we first prove a central limit theorem for independent identically distributed data and then extend it to the case of weakly dependent ones. Finally, a number of illustrating examples is presented.

1 Introduction

In the seminal paper Artzner et al. (1999) the authors introduced the concept of coherent risk measures as a mathematical tool to assess the risks of financial positions. Formally, these objects are functionals on sets of random variables expressing risks of financial positions. The functionals should fulfill some defining properties which are axiomatic in nature to give a foundation for a normative risk assessment from the viewpoint of a regulator. An alternative axiomatic approach from the perspective of financial investors has been provided by Föllmer and Schied (2004) leading to a more general notion of convex risk measures.

During the last decade coherent risk measures identifying risks of financial positions with identical distributions, the so called law-invariant coherent risk measures, have become popular in some applied fields. They are building blocks in quantitative risk management (see McNeil et al. (2005)), and they have been suggested as a systematic approach for calculations of insurance premia (cf. Kaas et al. (2008)). Moreover, viewed as statistical functionals on sets of distribution functions, they satisfy the property to be monotone w.r.t. second order stochastic dominance (cf. Bäuerle and Müller (2006), for general information on stochastic orders see Müller and Stoyan (2002)). This illustrates the genuine intuition of risk measures as indices of distributions emphasizing the downsize risk of underlying financial positions.

In practice, we are often facing the problem of estimating the values of law-invariant coherent risk measures from a time series. A customary approach is to replace the unknown distribution function with its empirical counterpart based on observed data and then to plug this estimate into the risk measure to obtain its estimate. In this paper we are going to study the asymptotic properties of the resulting plug-in estimates. Such asymptotic analysis might be, for example, helpful for constructing confidence sets or performing statistical tests. Asymptotic properties of the plug in estimates for coherent risk measures have been investigated in two recent works, namely in Pflug and Wozabal (2010) and Beutner and Zähle (2010). While Pflug and Wozabal (2010) provided general results for a class of coherent risk measures in the case of independent data, Beutner and Zähle (2010) used a new functional delta method to obtain limit distributions for the subclass of concave distortion risk measures in the case of strongly mixing data.

In both aforementioned articles the results are based on general methods which do not take into account specific properties of the law-invariant coherent risk measures, leading to unnecessary strong assumptions on the underlying distribution. The aim of this paper is to extend and systemize the results on central limit theorems for plug-in estimates of law-invariant coherent risk measures. The contribution of the paper is twofold. On the one side, we prove central limit theorems for plug-in estimates for a rather general class of coherent risk measures under less restrictive assumptions, taking into account the fact that the "loss" tails are more relevant than the "gain tails" for coherent risk measures. On the other side, in contrast to the previous literature our results do not rely on the knowledge of the specific representations for the risk measures,

expressing the assumptions just in terms of the functionals itself. The last but not the least, we extend our CLT also to the case dependent observations and discontinuous distributions.

The paper is organized as follows. After introducing the main setup in section 2 we shall present our main results in section 3 for independent data. These results will then be extended to the case of dependent data in section 4. Section 5 gathers some auxiliary results to prove the main results, whereas the section 6 gives their proofs. Then the following section 7 is devoted to the proofs of the main results. Some useful technical results will be formulated and shown in the appendix.

2   Main setup

Let $\mathbb{F}_{\mathcal{X}}$ be a set of distribution functions on $\mathbb{R}$ related to a vector space $\mathcal{X}$ of integrable random variables on some atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ enclosing all $\mathbb{P}-$essentially bounded ones. A mapping $\rho : \mathbb{F}_{\mathcal{X}} \to \mathbb{R}$ is called a *law-invariant coherent risk measure* if the following conditions are fulfilled.

Monotonicity: For any $X_1, X_2 \in \mathcal{X}$ with $F_{X_1}(x) \leq F_{X_2}(x)$, $x \in \mathbb{R}$,

$$\rho(F_{X_1}) \leq \rho(F_{X_2}).$$

Cash-invariance: For any $X \in \mathcal{X}$ and $c \in \mathbb{R}$,

$$\rho(F_{X+c}) = \rho(F_X) - c.$$

Sublinearity: For any $X_1, X_2 \in \mathcal{X}$ and $\lambda_1, \lambda_2 \geq 0$,

$$\rho(F_{\lambda_1 X_1 + \lambda_2 X_2}) \leq \lambda_1 \rho(F_{X_1}) + \lambda_2 \rho(F_{X_2}).$$

Here F_Z stands for the distribution function of the random variable Z. The defining properties of the coherent risk measures correspond to the well-known interpretations of them as representing risk attitudes of financial investors (cf. Föllmer and Schied (2004), Chapter 4). Let $(X_i)_{i \in \mathbb{N}}$ be an independent sequence of real random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F and related left-continuous quantile function q_F. Additionally, define, $q_F(0) := q_F(0+)$ as well as $q_F(1) := q_F(1-)$. Denote by F_n the empirical distribution function based on the sample $(X_1, \ldots, X_n)$ and set $\rho_n(F) := \rho(F_n)$. The main goal of this paper is to study the asymptotic properties of the process $(\sqrt{n}(\rho_n(F) - \rho(F)))_{n \in \mathbb{N}}$. As an important tool let us consider the following mapping

$$\psi_\rho : [0, 1] \to [0, 1], \ t \mapsto \rho(F_{-B(1,t)}),$$

where $B(1, t)$ stands for Bernoulli r.v. with expectation t. This mapping is a *distortion function*, i.e. it is nondecreasing with $\psi_\rho(0) = 0$ and $\psi_\rho(1) = 1$, suggesting the name *associated distortion function*.

3   Main results

In order to prove CLT for the process $(\sqrt{n}(\rho_n(F) - \rho(F)))_{n\in\mathbb{N}}$, we need the following two assumptions.

(AC) $\mathcal{X}$ is a *Stonean vector lattice*, i.e. here $X \wedge Y, X \vee Y \in \mathcal{X}$ for $X, Y \in \mathcal{X}$, and ρ satisfies

$$\lim_{k\to\infty} \rho(F_{-(X-k)^+}) = 0 \text{ for nonnegative } X \in \mathcal{X},$$
$$\lim_{t\to 0+} \psi_\rho(t) = 0.$$

(AI) The stationary distribution function F of the sequence $(X_i)_{i\in\mathbb{N}}$ fulfills the following integrability condition

$$\int_\mathbb{R} F(x)^{-1/2}(1 - F(x))^{1/2} \; \psi_\rho(\lambda F(x)) \, dx < \infty$$

for some $\lambda \in]0, 1/2[$.

The main result of our study is the following theorem giving the asymptotic distribution of the process $(\sqrt{n}(\rho_n(F) - \rho(F)))_{n\in\mathbb{N}}$.

Theorem 3.1. *Let F have a finite set $D(F)$ of discontinuity points such that the restriction of F to $]q_F(0), q_F(1)[\backslash D(F)$ is continuously differentiable with strictly positive derivative.*
Then under the assumptions (AC) and (AI) we may find a set $S(\rho(F))$ of continuous, concave distortion functions which is compact w.r.t. the uniform metric, and there exists some centered Gaussian process $(G_\psi)_{\psi\in S(\rho(F))}$ with continuous paths and

$$\mathbb{E}\big[G(\psi_1)G(\psi_2)\big] = \int_{\mathbb{R}^2} \psi_1'(F(x))\psi_2'(F(y))\left[F(x \wedge y) - F(x)F(y)\right] \; dx \; dy$$

for any $\psi_1, \psi_2 \in \Psi$ such that $\left(\sqrt{n}[\rho_n(F) - \rho(F)]\right)_{n\in\mathbb{N}}$ converges in law to $\max_{\psi\in S(\rho(F))} G_\psi$. Here ψ' denotes the right-sided derivative of ψ. Moreover, if

$$\mathbb{E}[G(\psi_1) - G(\psi_2)]^2 \neq 0$$

for any two different $\psi_1, \psi_2 \in S(\rho(F))$, then $\sup_{\psi\in S(\rho(F))} G(\psi) = G(Z)$ for some Borel-random element Z of $S(\rho(F))$.

The proof of Theorem 3.1 is postponed to Section 7.

Remark 3.2. As it will become clear from the proof of Theorem 3.1, $S(\rho(F))$ consists of continuous concave distortion functions ψ satisfying

$$\rho(F) = \int_{-\infty}^0 \psi(F(x)) \; dx - \int_0^\infty [1 - \psi(F(x))] \; dx.$$

In particular, $\psi \leq \psi_\rho$ for any $\psi \in S(\rho(F))$.

Remark 3.3. The condition (AC) is always fulfilled if there is some topologically complete semi-norm $\|\cdot\|$ on the Stonean vector lattice $\mathcal{X}$ such that the following properties are satisfied

(1) $$\|X\| \leq \|Y\| \text{ for } |X| \leq |Y| \ \mathbb{P}-\text{a.s.},$$

(2) $$\lim_{k\to\infty} \|X_k\| = 0 \text{ whenever } X_k \nearrow 0 \ \mathbb{P}-\text{a.s..}$$

(cf. Ruszczynski and Shapiro (2006)). General classes of random variables meeting these requirements are given by

$$\mathcal{L}^g(\Omega, \mathcal{F}, \mathbb{P}) := \{Y \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \mathbb{E}\big[g(|Y|/c)\big] < \infty \text{ for some } c > 0\},$$

and

$$\mathcal{M}^g(\Omega, \mathcal{F}, \mathbb{P}) := \{Y \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \mathbb{E}\big[g(|Y|/c)\big] < \infty \text{ for all } c > 0\},$$

where g denotes any continuous *Young function*, i.e. a continuous, nondecreasing, unbounded, convex function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $g(0) = 0$. Both classes may be equipped with the respective *Luxemburg* seminorm $\|\cdot\|_g$ defined by

$$\|Y\|_g := \inf \ \big\{c > 0 \mid \mathbb{E}\big[g(|Y|/c)\big] \leq 1\big\},$$

being complete, and satisfying the conditions (1), (2) (cf. Krätschmer and Zähle (2010)).

