Weierstraß-Institut

für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Properties of states of super-α-stable motion with branching of index $1+\beta$

Klaus Fleischmann[1] , Leonid Mytnik[2] , Vitali Wachtel[3]

submitted: October 13, 2010

[1] Weierstraß–Institut
für Angewandte Analysis und Stochastik
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: klaus.fleischmann@wias-berlin.de

[2] Faculty of Industrial
Engineering and Management
Technion Israel Institute of Technology
Haifa 32000
Israel
E-Mail: leonid@ie.technion.ac.il

[3] Mathematisches Institut
Ludwig–Maximilians–Universität München
Theresienstr. 39
80333 München
Germany
E-Mail: wachtel@mathematik.uni-muenchen.de

No. 1548
Berlin 2010

1991 *Mathematics Subject Classification.* 60G57.

Key words and phrases. Hölder continuity locally and at a given point, optimal exponent, multifractal spectrum, Hausdorff dimension, superprocess with Neveu's branching mechanism .

Supported by the German Israeli Foundation for Scientific Research and Development, Grant No. G-807-227.6/2003, Corresponding author: Klaus Fleischmann .

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. It has been well-known for a long time that the measure states of the process in the title are absolutely continuous at any fixed time provided that the dimension of space is small enough. However, besides the very special case of one-dimensional continuous super-Brownian motion, properties of the related density functions were not well understood. Only in 2003, Mytnik and Perkins [21] revealed that in the Brownian motion case and if the branching is discontinuous, there is a dichotomy for the densities: Either there are continuous versions of them, or they are locally unbounded. We recently showed, that the same type of fixed time dichotomy holds also in the case of discontinuous motion. Moreover, the continuous versions are locally Hölder continuous, and we determined the optimal index for them. Finally, we determine the optimal index of Hölder continuity at given space points which is strictly larger than the optimal index of local Hölder continuity.

1. MODEL: SUPER-α-STABLE MOTION WITH BRANCHING OF INDEX $1+\beta$

The process in the title, sometimes also called (α, d, β)-superprocess, is a finite measure-valued process $X = \{X_t : t \geq 0\}$ describing the evolution of populations of infinitesimally small individuals/particles. The process can be constructed as a limit of branching particle systems, where the particles move independently according to symmetric α-stable motions in Euclidean space R^d, and additionally they branch according to a branching mechanism in the domain of attraction of a stable law of index $1+\beta$. Here $\alpha \in (0,2]$ and $1+\beta \in (1,2]$. Of course, for the very special case of $\alpha = 2$ and $\beta = 1$ we have the famous continuous super-Brownian motion in R^d.

A convenient description of X can be given via the log-Laplace transition functional, which is determined by the log-Laplace equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t \;=\; \boldsymbol{\Delta}_\alpha u_t + a u_t - b u_t^{1+\beta}, \quad t > 0, \tag{1}$$

with fixed constants $a \in \mathsf{R}$ and $b > 0$. Here the fractional Laplacian $\boldsymbol{\Delta}_\alpha := -(-\boldsymbol{\Delta})^{\alpha/2}$ describes a symmetric stable motion in R^d of index $\alpha \in (0,2]$, whereas the two other terms reflect the continuous-state branching of index $1+\beta \in (1,2]$. Note that the branching is critical if $a = 0$. To be more specific, the log-Laplace transition functional of the homogeneous measure-valued Markov process X is defined as,

$$\log \mathbf{E}_\mu \exp\langle X_t, -\varphi\rangle \;=\; \big\langle \mu, -u\big(t,(\cdot)\big)\big\rangle, \quad t > 0. \tag{2}$$

Here $\mu \in \mathcal{M}_{\mathrm{f}}$ (the set of finite measures on R^d), $\varphi \geq 0$ is a test function, and the non-negative function $u = \big\{u(s,x) : s > 0, x \in \mathsf{R}^d\big\}$ solves the log-Laplace integral equation

$$\begin{aligned} u(s,x) \;=\; & \int_{\mathsf{R}^d} \mathrm{d}y\; p^\alpha_s(y-x)\,\varphi(y) \\ & + \int_0^s \mathrm{d}r \int_{\mathsf{R}^d} \mathrm{d}y\; p^\alpha_{s-r}(y-x)\Big[a u(r,y) - b\big(u(r,y)\big)^{1+\beta}\Big] \end{aligned} \tag{3}$$

which is the mild form the log-Laplace equation (1) with initial condition $u\big(0+,(\cdot)\big) = \varphi$. Also, p^α describes the transition kernel of the particles' α-stable motion.

