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# Inverse scattering of electromagnetic waves by multilayered structures: Uniqueness in TM mode

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**Abstract**

Assume a time-harmonic electromagnetic wave is scattered by an infinitely long cylindrical conductor surrounded by an unknown piecewise homogenous medium remaining invariant along the cylinder axis. We prove that, in TM mode, the far field patterns for all observation directions at a fixed frequency uniquely determine the unknown surrounding medium as well as the shape of the cylindrical conductor. A similar uniqueness result is obtained for the scattering by multilayered penetrable periodic structures in a piecewise homogenous medium. The periodic interfaces and refractive indices can be uniquely identified from the near field data measured only above (or below) the structure for all quasi-periodic incident waves with a fixed phase-shift. The proofs are based on the singularity of the Green function to a two dimensional elliptic equation with piecewise constant leading coefficients.

# 1 Introduction

The reconstruction of an obstacle from its far field pattern is of great importance in inverse scattering problems. In practical applications, the background might not be homogenous or known and then can be modeled as an unknown layered medium. In this paper, we consider the scattering of time-harmonic electromagnetic waves by a multilayered structure. Such a structure is allowed to be either an infinitely long cylinder or a penetrable multilayered periodic structure, which is stratified by an unknown piecewise homogeneous medium. All the media under consideration are supposed to be isotropic and invariant in $x_3$-direction. In TM mode where the magnetic field is transversal to the $(x_1, x_2)$-plane, this problem can be reduced to two dimensions and modeled by the Helmholtz equation with the TM transmission condition. The transmission coefficient on an interface in this model only depends on the refractive indices (or wave numbers) corresponding to the regions on both sides of the interface.

The first half of this paper investigates uniqueness in determining the shape of a cylindrical conductor and the unknown piecewise homogenous background medium. There have been few results on the inverse scattering of acoustic or electromagnetic waves by multilayered scatterers. If the wave numbers for characterizing the piecewise homogenous medium and the transmission coefficients on the interfaces are known, it was proved that the buried obstacle and the interfaces of the background can be uniquely determined from the measurements of far field for all incident directions at a fixed frequency; see [24, 29] for the scattering of acoustic waves and [25] for electromagnetic waves. If the background medium is unknown, Hähner [13] proved that, in TE mode, the Cauchy data of the scattered waves for all incident waves and an interval of frequencies uniquely determine an impenetrable obstacle and its surrounding inhomogeneity. We do not know other papers for reconstructing an obstacle (penetrable or impenetrable) buried in an unknown inhomogeneous medium. Note that an obstacle or a penetrable inhomogeneous media can always be uniquely determined by the far field data at a fixed frequency if the outside inhomogeneity is known in advance; see, e.g., [20, 23, 26].

One aim of this paper is to prove that, in the case of TM polarization and a piecewise homogeneous background, the far field data from all incident directions at a fixed frequency can uniquely determine the cross-section of the cylindrical conductor and its layered surroundings. Our proof is based on the Green function $G(x; y)$ to the scattering problem by multilayered obstacles (see [29]), which satisfies an elliptic equation with piecewise constant leading coefficients; see also [5] for using the fundamental solutions in inverse scattering problems. In the 2D case, we will investigate the asymptotic behavior of $G(x; y)$ as $x, y \to y_0$ when $y_0$ is located on an interface, analogously to the treatment by Ramm [1, 29] in $\mathbb{R}^3$. However, we deal with the problem in a completely unknown background, without establishing the orthogonality relation used in [1, 29, 33], and provide a rigorous mathematical analysis. Furthermore, we significantly simplify the existing proofs by avoiding the mixed reciprocity relation used in [23] and the *a priori* estimates of the solutions on the interfaces essentially required by [24] (see also [21]). The idea of this paper dates back to Druskin [8] in 1982 who used point sources to prove uniqueness in determining a piecewise constant conductivity for a three dimensional electrical surveying problem; see also [16, Theorem 5.7.1.]. In Section 2.4 of this paper, we will extend this idea to prove uniqueness under general transmission conditions with unknown transmission coefficients.

In the second half of this paper, the previous argument is carried over to the inverse scattering by a multilayered periodic structure. In the case of TE polarization and one periodic interface, Elschner and Yamamoto [11] proved that measurements corresponding to a finite number of refractive indices above or below the grating profile uniquely determine the periodic interface. This extended the uniqueness result by Hettlich and Kirsch on Schiffer's theorem [14] to the inverse transmission problem. For two periodic interfaces with an inhomogeneity between them, it was proved in [31] that the interfaces and transmission coefficients can be uniquely identified from the scattered waves for all quasi-periodic incident waves, and so can the refractive index of the inhomogeneity if it only depends on $x_1$ and the interfaces are parallel to the $x_2$-axis. Note that the measurements in [11, 31] must be taken both above and below the structure. In this paper, we prove that the scattered fields in the TM mode measured only above (or below) the structure for all incident quasi-periodic incident waves (with a fixed phase-shift) are enough to uniquely identify a multilayered diffraction grating, including all the interfaces and refractive indices.

For numerical aspects, we refer to [6, 32] and the references therein for reconstructing an obstacle buried in a layered background medium, and [2, 22] for recovering a periodic interface separating two homogenous materials in the TE mode via the optimization or factorization method. Note that the uniqueness issue is always required in order to proceed an efficient inversion algorithm.

The paper is organized as follows. In Section 2.1, mathematical formulations are presented for the inverse scattering by infinitely long multilayered cylinders. In Section 2.2, the Green function is introduced and its singularity is investigated. Our main uniqueness result under the TM transmission conditions for cylinders is proved in Section 2.3, and it is extended to general transmission conditions in Section 2.4. Finally, Section 3 is devoted to the uniqueness for multilayered periodic structures in a piecewise homogenous medium; see Section 3.1 for the mathematical model and the uniqueness result, and Section 3.2 for the proof.

# 2 Inverse scattering by infinitely long multilayered cylinders

Assume a time-harmonic electromagnetic wave (with time variation of the form $\exp(-i\omega t)$, $\omega > 0$) is incident on an infinitely long perfect cylindrical conductor surrounded by an unknown piecewise homogenous medium. The cylinder axis is supposed to coincide with the $x_3$-axis, so that the cylinder can be represented as $\mathcal{D} \times \mathbb{R}$ with the cross-section $\mathcal{D}$ belonging to the $(x_1, x_2)$-plane. For simplicity, and without loss of generality, we restrict ourselves to the case of three layered structures by assuming $\mathcal{D} = \overline{D}_1 \cup \overline{D}_2 \cup \overline{D}_3$ with two $C^2$-smooth interfaces $\Gamma_3 := \partial D_3, \Gamma_2 := \overline{D}_2 \cap \overline{D}_3$, where $D_3$ denotes the cross-section of the interior impenetrable perfect cylindrical conductor. Thus $\mathcal{D}$ can be also considered as a multilayered obstacle in $\mathbb{R}^2$ with the impenetrable core $D_3$. Let $\Gamma_1 := \partial\mathcal{D}$ be a $C^2$-smooth boundary, and let $D_0$ denote the complement of $\mathcal{D}$, that is, $D_0 := \mathbb{R}^2 \backslash \overline{\mathcal{D}}$; see Figure 1.

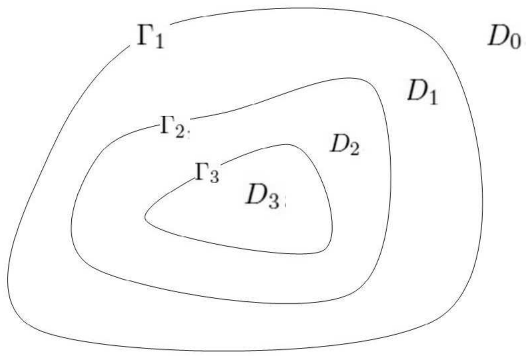


Figure 1: A multilayered obstacle $\mathcal{D} = \overline{D}_1 \cup \overline{D}_2 \cup \overline{D}_3$ with the impenetrable core $D_3$.

## 2.1 Mathematical formulations in TM mode

We focus on the TM mode of the above scattering problem by assuming all fields are propagating perpendicular to the $x_3$-axis. Let $u(x_1, x_2)$ be the third component of the magnetic field, i.e., $H = (0, 0, u(x_1, x_2))$. Then, we have

$$\Delta u + k_j^2 u = 0 \qquad \text{in } D_j,\ j = 0, 1, 2; \tag{2.1}$$

$$u_+ = u_-, \quad \frac{1}{k_{j-1}^2}\frac{\partial u_+}{\partial \mathbf{n}} = \frac{1}{k_j^2}\frac{\partial u_-}{\partial \mathbf{n}} \qquad \text{on } \Gamma_j,\ j = 1, 2; \tag{2.2}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \qquad \text{on } \Gamma_3. \tag{2.3}$$

Here, $k_j^2 = (\varepsilon_j + i\sigma_j/\omega)\mu_j\omega^2$ are distinct wave numbers corresponding to the regions $D_j$ $(j = 0, 1, 2)$ in terms of the space independent electric permittivity $\varepsilon_j > 0$, magnetic permeability $\mu_j > 0$ and electric conductivity $\sigma_j \geq 0$; the homogenous medium in $D_0$ has vanishing conductivity, that is, $\sigma_0 = 0$, implying that $k_0 > 0$; **n** denotes the unit outward normal to the boundary $\Gamma_j$; $u_+, \frac{\partial u_+}{\partial \mathbf{n}}$ (resp. $u_-, \frac{\partial u_-}{\partial \mathbf{n}}$) denote the limits of $u$ on $\Gamma_j$ from the exterior (resp. interior) of $D_j$. Note that the transmission conditions on $\Gamma_j$ $(j = 1, 2)$ in (2.2) and the Neumann condition (2.3) on $\Gamma_3$ are derived from the continuity of the tangential components of the electric and magnetic fields when getting across the interfaces in the case of TM polarization.