Let us turn now to some examples.

Example 3.4. An important class of law-invariant coherent risk measures consists of the so-called *concave distortion risk measures*. To recall, the concave distortion risk measure $\rho =: \rho_\psi$ w.r.t. a concave distortion function ψ is defined by

(3) $$\rho_\psi(F_X) = \int\limits_{-\infty}^{0} \psi(F_X(x)) \ dx - \int\limits_{0}^{\infty} [1 - \psi(F_X(x))] \ dx$$

(cf. e.g. Denneberg (1994) or Föllmer and Schied (2004)). Notice that $\psi_{\rho_\psi} = \psi$ holds. The risk measure may be viewed as a Choquet integral w.r.t. the set function $\psi\big(\mathbb{P}(\cdot)\big)$ (cf. Denneberg (1994)), and $\mathbb{F}_\mathcal{X}$ consists of all distribution functions on $\mathbb{R}$ such that each integral in the representation (3) is finite. The set $\mathcal{X}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ whose distribution functions belong to $\mathbb{F}_\mathcal{X}$ is indeed a linear space satisfying $X \wedge Y, X \vee Y \in \mathcal{X}$ for $X, Y \in \mathcal{X}$ (cf. Denneberg (1994), Proposition 9.5 with Proposition 9.3). If, in addition, ψ is continuous, then

$$\|X\|_\psi := \int_0^{\infty} \psi(1 - F_{|X|}(x)) \ dx$$

defines a topologically complete semi-norm on $\mathcal{X}$ satisfying conditions (1) and (2) (cf. Denneberg (1994), Theorems 9.5, 8.9).
The choice $\psi(u) = \frac{1}{\alpha} (u \wedge \alpha)$ with $\alpha \in (0, 1]$ leads to

$$\rho_\psi(F_X) := \int 1_{(0,\alpha]}(\beta)q_X(\beta) \ d\beta = AV@R_\alpha(X),$$

where q_X denote any quantile function of the distribution function F_X of X. It is known as the *average value at risk at level* α, and it is well-defined for $\mathcal{X} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

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If ψ_ρ is continuous, and if F is as in Theorem 3.1, then the application of Theorem 3.1 along with Remark 3.2 yields that under condition (AI), the sequence $(\sqrt{n}[\rho_n(F) - \rho(F)])_{n\in\mathbb{N}}$ converges in law to a centered normally distributed random variable with variance σ^2 satisfying

$$\sigma^2 = \int_{\mathbb{R}^2} \psi'_\rho(F(x))\psi'_\rho(F(y)) \left[F(x \wedge y) - F(x)F(y)\right] \ dx \ dy.$$

Example 3.5. Setting

$$\rho(X) = -\mathbb{E}[X] + a\|(X - \mathbb{E}[X])^-\|_p, \quad a \in [0,1], \quad p \in [1, \infty[,$$

for all $X \in L^p(\Omega, \mathcal{F}, \mathrm{P})$ we arrive at the so called *one-sided moment* coherent risk measure (see Fischer (2003)). The associated distortion function ψ_ρ satisfies $\psi_\rho(t) = t + a(1-t)t^{1/p}$. Hence the assumption (AI) reads as follows

$$\int_{\mathbb{R}} [F(x)(1 - F(x))]^{1/2} \left[1 + a(1 - \lambda F(x))(\lambda F(x))^{1/p-1}\right] dx < \infty \text{ for some } \lambda \in]0, 1/2[,$$

which is always fulfilled in the case of

$$\int_{\mathbb{R}} [F(x)(1 - F(x))]^{1/2} \ F(x)^{1/p-1} dx < \infty.$$

Example 3.6. Let g be a strictly increasing continuous Young function satisfying $g(1) = 1$, and let $\mathcal{X}$ be the space $\mathcal{M}^g(\Omega, \mathcal{F}, \mathbb{P})$ associated with g as in Remark 3.3. Moreover fix $\alpha \in]0, 1[$. It was shown in Goovaerts et al. (2004) that for every $X \in \mathcal{M}^g(\Omega, \mathcal{F}, \mathbb{P})$ and every $x \in \mathbb{R}$ with $1 - F_X(x) > 0$ there exists a unique real number $\pi^g_\alpha(X, x) > x$ such that

$$\mathbb{E}\left[g\left(\frac{(X - x)^+}{\pi^g_\alpha(X, x) - x}\right)\right] = 1 - \alpha.$$

Therefore we may define a functional $\rho^{H,g}_\alpha$ on the set $\mathbb{F}^g$ of all distribution functions F_X of random variables X from $\mathcal{M}^g(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\rho^{H,g}_\alpha(F_X) := \inf\left\{\pi^g_\alpha(-X, x) \ : \ x \in \mathbb{R} \text{ with } 1 - F_{-X}(x) > 0\right\}.$$

Indeed, $\rho^{H,g}_\alpha$ is a law-invariant coherent risk measure (cf. Bellini and Rosazza Gianin (2008) with Krätschmer and Zähle (2010)) which satisfies condition (AC) in view of Remark 3.3. Moreover, it is easy to check that we have for $t \in]0, 1[$

$$\psi_{\rho^{H,g}_\alpha}(t) \leq 1 \wedge \left(t + \frac{1 - t}{g^{-1}((1-\alpha)/t))}\right) =: \hat{\psi}_{\rho^{H,g}_\alpha}(t),$$

where g^{-1} denotes the inverse of g (recall that we assumed the Young function g to be strictly increasing). Hence we may replace $\psi_{\rho^{H,g}_\alpha}$ with $\hat{\psi}_{\rho^{H,g}_\alpha}$ when verifying condition (AI).

Recently, Müller has pointed out that *expectiles*, genuinely introduced in Newey and Powell (1987), may be viewed as law-invariant coherent risk measures (cf. Müller (2010)).

Example 3.7. The *expectiles* based risk measure w.r.t. to any fixed $\alpha \in [1/2, 1[$ is defined by

$$\rho(F_X) = \underset{x \in \mathbb{R}}{\operatorname{argmin}} \left[(1-\alpha) \| ((-X) - x)^- \|_2^2 + \alpha \| ((-X) - x)^+ \|_2^2 \right],$$

for all $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathrm{P})$. The associated distortion function ψ_ρ satisfies

$$\psi_\rho(t) = \frac{\alpha t}{1 - \alpha + t(2\alpha - 1)}.$$

In particular condition (AI) is equivalent with

$$\int_{\mathbb{R}} \frac{\sqrt{F(x)(1 - F(x))}}{1 - \alpha + \lambda F(x)(2\alpha - 1)} \, dx < \infty \text{ for some } \lambda \in]0, 1/2[.$$

Discussion Pflug and Wozabal (2010) studied CLT for distortion risk measures discussed in Example 3.4. Motivated by earlier results on limit theorems for L statistics they implicitly assumed that $\sup_{t \in]0,1[} \psi_\rho(t)/t^\beta < \infty$ for some $\beta \in]0, 1/2]$ and

(4)
$$|q_F(t)| \le C[t(1-t)]^{-d}, \quad t \in]0, 1[,$$

for some $d \in]-\infty, \beta - 1/2[$. First, note that as opposite to (4), our assumption (AI) concerns only the left tail of the distribution F. Furthermore, the next example shows that the tail condition (4) is substantially more restrictive than condition (AI). Define via $\psi(t) := \sqrt{t}[1 + \ln(100)]/[1 + \ln(100) - \ln(t)]$ a concave distortion function which induces a concave distortion risk measure say ρ_ψ as in Example 3.4. It is obvious that in this case the tail condition (4) is satisfied for distributions with lower-bounded support only, in contrast to condition (AI). Indeed for ρ_ψ the condition (AI) reads as follows

$$\int_{q_F(0)}^{q_F(1)} \frac{\sqrt{1 - F(x)}}{1 + \ln(100) - \ln(F(x))} \, dx < \infty.$$

Invoking the well-known expansions for the Gaussian error function, it may be seen that the above condition is satisfied for any normal distribution F.

4 Extension to dependent data

In this section we carry over the results of the previous section to the case of dependent observations $X_1, \ldots, X_n$. First, let us impose the following mixing assumption.

(AM) The sequence $(X_i)_{i \in \mathbb{N}}$ is strictly stationary and strongly mixing with the mixing coefficients $\alpha(i)$ satisfying

$$\alpha(i) \le \bar{\alpha}_0 \exp(-\bar{\alpha}_1 i), \quad i \in \mathbb{N},$$

for some constants $\bar{\alpha}_0 > 0$ and $\bar{\alpha}_1 > 0$.

Remark 4.1. As an example of stationary sequences fulfilling the mixing condition (AM) we may take ARMA processes with continuously distributed innovations (cf. Mokkadem (1988)) or GARCH processes with continuously distributed innovations and Lebesgue density being positive in a neighbourhood of zero (cf. Lindner (2008)). For further examples and general conditions see Masuda (2007).

In order to extend Theorem 3.1 to dependent data we also have to modify condition (AI) and replace it by the following one.

(AI') The common distribution function F of the sequence $(X_i)_{i \in \mathbb{N}}$ fulfills the following integrability condition:

$$\int_{q_F(0)}^{q_F(1)} F(x)^{-1/2 - 2\delta} (1 - F(x))^{1/2 - \delta} \psi_\rho\big(\lambda F(x)^{1+\delta}\big) \, dx < \infty.$$

for some $\delta, \lambda \in]0, 1/2[$.

We are now ready to formulate the main result of this section concerning the asymptotic distribution of $\sqrt{n}(\rho_n(F) - \rho(F))$.