This rather general model was introduced by Iscoe in his thesis 1980 at Carleton University, published in [10, 11], and investigated later by many authors. From the

beginning, one of the central issues was the question of the nature of the states of the process X.

2. Dichotomy of states at fixed times

Since 1988 (see Fleischmann [4]) it is known that at any fixed time $t > 0$, with probability one, the measure $X_t = X_t(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure, provided that the dimension d is sufficiently small: $d < \frac{\alpha}{\beta}$. To be more precise, in [4] it was assumed that $a = 0$, but $a \neq 0$ requires just the obvious changes. On the other hand, in all higher dimensions $d \geq \alpha/\beta$, the states are singular a.s. The singularity statement was proved in [4] only in the critical dimension, and for the case of $d > \alpha/\beta$ it follows from Theorem 7.3.4 of Dawson [1].

3. Absolutely continuous states

From now on assume $d < \frac{\alpha}{\beta}$, that is, for any time t, the measure $X_t(dx)$ has a density function $x \mapsto X_t(x)$, which by a slight abuse of notation is denoted by the same symbol X_t as the corresponding measure. How to characterize the density and what are its properties? In the very special case of one-dimensional continuous super-Brownian motion ($\alpha = 2$, $d = 1 = \beta$) it is well-known that a jointly continuous density field $\{X_t(x) : t > 0, x \in \mathsf{R}\}$ exists, and it satisfies a stochastic partial differential equation (SPDE); see Konno and Shiga [17] as well as Reimers [23]. However, it took a long time to make some progress in the case of $\beta < 1$. For the Brownian case $\alpha = 2$ (and $a = 0$, that is, critical branching), it was proved in Mytnik [20] that a version of the density $\{X_t(x) : t > 0, x \in \mathsf{R}^d\}$ of the measure $X_t(\mathrm{d}x)\mathrm{d}t$ exists that satisfies – in a weak sense – the following SPDE

$$\frac{\partial}{\partial t} X_t(x) = \boldsymbol{\Delta} X_t(x) + \big(X_{t-}(x)\big)^{1/(1+\beta)} \dot{L}(t,x), \tag{4}$$

where $\dot{L}$ is a $(1+\beta)$-stable noise without negative jumps. Of course, this is a counterpart of the SPDE result for the continuous super-Brownian motion mentioned above.

From now on assume that $\beta < 1$. In other words, we restrict our discussion to the case of a discontinuous branching mechanism and ask for properties of the density function at fixed times.

3.1. Dichotomy of density function.

The particular case of $\alpha = 2$ was first treated in [21], where regularity and irregularity properties of the density at fixed times t had been revealed. More precisely, it was shown that

- these densities have continuous versions if $d = 1$,
- they are locally unbounded on open sets of positive $X_t(\mathrm{d}x)$-measure in all higher dimensions.

However the case of discontinuous motion ($\alpha < 2$) was not considered in [21].

Here we take care of the general case of $\alpha \leq 2$, that is, we include also discontinuous underlying motions. We show that the same type of fixed time dichotomy still holds, and this is our first result, taken from Fleischmann et al. [5].

Theorem 1 (**Fixed time dichotomy of density function**). *Recall that* $\alpha \leq 2$, $\beta < 1$, *and* $d < \frac{\alpha}{\beta}$. *Fix an initial state* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$ *and any time* $t > 0$.

(a) (Continuity): *If* $d=1$ *and* $1+\beta<\alpha$, *then a.s. there exists a continuous version, say* $\widetilde{X}_t$, *of the density function of the measure* $X_t(\mathrm{d}x)$.