The total field $u(x_1, x_2)$ can be decomposed as the sum of the incident plane wave $u^i$ and the scattered wave $u^s$, i.e.,

$$u = u^i + u^s \qquad \text{in } \mathbb{R}^2\backslash\overline{D}_3, \tag{2.4}$$

where $u^i$ takes the form of $u^i = \exp(ik_0 x \cdot d)$ for some incident direction $d = (\cos\theta, \sin\theta)$ with the incident angle $\theta \in [0, 2\pi)$, and $u^s$ is required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - ik_0 u^s\right) = 0 \qquad \text{with } r = ||x||, \tag{2.5}$$

uniformly in all directions $\hat{x} := x/||x||$. The radiation condition (2.5) gives rise to the following asymptotic behavior of the scattered field

$$u^s(x; d) = \frac{e^{ik_0||x||}}{\sqrt{||x||}}\left\{u^\infty(\widehat{x}; d) + O\left(\frac{1}{||x||}\right)\right\}, \qquad \text{as } ||x|| \to \infty, \tag{2.6}$$

where the function $u^\infty(\widehat{x}; d)$ defined on the unit sphere $S := \{x \in \mathbb{R}^2 : ||x|| = 1\}$ is known as the far field pattern for the observation direction $\widehat{x} \in S$ and the incident direction $d \in S$.

There always exists a unique solution $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{D}_3)$ to the above scattering problem (2.1)-(2.5); see [1, 23, 24, 25] for the acoustic or electromagnetic scattering problem by a piecewise homogenous medium with a penetrable or impenetrable core. For notational simplicity, we write $D = (\Gamma_1, \Gamma_2, \Gamma_3, k_1, k_2)$ to indicate the dependence of the obstacle $D$ on the outmost boundary $\Gamma_1$, the interior interfaces $\Gamma_2$, $\Gamma_3$ and the wave numbers $k_1$, $k_2$.

Now we formulate the inverse scattering problem as follows:

**Inverse Problem (IP):** Given the wave number $k_0$ and the far field pattern data $u^\infty(\hat{x}; d)$ for all observation directions $\hat{x} \in S$ and all incident directions $d \in S$, determine the multilayered obstacle $D = (\Gamma_1, \Gamma_2, \Gamma_3, k_1, k_2)$.

The main theorem of this section is

**Theorem 2.1.** *Assume* $D = (\Gamma_1, \Gamma_2, \Gamma_3, k_1, k_2)$ *and* $\tilde{D} = (\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3, \tilde{k}_1, \tilde{k}_2)$ *are two multi-layered obstacles, and* $u^\infty(\hat{x}; d), \tilde{u}^\infty(\hat{x}; d)$ *are the far field patterns corresponding to* $D, \tilde{D}$*, respectively. If*

$$u^\infty(\hat{x}; d) = \tilde{u}^\infty(\hat{x}; d) \qquad \textit{for all } \hat{x}, d \in S, \tag{2.7}$$

*then* $D = \tilde{D}$*, that is,* $\Gamma_j = \tilde{\Gamma}_j$*,* $j = 1, 2, 3$*, and* $k_i = \tilde{k}_i$*,* $i = 1, 2$.

## 2.2 Green's function of the scattering problem

Before proving the theorem, we notice that the equation (2.1) together with the transmission conditions in (2.2) can be reformulated as follows:

Find a weak solution $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D}_3)$ such that

$$Lu = L(u, \partial) := \nabla \cdot (a \nabla u) + u = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}_3, \tag{2.8}$$

$$u_+ = u_-, \quad a^+ \frac{\partial u_+}{\partial \mathbf{n}} = a^- \frac{\partial u_-}{\partial \mathbf{n}} \qquad \text{on } \Gamma_j, \ j = 1, 2, \tag{2.9}$$

where

$$a(x) = \frac{1}{k_j^2}, \qquad x \in D_j. \tag{2.10}$$

This motivates us to introduce the Green function $G(x; y)$ to the scattering problem (2.1)-(2.5), which satisfies the radiation condition (2.5), the transmission conditions in (2.9) and

$$L_x G(x; y) = -\delta(x - y), \qquad x, y \in \mathbb{R}^2 \backslash \overline{D}_3, \ x \neq y, \ y \notin \Gamma_1 \cup \Gamma_2,$$

$$\frac{\partial G(x; y)}{\partial \mathbf{n}} = 0 \qquad \text{on} \qquad \Gamma_3,$$

where $L_x(\cdot) := L(\cdot, \partial_x)$, $\frac{\partial G(x;y)}{\partial \mathbf{n}} := \mathbf{n}(x) \cdot \nabla_x G(x; y)$ with $\mathbf{n}(x)$ being the unit normal on $\Gamma_3$ pointing into $D_2$. We assume that, for all $y \in \mathbb{R}^2 \backslash \overline{D}_3, y \notin \Gamma_1 \cup \Gamma_2$, the function

$$x \mapsto (1 - \chi(||x - y|| \epsilon^{-1})) G(x; y)$$

belongs to $H^1_{loc}(\mathbb{R}^2 \backslash \overline{D}_3) \cap H^2_{loc}(D_j)$ $(j = 0, 1, 2)$ for each $\epsilon > 0$. Here $\chi(t)$ is a smooth function on $[0, +\infty)$ satisfying $\chi(t) = 1$ for $t \leq 1/2$ and $\chi(t) = 0$ for $t \geq 1$.

**Lemma 2.2.** *For* $y \in \mathbb{R}^2 \backslash \overline{D}_3, y \notin \Gamma_1 \cup \Gamma_2$*, the Green function* $G(x; y)$ *exists and is unique.*

**Proof.** If $G_1(x; y)$ and $G_2(x; y)$ are two Green functions for a fixed $y \in \mathbb{R}^2 \backslash \overline{D}_3$, $y \notin \Gamma_1 \cup \Gamma_2$, then $\tilde{G} = G_1 - G_2$ is infinitely smooth in a small neighborhood of $y$ and satisfies the radiation condition (2.5), the transmission conditions in (2.9) and the Neumann condition on $\Gamma_3$. It follows from Green's second theorem applied to each domain $D_j$ $(j = 0, 1, 2)$ and the Rellich identity that $\tilde{G} = 0$ in $\mathbb{R}^2 \backslash \overline{D}$, whence one obtains $\tilde{G} = 0$ $\mathbb{R}^2 \backslash \overline{D}_3$ as a consequence of Holmgren's uniqueness theorem. To verify the existence of the Green function, we may assume $y \in D_0$ without loss of generality, and make the ansatz

$$G(x;y) = H(x;y) + k_0^2 \tilde{\Psi}(x;y) \quad \text{with} \quad \tilde{\Psi}(x;y) := \begin{cases} \Psi(x;y) & \text{for} \quad x \in D_0 \backslash \{y\}, \\ 0 & \text{for} \quad x \notin D_0, \end{cases}$$

where $\Psi(x;y)$ denotes the fundamental solution to the Helmholtz equation $\triangle u + k_0^2 u = 0$ in the whole two dimensional space given by

$$\Psi(x;y) := \frac{i}{4} H_0^{(1)}(k_0|x-y|). \tag{2.11}$$

Note that $H_0^{(1)}(t)$ is the *Hankel function* of the first kind of order zero. We observe that $H(\cdot;y)$ satisfies the boundary value problem

$$\begin{aligned}
&\triangle H + k_j^2 H = 0 \quad \text{in} \quad D_j,\ j = 0,1,2,\\
&H_+ - H_- = k_0^2 \Psi(\cdot;y), \quad \frac{\partial H_+}{\partial \mathbf{n}} - \frac{k_0^2}{k_1^2}\frac{\partial H_-}{\partial \mathbf{n}} = k_0^2 \frac{\partial \Psi_+(\cdot;y)}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma_1,\\
&H_+ - H_- = 0, \quad \frac{\partial H_+}{\partial \mathbf{n}} - \frac{k_0^2}{k_1^2}\frac{\partial H_-}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_2,\\
&\frac{\partial H}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_3.
\end{aligned}$$

Since $H(\cdot;y)$ satisfies the radiation condition and $\Gamma_j$ is $C^2$-smooth, the above boundary value problem for $H(\cdot;y)$ can be transformed into an equivalent boundary integral equation system, and the existence of the solution in the Hölder space $C^2(D_j) \cap C^{1,\alpha}(\overline{D}_j)$ for $j = 0,1,2$ can always be guaranteed by the Fredholm alternative and the uniqueness of $G(\cdot;y)$. We refer to [24, Theorem 2.3] for a treatment in the case of one interface and [25] using the integral equation method applied to the Maxwell equations with general inhomogeneous transmission conditions for several interfaces. The existence and uniqueness of $G(x;y)$ when $y \notin D_0$ can be proved analogously. □

We denote by $G^\infty(\hat{x};y)$ the far field pattern of $G(x;y)$ as $||x|| \to +\infty$, and similarly by $\tilde{G}(x;y), \tilde{G}^\infty(\hat{x};y)$ the Green function and its far field pattern corresponding to another obstacle $\tilde{D}$. Let $u(y;-\hat{x}) := \exp(-ik_0 y \cdot \hat{x}) + u^s(y;-\hat{x})$ be the unique solution to the direct scattering problem (2.1)-(2.5) for the incident wave with the direction $-\hat{x}$. The far field pattern $G^\infty(\hat{x};y)$ is related to $u(y;-\hat{x})$ via the following lemma.

**Lemma 2.3.** *For all* $y \in D_0$, $G^\infty(\hat{x};y) = \eta k_0^2 u(y;-\hat{x})$, *where* $\eta = \frac{e^{i\pi/4}}{\sqrt{8\pi k_0}}$.