Theorem 4.2. *Let F have a finite set $D(F)$ of discontinuity points such that the restriction of F to $]q_F(0), q_F(1)[\backslash D(F)$ is continuously differentiable with strictly positive derivative. Then under assumptions (AC), (AI') and (AM), we may find a set $S(\rho(F))$ of continuous, concave distortion functions which is compact w.r.t. the uniform metric, and there exists some centered Gaussian process $(G_\psi)_{\psi \in S(\rho(F))}$ with continuous paths and*

$$
\begin{aligned}
\mathbb{E}\big[G(\psi_1) G(\psi_2)\big] \;=\; & \int_{\mathbb{R}^2} \psi_1'(F(x)) \psi_2'(F(y)) \left[F(x \wedge y) - F(x) F(y) \right. \\
& \left. + 2 \sum_{k=1}^\infty \big(\mathbb{P}(X_1 \le x, X_k \le y) - F(x) F(y) \big) \right] \, dx \, dy
\end{aligned}
$$

for any $\psi_1, \psi_2 \in \Psi$ such that $\big(\sqrt{n}[\rho_n(F) - \rho(F)]\big)_{n \in \mathbb{N}}$ converges in law to $\max_{\psi \in S(\rho(F))} G(\psi)$. Moreover, if

$$\mathbb{E}[G(\psi_1) - G(\psi_2)]^2 \ne 0$$

for any two different $\psi_1, \psi_2 \in S(\rho(F))$, then $\sup_{\psi \in S(\rho(F))} G(\psi) = G(Z)$ for some Borel-random element Z of $S(\rho(F))$.

The proof of Theorem 4.2 may be found in section 7.

5 Auxiliary results

In this section we formulate some auxiliary results needed to prove Theorems 3.1 and 4.2.

Proposition 5.1. *Under condition (AC) there exists a set Ψ of continuous concave distortion functions which is compact w.r.t. the uniform metric on $[0,1]^{[0,1]}$ such that*

$$\rho = \sup_{\psi \in \Psi} \rho_\psi.$$

The proof is delegated to Appendix B.

According to Proposition 5.1 we may restrict considerations to the risk measure ρ admitting representation $\rho = \sup_{\psi \in \Psi} \rho_\psi$ for some set Ψ of continuous concave distortion functions which is compact w.r.t. the uniform metric on $[0,1]^{[0,1]}$. Then we may write $\sqrt{n}[\rho_n(F) - \rho(F)] = \sqrt{n}[\sup_{\psi \in \Psi} \rho_\psi(F_n) - \sup_{\psi \in \Psi} \rho_\psi(F)]$. Let us now consider the auxiliary stochastic processes $(\mathcal{D}_n(\psi))_{\psi \in \Psi}$ $(n \in \mathbb{N})$, where

$$\mathcal{D}_n(\psi) := \sqrt{n}[\rho_\psi(F_n) - \rho_\psi(F)] = \sqrt{n} \int_{\mathbb{R}} [\psi(F_n(x)) - \psi(F(x))]\, dx, \quad \psi \in \Psi.$$

They have paths in the space $l^\infty(\Psi)$ defined to consist of all bounded, real-valued mappings on Ψ. Endowing $l^\infty(\Psi)$ with the uniform topology, we shall show next that the mapping $\mathcal{D}_n : \Psi \mapsto \mathbb{R}^\Omega$ can be viewed as a Borel random element of $l^\infty(\Psi)$. The idea behind is to reduce the proof of the Theorems 3.1, 4.2 to a convergence in law of the sequence of $(\mathcal{D}_n)_n$ in $l^\infty(\Psi)$. This would allow to apply the functional delta method for sup functionals to obtain the desired convergence results for $\left(\sqrt{n}[\rho_n(F) - \rho(F)]\right)_n$ (see Römisch (2006)).

Firstly, we have

(5) $$\qquad\qquad\qquad |\psi(t) - \psi(s)| \leq \psi_\rho(|t - s|) \text{ for } t, s \in [0,1]$$

(cf. Krätschmer and Zähle (2010)). Moreover, observe that concavity of each $\psi \in \Psi$ implies that

(6) $$|\psi(t) - \psi(s)| = \left| \int_s^t \psi'(u)\, du \right| \leq |\psi'(s)||s - t| \leq |s - t| \frac{\psi(s) - \psi(\gamma s)}{(1 - \gamma)s} \leq |s - t| \frac{\psi_\rho((1 - \gamma)s)}{(1 - \gamma)s}$$

holds for $s, \gamma \in\,]0,1[$ and $t \in [s, 1]$, where henceforth ψ' denotes the right-sided derivative of ψ. The following technical auxiliary result will turn out to be useful later on.

Lemma 5.2. *If either (AI) or (AI') is satisfied, then the set $\{\psi(F)1_{]-\infty,0]} - [1 - \psi(F)]1_{]0,\infty[} \mid \psi \in \Psi\}$ is dominated by a mapping which is integrable w.r.t. the ordinary Lebesgue-Borel measure on $\mathbb{R}$.*

Proof:
We shall restrict ourselves to show the statement of Lemma 5.2 under condition (AI'), the respective proof under condition (AI) follows the same line of reasoning.
Let $\delta, \lambda \in\,]0, 1/2]$ as in (AI'). By concavity of ψ we have

$$\begin{aligned} F(x)^\delta \psi(F(x)) &\leq (1/\lambda)\, \psi\left(\lambda F(x)^{1+\delta}\right) \\ &\leq (1/\lambda)\, F(x)^{-1/2-\delta}(1 - F(x))^{1/2-\delta} \psi\left(\lambda F(x)^{1+\delta}\right) F(x)^{1/2+\delta}(1 - F(x))^{\delta-1/2} \end{aligned}$$

Hence in view of (5) we obtain for $x < q_F(1/2)$

(7) $$\qquad\qquad \psi(F(x)) \leq (2/\lambda)\, F(x)^{-1/2-2\delta}(1 - F(x))^{1/2-\delta} \psi_\rho\left(\lambda F(x)^{1+\delta}\right).$$

Furthermore by (6) and concavity of ψ

$$1 - \psi(F(x)) = \int_{F(x)}^1 \psi'(u)\, du \leq (1 - F(x))\psi'(F(x)) \leq (1 - F(x))\psi'\left(\lambda F(x)^{1+\delta}\right)$$

$$\overset{(6)}{\leq} 2 \frac{\psi_\rho\left(\lambda F(x)^{1+\delta}/2\right)}{\lambda F(x)^{1+\delta}} (1 - F(x)) \leq 2 \frac{\psi_\rho\left(\lambda F(x)^{1+\delta}\right)}{\lambda F(x)^{1+\delta}} (1 - F(x))$$

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for $F(x) > 0$. This implies for $x > q_F(1/2)$

$$(8) \qquad 1 - \psi(F(x)) \le (2/\lambda) \; F(x)^{-1/2-2\delta}(1 - F(x))^{1/2-\delta}\psi_\rho\big(\lambda F(x)^{1+\delta}\big).$$

Since $\{\psi(F)1_{]-\infty,0]} - [1 - \psi(F)]1_{]0,\infty[} \mid \psi \in \Psi\}$ is uniformly bounded, we may conclude the statement of Lemma 5.2 from (7), (8) and condition (AI'). $\qquad \square$

As a first consequence of Lemma 5.2 we may show that within our setting the paths of the processes $\big(\mathcal{D}_n(\psi)\big)_{\psi \in \Psi}$ are uniformly continuous.

Lemma 5.3. *If either (AI) or (AI') is satisfied, then each process $\big(\mathcal{V}_n(\psi)\big)_{\psi \in \Psi}$, $\big(\mathcal{V}(\psi)\big)_{\psi \in \Psi}$, $\big(\mathcal{D}_n(\psi)\big)_{\psi \in \Psi}$ has uniformly continuous paths w.r.t. the uniform metric. Here we use notation*

$$\mathcal{V}_n(\psi) := \sqrt{n} \; \Big(\int_{-\infty}^{0} \psi(F_n(x)) \; dx + \int_{0}^{\infty} [1 - \psi(F_n(x))] \; dx \Big)$$

$$\mathcal{V}(\psi) := \sqrt{n} \; \Big(\int_{-\infty}^{0} \psi(F(x)) \; dx + \int_{0}^{\infty} [1 - \psi(F(x))] \; dx \Big).$$

Proof:
Since Ψ is compact, the paths of any process $\big(\mathcal{V}_n(\psi)\big)_{\psi \in \Psi}$, $\big(\mathcal{V}(\psi)\big)_{\psi \in \Psi}$, $\big(\mathcal{D}_n(\psi)\big)_{\psi \in \Psi}$ are uniformly continuous if and only if they are continuous. So it suffices to show the continuity of the paths. Let $(\psi_k)_k$ denote any sequence in Ψ which converges to some $\psi \in \Psi$ w.r.t. the uniform metric. Denoting the sample minimum and maximum of $(X_1, ..., X_n)$ by $X_{n:1}$ and $X_{n:n}$ respectively, we may observe

$$\big| \; \psi_k(F_n)1_{]-\infty,0]} - [1 - \psi_k(F_n)]1_{]0,\infty[} \; \big| \; \le \; 1_{[X_{n:1} \wedge 0, X_{n:n} \vee 0]}.$$

Hence in view of Lemma 5.2, $\{\psi_k(F_n) - \psi_k(F) \mid k \in \mathbb{N}\}$ is $\mathbb{P}$−a.s. dominated by mappings which are integrable w.r.t. the ordinary Lebesgue-Borel measure λ^1 on $\mathbb{R}$. This shows continuity of the paths of $\big(\mathcal{D}_n(\psi)\big)_{\psi \in \Psi}$ due to the dominated convergence theorem. The proofs of the continuity of the paths of $\big(\mathcal{V}_n(\psi)\big)_{\psi \in \Psi}$ and $\big(\mathcal{V}(\psi)\big)_{\psi \in \Psi}$ follow the same line of reasoning. $\qquad \square$

The uniform metric on Ψ is separable due to compactness, so by Lemma 5.3 the mappings $\mathcal{D}_n$ are Borel random elements of $UCB(\Psi)$, the space of bounded real-valued mappings on Ψ which are uniformly continuous w.r.t. the supremum metric, where $UCB(\Psi)$ is equipped with the supremum norm $\| \cdot \|_\infty$. Hence, the map $\mathcal{D}_n : \Psi \mapsto \mathbb{R}$ can be viewed as a Borel random element of $l^\infty(\Psi)$.
We shall show the following result concerning the convergence of $(\mathcal{D}_n)_n$.

