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On the stability of solutions and absence of Arnol'd diffusion in a nonintegrable Hamiltonian system of a general form with three degrees of freedom

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We consider the function

$$H = H(x,y,q,p,\varphi) = \sqrt{1+(q+\frac{\Omega y}{2})^2+(p-\frac{\Omega x}{2})^2} + \widehat{D}(x,y,\varphi) \tag{1}$$

and the Hamiltonian system with Hamiltonian H:

$$\frac{dx}{d\varphi} = \frac{\partial H}{\partial q}, \frac{dy}{d\varphi} = \frac{\partial H}{\partial p}, \frac{dq}{d\varphi} = -\frac{\partial H}{\partial x}, \frac{dp}{d\varphi} = -\frac{\partial H}{\partial y}, \tag{2}$$

where φ is an independent variable, $\Omega = const > 0$.

It is assumed that the following conditions hold.

I. $\widehat{D}(x,y,\varphi)$ is a smooth function with respect to all variables in the space $-\infty < x,y,\varphi < \infty$ which vanishes outside of the rectangle $\widehat{\Pi} = \{(x,y) : |x| \le \hat{a}, |y| \le b\}$ and has the period 2π with respect to φ.

II. $\widehat{D}(x,y,\varphi) = D(y,\varphi)$, if $(x,y) \in \Pi \subset \widehat{\Pi}$, where $\Pi = \{(x,y) : |x| \le a, |y| \le b\}$, a and b are constsnts, $a > 0, b > 0$ and $D(y,\varphi)$ is an analytic function with respect to y,φ.

Theorem 1. *It is assumed that the function $D(y,\varphi)$ satisfies the following conditions. There exist an integer $k > 0$ and a real φ_0, such* *that* $0 \le \varphi_0 < 2\pi$, *and if* $E(\tau) = -\frac{\partial D}{\partial y}(\tau - \varphi_0 + b, \tau)$, *then*

$$1) \qquad \int_{\varphi_0 - 2b}^{\varphi_0} E(\tau) d\tau = \Omega k,$$

$$-\frac{\Omega}{\pi} < \frac{1}{2}(E(\varphi_0) - E(\varphi_0 - 2b) = \beta < 0;$$

$$2) \qquad \beta \ne -\frac{\Omega}{2\pi} + \frac{\Omega}{2\pi} \cos\left(\frac{2\pi t}{\mu}\right), \ \textit{where } t = 0, \pm 1, \dots, \pm\mu;\ \mu = 1, 2, 3, 4.$$

$$3) \qquad a_0(\beta)\left(\frac{d^2 E}{d\tau^2}(\varphi_0) - \frac{d^2 E}{d\tau^2}(\varphi_0 - 2b)\right) +$$

$$+ a_1(\beta)\left(\frac{dE}{d\tau}(\varphi_0) - \frac{dE}{d\tau}(\varphi_0 - 2b)\right)^2 \ne 0,$$

where $a_0(s), a_1(s)$ *are some fixed functions, not depending on the form of the function* $D(y, \varphi)$, *and* $a_0(\beta) \ne 0$. *Then there exist a natural number* n_0, *a real number* $\kappa > 0$ *and a domain* $\Gamma = \Big\{\tilde{x}, \tilde{y}, \tilde{q}, \tilde{p}, \tilde{\varphi} : |\tilde{x}| \le \kappa,\ |\tilde{y} - b| \le \kappa,\ |\tilde{q} + \frac{\Omega\tilde{y}}{2} - \varphi_0\Omega| \le \kappa,\ |\varphi - \varphi_0| \le \kappa,$ $\left|\sqrt{1 + (\tilde{q} + \frac{\Omega\tilde{y}}{2})^2 + (\tilde{p} - \frac{\Omega\tilde{x}}{2})^2} - n_0\Omega\right| \le \kappa\Big\}$, *such that the following assertions are true for solutions of the system (2) with initial data* $x(\tilde{\varphi}) = \tilde{x}$, $y(\tilde{\varphi}) = \tilde{y}$, $q(\tilde{\varphi}) = \tilde{q}$, $p(\tilde{\varphi}) = \tilde{p}$ *in the domain* Γ *for* $\varphi \ge \tilde{\varphi}$:

1. *Every such solution intersects the domain* Π *an infinite number of times.*
2. $\lim \sqrt{1 + (q(\varphi) + \frac{\Omega y(\varphi)}{2})^2 + (p(\varphi) - \frac{\Omega x(\varphi)}{2})^2} = \infty$, $\varphi \to \infty$.
3. *For any* $\varepsilon > 0$ *there exists a nonempty open five-dimensional region* $U_\varepsilon \subset \Gamma$ *such that if* $(x_1(\varphi), y_1(\varphi), q_1(\varphi), p_1(\varphi))$ *and* $(x_2(\varphi), y_2(\varphi), q_2(\varphi), p_2(\varphi))$ *are any two distinct solutions of the system (2) with initial data belonging to* U_ε *for* $\varphi = \tilde{\varphi}$, *then for all* $\varphi \ge \tilde{\varphi}$

$$|x_1(\varphi) - x_2(\varphi)| + |y_1(\varphi) - y_2(\varphi)| + |q_1(\varphi) - q_2(\varphi)| + |p_1(\varphi) - p_2(\varphi)| < \varepsilon.$$

Let m be an integer, δ a real number, $\delta > 0, k$ and φ_0 satisfy the condition of theorem 1. For any natural n we introduce the domains

$$\Gamma_n = \{x, y, q, p, \varphi : \left|\sqrt{1 + (q + \frac{\Omega y}{2})^2 + (p - \frac{\Omega x}{2})^2} - (m + nk)\Omega\right| \le \delta\},$$

$$\begin{aligned} K_n = \Big\{ & x, y, q, p, \varphi : -\infty < \varphi < \infty, (x, y) \in \Pi, \\ & - 2\Omega b - \delta + \varphi_0\Omega \le q + \frac{\Omega y}{2} \le \delta + \varphi_0\Omega, \\ & (m + (n-1))\Omega - \delta \le \sqrt{1 + (q + \frac{\Omega y}{2})^2 + (p - \frac{\Omega x}{2})^2} \\ & \le (m + nk)\Omega + \delta \Big\}, \end{aligned}$$

and for any smooth function $G = G(x, y, q, p, \varphi)$ we introduce the norms

$$h_n(G) = \sup_{(x,y,q,p,\varphi)\in\Gamma_n} |\nabla G|, \sigma_n(G) = \sup_{(x,y,q,p,\varphi)\in K_n} |\nabla G|,$$

where $\nabla G = (\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial q}, \frac{\partial G}{\partial p},)$ is the gradient of the function G with respect to the variables x, y, q, p and Π is the domain introduced in condition II.

Theorem 2. *We consider the function $W = W(x, y, q, p, \varphi) = H + \varepsilon_1\widehat{D}_1(x, y, \varphi) + G$, where H is the function introduced in (1), for which the conditions of theorem 1 are satisfied, $\widehat{D}_1(x, y, \varphi)$ is a function that satisfies the conditions I and II for some function $D_1(y, \varphi)$ instead of $D(y, \varphi), \varepsilon_1$ is a parameter, $G = G(x, y, q, p, \varphi)$ is a smooth function that fulfils $\sum_{m=1}^{\infty} \sigma_n(G) < \infty, h_n(G) \le \frac{C}{n^{2+\rho}}, \rho > 0, \rho$ and C are constants not depending on n. Then if the quantity $|\varepsilon_1| \ne 0$ and is sufficiently small, all assertions of theorem 1 are satisfied for solutions of the system*

$$\frac{dx}{d\varphi} = \frac{\partial W}{\partial q}, \frac{dy}{d\varphi} = \frac{\partial W}{\partial p}, \frac{dq}{d\varphi} = -\frac{\partial W}{\partial x}, \frac{dp}{d\varphi} = -\frac{\partial W}{\partial y}. \quad (3)$$

Theorem 2 shows that there exists a stability tube of trajectories for Hamiltonian systems from some region in the functional space. For the systems (2) and (3) it is possible to introduce a small parameter $\nu = (1+(q+\frac{\Omega y}{2})^2+(p-\frac{\Omega x}{2})^2)^{-\frac{1}{2}}$ such that the difference between two distinct solutions of the system (2) or (3) is also a solution of some Hamiltonian system which is integrable in the limit $\nu = 0$. Therefore in the situation considered here there is no Arnol'd diffusion [1], that is also found for five-dimensional system in which the Hamiltonian function depend on independent variable periodically, and thus the Arnol'd conjecture [2] on topological instability of Hamiltonian systems with more than two degrees of freedom is disproved. Proofs of theorems 1 and 2 are found by reducing the system (2) to the system that is investigated in [3], but with stronger restrictions to the function $D(y, \varphi)$.

The main difficulties of proofs are overcome in [3].

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