**Proof.** For a fixed $y \in D_0$, $G(x;y)$ satisfies the equation

$$\triangle_x G(x;y) + k_0^2 G(x;y) = -k_0^2 \delta(x-y) \quad \text{in } \mathbb{R}^2 \backslash \overline{D}_3, \tag{2.12}$$

in a distributional sense. It follows from Green's second theorem and the Sommerfeld radiation condition that

$$\begin{aligned}
u^s(y;-\hat{x}) &= \frac{1}{k_0^2} \int_{\Gamma_1} u_+^s(z;-\hat{x}) \frac{\partial G_+(z;y)}{\partial \mathbf{n}} - G_+(z;y) \frac{\partial u_+^s(z;-\hat{x})}{\partial \mathbf{n}} ds(z)\\
&= \frac{1}{k_0^2} \int_{\Gamma_1} u_+(z;-\hat{x}) \frac{\partial G_+(z;y)}{\partial \mathbf{n}} - G_+(z;y) \frac{\partial u_+(z;-\hat{x})}{\partial \mathbf{n}} ds(z)\\
&\quad - \frac{1}{k_0^2} \int_{\Gamma_1} e^{-ik_0 \hat{x}\cdot z} \frac{\partial G_+(z;y)}{\partial \mathbf{n}} - G_+(z;y) \frac{\partial e^{-ik_0 \hat{x}\cdot z}}{\partial \mathbf{n}} ds(z).
\end{aligned} \tag{2.13}$$

Applying Green's second theorem to the region $D$ and making use of the transmission conditions for $u(z;-\hat{x}), G(z;y)$ on $\Gamma_j$ $(j=1,2)$ and the Neumann condition on $\Gamma_3$, we obtain

$$\begin{aligned} & \int_{\Gamma_1} u_+(z;-\hat{x})\frac{\partial G_+(z;y)}{\partial \mathbf{n}} - G_+(z;y)\frac{\partial u_+(z;-\hat{x})}{\partial \mathbf{n}} ds(z) \\ = \ & \frac{k_0^2}{k_1^2}\int_{\Gamma_2} u_+(z;-\hat{x})\frac{\partial G_+(z;y)}{\partial \mathbf{n}} - G_+(z;y)\frac{\partial u_+(z;-\hat{x})}{\partial \mathbf{n}} ds(z) \\ = \ & \frac{k_0^2}{k_2^2}\int_{\Gamma_3} u_+(z;-\hat{x})\frac{\partial G_+(z;y)}{\partial \mathbf{n}} - G_+(z;y)\frac{\partial u_+(z;-\hat{x})}{\partial \mathbf{n}} ds(z) \\ = \ & 0, \end{aligned}$$

which together with (2.13) leads to

$$k_0^2 u^s(y;-\hat{x}) = \int_{\Gamma_1} G_+(z;y)\frac{\partial e^{-ik_0\hat{x}\cdot z}}{\partial \mathbf{n}} - e^{-ik_0\hat{x}\cdot z}\frac{\partial G_+(z;y)}{\partial \mathbf{n}} ds(z). \qquad (2.14)$$

It is seen from (2.12) and Green's second theorem applied to $G(x;y)$ that

$$G(x;y) = \int_{\Gamma_1} G_+(z;y)\frac{\partial \Psi(x;z)}{\partial \mathbf{n}} - \frac{\partial G_+(z;y)}{\partial \mathbf{n}}\Psi(x;z) ds(z) + k_0^2\Psi(x,y), \quad x \in D_0$$

where $\Psi(x;y)$, which is defined by (2.11), has the asymptotic behavior

$$\Psi(x;y) = \eta\frac{e^{ik_0||x||}}{\sqrt{||x||}}\left\{e^{-ik_0\hat{x}\cdot y} + O(\frac{1}{||x||})\right\} \quad \text{as} \quad ||x|| \to \infty. \qquad (2.15)$$

Inserting (2.15) into the above representation of $G(x;y)$, we obtain the following asymptotic behavior of $G(x;y)$ as $||x|| \to \infty$:

$$\eta\frac{e^{ik_0||x||}}{\sqrt{||x||}}\left\{\int_{\Gamma_1} G_+(z;y)\frac{\partial e^{-ik_0\hat{x}\cdot z}}{\partial \mathbf{n}} - e^{-ik_0\hat{x}\cdot z}\frac{\partial G_+(z;y)}{\partial \mathbf{n}} ds(z) + k_0^2 e^{-ik_0\hat{x}\cdot y} + O(\frac{1}{||x||})\right\}.$$

From (2.14) and the definition of the far field pattern in (2.6), we conclude that

$$\begin{aligned} G^\infty(\hat{x};y) &= \eta\left\{\int_{\Gamma_1} G_+(z;y)\frac{\partial e^{-ik_0\hat{x}\cdot z}}{\partial \mathbf{n}} - e^{-ik_0\hat{x}\cdot z}\frac{\partial G_+(z;y)}{\partial \mathbf{n}} ds(z) + k_0^2 e^{-ik_0\hat{x}\cdot y}\right\} \\ &= \eta k_0^2 u(y;-\hat{x}). \end{aligned}$$

The proof is thus complete. □

Based on Lemma 2.3, we may establish a relation between the fundamental solutions $G(x;y)$ and $\tilde{G}(x;y)$ for the two multilayered obstacles $D$ and $\tilde{D}$.

**Lemma 2.4.** *If $u^\infty(\hat{x};d) = \tilde{u}^\infty(\hat{x};d)$ for all $\hat{x}, d \in S$, then*

$$G(x;y) = \tilde{G}(x;y) \quad \textit{for all} \ \ x \neq y, \ x, y \in \Omega,$$

*where $\Omega$ denotes the unbounded connected component of $\mathbb{R}^2\backslash\overline{D \cup \tilde{D}}$.*

**Proof.** By Rellich's lemma [4], the assumption $u^\infty(\hat{x}; d) = \tilde{u}^\infty(\hat{x}; d)$ for all $\hat{x}, d \in S$ implies that $u(y; -\hat{x}) = \tilde{u}(y; -\hat{x})$ for all $y \in \Omega, \hat{x} \in S$. Recalling Lemma 2.3, we have $G^\infty(\hat{x}; y) = \tilde{G}^\infty(\hat{x}; y)$ for all $y \in \Omega$, and thus applying Rellich's lemma again gives the relation $G(x; y) = \tilde{G}(x; y)$ for all $x \neq y,\ x, y \in \Omega$. □

Given two functions $f(x)$ and $g(x)$, we say that $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x) = 1$. Obviously, if $f(x), g(x) \to \infty$ as $x \to x_0$ and $f(x) - g(x)$ is bounded in a neighborhood of $x_0$, then $f(x) \sim g(x)$ as $x \to x_0$. Analogously, given two sequences $f_n$ and $g_n$, we say that $f_n \sim g_n$ as $n \to +\infty$ if $\lim_{n\to\infty} f_n/g_n = 1$.

Our idea of proving Theorem 2.1 is to analyze the singularity of $G(x; y)$ as $y \to y_0, x \to y_0$ for some $y_0 \in \mathbb{R}^2 \backslash \overline{D}_3$. If $y_0 \in D_j$ for some $j \in \{0, 1, 2\}$, it can be readily deduced from the fundamental solution to the two dimensional Laplace equation that

$$G(x; y_0) \sim -\frac{k_j^2}{2\pi} \ln ||x - y_0|| \quad \text{as} \quad x \to y_0,$$

only depending on the wave number $k_j$ corresponding to $D_j$. In the following we are going to investigate the singularity of that when $y_0 \in \Gamma_j\ (j = 1, 2)$, which turns out to depend on both $k_j$ and $k_{j-1}$. Thus, with the help of Lemma 2.4, a contradiction can always be derived if two different multilayered obstacles generate the same far field data for all incident directions. This will be carried out in Section 2.3.

We need to pay attention to the Green function $G(x; y)$, which exists if $y$ does not belong to the interfaces $\Gamma_j\ (j = 1, 2)$. In the case of $y_0 \in \Gamma_j$ for some $j \in \{1, 2\}$, we define a sequence $y_n$ by

$$y_n = y_0 + \frac{1}{n}\mathbf{n}(y_0), \qquad n = 1, 2, \cdots. \tag{2.16}$$

By the symmetry of $G(x; y)$, we can define $G(y_n; y_0)$ with some fixed $n$ in the following way

$$G(y_n; y_0) := G(y_0; y_n) = \lim_{m\to+\infty} G(y_0 + \frac{1}{m}\mathbf{n}(y_0); y_n);$$

note that the limit exists because $\Gamma_j$ is $C^2$-smooth and the function $G(\cdot; y_n)$ is continuous up to $\Gamma_j$. The following lemma plays an important role in this paper.

**Lemma 2.5.** *For a fixed $y_0 \in \Gamma_j$ with $j \in \{1, 2\}$, we have*

$$G(y_n; y_0) \sim -\frac{k_j^2 k_{j-1}^2}{\pi(k_{j-1}^2 + k_j^2)} \ln ||y_n - y_0|| \quad \textit{as} \quad n \to +\infty,$$

*where the sequence $y_n$ is defined by (2.16).*

Before proving Lemma 2.5, we introduce the following auxiliary transmission problem in a half-space for the Laplace equation

$$\triangle_x G(x; y) = -k_{j-1}^2 \delta(x - y), \quad x \in \mathbb{R}^2_+ := \{x_2 > 0\}, \tag{2.17}$$

$$\triangle_x G(x; y) = -k_j^2 \delta(x - y), \quad x \in \mathbb{R}^2_- := \{x_2 < 0\}, \tag{2.18}$$

$$G(x; y)_+ = G(x; y)_-, \quad \frac{1}{k_{j-1}^2}\frac{\partial G(x; y)_+}{\partial x_2} = \frac{1}{k_j^2}\frac{\partial G(x; y)_-}{\partial x_2} \qquad \text{on } x_2 = 0, \tag{2.19}$$

with the following condition at infinity

$$\lim_{||x||\to+\infty} G(x;y)=0. \tag{2.20}$$

**Lemma 2.6.** *The unique solution $G(x;y)$ to (2.17)-(2.20) with $y=O=(0,0)$ is given by*

$$G(x;O)=-\frac{1}{\pi^2}\frac{k_j^2k_{j-1}^2}{k_j^2+k_{j-1}^2}\left\{\int_{\mathbb{R}}\frac{\ln(|t|x_2|+x_1|)}{1+t^2}dt+\gamma\pi\right\},\quad x\neq O,$$

*where $\gamma$ denotes the Euler-Mascheroni constant. In particular,*

$$G((0,x_2);(0,0))=-\frac{1}{\pi}\frac{k_j^2k_{j-1}^2}{k_j^2+k_{j-1}^2}(\ln|x_2|+\gamma).$$

**Proof.** Throughout the paper, we define the Fourier and inverse Fourier transformation of an integrable function $f(t)$ by

$$F[f](\xi)=\hat{f}(\xi):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(t)e^{-i\xi\cdot t}dt,\quad F^{-1}[\hat{f}](t)=f(t):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(\xi)e^{i\xi t}d\xi.$$

Denote by $\mathcal{S}$ the Schwartz space or space of rapidly decreasing functions on $\mathbb{R}$ and by $\mathcal{S}'$ its dual space. The Fourier transform gives a homeomorphism of $\mathcal{S}$ onto itself. Given a tempered distribution $T\in\mathcal{S}'$, we define the Fourier transform of $T$ by

$$\hat{T}(\varphi)=T(\hat{\varphi})\quad\text{for all}\quad\varphi\in\mathcal{S}.$$

We refer to [18] for basic properties of the Fourier transformation of tempered distributions, and [9, Appendices 2 and 3] for some elementary calculations related to the Fourier and inverse Fourier transforms in the subsequent analysis.