Theorem 5.4. *Let the assumptions of either Theorem 3.1 or Theorem 4.2 be fulfilled. Then there exists a tight centered Gaussian Borel random element G of $UCB(\Psi)$ with*

$$\mathbb{E}\big[G(\psi_1)G(\psi_2)\big] \;\; = \;\; \int_{\mathbb{R}^2} \psi_1'(F(x))\psi_2'(F(y)) \, \Big[F(x \wedge y) - F(x)F(y)$$

$$+ 2\sum_{k=1}^{\infty} \big(\mathbb{P}(X_1 \le x, X_k \le y) - F(x)F(y)\big)\Big] \; dx \; dy$$

for any $\psi_1, \psi_2 \in \Psi$ such that $\big(\mathcal{D}_n(\psi)\big)_{\psi \in \Psi}$ converges in law to G.

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For the proof of Theorem 5.4 we shall verify the following two results whose formulations need some preparation. By assumption on F we may find $q_F(0) =: a_0 < a_1 < ... < a_{r+1} := q_F(1)$ such that $F_{]|a_{i-1}, a_i[}$ is continuously differentiable with derivative $f_i > 0$. Let us select any strictly decreasing sequence $(t_k)_{k \in \mathbb{N}}$ in $]0, F(a_{1-})[$ which converges to $\inf\{F(x) \mid F(x) > 0\}$.

For any k we may find a vector $(\alpha_{ko}, ..., \alpha_{kr}, \beta_{k0}, ..., \beta_{kr})$ satisfying

(9) $t_k = \alpha_{k0} < \beta_{k0} < F(a_{1-})$ with $F(a_{1-}) - \beta_{k0} < \dfrac{1}{k}$,

(10) $F(a_i) < \alpha_{ki} < \beta_{ki} < F(a_{(i+1)-})$ with $\max\{\alpha_{ki} - F(a_i), F(a_{(i+1)-}) - \beta_{ki}\} < \dfrac{1}{k}$

for $i \in \{1, ..., r\}$.

Setting $I_k := \bigcup\limits_{i=0}^{r}]\alpha_{ki}, \beta_{ki}[$, we consider the mapping

$$\mathcal{D}_{nk} : \Psi \to \mathbb{R}^\Omega, \ \psi \mapsto \sqrt{n} \int_{\mathbb{R}} [\Pi_k(\psi)(F_n(x)) - \Pi_k(\psi)(F(x))] \ dx,$$

where $\Pi_k(\psi) : [0, 1] \to [0, 1]$ is defined via $\Pi_k(\psi)(t) := \int_0^t 1_{I_k}(u)\psi'(u) \ du$.

The mapping $\mathcal{D}_{nk}$ may be viewed as a Borel random element of $UCB(\Psi)$, following an argumentation analogously to that used for the mapping $\mathcal{D}_n$. We are now ready to formulate the auxiliary results which will be used to prove Theorems 3.1, 4.2.

Proposition 5.5. *Let the assumptions of either Theorem 3.1 or Theorem 4.2 be fulfilled. Then* $\sup\limits_{\psi \in \Psi} |\mathcal{D}_n(\psi) - \mathcal{D}_{nk}|$ *is a real-valued random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$ *for arbitrary* $n, k \in \mathbb{N}$, *and*

$$\lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{P}\big(\{\sup_{\psi \in \Psi} |\mathcal{D}_n(\psi) - \mathcal{D}_{nk}(\psi)| > \varepsilon\}\big) = 0$$

holds for arbitrary $\varepsilon > 0$.

Proposition 5.6. *Let the assumptions of either Theorem 3.1 or Theorem 4.2 be fulfilled, and let* $l^\infty(\mathbb{R})$ *denote the set of bounded real-valued mappings on* $\mathbb{R}$ *which is equipped with the uniform metric. Then there exists some tight centered Gaussian Borel random element* B_F *of* $l^\infty(\mathbb{R})$ *satisfying*

$$\mathbb{E}\big[B_F(x)B_F(y)\big] \quad = \quad F(x \wedge y) - F(x)F(y) + 2\sum_{k=1}^\infty \big(\mathbb{P}(X_1 \le x, X_k \le y) - F(x)F(y)\big)$$

for $x, y \in \mathbb{R}$ *such that for any* $k \in \mathbb{N}$, *the sequence* $\big(\mathcal{D}_{nk}\big)_n$ *converges in law to the centered Gaussian Borel random element* G_k *of* $UCB(\Psi)$ *defined by*

$$\mathbb{E}\big[G(\psi_1)G(\psi_2)\big] \quad = \quad \int_{\mathbb{R}^2} I_k(F(x))\psi_1'(F(x))I_k(F(y))\psi_2'(F(y)) \left[F(x \wedge y) - F(x)F(y)\right.$$
$$\left. +2\sum_{k=1}^\infty \big(\mathbb{P}(X_1 \le x, X_k \le y) - F(x)F(y)\big)\right] \ dx \ dy$$

for every $\psi_1, \psi_2 \in \Psi$.

Propositions 5.5, 5.6 and Theorem 5.4 will be shown sequentially in the following section 6.

6 Proofs of Propositions 5.5, 5.6 and Theorem 5.4

Let us retake assumptions and notations from section 5. We want to carry out the announced proofs by considering the assumptions of Theorems 3.1, 4.2 simultaneously. For that purpose we shall replace respectively (AI) and (AI') with the following condition.

(AI!) The distribution function F fulfills

$$\int_{q_F(0)}^{q_F(1)} F(x)^{-1/2-2\delta}(1 - F(x))^{1/2-\delta}\psi_\rho\big(\lambda F(x)^{1+\delta}\big)\, dx < \infty.$$

for some $\lambda \in\,]0, 1/2[,\ \delta \in [0, 1/2[$.

For $\delta = 0$ condition (AI!) reduces to (AI), whereas we have (AI') if $\delta > 0$.

The assumptions of independent $(X_i)_{i\in\mathbb{N}}$ or strictly stationary $(X_i)_{i\in\mathbb{N}}$ with mixing coefficients $(\alpha(i))_{i\in\mathbb{N}}$ satisfying condition (AM) may be described simultaneously by the following condition.

(AM!) The sequence $(X_i)_{i\in\mathbb{N}}$ is strictly stationary and strongly mixing with the mixing coefficients $\alpha(i)$ satisfying

$$\alpha(i) \le \bar{\alpha}_0 \exp(-\bar{\alpha}_1 i), \quad i \in \mathbb{N},$$

for some constants $\bar{\alpha}_0 \ge 0$ and $\bar{\alpha}_1 > 0$.

In the case of independent $(X_i)_{i\in\mathbb{N}}$ we may choose $\bar{\alpha}_0 = 0$.

As a starting point we may conclude from (AM!) that there is a centered Gaussian process $B_F := (B_F(x))_{x\in\mathbb{R}}$ satisfying

$$\mathrm{Cov}(B_F(x), B_F(y)) = \mathbb{E}\left[B_F(x)B_F(y)\right]$$

$$(11) \qquad = F(x \wedge y) - F(x)F(y) + 2\sum_{k=1}^{\infty}\left[\mathbb{P}(X_1 \le x, X_{k+1} \le y) - F(x)F(y)\right],$$

and which is a tight Borel random element of the space $D(\mathbb{R})$ of all cadlag functions on $\mathbb{R}$ w.r.t the sup norm such that the sequence $\left(\left(\sqrt{n}[F_n(x) - F(x)]\right)_{x\in\mathbb{R}}\right)_n$, viewed as a sequence of Borel random elements of $D(\mathbb{R})$, converges in law to B_F (see e.g. Ben Hariz (2005), Corollary 1). Moreover, the induced stochastic process $\left(B_F(x)\right)_{x\in\mathbb{R}}$ has paths which are continuous at every continuity point of F (Corollary 1 in Ben Hariz (2005) again).

Let $q_F(0) =: a_0 < a_1 < ... < a_{r+1} =: q_F(1)$ be as in the discussion preceding Proposition 5.5. Possibly changing to a suitable probability space we may assume without loss of generality that there is a set $\{Z_{ij} \mid i \in \mathbb{N}, j \in \{0, ..., r + 1\}\}$ of independent random variables all having the uniform distribution on $]0, 1[$ as common distribution such that $\{Z_{ij} \mid i \in \mathbb{N}, j \in \{0, ..., r + 1\}\}$ and $(X_i)_{i\in\mathbb{N}}$ are independent. This allows us to prove the following result on bounds for empirical distribution functions which will be crucial for our line of reasoning.

Lemma 6.1. *Let conditions (AI!), (AM!) be satisfied, and let $\lambda \in\,]0, 1/2[$ as well as $\delta \in [0, 1/2[$ be as in (AI!). The sample minimum of $(X_1, ... X_n)$ will be denoted by $X_{n:1}$. Then for any $\eta \in\,]0, 1[$, we may find a constant $\gamma_\eta \in\,]0, \lambda[$, and a sequence $(A_{n\eta})_{n\in\mathbb{N}}$ in $\mathcal{F}$ with $\mathbb{P}(A_{n\eta}) \ge 1 - \eta$ such that*

$$\gamma_\eta 1_{[X_{n:1}, 1]}(x)F(x)^{1+\delta}1_{A_{n\eta}} \le 1_{[X_{n:1}, 1]}(x)F_n(x)1_{A_{n\eta}}$$

for any $x \in \mathbb{R}$.