(b) (Local unboundedness): *If* $d>1$ *or* $1+\beta\geq\alpha$, *then a.s., for all open* $U\subseteq\mathsf{R}^d$,

$$\|X_t\|_U \;:=\; \operatorname*{ess\,sup}_{x\in U} X_t(x) \;=\; \infty \;\; \textit{whenever} \;\; X_t(U)>0. \tag{5}$$

The proof of **(b)** is rather technical, heavily uses ideas of [21], and roughly goes as follows. Let U be a fixed open ball. One first shows that on the event $\{X_t(U)>0\}$ there are always sufficiently "big" jumps of X that occur in U close to time t. Then with the help of properties of solutions of the log-Laplace equation one is able to show that the "big" jumps are large enough to ensure the unboundedness of the density at time t in U. Loosely speaking, the density is getting unbounded in the proximity of "big" jumps. Finally, the exceptional set concerning the a.s. statement in **(b)** can be chosen uniformly in U since each U contains a non-empty ball with rational center and radius.

As for **(a)**, the continuity of the density is verified via the Kolmogorov criterion. Note that besides the continuity, this criterion gives also some Hölder exponent of continuity. This immediately raises the question of determining the optimal Hölder index for the density and this question is addressed in the next subsection. Here we just mention that in the case of one-dimensional continuous super-Brownian motion ($\alpha=2$, $d=1=\beta$), the densities are locally Hölder continuous (in the spatial variable) for any index $\eta<\frac{1}{2}$, and the bound $\frac{1}{2}$ is moreover optimal.

3.2. **Local Hölder continuity of continuous density functions.** Here is our next result, again taken from [5].

Theorem 2 (Local Hölder continuity). *Fix* $X_0=\mu\in\mathcal{M}_{\mathrm{f}}$ *and* $t>0$. *Suppose* $d=1$ *and* $1+\beta<\alpha$.

(a) (Local Hölder continuity): *For each* $\eta<\eta_{\mathrm{c}}:=\frac{\alpha}{1+\beta}-1$, *with probability one, the continuous version* $\widetilde{X}_t$ *of the density function* X_t *is locally Hölder continuous of index* η:

$$\sup_{x_1,x_2\in K,\,x_1\neq x_2}\frac{\bigl|\widetilde{X}_t(x_1)-\widetilde{X}_t(x_2)\bigr|}{|x_1-x_2|^{\eta}}<\infty, \quad \textit{compact } K\subset\mathsf{R}. \tag{6}$$

(b) (Optimality of η_{c}): *With probability one, for any open* $U\subseteq\mathsf{R}$,

$$\sup_{x_1,x_2\in U,\,x_1\neq x_2}\frac{\bigl|\widetilde{X}_t(x_1)-\widetilde{X}_t(x_2)\bigr|}{|x_1-x_2|^{\eta_{\mathrm{c}}}}=\infty \quad \textit{whenever } X_t(U)>0. \tag{7}$$

However, how the optimal index η_{c} is related to the optimal index $\frac{1}{2}$ in the excluded boundary case of continuous super-Brownian motion?

3.3. **Some transition curiosity.** Suppose for the moment that $\alpha=2$, and let $\beta\uparrow 1$. Then

$$\eta_{\mathrm{c}}=\frac{2}{1+\beta}-1\downarrow 0\neq\frac{1}{2}. \tag{8}$$

That is, we have some surprising discontinuity while passing to the boundary case of continuous super-Brownian motion. How to understand this phenomenon? An

explanation can be given using the notion of Hölder continuity at a point. The latter notion is recalled in the next subsection, and related results concerning our process are given.

3.4. **Hölder continuity at a given point.** Recall that a function f is Hölder continuous with index $\eta \in (0,1)$ *at a point* x_0 if there exists a neighborhood $U(x_0)$ such that

$$\big|f(x) - f(x_0)\big| \le C\,|x - x_0|^\eta, \quad x \in U(x_0). \tag{9}$$

The *optimal* Hölder index, say $H(x_0)$, of f at the point x_0 is defined as the supremum over all such η. Of course, there exist functions f where $H(x_0)$ indeed depends on x_0. Clearly, the optimal index of local Hölder continuity in a domain is determined by the infimum of $H(x_0)$ over points x_0 in the domain.