Following Ramm's approach in the three dimensional space (see [29]), we take the Fourier transformation of (2.17) and (2.18) with respect to $x_1$ to get

$$\frac{d^2\hat{G}(\xi,x_2)}{dx_2^2}-|\xi|^2\hat{G}(\xi,x_2)=\begin{cases}-\frac{k_{j-1}^2}{\sqrt{2\pi}}\delta(x_2-y_2) & \text{if } x_2>0,\\ -\frac{k_j^2}{\sqrt{2\pi}}\delta(x_2-y_2) & \text{if } x_2<0,\end{cases} \tag{2.21}$$

with the following boundary conditions on $x_2=0$ and at $x_2=\pm\infty$:

$$\lim_{x_2\to0^+}\hat{G}(\xi,x_2)=\lim_{x_2\to0^-}\hat{G}(\xi,x_2),\ \lim_{x_2\to0^+}\frac{1}{k_{j-1}^2}\frac{\partial\hat{G}(\xi,x_2)}{\partial x_2}=\lim_{x_2\to0^-}\frac{1}{k_j^2}\frac{\partial\hat{G}(\xi,x_2)}{\partial x_2}, \tag{2.22}$$

$$\lim_{|x_2|\to\infty}\hat{G}(\xi,x_2)=0. \tag{2.23}$$

Note that in (2.21)-(2.23) we write $\hat{G}(\xi,x_2)=\hat{G}((\xi,x_2);(0,y_2))$ for simplicity. Assume $y_2>0$. Then the generalized solution to (2.21) and (2.23) is given by

$$\hat{G}(\xi,x_2)=v(\xi,x_2)+Ce^{-|\xi||x_2|} \tag{2.24}$$

with a constant $C \in \mathbb{C}$ and the distribution $v(\xi, x_2)$ satisfying

$$\frac{d^2 v(\xi, x_2)}{dx_2^2} - |\xi|^2 v(\xi, x_2) = -\frac{k_{j-1}^2}{\sqrt{2\pi}} \delta(x_2 - y_2), \ \ x_2 \in \mathbb{R}.$$

Taking the Fourier transformation of the above equation with respect to $x_2$ yields

$$\hat{v}(\xi, \eta) = F[v(\xi, x_2)](\eta) = \frac{k_{j-1}^2}{2\pi} \frac{e^{i\eta y_2}}{\eta^2 + \xi^2},$$

and then by the inverse Fourier transformation with respect to $\eta$, we have

$$\begin{aligned} v(\xi, x_2) &= F^{-1}[\hat{v}(\xi, \eta)](x_2) \\ &= \frac{k_{j-1}^2}{2\pi\sqrt{2\pi}} F^{-1}[e^{i\eta y_2}](x_2) * F^{-1}[\frac{1}{\eta^2 + \xi^2}](x_2) \\ &= \frac{k_{j-1}^2}{2\pi\sqrt{2\pi}} \sqrt{2\pi}\delta(x_2 - y_2) * \sqrt{\frac{\pi}{2}} \frac{1}{|\xi|} e^{-|\xi||x_2|} \\ &= \frac{k_{j-1}^2}{2\sqrt{2\pi}} \frac{1}{|\xi|} e^{-|\xi||x_2 - y_2|}, \end{aligned}$$

where $*$ denotes convolution. Inserting the above function $v(\xi, x_2)$ into (2.24), we deduce from the transmission conditions (2.22) that

$$C = \frac{k_{j-1}^2}{2\sqrt{2\pi}} \frac{k_j^2 - k_{j-1}^2}{k_j^2 + k_{j-1}^2} \frac{e^{-|\xi| y_2}}{|\xi|},$$

and thus

$$\hat{G}(\xi, x_2) = \frac{k_{j-1}^2}{2\sqrt{2\pi}} \frac{1}{|\xi|} (e^{-|\xi||x_2 - y_2|} + \frac{k_j^2 - k_{j-1}^2}{k_j^2 + k_{j-1}^2} e^{-|\xi|(|x_2| + y_2)}) \quad \text{for } y_2 > 0. \tag{2.25}$$

Analogously, we obtain

$$\hat{G}(\xi, x_2) = \frac{k_j^2}{2\sqrt{2\pi}} \frac{1}{|\xi|} (e^{-|\xi||x_2 - y_2|} + \frac{k_j^2 - k_{j-1}^2}{k_j^2 + k_{j-1}^2} e^{-|\xi|(|x_2| - y_2)}) \quad \text{for } y_2 < 0. \tag{2.26}$$

Next we need to calculate $G((x_1, x_2); (0, y_2))$ by taking the inverse Fourier transformations of (2.25) and (2.26) with respect to $\xi$, and then to analyze the limit of $G((x_1, x_2); (0, y_2))$ as $y_2 \to 0, x_1 \to 0$.

By properties of the inverse Fourier transformation for tempered distributions, we first note that, for $\tau \in \mathbb{R}^+$,

$$\begin{aligned} J_1(x_1, \tau) := F^{-1}[\frac{e^{-|\xi|\tau}}{\tau}](x_1) &= \frac{1}{\sqrt{2\pi}} F^{-1}[\frac{1}{|\xi|}](x_1) * F^{-1}[e^{-|\xi|\tau}](x_1) \\ &= \frac{1}{\sqrt{2\pi}} \frac{-2(\ln|x_1| + \gamma)}{\sqrt{2\pi}} * \sqrt{\frac{2}{\pi}} \frac{\tau}{|x_1|^2 + \tau^2} \\ &= \frac{-2\tau}{\pi\sqrt{2\pi}} (\int_{\mathbb{R}} \frac{\ln|t| + \gamma}{\tau^2 + |x_1 - t|^2} dt), \end{aligned}$$

where $\gamma$ denotes the Euler-Mascheroni constant. Note that

$$\int_{\mathbb{R}} \frac{\ln|t|+\gamma}{\tau^2+|x_1-t|^2}dt < +\infty, \qquad \text{for} \quad \tau\in\mathbb{R}^+, \tau\neq 0,\ x_1\in\mathbb{R}.$$

Then, taking the inverse transformation of (2.25) gives

$$\begin{aligned}
& \lim_{y_2\to 0^+} G((x_1,x_2);(0,y_2)) \\
= & \lim_{y_2\to 0^+} F^{-1}[G((\xi,x_2);(0,y_2))](x_1) \\
= & \lim_{y_2\to 0^+} \frac{k_{j-1}^2}{2\sqrt{2\pi}}\left\{J_1(x_1,|x_2-y_2|)+\frac{k_j^2-k_{j-1}^2}{k_j^2+k_{j-1}^2}J_1(x_1,|x_2|+y_2)\right\} \\
= & \frac{k_{j-1}^2}{2\sqrt{2\pi}}\left\{J_1(x_1,|x_2|)+\frac{k_j^2-k_{j-1}^2}{k_j^2+k_{j-1}^2}J_1(x_1,|x_2|)\right\} \\
= & \frac{1}{\sqrt{2\pi}}\frac{k_j^2k_{j-1}^2}{k_j^2+k_{j-1}^2}J_1(x_1,|x_2|).
\end{aligned} \tag{2.27}$$

The same result as in (2.27) remains true when $y_2\to 0^-$ by taking the inverse transformation of (2.26). Thus, employing some simple calculations we arrive at

$$G(x;O) = -\frac{1}{\pi^2}\frac{k_j^2k_{j-1}^2}{k_j^2+k_{j-1}^2}\left\{\int_{\mathbb{R}}\frac{\ln(|t|x_2|+x_1|)}{1+t^2}dt+\gamma\pi\right\}. \tag{2.28}$$

We end up the proof by calculating $G((0,x_2);(0,0))$. Clearly,

$$\begin{aligned}
\int_{\mathbb{R}}\frac{\ln(|t|x_2|+x_1|)}{1+t^2}dt|_{x_1=0} &= \int_{\mathbb{R}}\frac{\ln(|tx_2|)}{1+t^2}dt \\
&= \int_{\mathbb{R}}\frac{\ln|t|}{1+t^2}dt+\ln|x_2|\int_{\mathbb{R}}\frac{1}{1+t^2}dt \\
&= \pi\ln|x_2|,
\end{aligned}$$

which together with (2.28) yields the second assertion of Lemma 2.6 for $G((0,x_2);(0,0))$. □

Now, we are in a position to prove Lemma 2.5.