Proof:

Let $\{Z_{ij} \mid i \in \mathbb{N}, j \in \{0, ..., r+1\}\}$ be as discussed above. Then we may invoke the randomized probability integral transformation U_i of each X_i, i.e.

$$U_i := F(X_i) - \sum_{j=0}^{r+1} 1_{\{a_j\}}(X_i)\mathbb{P}(\{X_i = a_j\})Z_{ij}.$$

In this way we obtain a strictly stationary sequence $(U_i)_{i \in \mathbb{N}}$ of random variables with the uniform distribution on $]0, 1[$ as common distribution and mixing coefficients $\alpha^U(i) \leq \bar{\alpha}_0 \exp(-\bar{\alpha}_1 i)$ with $\bar{\alpha}_0, \bar{\alpha}_1$ as in (AM!). Moreover, $X_i = q_F(U_i)$ a.s. so that $F_n(x) = 1/n \sum_{i=1}^{n} 1_{]-\infty, F(x)]}(U_i)$ a.s.. The statement of Lemma 6.1 is then a direct consequence of Inequality 12.11.2 in Shorack (2000) if $\bar{\alpha}_0 = 0$, and it may be concluded from Theorem 1.3 in Puri and Tran (1980) otherwise. $\qquad \square$

Let us now turn over to the proof of Proposition 5.5.

Proof of Proposition 5.5:

Let $\lambda \in]0, 1/2[, \delta \in [0, 1/2[$ be as in (AI!). Firstly, $\sup_{\psi \in \Psi} |\mathcal{D}_n(\psi) - \mathcal{D}_{nk}(\psi)|$ is a real-valued random variable as a continuous transformation of a Borel random element of $UCB(\Psi)$.

Since Ψ is compact w.r.t. the uniform metric it has some at most countable dense subset Ψ_0. Then we have $\sup_{\psi \in \Psi} |\mathcal{D}_n(\psi) - \mathcal{D}_{nk}(\psi)| = \sup_{\psi \in \Psi_0} |\mathcal{D}_n(\psi) - \mathcal{D}_{nk}(\psi)|$ because the paths of $\mathcal{D}_n$ and $\mathcal{D}_{nk}$ are continuous. In particular, for any fixed $\varepsilon \in]0, 1[$

$$B_{nk\varepsilon} := \{\sup_{\psi \in \Psi} |\mathcal{D}_n - \mathcal{D}_{nk}| > \varepsilon\} = \{\sup_{\psi \in \Psi_0} |\mathcal{D}_n - \mathcal{D}_{nk}| > \varepsilon\} \in \mathcal{F},$$

and

$$
\begin{aligned}
g_{nk}(x) &:= \sqrt{n} \sup_{\psi \in \Psi_0} |[\psi(F_n(x)) - \psi(F(x))] - [\Pi_k(\psi)(F_n(x)) - \Pi_k(\psi)(F(x))]| \\
&= \sqrt{n} \sup_{\psi \in \Psi_0} \left| \int_{F(x)}^{F_n(x)} 1_{]0,1[\setminus I_k}(t)\psi'(t) \, dt \right|
\end{aligned}
$$

is indeed a random variable. The important part of the proof is to show the following statement.

(*) For any $\eta \in]0, 1[$, there exist $C > 0$ and a sequence $(A_{n\eta})_{n \in \mathbb{N}}$ in $\mathcal{F}$ with $\mathbb{P}(A_{n\eta}) \geq 1 - \eta$ such that

$$\sqrt{\mathbb{E}\left[|1_{A_{n\eta}}g_{nk}(x)|^2\right]} \leq CF(x)^{-1/2-2\delta}(1-F(x))^{1/2-\delta}\psi_\rho\left(\lambda F(x)^{1+\delta}\right)$$

for any $x \in]q_F(0), q_F(1)[$ and every $n \in \mathbb{N}$.

Let us first see how we may conclude the statement of Proposition 5.5 from (*).

For arbitrary $\eta \in]0, 1[$ choose $C, (A_{n\eta})_{n \in \mathbb{N}}$ as in (*), and for $x \in]q_F(0), q_F(1)[$ use notation $h(x) := F(x)^{-1/2-2\delta}(1-F(x))^{1/2-\delta}\psi_\rho\left(\lambda F(x)^{1+\delta}\right)$. Then

$$(12) \qquad \int_{q_F(0)}^{q_F(1)} \mathbb{E}\left[1_{A_{n\eta}}1_{]0,1[\setminus I_k}(F(x))g_{nk}(x)\right] dx \leq C \int_{q_F(0)}^{q_F(1)} 1_{]0,1[\setminus I_k}(F(x))h(x) \, dx$$

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and

(13) $\qquad \left(\mathbb{E}\big[|1_{A_{n\eta}}g_{nk}(x)|\big]\right)^2 \leq \mathbb{E}\big[|1_{A_{n\eta}}g_{nk}(x)|^2\big] \leq C^2 h(x)^2$ for any $x \in]q_F(0), q_F(1)[$.

By continuous mapping theorem the convergence in law of $(\sqrt{n}[F_n - F])_{n\in\mathbb{N}}$ implies the convergence of law of $(\sup_{x\in\mathbb{R}} |\sqrt{n}[F_n(x) - F(x)]|)_{n\in\mathbb{N}}$. In particular the latter sequence is uniformly tight which implies that for any every $\beta \in]0, 1[$, there is some $A_\beta \in \mathcal{F}$ with $\mathbb{P}(A_\beta) \geq 1 - \beta$ such that $(1_{A_\beta} \sup_{x\in\mathbb{R}} |[F_n(x) - F(x)]|)_{n\in\mathbb{N}}$ converges uniformly to 0. Since any I_k is a finite union of open intervals of $\mathbb{R}$, it is then easy to verify that $\big(1_{I_k}(F(x))g_{nk}(x)\big)_{n\in\mathbb{N}}$ converges in probability to 0 for any $x \in]q_F(0), q_F(1)[$. Moreover, (13) means that $\big(1_{I_k}(F(x))g_{nk}(x)1_{A_{n\eta}}\big)_{n\in\mathbb{N}}$ is uniformly integrable for $x \in]q_F(0), q_F(1)[$, implying that $\big(1_{I_k}(F(x))g_{nk}(x)1_{A_{n\eta}}\big)_{n\in\mathbb{N}}$ converges in mean to 0 for $x \in]q_F(0), q_F(1)[$. Furthermore, by (13) and condition (AI!), we may apply the dominated convergence theorem yielding

$$\lim_{n\to\infty} \int_{q_F(0)}^{q_F(1)} \mathbb{E}\big[|1_{I_k}(F(x))g_{nk}(x)1_{A_{n\eta}}|\big]\ dx = 0 \text{ for } k \in \mathbb{N}, \eta \in]0, 1[.$$

Thus by (12) and Markov's inequality along with Tonelli's theorem

(14) $\qquad \lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(B_{nk\varepsilon} \cap A_{n\eta}) \leq \lim_{k\to\infty} 2/\varepsilon \int_{q_F(0)}^{q_F(1)} 1_{]0,1[\setminus I_k}(F(x))h(x)\ dx.$

Furthermore, $\lim_{k\to\infty} 1_{]0,1[\setminus I_k}(F(x)) = 0$ for every $x \in]q_F(0), q_F(1)[$. Then in view of condition (AI!) we may apply the dominated convergence theorem to conclude from (14)

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(B_{nk\varepsilon}) \leq \lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(B_{nk\varepsilon} \cap A_{n\eta}) + \eta = \eta.$$

So it remains to show (*).

proof of (*):

Let for $\eta \in]0, 1[$ choose $C_\eta > 0, \gamma_\eta \in]0, \lambda[$ and $(A_{n\eta})_{n\in\mathbb{N}}$ as in Lemma 6.1. First of all, since every $\psi \in \Psi$ is concave with $\psi(0) = 0$, we have $\psi\big(\lambda F(x)^{1+\delta}\big) \geq \lambda F(x)^\delta \psi(F(x))$. Hence

$$\sup_{\psi\in\Psi_0} \psi(F(x)) \leq \frac{1}{\lambda}\ F(x)^{-\delta} \sup_{\psi\in\Psi_0} \psi\big(\lambda F(x)^{1+\delta}\big) \leq \frac{2}{\gamma_\eta} F(x)\ \frac{\psi_\rho\big(\lambda F(x)^{1+\delta}\big)}{F(x)^{1+\delta}}$$

for $x \in]q_F(0), q_F(1)[$. Then we obtain for $\omega \in A_{n\eta}$ and $q_F(0) < x < X_{n:1}(\omega) := \min_{i\in\{1,\dots,n\}} X_i(\omega)$

(15) $\qquad g_{nk}(x)(\omega) \leq \sqrt{n} \sup_{\psi\in\Psi_0} \Psi(F(x)) \leq \frac{2}{\gamma_\eta} \sqrt{n}|F_n(x)(\omega) - F(x)|\ \frac{\psi_\rho\big(\lambda F(x)^{1+\delta}\big)}{F(x)^{1+\delta}}$

Since the right-side derivative of any $\psi \in \Psi$ is nonincreasing, we may conclude from Lemma 6.1 along with (6)

$$
\begin{aligned}
g_{nk}(x)(\omega) \leq \sqrt{n} \sup_{\psi\in\Psi_0} |\int_{F(x)}^{F_n(x)(\omega)} \psi'(t)\ dt\ | \quad &\leq\quad \sqrt{n}|F_n(x)(\omega) - F(x)| \sup_{\psi\in\Psi_0} \psi'\big(\gamma_\eta F(x)^{1+\delta}\big) \\[2mm]
&\leq\quad \sqrt{n}|F_n(x)(\omega) - F(x)|\ \frac{2\psi_\rho\big((\gamma_\eta/2)F(x)^{1+\delta}\big)}{\gamma_n F(x)^{1+\delta}} \\[2mm]
(16) \qquad &\leq\quad \frac{2}{\gamma_\eta} \sqrt{n}|F_n(x)(\omega) - F(x)|\ \frac{\psi_\rho\big(\lambda F(x)^{1+\delta}\big)}{F(x)^{1+\delta}}
\end{aligned}
$$

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for $\omega \in A_{n\eta}$ and $F(x) \geq X_{n:1}(\omega)$.