This phenomenon of difference of optimal index of local Hölder continuity from optimal Hölder index at some points can be observed in our case of the continuous densities of superprocesses with discontinuous branching ($\beta < 1$). Here is our result, presented in Fleischmann et al. [6].

Theorem 3 (**Hölder continuity at a given point**). *Fix* $X_0 = \mu \in \mathcal{M}_f$ *and* $t > 0$ *as well as* $x_0 \in \mathsf{R}$. *Suppose* $d = 1$ *and* $1 + \beta < \alpha$.

(a) (Hölder continuity at a given point): *For each* $\eta > 0$ *satisfying*

$$\eta < \bar\eta_c := \min\Big\{\frac{1+\alpha}{1+\beta} - 1,\, 1\Big\}, \tag{10}$$

with probability one the continuous version $\widetilde{X}_t$ *of the density is Hölder continuous of order* η *at the point* x_0. *That is, for all neighborhoods* $U(x_0)$ *of* x_0,

$$\sup_{x \in U(x_0),\, x \ne x_0} \frac{\big|\widetilde{X}_t(x) - \widetilde{X}_t(x_0)\big|}{|x - x_0|^\eta} < \infty. \tag{11}$$

(b) (Optimality of $\bar\eta_c$): *If additionally* $\beta > (\alpha - 1)/2$, *then* $\bar\eta_c$ *is optimal. That is, with probability one for all neighborhoods* $U(x_0)$ *of* x_0,

$$\sup_{x \in U(x_0),\, x \ne x_0} \frac{\big|\widetilde{X}_t(x) - \widetilde{X}_t(x_0)\big|}{|x - x_0|^{\bar\eta_c}} = \infty \quad \textit{whenever } X_t(x_0) > 0. \tag{12}$$

Note that, in fact,

$$\bar\eta_c > \eta_c. \tag{13}$$

The above results imply that for the density $\widetilde{X}_t$ the optimal Hölder index H varies from point to point. The optimal local Hölder index $\eta_c = \frac{\alpha}{1+\beta} - 1$ equals to the infimum of H over an open domain. Therefore, there have to be (random) points x_0 in the domain with $H(x_0)$ arbitrary close or equal to η_c. On the other hand by Theorem 3 there are also points x_0 with the optimal Hölder index $H(x_0) = \bar\eta_c > \eta_c$ and hence we can conclude that H varies from point to point.

Heuristically the reason to the fact that there exist points in an open domain with different Hölder indexes is as follows. It can be seen from the proofs that the Hölder index at a point is highly influenced by relatively "big" jumps of the superprocess that occur close to time t in the proximity of the point. Therefore when we choose any fixed point in space, the size of the "biggest" jump close to it may be, and in fact is, much smaller than the "biggest" jump somewhere in an

open domain which in turn influences the index of continuity at some exceptional random point in the domain. This consequently implies that the local Hölder index of continuity in an open domain is smaller than the modulus of continuity at a fixed point.

Now note that, if $\alpha = 2$, then as $\beta \uparrow 1$,

$$\bar{\eta}_{\mathrm{c}} = \big(\frac{3}{1+\beta} - 1\big) \wedge 1 \downarrow \frac{1}{2}. \tag{14}$$

That is, for Hölder continuity at fixed points we have continuity for the optimal index $\bar{\eta}_{\mathrm{c}}$ while approaching the boundary case of $\beta = 1$, whereas earlier we observed discontinuity of η_{c}. Heuristically this can be explained as follows. We conjecture that the fact that the optimal Hölder index of continuity at fixed points equals to $\bar{\eta}_{\mathrm{c}}$ implies that, with probability one, the optimal Hölder index $H(x)$ of continuity at $X_t(dx)$-a.e. point x equals to $\bar{\eta}_{\mathrm{c}}$, and moreover the Lebesgue measure of the points with optimal Hölder index of continuity η_{c} equals to 0. On the other hand for continuous super-Brownian motion we have that almost surely $H(x) = \frac{1}{2}$ for all x, and so we see that the continuity at the boundary case $\beta = 1$ holds for the optimal Hölder index $\bar{\eta}_{\mathrm{c}}$ that describes the modulus of continuity at a.e. point and not just at exceptional points.