**Proof of Lemma 2.5.** Let $y_0\in\Gamma_j$ for some $j\in\{1,2\}$ be fixed. Since the Helmholtz equation remains invariant under coordinate translations and rotations, we may suppose that the origin is located at $y_0$ and the $x_2$-axis is tangent to $\Gamma_j$ at $y_0$. Furthermore, without loss of generality, the unit normal $\mathbf{n}(y_0)$ to $\Gamma_j$ at $y_0$ is supposed to coincide with $e_2:=(0,1)$ so that the sequence $y_n$ defined by (2.16) can be written as $y_n=(0,1/n)$. To prove Lemma 2.5, we only need to show that

$$G((0,y_2);(0,0)) \sim -\frac{k_j^2k_{j-1}^2}{\pi(k_{j-1}^2+k_j^2)}\ln|y_2|, \quad \text{as} \quad |y_2|\to 0. \tag{2.29}$$

From the assumption on the regularity of $\Gamma_j$, it follows that the curve $B_\delta(y_0)\cap\Gamma_j$ for some sufficiently small $\delta>0$ can be represented as a $C^2$-smooth function $x_2=f(x_1)$, $x_1\in(-a,a)$ for some small $a>0$, satisfying $f(0)=0, f'(0)=0$. We next prove the lemma by flattening the curve in a neighborhood of $O$.

Set $V(y_1, y_2) := G(y_1, y_2 + f(y_1); O)$,where $G(x_1, x_2; O)$ is the fundamental solution of the scattering problem (2.8)-(2.9) in the new coordinate system with the origin centered at $y_0 \in \Gamma_j$ for some $j \in \{1, 2\}$. After some calculations, we see that $V(y)$ fulfills

$$\tilde{L}V(y) = -k^2\delta(y), \quad \text{in } \tilde{D} = \tilde{D}^+ \cup \tilde{D}^-, \tilde{D}^+ = B_a(O) \cap \{y_2 > 0\}, \tilde{D}^- = B_a(O) \cap \{y_2 < 0\},$$
$$V(y) \text{ satisfies the transmission conditions in (2.9) for } y \in (B_a(O) \cap \{y_2 = 0\})\backslash\{O\},$$

where

$$\tilde{L}V = \tilde{L}(V, \partial_y) := \frac{\partial^2 V}{\partial y_1^2} + \frac{\partial^2 V}{\partial y_2^2}(1 + f^{'}(y_1)^2) - 2f^{'}(y_1)\frac{\partial^2 V}{\partial y_1 \partial y_2} - f^{''}(y_1)\frac{\partial V}{\partial y_2} + k^2 V,$$

with $k = k_{j-1}$ in $\tilde{D}^+$ and $k = k_j$ in $\tilde{D}^-$. Let $U(x_1, x_2) := U(x; O)$ be the unique solution to (2.17)-(2.20) obtained in Lemma 2.6, and set $W(y) = V(y) - U(y)$. Then, we see that

$$\tilde{L}W = g, \quad \text{in} \quad \tilde{D} = \tilde{D}^+ \cup \tilde{D}^-,$$
$$W(y) \text{ satisfies the transmission conditions in (2.9) for } y \in (B_a(O) \cap \{y_2 = 0\})\backslash\{O\},$$

where

$$g(y) = -f^{'}(y_1)^2\frac{\partial^2 U}{\partial y_2^2} + 2f^{'}(y_1)\frac{\partial^2 U}{\partial y_1 \partial y_2} + f^{''}(y_1)\frac{\partial U}{\partial y_2} - k^2 U,$$

Since $U(y_1, y_2)$ is an analytic function in $\tilde{D}^+ \cup \tilde{D}^-$, making use of the explicit form of $U$ as shown in Lemma 2.6, by direct computations one may check that

$$|\frac{\partial^2 U}{\partial y_2^2}|, |\frac{\partial^2 U}{\partial y_1 \partial y_2}| \le \frac{C}{r^2}, \quad |\frac{\partial U}{\partial y_2}| \le \frac{C}{r}, \quad |k^2 U| \le C \ln\frac{1}{r}, \quad \text{in} \quad B_a(O),$$

for some $C > 0$, with $r = (y_1^2 + y_2^2)^{1/2}$. On the other hand, there exists some positive constant $M(a) > 0$ such that

$$f^{''}(y_1) \le M, \quad f^{'}(y_1) \le Mr, \quad \text{for} \quad |y_1| < a.$$

Combining the previous estimates, one obtains $|g(y)| \le C^{'}\frac{1}{r}$ for some $C^{'} > 0$, leading to $g(y) \in H^{-\epsilon}(\tilde{D}^+) \cap H^{-\epsilon}(\tilde{D}^-)$ for some $\epsilon > 0$. Since the differential operator $\tilde{L}$ is uniformly elliptic in $B_a(O)$ for sufficiently small $a > 0$, the standard elliptic regularity implies that $W(y) \in H^{2-\epsilon}(\tilde{D}^+) \cap H^{2-\epsilon}(\tilde{D}^-)$ (see [12]). Applying the Sobolev imbedding theorem and recalling the transmission conditions for $U$ and $V$ on $\{y_2 = 0\} \cap B_a(O)$ yield that $W(y) \in C(B_a(O))$, i.e., $W(y)$ is continuous across the interface $\{y_2 = 0\} \cap B_a(O)$. This implies that $V(y) \sim U(y)$ as $||y|| \to 0$, and in particular $V(0, y_2) \sim U(0, y_2)$ as $y_2 \to 0$. Noting that $V(0, y_2) = G((0, y_2); (0, 0))$ , we have proved (2.29) as a consequence of the second assertion of Lemma 2.6. The proof is thus complete.

□

## 2.3 Proof of Theorem 2.1

Relying on the asymptotic behavior of $G(x; y)$ as $x \to y$, we next prove Theorem 2.1 by the following steps.

**Step 1:** Proof of $\Gamma_1 = \tilde{\Gamma}_1$.

Assume $\Gamma_1 \neq \tilde{\Gamma}_1$. Let $\Omega$ be the unbounded connected component of $\mathbb{R}^2\backslash(\overline{D \cup \tilde{D}})$. Without loss of generality, we may assume that there exists $y_0 \in \tilde{\Gamma}_1 \cap (\mathbb{R}^2\backslash\overline{D}) \cap \partial\Omega$. Let $y_n$ be defined as in (2.16) and define two functions $F(x), \tilde{F}(x)$ by

$$F(x) := -\frac{2\pi G(x; y_0)}{\ln ||x - y_0||}, \quad \tilde{F}(x) := -\frac{2\pi \tilde{G}(x; y_0)}{\ln ||x - y_0||}, \tag{2.30}$$

where $G(x; y)$ and $\tilde{G}(x; y)$ are the Green functions corresponding to $D = (\Gamma_1, \Gamma_2, \Gamma_3, k_1, k_2)$ and $\tilde{D} = (\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3, \tilde{k}_1, \tilde{k}_2)$, respectively. Since $y_n$ is contained in $D_0 \cap \Omega$ for sufficiently large $n$, it follows from Lemma 2.5 that

$$\lim_{n\to+\infty} F(y_n) = k_0^2, \quad \lim_{n\to+\infty} \tilde{F}(y_n) = \frac{2k_0^2\tilde{k}_1^2}{k_0^2 + \tilde{k}_1^2},$$

leading to

$$\lim_{n\to+\infty} [F(y_n) - \tilde{F}(y_n)] = \frac{k_0^2(k_0^2 - \tilde{k}_1^2)}{k_0^2 + \tilde{k}_1^2}. \tag{2.31}$$

However, by Lemma 2.4 we have

$$\tilde{F}(y_n) = F(y_n) \quad \text{for all sufficiently large } n,$$

which contradicts (2.31) because $k_0 \neq \pm\tilde{k}_1$. Hence $\Gamma_1 = \tilde{\Gamma}_1$.

**Step 2:** Proof of $k_1 = \tilde{k}_1$.

Choose $y_0 \in \Gamma_1 = \tilde{\Gamma}_1$, and define $y_n$, $F(x)$, $\tilde{F}(x)$ in the same way as in (2.16) and (2.30). Combining Lemma 2.4 and Lemma 2.5 gives the identity

$$0 = \lim_{n\to+\infty} [F(y_n) - \tilde{F}(y_n)] = \frac{2k_0^2 k_1^2}{k_0^2 + k_1^2} - \frac{2k_0^2\tilde{k}_1^2}{k_0^2 + \tilde{k}_1^2} = \frac{2k_0^4(k_1^2 - \tilde{k}_1^2)}{(k_0^2 + k_1^2)(k_0^2 + \tilde{k}_1^2)},$$

from which $k_1 = \tilde{k}_1$ follows.

**Step 3:** Proof of $\Gamma_2 = \tilde{\Gamma}_2$, $k_2 = \tilde{k}_2$.

Recall that $\Gamma_1 = \tilde{\Gamma}_1$ and $k_1 = \tilde{k}_1$. It follows from Holmgren's uniqueness theorem and Lemma 2.4 that $G(x; y) = \tilde{G}(x; y)$ for all $x \neq y$, $y \in D_0 = \tilde{D}_0$ and $x \in \Omega_0$, where $\Omega_0$ denotes the unbounded connected component of $\mathbb{R}^2\backslash(\overline{(D_2 \cup D_3) \cup (\tilde{D}_2 \cup \tilde{D}_3)})$. Making use of symmetries of $G(x; y)$ and $\tilde{G}(x; y)$, which can be readily proved by applying Green's formula, we arrive at $G(x; y) = \tilde{G}(x; y)$ for all $x \neq y$, $x, y \in \Omega$. Thus, analogously to Steps 1 and 2, one can prove $\Gamma_2 = \tilde{\Gamma}_2$ and $k_2 = \tilde{k}_2$ using Lemma 2.5.

**Step 4:** Proof of $\Gamma_3 = \tilde{\Gamma}_3$.

Combining Steps 1-3 and Holmgren's uniqueness theorem, we see that $G(x; y) = \tilde{G}(x; y)$ for all $x \neq y$, $x, y \in \Omega_1$, where $\Omega_1$ denotes the unbounded connected component of $\mathbb{R}^2\backslash(\overline{D_3 \cup \tilde{D}_3})$.