Finally, by Lemma C.1 (cf. appendix C), we may find a constant $\overline{C} > 0$ such that

$$\mathbb{E}\big[n[F_n(x) - F(x)]^2\big] \leq \overline{C}^2[F(x)(1 - F(x))]^{1-2\delta}$$

holds for any $x \in]q_F(0), q_F(1)[$. Setting $C := (2\overline{C})/\gamma_\eta$, then $(*)$ follows immediately from (15) and (16). The proof of Proposition 5.5 is complete. $\qquad\square$

Proof of Proposition 5.6:

Lemma A.1 (cf. appendix A) gives the following representation of $\mathcal{D}_{nk}$

$$(17) \qquad \mathcal{D}_{nk}(\psi) = -\sum_{i=0}^{r} \int_{\alpha_{ki}}^{\beta_{ki}} \sqrt{n}\ [q_{F_n}(t) - q_F(t)]\ \psi'(t)\ dt \text{ for } k \in \mathbb{N},\ \psi \in \Psi,$$

where q_{F_n} denotes the left-continuous quantile function of F_n. Representation (17) suggests to apply already known asymptotic results for the quantile processes $(\sqrt{n}[q_{F_n}(t) - q_F(t)])_{t \in]0,1[}$.

Firstly, we already have convergence in law of $(\sqrt{n}[F_n - F])_{n \in \mathbb{N}}$ to some tight centered Borel random element B_F of $D(\mathbb{R})$ with covariance function satisfying (11), and whose paths are continuous at every continuity point of F. Furthermore, by construction, we may find for any $k \in \mathbb{N}$ some positive constant $\varepsilon_k > 0$ such that $F\lvert]q_F(\alpha_{ki}) - \varepsilon_k, q_F(\beta_{ki}) + \varepsilon_k[$ is continuously differentiable with derivative $f_{ki} > 0$ for $i = 0, ..., r$.

Before proceeding, we need some notations. Setting $a_{ki} := q_F(\alpha_{ki}) - \varepsilon_k$ and $b_{ki} := q_F(\beta_{ki}) + \varepsilon_k$, we denote the real vector space of restrictions of members of $D(\mathbb{R})$ to $J_k := \bigcup_{i=0}^{r} [a_{ki}, b_{ki}]$ by $D(J_k)$, and we endow it with the sup norm. The subset $D_1(J_k) \subseteq D(J_k)$ is defined to consist of all restrictions of distribution functions on $\mathbb{R}$ to J_k. Finally, $l^\infty([\alpha_{k0}, \beta_{k0}]) \times ... \times l^\infty([\alpha_{kr}, \beta_{kr}])$ stands for the set of mappings $(g_0, ..., g_r) : [\alpha_{k0}, \beta_{k0}] \times ... \times [\alpha_{kr}, \beta_{kr}] \to \mathbb{R}^{r+1}$, whose components are bounded. It will be equipped with the metric d, defined by $d((g_0, ..., g_r), (h_0, ..., h_r)) := \sum_{i=0}^{r} \sup_{t \in [\alpha_{ki}, \beta_{ki}]} |g_i(t) - h_i(t)|$.

Next, we obtain from the continuous mapping theorem that $(\sqrt{n}[F_n - F]|J_k)_{n \in \mathbb{N}}$, as a sequence of Borel random elements of $D(J_k)$ converges in law to the tight centered Gaussian Borel random element $B_{Fk} := B_F|J_k$ which has continuous paths. Therefore, in view of Lemma 21.4 in van der Vaart (1998), we may apply the functional delta method (see Theorem 20.8 in van der Vaart (1998)) to the mapping

$$\Phi_k : D_1(J_k) \to l^\infty([\alpha_{k0}, \beta_{k0}]) \times ... \times l^\infty([\alpha_{kr}, \beta_{kr}]),\ G|J_k \mapsto (q_G|[\alpha_{k0}, \beta_{k0}], ..., q_G|[\alpha_{kr}, \beta_{kr}]),$$

where q_G denotes the left-continuous quantile function of G, to conclude that the sequence $\big(\sqrt{n}[\Phi_k(F_n|J_k) - \Phi_k(F|J_k)]\big)_{n \in \mathbb{N}}$ converges in law to the tight Borel random element

$$\left(-\frac{B_F}{f_{k0}} \circ q_F|[\alpha_{k0}, \beta_{k0}], ..., -\frac{B_F}{f_{kr}} \circ q_F|[\alpha_{kr}, \beta_{kr}] \right).$$

Then by (17), the application of the continuous mapping theorem yields that $(\mathcal{D}_{nk})_{n \in \mathbb{N}}$ converges in law to some tight Borel random element G_k of $l^\infty(\Psi)$, defined by

$$G_k(\psi) := \sum_{i=0}^{r} \int_{\alpha_{ki}}^{\beta_{ki}} \frac{B_F(q_F(t))}{f_{ki}(q_F(t))}\ \psi'(t)\ dt.$$

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Since by construction, $F|]a_{ki}, b_{ki}[$ is invertible for every $i \in \{0, ..., r\}$, we obtain by change of variable formula

$$G_k(\psi) = \int_{\mathbb{R}} B_F(x) \sum_{i=0}^{r} 1_{]q_F(\alpha_{ki}), q_F(\beta_{ki})[}(x) \psi'(F(x)) \, dx = \int_{\mathbb{R}} B_F(x) 1_{I_k}(F(x)) \psi'(F(x)) \, dx.$$

Moreover, the set of Borel probability measures on $UCB(\Psi)$ is a Polish space because $UCB(\Psi)$, equipped with the uniform metric, is a Polish space too. Since, each $\mathcal{D}_{nk}$ is a Borel random elements of $UCB(\Psi)$, the stochastic process $(G_k(\psi))_{\psi \in \Psi}$ has continuous paths a.s., and then G_k is as required. $\qquad\square$

Theorem 5.4 may be concluded from Propositions 5.5, 5.6 in the following way.

Proof of Theorem 5.4:
Let $l^{\infty}(\mathbb{R})$ be the space of bounded real-valued mappings on $\mathbb{R}$ which is equipped with the uniform metric. Furthermore let B_F be the Gaussian Borel random element of $l^{\infty}(\mathbb{R})$ from Proposition 5.6, inducing a sequence $(G_k)_{k \in \mathbb{N}}$ of Gaussian Borel random elements of $UCB(\Psi)$ as in Proposition 5.6.
Since the mappings $\mathcal{D}_n, \mathcal{D}_{nk}$ are Borel random elements of a separable metric space we may apply Theorem 4.2 in Billingsley (1968). Therefore, in view of Propositions 5.5, 5.6 it remains to show that the mapping $G(\psi) := \int_{q_F(0)}^{q_F(1)} \psi'(F(x)) B_F(x) \, dx$ defines a Borel random element G of $UCB(\Psi)$ such that $(G_k)_k$ converges in law to G.

Let $\lambda, \delta \in [0, 1/2[$ as in condition (AI!). Then by Lemma C.1 (cf. appendix C), there exists some constant $C \geq 0$ such that $\text{Var}(B_F(x)) \leq C^2[F(x)(1 - F(x))]^{1-2\delta}$ for every $x \in \mathbb{R}$. Then we may conclude from (6) along with (AI!)

$$
\begin{aligned}
\int_{q_F(0)}^{q_F(1)} \sqrt{\text{Var}\left(\psi'(F(x)) B_F(x)\right)} \, dx \quad &\leq \quad \int_{q_F(0)}^{q_F(1)} \sqrt{\text{Var}\left(\psi'\left(2\lambda F(x)^{1+\delta}\right) B_F(x)\right)} \, dx \\
&\overset{(6)}{\leq} \quad \frac{2}{\lambda} \int_{q_F(0)}^{q_F(1)} \sqrt{\text{Var}\left(\frac{\psi_{\rho}\left(\lambda F(x)^{1+\delta}\right)}{F(x)^{1+\delta}} B_F(x)\right)} \, dx \\
&\leq \quad \frac{2C}{\lambda} \int_{q_F(0)}^{q_F(1)} F(x)^{-1/2-2\delta}(1 - F(x))^{1/2-\delta} \psi_{\rho}\left(\lambda F(x)^{1+\delta}\right) \, dx \\
&\overset{(\text{AI!})}{<} \quad \infty.
\end{aligned}
$$

By Lemma 3.3 in Rajput (1972), this means that B_F has paths in V almost surely, where V denotes the set of all $g \in l^{\infty}(\mathbb{R})$ such that $g \psi_{\rho}(\lambda F^{1+\delta})/F^{1+\delta}$ is integrable w.r.t. the ordinary Lebesgue-Borel measure λ^1 on $\mathbb{R}$. By the same argument from Rajput (1972), $(G(\psi))_{\psi \in \Psi}$ is a well-defined centered Gaussian process. Moreover, $\lim_{k \to \infty} 1_{]0,1[\backslash I_k}(F(x)) = 0$ holds for every $x \in]q_F(0), q_F(1)[$. Then, an application of the dominated convergence theorem along with (6) yields

$$(18) \quad \lim_{k \to \infty} \sup_{\psi \in \Psi} |G(\psi) - G_k(\psi)| \leq \lim_{k \to \infty} \int_{q_F(0)}^{q_F(1)} 1_{]0,1[\backslash I_k}(F(x)) \mid B_F(x) \frac{\psi_{\rho}\left(\lambda F(x)^{1+\delta}\right)}{\lambda F(x)^{1+\delta}} \mid dx = 0 \text{ a.s.}.$$

Since every process $(G_k(\psi))_{\psi \in \Psi}$ has paths in $UCB(\Psi)$, (18) tells us that $(G(\psi))_{\psi \in \Psi}$ has paths in $UCB(\Psi)$ a.s.. So we may choose an indistinguishable version of $(G(\psi))_{\psi \in \Psi}$ as a centered

Gaussian Borel random element of $UCB(\Psi)$, denoted by G, which is in addition tight because the uniform topology on $UCB(\Psi)$ is separably and completely metrizable. Finally, (18) also implies that $(G_k)_{k \in \mathbb{N}}$ convergese in law to G. The proof is complete now. $\qquad \square$

7 Proof of the main results

Let us retake notions and notations from sections 3, 4.