Assume $\Gamma_3 \neq \tilde{\Gamma}_3$. Without loss of generality, we may assume that there exists $y_0 \in \tilde{\Gamma}_3 \cap (\mathbb{R}^2 \backslash \overline{D_3}) \cap \partial\Omega_1$. Define a sequence $y_n$ by

$$y_n := y_0 + \frac{1}{n}\mathbf{n}(y_0), \quad n = 1, 2, \cdots, \tag{2.32}$$

where $\mathbf{n}(y_0)$ is the outward unit normal to $\tilde{\Gamma}_3$ at $y_0$, and define two functions $F_1(y), \tilde{F}_1(y)$ by

$$\tilde{F}_1(y) = \mathbf{n}(y_0) \cdot \nabla_x \tilde{G}(x;y)|_{x=y_0}, \qquad F_1(y) = \mathbf{n}(y_0) \cdot \nabla_x G(x;y)|_{x=y_0}.$$

It follows from the Neumann boundary condition for $\tilde{G}(x;y)$ on $\tilde{\Gamma}_3$ that

$$\tilde{F}_1(y_n) = 0 \qquad \text{for all sufficiently large } n \in \mathbb{N},$$

and from Lemma 2.5 that

$$|F_1(y_n)| \to +\infty \qquad \text{as } n \to +\infty.$$

This contradiction implies that $\Gamma_3 = \tilde{\Gamma}_3$. The proof is complete. □

## 2.4 Uniqueness under general transmission conditions

In acoustic scattering problems, one needs to consider a problem modeled by (2.1) and (2.4) with the following conditions

$$u_+ = u_-, \ \frac{\partial u_+}{\partial \mathbf{n}} = \lambda_j \frac{\partial u_-}{\partial \mathbf{n}} \text{ on } \Gamma_j, \ j = 1, 2; \tag{2.33}$$

$$B(u) = 0 \quad \text{on } \Gamma_3; \tag{2.34}$$

$$\lim_{r\to\infty} r^{\frac{N-1}{2}} (\frac{\partial u^s}{\partial r} - ik_0 u^s) = 0. \tag{2.35}$$

Here, the transmission coefficient $\lambda_j$ denotes the ratio of mass densities in $D_j$ and $D_{j+1}$ satisfying $\lambda_j \neq 1$ and $\lambda_j > 0$; $N$ represents the dimension of the space ($N = 2$ or $N = 3$); and the boundary condition on $\Gamma_3$ may take one of the following forms:

$$B(u) := \begin{cases} u & \text{if the pressure vanishes on } \Gamma_3 \ (D_3 \text{ is a sound-soft obstacle}), \\ \frac{\partial u}{\partial \mathbf{n}} & \text{if the normal velocity vanishes on } \Gamma_3 \ (D_3 \text{ is a sound-hard obstacle}), \\ \frac{\partial u}{\partial \mathbf{n}} + i\eta u & \text{if the normal velocity is proportional to the pressure on } \Gamma_3, \end{cases}$$

where $\eta > 0$ is a constant. In this section, we extend the argument in Sections 2.1-2.3 to prove uniqueness under the general transmission conditions (2.33). Note that the results in this section are not limited to two dimensions.

The Green function $G(x;y)$ in this case is defined as follows:

$$L_x G(x;y) = \nabla \cdot (a(x) \nabla G(x;y)) + b(x)G(x;y) = -\delta(x-y), \ x \in \mathbb{R}^N \backslash \overline{D}_3, \ y \notin \Gamma_j, \tag{2.36}$$

$$G_+ = G_-, \ a^+ \frac{\partial G_+}{\partial \mathbf{n}} = a^- \frac{\partial G_-}{\partial \mathbf{n}} \qquad \text{on } \Gamma_j, \ j = 1, 2, \tag{2.37}$$

$$G(x;y) \text{ satisfies the boundary condition on } \Gamma_3 \text{ and the radiation condition,} \tag{2.38}$$

where

$$a(x) = \begin{cases} 1, & x \in D_0, \\ \lambda_1, & x \in D_1, \\ \lambda_1\lambda_2, & x \in D_2; \end{cases} \qquad b(x) = \begin{cases} k_0^2, & x \in D_0, \\ \lambda_1 k_1^2, & x \in D_1, \\ \lambda_1\lambda_2 k_2^2, & x \in D_2. \end{cases}$$

If $N = 2$, it follows from Lemma 2.5 that

$$G(x; y_0) \sim -\frac{\ln ||x - y_0||}{2\pi a(y_0)} \text{ as } x \to y_0, \text{ if } y_0 \in D_j, \quad j = 0, 1, 2,$$
$$G(y_n; y_0) \sim -\frac{\ln ||y_n - y_0||}{\pi[a^+(y_0) + a^-(y_0)]} \text{ as } n \to +\infty, \quad \text{if } y_0 \in \Gamma_j, \quad j = 1, 2;$$

if $N = 3$, using an argument similar to Lemma 2.5 one obtains that (see also [29])

$$G(x; y_0) \sim \frac{1}{4\pi a(y_0)||x - y_0||} \text{ as } x \to y_0, \text{ if } y_0 \in D_j, \quad j = 0, 1, 2,$$
$$G(y_n; y_0) \sim \frac{1}{2\pi[a^+(y_0) + a^-(y_0)]||x - y_0||} \text{ as } n \to +\infty, \quad \text{if } y_0 \in \Gamma_j, \quad j = 1, 2;$$

where $y_n$ is a sequence defined as in (2.16), and

$$a^+(y_0) = \lim_{j \to +\infty} a(y_0 + \frac{1}{j}\mathbf{n}(y_0)), \quad a^-(y_0) = \lim_{j \to +\infty} a(y_0 - \frac{1}{j}\mathbf{n}(y_0)).$$

Recall that $\mathbf{n}(y_0)$ stands for the unit outward normal at $y_0 \in \Gamma_j$. Our inverse problem corresponding to (2.1), (2.4), (2.33)-(2.35) is:

**(IP')** Given the wave numbers $k_j$ $(j = 0, 1, 2)$ and the far field pattern $u^\infty(\hat{x}; d)$ for all $\hat{x}, d \in S$, determine the interfaces $\Gamma_j$ $(j = 1, 2, 3)$, the transmission coefficients $\lambda_j$ $(j = 1, 2)$ and the boundary condition on $\Gamma_3$.

Note that the boundary condition on $\Gamma_3$ tells us the physical property of the impenetrable core $D_3$. Let $D = (\Gamma_1, \Gamma_2, \Gamma_3, \lambda_1, \lambda_2, B)$ and $\tilde{D} = (\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{B})$ denote two multilayered obstacles with the boundary conditions $B$, $\tilde{B}$ on $\Gamma_3$, $\tilde{\Gamma}_3$, respectively. Following the approach in Sections 2.2 and 2.3, we establish the uniqueness to (IP') under the general transmission conditions (2.33).

**Corollary 2.7.** *Assume $N = 2$ or $N = 3$, and $\lambda_j, \tilde{\lambda}_j \neq 1$ for $j = 1, 2$. Suppose $u^\infty(\hat{x}; d)$, $\tilde{u}^\infty(\hat{x}; d)$ are the far field patterns corresponding to $D$, $\tilde{D}$, respectively. If*

$$u^\infty(\hat{x}; d) = \tilde{u}^\infty(\hat{x}; d) \quad \textit{for all } \hat{x}, d \in S,$$

*then $D = \tilde{D}$, that is, $\Gamma_j = \tilde{\Gamma}_j$ $(j = 1, 2, 3)$, $\lambda_i = \tilde{\lambda}_i$ $(i = 1, 2)$ and $B = \tilde{B}$.*

**Proof.** From Rellich's lemma, we see that Lemma 2.4 remains valid under the general transmission condition (2.33). Then, using the assumptions that $\lambda_j \neq 1, \tilde{\lambda}_j \neq 1$ and repeating Step 1 of the proof of Theorem 2.1, we have $\Gamma_1 = \tilde{\Gamma}_1$ and $\lambda_1 = \tilde{\lambda}_1$. Since the wave numbers $k_1$ and $k_2$ are given, we may proceed to justify that $\Gamma_2 = \tilde{\Gamma}_2$ and $\lambda_2 = \tilde{\lambda}_2$. This implies that the surrounding media around $D_3$ and $\tilde{D}_3$ can be uniquely identified. To prove $\Gamma_3 = \tilde{\Gamma}_3$, we may define $F_2(y_n) := B(G(x; y_n))|_{x=y_0}$ and $\tilde{F}_2(y_n) := \tilde{B}(\tilde{G}(x; y_n))|_{x=y_0}$ with $y_n$, $y_0$ defined in the same way as (2.32). Then, we obtain $\Gamma_3 = \tilde{\Gamma}_3$ by an argument analogous to Step 4 of the proof of Theorem 2.1 and $B = \tilde{B}$ as a consequence of Holmgren's uniqueness theorem. □

**Remark 2.8.** *In the case of the TM mode, Theorem 3.2 improves the uniqueness results in [23, 33] which both require a known piecewise homogenous background, while in three dimensions Corollary 2.7 improves those in [1, 24, 33] which suppose that the transmission coefficients $\lambda_j$ are known. In addition, for recovering the interfaces, the orthogonality relation used in [1, 33] and the a priori estimates of solutions on the interface essentially required by [24] are both avoided. If the background refractive indices and the transmission coefficients are not available in advance, we do not know how to prove uniqueness from the knowledge of the far field at a fixed frequency. We refer to Isakov [15, 17] and Kirsch $\&$ Kress [21] for uniqueness on the inverse scattering by a penetrable obstacle in a known homogeneous background medium.*

# 3 Inverse scattering by multilayered periodic structures

In this section, we assume that a time-harmonic electromagnetic wave is scattered by a multi-layered diffraction grating in a piecewise homogeneous isotropic medium. Suppose further that the grating is periodic in $x_1$-direction and constant in $x_3$-direction. We still restrict ourselves to the TM mode (transverse magnetic polarization), which means that the time-harmonic Maxwell equation can be reduced to a two dimensional scalar Helmholtz equation $(\triangle + k^2)u = 0$ where $u = u(x_1, x_2)$ is the third component of the magnetic field.