First of all, assumption (AC) on the risk measure ρ allows us to apply Proposition 5.1. Therefore, $\rho = \sup_{\psi \in \Psi} \rho_\psi$ for some set Ψ of continuous concave distortion functions which is compact w.r.t. the uniform metric on $[0,1]^{[0,1]}$. The compactness of Ψ implies by an exercise of dominated convergence theorem along with Lemma 5.2

$$(19) \qquad S(\rho(F)) := \{\psi \in \Psi \mid \rho(F) = \rho_\psi(F)\} \neq \emptyset \text{ and compact w.r.t. uniform metric}$$

under (AI) or (AI').

Now, let $(\mathcal{D}_n)_{n \in \mathbb{N}}$ be the sequence of Borel random elements of $UCB(\Psi)$ defined as in section 5. Each of them may be decomposed in the following way

$$(20) \qquad \qquad D_n(\psi) = \sqrt{n}[\rho_\psi(F_n) - \rho_\psi(F)].$$

According to Theorem 5.4, if the assumptions of either Theorem 3.1 or Theorem 4.2 are satisfied, then there exists a tight centered Gaussian Borel random element G of $UCB(\Psi)$ with

$$
\begin{aligned}
\mathbb{E}\big[G(\psi_1)G(\psi_2)\big] \quad = \quad & \int_{\mathbb{R}^2} \psi_1'(F(x))\psi_2'(F(y)) \big[F(x \wedge y) - F(x)F(y) \\
& + 2\sum_{k=1}^{\infty} \big(\mathbb{P}(X_1 \leq x, X_k \leq y) - F(x)F(y) \big) \big] \ dx \ dy
\end{aligned}
$$

for any $\psi_1, \psi_2 \in \Psi$ such that $\big(\mathcal{D}_n(\psi)\big)_{\psi \in \Psi}$ converges in law to G. As a further consequence, (20) along with representation $\rho = \sup_{\psi \in \Psi} \rho_\psi$ and (19) allows us to apply the functional delta method for sup functionals (cf. Römisch (2006)) to conclude that $\big(\sqrt{n}[\rho_n(F) - \rho(F)]\big)_{n \in \mathbb{N}}$ converges in law to $\sup_{\psi \in S(\rho)} G_\psi$. Finally, if $\mathbb{E}\big[(G(\psi_1) - G(\psi_2))^2\big] \neq 0$ for different $\psi_1, \psi_2 \in S(\rho(F))$, then it is well-known that the paths of $G|S(\rho(F))$ have unique maximizers a.s. (cf. Lifshits (1982)). Then by measurable selection we may find a Borel random element Z of $S(\rho(F))$ such that $G(Z)$ is distributed as $\sup_{\psi \in S(\rho)} G_\psi$. This completes the proof. $\qquad \square$

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A Appendix

Lemma A.1. *Let $\psi : [0,1] \to [0,1]$ be a nondecreasing, continuous mapping with $\psi(0) = 0$, and let G be any distribution function on $\mathbb{R}$ such that $\int_{-\infty}^{0} \psi(G(x))\ dx, \int_{0}^{\infty}[\psi(1) - \psi(G(x))]\ dx < \infty$. Then*

$$\int_{-\infty}^{0} \psi(G(x))\ dx - \int_{0}^{\infty}[\psi(1) - \psi(G(x))]\ dx = -\int_{0}^{1} q_G\ d\mu_\psi,$$

where q_G and μ_ψ denote respectively the left-continuous quantile function of G and the Borel probability measure on $[0,1]$ induced by ψ.

Proof:
Let $\mu_{\psi \circ G}$ denote the Borel probability measure on $\mathbb{R}$ induced by the right-continuous mapping

$\psi \circ G$. It coincides with the image measure of μ_ψ under q_G, implying $\int_0^1 q_G \, d\mu_\psi = \int_{\mathbb{R}} x \, \mu_{\psi \circ G}(dx)$. Furthermore, by right-continuity of $\psi \circ G$

$$
\int_{-\infty}^0 x \, \mu_{\psi \circ G}(dx) \;=\; -\int_{\mathbb{R}} [-1_{]-\infty,0]}(x)x] \, \mu_{\psi \circ G}(dx) = -\int_0^\infty \mu_{\psi \circ G}(]-\infty,-\beta]) \, d\beta
$$

$$
(21) \qquad\qquad\qquad =\; -\int_0^\infty \psi(G(-\beta)) \, d\beta
$$

and

$$
\int_0^\infty x \, \mu_{\psi \circ G}(dx) \;=\; \int_{\mathbb{R}} 1_{]0,\infty[}(x)x \, \mu_{\psi \circ G}(dx) = \int_0^\infty \mu_{\psi \circ G}(]\beta,\infty[) \, d\beta
$$

$$
(22) \qquad\qquad\qquad =\; \int_0^\infty [\psi(1) - \psi(G(\beta))] \, d\beta
$$

Then we may conclude the statement of Lemma A.1 from (21) and (22) by applying the change of variable formula to (21). $\qquad\square$

B Proof of Proposition 5.1

The proof of Proposition 5.1 relies on the following lemma.

Lemma B.1. *If in addition $X \wedge Y, X \vee Y \in \mathcal{X}$ for $X, Y \in \mathcal{X}$, then the first property of condition (AC) implies*

$$
\rho(X) = \sup_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \rho(F_{[X^+ \wedge k] - [X^- \wedge m]}) \qquad \forall X \in \mathcal{X}.
$$

Proof:
In view of Proposition 6.6 from Krätschmer (2007) the first property of condition (AC) allows us to apply Lemma 6.5 from the same paper. According to Lemma 6.5, .1, we have $\sup_{m \in \mathbb{N}} \rho(F_{X^+ - [X^- \wedge m]}) = \rho(F_X)$, whereas $\inf_{k \in \mathbb{N}} \rho(F_{[X^+ \wedge k] - [X^- \wedge m]}) = \rho(F_{X^+ - [X^- \wedge m]})$ holds for any $m \in \mathbb{N}$ due to Lemma 6.5, .2. The statement of Lemma B.1 is obvious now. $\qquad\square$

Lemma B.1 enables us to conclude a robust representation of ρ by concave distortion risk measures when its restriction to $\{F_X \mid X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})\}$ admits such a representation.

Lemma B.2. *Let Ψ be any set of concave distortion functions such that $\rho(F_X) = \sup_{\psi \in \Psi} \rho_\psi(F_X)$ holds for $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$.*
If $X \wedge Y, X \vee Y \in \mathcal{X}$ for $X, Y \in \mathcal{X}$, and if ρ satisfies the first property of condition (AC), then $\rho(F) = \sup_{\psi \in \Psi} \rho_\psi(F)$ is valid for arbitrary $F \in \mathbb{F}_{\mathcal{X}}$.

Proof:
Let us set $\widetilde{\rho} := \sup_{\psi \in \Psi} \rho_\psi$. The proof is divided into two steps: First we will show that $\widetilde{\rho}$ is well-defined and defines a law-invariant coherent risk measure on $\mathbb{F}_{\mathcal{X}}$, which obviously concide with ρ on $\{F_X \mid X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})\}$. Second we shall prove that both risk measures are even identical. *Step 1.* If we can show that $\rho_\psi(F_X) \in \mathbb{R}$ (for all $\psi \in \Psi$) and $\sup_{\psi \in \Psi} \rho_\psi(F_X) < \infty$ for all $X \in \mathcal{X}$, then it follows easily that $\widetilde{\rho}$ defines a law-invariant coherent risk measure on $\mathcal{X}$, since

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every concave distortion risk measure ρ_ψ is a law-invariant coherent risk measure. Of course, the mentioned conditions hold if we can show

$$(23) \qquad \sup_{\psi\in\Psi} \int_{-\infty}^0 \psi(F_X(x))\, dx \leq \rho(F_{-X^-}) \quad \text{and} \quad \forall\, \psi\in\Psi: \quad \int_0^\infty [1-\psi(F_X(x))]\, dx < \infty$$

for all $X \in \mathcal{X}$ with distribution function F_X. To verify the first statement in (23), we pick $X \in \mathcal{X}$. For every $\psi\in\Psi$ we have

$$\psi(F_{-X^-}(x)) \leq \liminf_{m\to\infty} \psi(F_{-[X^-\wedge m]}(x))$$

at every continuity point $x < 0$ of the distribution function F_{-X^-} of $-X^-$, since ψ as a concave function is lower semicontinuous. Using this and applying Fatou's lemma, we obtain

$$
\begin{aligned}
\sup_{\psi\in\Psi} \int_{-\infty}^0 \psi(F_X(x))\, dx \;&\leq\; \sup_{\psi\in\Psi} \int_{-\infty}^0 \psi(F_{-X^-}(x))\, dx \leq \sup_{\psi\in\Psi} \int_{-\infty}^0 \liminf_{m\to\infty} \psi(F_{-[X^-\wedge m]}(x))\, dx \\[2mm]
&\leq\; \sup_{\psi\in\Psi} \liminf_{m\to\infty} \int_{-\infty}^0 \psi(F_{-[X^-\wedge m]}(x))\, dx = \sup_{\psi\in\Psi} \liminf_{m\to\infty} \rho_\psi(F_{-[X^-\wedge m]}) \\[2mm]
&\leq\; \liminf_{m\to\infty} \rho(F_{-[X^-\wedge m]}) \leq \rho(F_{-X^-}).
\end{aligned}
$$

Hence the first statement in (23) holds indeed. To verify the second statement in (23), we pick $X \in \mathcal{X}$. As ψ is nondecreasing and concave its restriction to $]0,1]$ is continuous, so that

$$1 - \psi(F_X(x)) = \psi(1) - \psi(F_X(x)) \leq \psi'(F_X(x_0))[1 - F_X(x)] \qquad \forall\, x \geq x_0,$$

for any $x_0 > 0$ such that $F_X(x_0) > 0$. Moreover, the integral $\int_0^\infty [1-F_X(x)]dx$ exists since $X \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$. Hence,

$$\int_0^\infty [1-\psi(F_X(x))]\, dx \leq \int_0^{x_0} [1-\psi(F_X(t))]\, dx + \psi'(F_X(x_0)) \int_{x_0}^\infty [1-F_X(x)]\, dx < \infty.$$

This shows that the second statement in (23) holds, too.

Step2. The first property of assumption (AC) on ρ ensures that the right-hand side of

$$0 \leq \widetilde{\rho}(F_{-(X-r)^+}) \overset{(23)}{\leq} \rho(F_{-(X-r)^+})$$

converges to 0, as $r \to \infty$, for every nonnegative $X \in \mathcal{X}$. Therefore the first property of condition (AC) is fulfilled by $\widetilde{\rho}$ too, and Lemma B.1 applied to $\widetilde{\rho}$ implies $\rho = \widetilde{\rho}$ on $\mathcal{X}$. The proof is now complete. $\qquad\square$

Now we are ready for the proof of Proposition 5.1.

Proof of Proposition 5.1:
Possibly changing to a suitable probability space we may assume that $L^2(\Omega,\mathcal{F},\mathbb{P})$ is separable. Then in the specified setting, Corollary 4.72 in Föllmer and Schied (2004) along with Theorem 2.1 in Jouini et al. (2006) yield the existence of some set $\widetilde{\Psi}$ of concave distortions such that $\rho(F_X) = \sup_{\psi\in\widetilde{\Psi}} \rho_\psi(F_X)$ holds for $X \in \mathcal{L}^\infty(\Omega,\mathcal{F},\mathbb{P})$. Notice that all members of the topological

closure Ψ of $\widetilde{\Psi}$ w.r.t. the uniform metric are concave distortion functions again. Therefore, in view of (5)

$$|\psi(q) - \psi(q)| \leq \psi_\rho(q - p) \text{ for } \psi \in \Psi \text{ and } 0 \leq p < q \leq 1.$$

Since $\lim_{q \to 0_+} \psi_\rho(q) = 0$ by the second property of condition (AC), we may conclude that Ψ is a uniformly equicontinuous w.r.t. the uniform metric, which means by Arzela-Ascoli theorem that it is not only closed but also compact w.r.t. the sup metric. We want to show that ρ admits a robust representation by concave distortion risk measures with concave distortions from Ψ. For this purpose by Lemma B.2 it suffices to show that $\rho(F_X) = \sup_{\psi \in \Psi} \rho_\psi(F_X)$ is valid for every $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

Indeed for any fixed $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with distribution function F_X there exists some $\varepsilon > 0$ such

$$\rho_\psi(F_X) = \int_{-\varepsilon}^0 \psi(F_X)(x))\ dx - \int_0^\varepsilon [1 - \psi(F_X(x))]\ dx \text{ for all } \psi \in \Psi.$$

Then a routine application of the dominated convergence theorem yields the continuity of the mapping

$$\Phi : \Psi \to \mathbb{R}, \ \psi \mapsto \rho_\psi(F_X)$$

w.r.t. the uniform metric. Therefore $\lim_{k \to \infty} \rho_{\psi_k}(F_X) = \rho_\psi(F_X)$ holds for any sequence $(\psi_k)_{k \in \mathbb{N}}$ in $\widetilde{\Psi}$ which converges to some ψ w.r.t. the uniform metric. Hence obviously, $\rho(F_X) = \sup_{\psi \in \Psi} \rho_\psi(F_X)$, and thus $\rho = \sup_{\psi \in \Psi} \rho_\psi$ due to Lemma B.2. The proof is complete.

$\square$

C   Appendix

Lemma C.1. *Let $(Z_i)_{i \in \mathbb{N}}$ be a strictly stationary, strongly mixing sequence of random variables on some probability space $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ with common distribution function H and mixing coefficients $\alpha(i)$ satisfying*

$$\alpha(i) \leq \bar{\alpha}_0 \exp(-\bar{\alpha}_1 i), \quad i \in \mathbb{N},$$

for some constants $\bar{\alpha}_0 > 0$ and $\bar{\alpha}_1 > 0$. Then for any $\eta \in]0, 1[$, there is some constant C_η such that

$$\left| H(x)(1 - H(x)) + 2 \sum_{i=1}^\infty \left[\widetilde{\mathbb{P}}(\{Z_1 \leq x, Z_{i+1} \leq x\}) - H(X)^2 \right] \leq C_\eta [H(x)(1 - H(x))]^{1-\eta}$$

for every $x \in \mathbb{R}$.

Proof:
Let $\eta \in]0, 1[$, $x \in \mathbb{R}$, and define $Y_i(x) := 1_{]-\infty, x]} \circ Z_i$. Without loss of generality we may assume $\bar{\alpha}_0 \geq 1$.

Firstly observe

$$1/n \ \mathrm{Var}(\sum_{i=1}^{n} Y_i(x)) \ = \ \mathrm{Var}(Z_1) + 2 \sum_{i=1}^{n-1} (n-i)/n \ \mathrm{Cov}(Z_1, Z_{i+1})$$

$$= \ H(x)(1 - H(x)) + 2 \sum_{i=1}^{n-1} (n-i)/n \ \left[\widetilde{\mathbb{P}}(\{Z_1 \leq x, Z_{i+1} \leq x\}) - H(x)^2 \right]$$

for any $n \geq 2$. By assumption on $(\alpha(i))_{i \in \mathbb{N}}$ the series $\sum_{i=1}^{\infty} \mathrm{Cov}(Z_i, Z_{i+1})$ converges absolutely (c.f. e.g. Athreya an Lahiri (2006), Proposition 16.3.1) so that by dominated convergence theorem

$$(24) \quad \lim_{n \to \infty} 1/n \ \mathrm{Var}(\sum_{i=1}^{n} Y_i(x)) = H(x)(1 - H(x)) + 2 \sum_{i=1}^{\infty} \left[\widetilde{\mathbb{P}}(\{Z_1 \leq x, Z_{i+1} \leq x\}) - H(x)^2 \right].$$

Moreover, we may apply Theorem 1.2 in Rio (1993) to $\mathrm{Var}(\sum_{i=1}^{n} Y_i(x))$ yielding

$$1/n \ \mathrm{Var}(\sum_{i=1}^{n} Y_i(x)) \leq 4 \int_0^1 \alpha^{-1}(u/2) Q(u)^2 \ du,$$

where $Q(u) := \sup \{y \in \mathbb{R} \mid \widetilde{\mathbb{P}}(\{|Y_1(x)| > y\}) > u\}$ and $\alpha^{-1}(u/2) := \sup\{i \in \mathbb{N} \mid \alpha(i) > u/2\}$ ($\sup \emptyset := 0$).

It is easy to check that $Q(u) = 1$ if $H(x) > u$ and $Q(u) = 0$ otherwise. Moreover, by assumption on $(\alpha(i))_{i \in \mathbb{N}}$, we obtain $\alpha^{-1}(u/2) \leq [\ln(2\bar{\alpha}_0) - \ln(u)]/\bar{\alpha}_1$. Thus

$$(25) \quad 1/n \ \mathrm{Var}(\sum_{i=1}^{n} Y_i(x)) \leq 4 \int_0^{H(x)} [\ln(2\bar{\alpha}_0) - \ln(u)]/\bar{\alpha}_1 \ du = 4 \ \frac{H(x)[1 - \ln(H(x)/(2\bar{\alpha}_0))]}{\bar{\alpha}_1}.$$

Using an analogous line of reasoning, an additional application of Theorem 1.2 in Rio (1993) to $\mathrm{Var}(\sum_{i=1}^{n} [1 - Y_i(x)])$ leads to

(26)

$$1/n \ \mathrm{Var}(\sum_{i=1}^{n} Y_i(x)) \leq 4 \int_0^{1-H(x)} [\ln(2\bar{\alpha}_0) - \ln(u)]/\bar{\alpha}_1 \ du = 4 \ \frac{[1 - H(x)][1 - \ln((1 - H(x))/(2\bar{\alpha}_0))]}{\bar{\alpha}_1}.$$

Since $\lim_{\gamma \to 0_+} \exp([\gamma - 1]/\gamma) = 0$, we may find some $\gamma \in]0, \eta[$ such that $t_\gamma := 2\bar{\alpha}_0 \exp([\gamma - 1]/\gamma) \in]0, 1[$. Then routine considerations yield

$$\max_{t \in]0,1[} t^\gamma [1 - \ln(t/(2\bar{\alpha}_0))] = t_\gamma^\gamma [1 - \ln(t_\gamma/(2\bar{\alpha}_0))] = \frac{(2\bar{\alpha}_0)^\gamma \exp(\gamma - 1)}{\gamma} \leq \frac{2\bar{\alpha}_0 \exp(\gamma - 1)}{\gamma} .$$

Hence by (25), (26)

$$1/n \ \mathrm{Var}(\sum_{i=1}^{n} Y_i(x)) \ \leq \ \frac{8\bar{\alpha}_0 \exp(\gamma - 1)}{\gamma} \left(H(x)^{1-\gamma} \wedge (1 - H(x))^{1-\gamma} \right)$$

$$\leq \ \frac{8\bar{\alpha}_0 \exp(\gamma - 1)}{\gamma} 2^{1-\gamma} [H(x)(1 - H(x))]^{1-\gamma}$$

$$\leq \ \frac{16\bar{\alpha}_0}{\gamma} [H(x)(1 - H(x))]^{1-\eta}.$$

Then in view of (24) we may conclude

$$H(x)(1 - H(x)) + 2\sum_{i=1}^{\infty} \left[\widetilde{\mathbb{P}}(\{Z_1 \leq x, Z_{i+1} \leq x\}) - H(x)^2\right] \leq \frac{16\bar{\alpha}_0}{\gamma} \left[H(x)(1 - H(x))\right]^{1-\eta}$$

for every $x \in \mathbb{R}$, which completes the proof. $\qquad\square$