## 3.1 Mathematical formulations

Without loss of generality, we assume the cross-sections of the grating profiles in the $(x_1, x_2)$-plane are given by two $C^2$-smooth disjoint graphs $\Gamma_j := \{x_2 = f_j(x_1), x_1 \in \mathbb{R}\}$, $j = 1, 2$, which are $2\pi$-periodic with respect to $x_1$. Denote the region above $\Gamma_1$ by $D_0$, the one below $\Gamma_2$ by $D_2$, and that between $\Gamma_1$ and $\Gamma_2$ by $D_1$; see Figure 2. The three distinct constant refractive

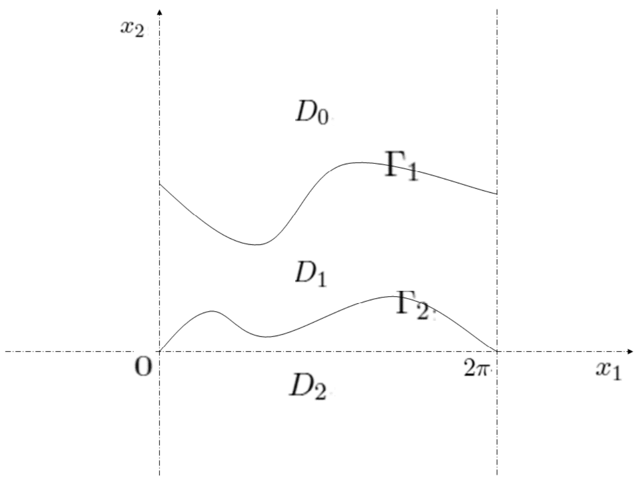


Figure 2: Multilayered periodic structures

indices corresponding to $D_i$ are denoted by $k_i$ $(i = 0, 1, 2)$, respectively, satisfying $k_0, k_2 > 0$, Re$k_1 > 0$ and Im$k_1 \geq 0$. Let

$$\Gamma_1^+ := \max_{x_1 \in \mathbb{R}} \{f_1(x_1)\}, \quad \Gamma_2^- := \min_{x_1 \in \mathbb{R}} \{f_2(x_1)\}.$$

Suppose that a plane wave in the $(x_1, x_2)$-plane given by

$$u^i = \exp(i(\alpha x_1 - \beta x_2)),$$

with $(\alpha, \beta) = k_0(\sin\theta, -\cos\theta)$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, is incident upon the grating from the top. Then, the total field $u = u(x_1, x_2)$ satisfies

$$\Delta u + k_j^2 u = 0 \quad \text{in } D_j, \; j = 0, 1, 2, \tag{3.39}$$

$$u_+ = u_-, \; \frac{1}{k_{j-1}^2} \frac{\partial u_+}{\partial \mathbf{n}} = \frac{1}{k_j^2} \frac{\partial u_-}{\partial \mathbf{n}} \quad \text{on } \Gamma_j, \; j = 1, 2, \tag{3.40}$$

$$u = u^i + u^s \quad \text{in } D_0, \tag{3.41}$$

with the following two radiation conditions as $x_2 \to \pm\infty$:

$$u^s = \sum_{n \in \mathbb{Z}} A_n^+ \exp(i\alpha_n x_1 + i\beta_n^+ x_2), \quad \text{for } x_2 > \Gamma_1^+, \tag{3.42}$$

$$u = \sum_{n \in \mathbb{Z}} A_n^- \exp(i\alpha_n x_1 - i\beta_n^- x_2), \quad \text{for } x_2 < \Gamma_2^-, \tag{3.43}$$

where $\alpha_n = n + \alpha$ and

$$\beta_n^+ := \begin{cases} (k_0^2 - \alpha_n^2)^{\frac{1}{2}} & \text{if } |\alpha_n| \leq k_0, \\ i(\alpha_n^2 - k_0^2)^{\frac{1}{2}} & \text{if } |\alpha_n| > k_0; \end{cases} \quad \beta_n^- := \begin{cases} (k_2^2 - \alpha_n^2)^{\frac{1}{2}} & \text{if } |\alpha_n| \leq k_2, \\ i(\alpha_n^2 - k_2^2)^{\frac{1}{2}} & \text{if } |\alpha_n| > k_2, \end{cases}$$

with $i = \sqrt{-1}$. Here $\mathbf{n}$ denotes the unit normal to $\Gamma_j$ with a non-negative $x_2$-component; the expansions in (3.42) and (3.43) are the well-known Rayleigh expansions; $A_n^\pm \in \mathbb{C}$ $(n \in \mathbb{Z})$ are called the Rayleigh coefficients. Obviously, in $x_2 > \Gamma_1^+$ resp. $x_2 < \Gamma_2^-$, the scattered field $u^s$ resp. $u$ can be split into a finite sum of outgoing plane waves propagating into the far field and an infinite sum of exponentially decreasing functions as $x_2 \to +\infty$ resp. $x_2 \to -\infty$ which are called surface or evanescent waves. Thus, the inverse diffraction grating problem always requires near-field measurement in order to reconstruct the grating profile. Note that the series in (3.42) resp. (3.43) and each derivative of it are uniformly convergent on the half space $\{x_2 \geq c\}$ for all $c > \Gamma_1^+$ resp. $\{x_2 \leq c\}$ for all $c < \Gamma_2^-$. The periodic structure together with the form of incident waves motivates us to seek $\alpha$-quasiperiodic solutions satisfying

$$u(x_1 + 2\pi, x_2) = \exp(2i\alpha\pi) u(x_1, x_2). \tag{3.44}$$

For a fixed $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, let the admissible class of incident waves with the phase-shift $\alpha$ be given by

$$\mathcal{I} := \{u_n^i = \exp[i(\alpha_n x_1 - \beta_n^+ x_2)] : n \in \mathbb{Z}\},$$

which consists of a finite number of incoming plane waves and infinitely many surface waves.

We recall the following existence and uniqueness result for two periodic interfaces.

**Lemma 3.1.** *Suppose $\Gamma_j$ $(j = 1, 2)$ are given by periodic graphs and $k_0, k_2 > 0$, Re $k_1 > 0$, Im $k_1 \geq 0$ satisfy one the following conditions*

$$(i)\ \text{Im}\, k_1 > 0; \quad (ii)\ \text{Im}\, k_1 = 0,\ k_0 > k_1 > k_2; \quad (iii)\ \text{Im}\, k_1 = 0,\ k_0 < k_1 < k_2.$$

*Then, for each incident wave $u_n^i \in \mathcal{I}$, there always exists a unique solution $u \in H_\alpha^1((0, 2\pi) \times (-c, c))$ for all $c > \max\{|\Gamma_1^+|, |\Gamma_2^-|\}$. Here $H_\alpha^1(K)$ denotes the quasi-periodic Sobolev space with phase-shift $\alpha$ defined by*

$$H_\alpha^1(K) := \{u(x) : \exp(-i\alpha x_1) u(x_1, x_2) \in H^1(K)\}, \quad K = (0, 2\pi) \times (-c, c).$$

To prove Lemma 3.1, one can first establish a variational formulation in a bounded truncated periodic cell in $\mathbb{R}^2$ by enforcing the TM transmission conditions and the Rayleigh expansions, and then prove that the sesquilinear form generated by the variational form is strongly elliptic. If Im $k_1 > 0$, the uniqueness follows using a simple integration by parts. If all the refractive indices are real, the uniqueness is obtained by applying a periodic version of the Rellich identity (see [3, 10]), the monotonicity condition $(ii)$ or $(iii)$ imposed on the refractive indices and the fact that the $x_2$-component of the normal **n** does not change sign on $\Gamma_j$. Since this can be easily achieved in a piecewise homogenous medium, we omit the proof, referring to [3, 10, 30, 31] for a detailed presentation. Note that the above lemma is a special case of [30, 31] for two periodic interfaces and can be easily extended to multilayered diffraction gratings with piecewise refractive indices (see e.g. [10]).

Suppose the assumptions in Lemma 3.1 are fulfilled, and denote by $u(x_1, x_2; n)$ $(n = 1, 2 \cdots)$ the unique solution to the scattering problem (3.39)-(3.44) corresponding to the incident wave $u_n^i \in \mathcal{I}$. We assume $k_0$ is given, so that the multilayered diffraction grating can be written as $D = (\Gamma_1, \Gamma_2, k_1, k_2)$. Now we formulate the inverse problem as follows:

$(\mathbf{IP}")$ Let $b > \Gamma_1^+$ be a fixed constant. Given a fixed wave number $k_0 > 0$, determine the periodic interfaces $\Gamma_j$ $(j = 1, 2)$ and the refractive indices $k_j$ $(j = 1, 2)$ from the knowledge of the near field data $u(x_1, b; n)$ $(n = 1, 2 \cdots)$ for all $x_1 \in (0, 2\pi)$ corresponding to all incident plane waves $u_n^i$ from $\mathcal{I}$.

Assuming $\tilde{D} := (\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{k}_1, \tilde{k}_2)$ is another multilayered grating, we denote analogously by $\tilde{u}(x_1, b; n)$ the unique total field corresponding to $u_n^i \in \mathcal{I}$ and $\tilde{D}$. The main result of this section is

**Theorem 3.2.** *Let $b > \max\{\Gamma_1^+, \tilde{\Gamma}_1^+\}$, and assume $\beta_n^+ \neq 0$ for all $n \in \mathbb{Z}$. If the identity*

$$u(x_1, b; n) = \tilde{u}(x_1, b; n) \quad \textit{for all } x_1 \in (0, 2\pi) \tag{3.45}$$

*holds for all incident waves $u_n^i \in \mathcal{I}$, then $\Gamma_j = \tilde{\Gamma}_j$ and $k_j = \tilde{k}_j$ for $j = 1, 2$.*

### 3.2 Proof of Theorem 3.2

To prove the theorem, we need the free-space $\alpha$-quasi-periodic Green function $\Phi(x;y)$ defined by

$$\Phi(x;y) = \sum_{n\in\mathbb{Z}} \frac{i}{4\pi\beta_n^+} e^{i[\alpha_n(x_1-y_1)+\beta_n^+|x_2-y_2|]} \tag{3.46}$$

for $x, y \in \mathbb{R}^2$ with $x \neq y$, noting that $\beta_n^+ \neq 0$ by assumption. It is known that $\Phi(x;y)$ is weakly singular at $x = y$ and satisfies the Helmholtz equation $\triangle\Phi + k_0^2\Phi = 0$ in $\mathbb{R}^2$ when $x \neq y$. In addition, $\Phi$ has the same singularity as the fundamental solution $\Psi$ of the two dimensional Helmholtz equation and the difference $\Psi - \Phi$ is even analytic in $[(0, 2\pi) \times \mathbb{R}] \times [(0, 2\pi) \times \mathbb{R}]$; see [27].

Let $\Omega_b := \{x \in \mathbb{R}^2 : x_2 > b\}$, and let $y = (y_1, y_2) \in \Omega_b$ be fixed with $0 < y_1 < 2\pi$. Define the incident wave $u^i(x;y) := \Phi(x;y)$, $x \in \mathbb{R}^2$, due to a point source at $y$. By (3.46), $u^i(x;y)$ can be written as

$$u^i(x;y) = \sum_{n\in\mathbb{Z}} B_n u_n^i \quad \text{with } B_n = \frac{i}{4\pi\beta_n^+} e^{i(-\alpha_n y_1+\beta_n^+ y_2)} \text{ for } x_2 < b, \tag{3.47}$$

which propagates downward from $D_0$. Let $u^s(x;y), u(x;y)$ resp. $\tilde{u}^s(x;y), \tilde{u}(x;y)$ denote the scattered and total fields corresponding to $D = (\Gamma_1, \Gamma_2, k_1, k_2)$ resp. $\tilde{D} = (\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{k}_1, \tilde{k}_2)$. We conclude from (3.47) and the assumption (3.45) that

$$u(x_1, b; y) = \tilde{u}(x_1, b; y) \quad \text{for all } x_1 \in (0, 2\pi),\ y \in \Omega_b.$$

From the uniqueness of the exterior Dirichlet problem (see, e.g., [19]) and the unique continuation of solutions to the Helmholtz equation, it follows that

$$u(x;y) = \tilde{u}(x;y) \quad \text{for all } x \in \Omega := D_0 \cap \tilde{D}_0,\ y \in \Omega_b. \tag{3.48}$$

Let the $(-\alpha)$-quasiperiodic Green solution $G(x;y)$ to the scattering problem (3.39)-(3.44) be defined by

$$\left.\begin{array}{l} L_x G(x;y) = \nabla\cdot(a\nabla G(x;y)) + G(x;y) = -\delta(x-y), \\ G_+ = G_-,\ \ a^+\frac{\partial G_+}{\partial \mathbf{n}} = a^-\frac{\partial G_-}{\partial \mathbf{n}}, \qquad \text{on } \Gamma_j,\ j = 1, 2, \\ G(x;y) \text{ satisfies the } (-\alpha)\text{-quasiperiodic Rayleigh expansions } (3.42), (3.43) \\ \text{and the } (-\alpha)\text{-quasiperiodic condition (3.44),} \end{array}\right\} \tag{3.49}$$

where $a(x) = 1/k_j^2$ for $x \in D_j$, $j = 0, 1, 2$. Denote by $\tilde{G}(x;y)$ the $(-\alpha)$-quasiperiodic Green function corresponding to $\tilde{D}$. To reduce the argument to one periodic cell, we need the following notations

$$\begin{array}{ll} \Omega^* := \{x \in \Omega : 0 < x_1 < 2\pi\}, & \Omega_b^* := \{x \in \Omega_b : 0 < x_1 < 2\pi\}, \\ \Gamma_j^* := \{x \in \Gamma_j : 0 < x_1 < 2\pi\}, & \Sigma_b := \{x : 0 < x_1 < 2\pi, f_1(x_1) < x_2 < b\}. \end{array}$$

Analogously to Lemma 2.4, we are going to prove the following lemma:

**Lemma 3.3.** *Under the assumptions of Theorem 3.2, we have*

$$G(x;y) = \tilde{G}(x;y) \qquad \textit{for all } x, y \in \Omega^*, x \neq y.$$

**Proof.** For $x, y \in \Omega^*$, it follows from Green's second theorem applied to the periodic cell $\Sigma_b$ for some $b > \Gamma_1^+$ and the Rayleigh expansions for $u^s(x;y)$ and $G(x;y)$ in $x_2 > \Gamma_1^+$ that

$$\begin{aligned} k_0^2 u^s(x;y) &= \int_{\Gamma_1^*} u_+^s(z;y)\frac{\partial G_+(z;x)}{\partial \mathbf{n}} - G_+(z;x)\frac{\partial u_+^s(z;y)}{\partial \mathbf{n}} ds(z) &(3.50)\\ &= \int_{\Gamma_1^*} u_+(z;y)\frac{\partial G_+(z;x)}{\partial \mathbf{n}} - G_+(z;x)\frac{\partial u_+(z;y)}{\partial \mathbf{n}} ds(z) \\ &\quad - \int_{\Gamma_1^*} \Phi(z;y)\frac{\partial G_+(z;x)}{\partial \mathbf{n}} - G_+(z;x)\frac{\partial \Phi(z;y)}{\partial \mathbf{n}} ds(z). &(3.51)\end{aligned}$$

Note that in obtaining (3.50), we have used the identity

$$\int_{\Gamma_b^*} u_+^s(z;y)\frac{\partial G_+(z;x)}{\partial \mathbf{n}} - G_+(z;x)\frac{\partial u_+^s(z;y)}{\partial \mathbf{n}} ds(z) = 0, \tag{3.52}$$

and the fact that the integrals over the vertical lines of $\partial\Sigma_b$ cancel because of the periodicity. The relation (3.52) follows from the $\alpha$-quasiperiodic Rayleigh expansions for $u_+^s(z;y)$ and the $(-\alpha)$-quasiperiodic Rayleigh expansions for $G_+(z;y)$ in $z_2 > \Gamma_1^+$. Similarly,

$$G(y;x) = \int_{\Gamma_1^*} G_+(z;x)\frac{\partial \Phi(z;y)}{\partial \mathbf{n}} - \frac{\partial G_+(z;x)}{\partial \mathbf{n}}\Phi(z;y) ds(z) + k_0^2\Phi(x;y). \tag{3.53}$$

Using the transmission conditions for $G(z;x)$ and $u(z;y)$ on $\Gamma_j$ $(j = 1,2)$ and their Rayleigh expansions in $z_2 < \Gamma_2^+$, we obtain analogously by Green's second theorem that

$$\begin{aligned} &\int_{\Gamma_1^*} u_+(z;y)\frac{\partial G_+(z;x)}{\partial \mathbf{n}} - G_+(z;x)\frac{\partial u_+(z;y)}{\partial \mathbf{n}} ds(z) \\ = &\frac{k_0^2}{k_3^2}\int_{\Gamma_3^*} u_-(z;y)\frac{\partial G_-(z;x)}{\partial \mathbf{n}} - G_-(z;x)\frac{\partial u_-(z;y)}{\partial \mathbf{n}} ds(z) \\ = &\ 0. &(3.54)\end{aligned}$$

Combining (3.51)-(3.54) yields the relation $G(y;x) = k_0^2 u(x;y)$ for all $x, y \in \Omega^*, x \neq y$. Similarly, there holds $\tilde{G}(y;x) = k_0^2\tilde{u}(x;y)$ for all $x, y \in \Omega^*, x \neq y$. In view of (3.48), we conclude that

$$G(y;x) = \tilde{G}(y;x) \quad \text{for all } x \in \Omega^*,\ y \in \Omega_b^*,\ x \neq y.$$

As functions of $y$, both $G(y;x)$ and $\tilde{G}(y;x)$ satisfy the Helmholtz equation $(\Delta + k_0^2)u = 0$ in $\Omega^*\backslash\{x\}$. Recalling the unique continuation of solutions to the Helmholtz equation and the fact that $\Omega_b^* \subset \Omega^*$, we obtain $G(y;x) = \tilde{G}(y;x)$ for all $x, y \in \Omega^*, x \neq y$. □

For a fixed $y_0 \in \Omega^*\backslash(\Gamma_1^* \cup \Gamma_2^*)$, the Green function $G(x;y_0)$ defined in (3.49) satisfies

$$(\triangle_x + k_j^2)G(x;y_0) = -k_j^2\delta(x - y_0) \quad \text{in} \quad D_j.$$

By the singularity of the free-space quasi-periodic Green function $\Phi(x;y)$, as mentioned at the beginning of Section 3.2, we know that

$$\Phi(x;y_0) \sim -\frac{1}{2\pi}\ln||x-y_0|| \quad \text{as} \quad x \to y_0,$$

implying that

$$G(x;y_0) \sim -\frac{k_j^2}{2\pi}\ln||x-y_0|| \quad \text{as} \quad x \to y_0,$$

since the difference $k_j^2\Phi(x;y_0) - G(x;y_0)$ is smooth in a neighborhood of $y_0$. By arguing as in Lemma 2.5, one can further obtain that

$$G(y_n;y_0) \sim -\frac{k_j^2 k_{j-1}^2}{\pi(k_{j-1}^2 + k_j^2)}\ln||y_n-y_0|| \quad \text{if } y_0 \in \Gamma_j^*, \quad j=1,2,$$

as $n \to +\infty$, where $y_n := y_0 + \frac{1}{n}\mathbf{n}(y_0)$. Thus, relying on Lemma 3.3 and the above asymptotic properties of $G(x;y)$ as $y \to y_0, x \to y_0$, we can carry over the arguments from Section 2.3 to the periodic case to complete the proof of Theorem 3.2. $\square$

**Remark 3.4.** *(i) From Theorem 3.2, we see that the near field measurements only above the grating are enough to determine the periodic interfaces as well as the piecewise constant refractive indices. This remains true if the measurements are taken only below the grating.*

*(ii) Under the general transmission conditions (2.33), a uniqueness result similar to Corollary 2.7 can be obtained on identifying the interfaces and transmission coefficients if the waves numbers* $k_i$ $(i=0,1,2)$ *are known.*

*(iii) Using point sources as incident waves, the argument in this section can be extended to prove uniqueness for inverse scattering by general non-periodic* $C^2$*-smooth profiles which are given by graphs. Note that we require the regularity of the profile in order to tackle the singularity of the Green function in the half space.*

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