3.5. **Some open problems.** We would like to list here some open problems.

At the first sight, in Theorem 3(b) there is the additional assumption $\beta > (\alpha - 1)/2$. But note that the opposite case $\beta \le (\alpha-1)/2$ implies that $\bar{\eta}_{\mathrm{c}} = 1$. Therefore the optimality of $\bar{\eta}_{\mathrm{c}}$ follows here automatically from the definition of $H(x_0)$. Our first conjecture deals with the finer analysis of the case $\beta < (\alpha - 1)/2$, and note that we exclude here the boundary case of $\beta = (\alpha - 1)/2$.

Conjecture 4 (Lipschitz). Let $\beta < (\alpha - 1)/2$. Then at any given point x_0, with probability one, the density function $\widetilde{X}_t$ is Lipschitz continuous at x_0. $\diamond$

Next we turn to the topic of so-called multifractal spectrum.

Conjecture 5 (Multifractal spectrum). Fix $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$ and $t > 0$. Let $d = 1$ and $1 + \beta < \alpha$.

- **(a) (Full Spectrum):** We conjecture that for any $\eta \in \big(\eta_{\mathrm{c}}, \bar{\eta}_{\mathrm{c}}\big)$ with probability one there are (random) points $x_0 \in \mathsf{R}$ such that the optimal Hölder index $H(x_0)$ of $\widetilde{X}_t$ at x_0 is exactly η.
- **(b) (Hausdorff dimension):** For $\eta \in \big(\eta_{\mathrm{c}}, \bar{\eta}_{\mathrm{c}}\big)$, let $D(\eta)$ denote the Hausdorff dimension of the (random) set $\big\{x_0 : H(x_0) = \eta\big\}$. We conjecture that

$$\lim_{\eta \downarrow \eta_{\mathrm{c}}} D(\eta) = 0 \quad \text{and} \quad \lim_{\eta \uparrow \bar{\eta}_{\mathrm{c}}} D(\eta) = 1 \quad \text{a.s.} \qquad \diamond$$

The function $\eta \mapsto D(\eta)$ reveals the so-called multifractal spectrum concerning the optimal Hölder index H for the densities of superprocesses with branching of index $1 + \beta < \alpha$ and it is definitely worth studying.

We would like to mention that the multifractal spectrum of random functions and measures has attracted attention already for a while and has been studied for example by the following authors: Dembo et al. [2], Durand [3], Hu and Taylor [9], Klenke and Mörters [16], Le Gall and Perkins [18], Mörters and Shieh [19] and Perkins and Taylor [22]. The multifractal spectrum of singularities that describe the Hausdorff dimension of sets of different Hölder exponents of functions was

investigated for deterministic and random functions in Jaffard [12]-[14] as well as in Jaffard and Meyer [15].

We would also like to mention here that we got ideas to study regularity properties of the densities of (α, d, β)-superprocesses when we had been dealing with super-α-stable motions, say X^0, with Neveu's branching mechanism. This process is defined via the log-Laplace equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t \;=\; \boldsymbol{\Delta}_\alpha u_t + au_t - u_t \log u_t, \quad t > 0, \tag{15}$$

$0 < \alpha \leq 2$, and consequently formally corresponds to the earlier excluded boundary case of $\beta = 0$. By the way, this equation (15) is interesting in itself since the non-linear term has a local non-Lipschitz property. Despite the fact that the branching mechanism has infinite expectation here, that process X^0 exists and was constructed in Fleischmann and Sturm [7]. It was also shown there, that the process is immortal and propagates mass instantaneously everywhere in space, opposed, for instance, to supercritical super-Brownian motions with finite expectation; see [7, Proposition 16]. The large-scale behavior of X^0 is also not at all typical for supercritical spatial branching processes. In fact, in Fleischmann and Wachtel [8, Theorem 1] it was shown, that X_t^0 normalized by its total masses $X_t^0(\mathsf{R}^d)$, with time t speeded up by a factor k, and contracted in space by $k^{1/\alpha}$, converges as $k \uparrow \infty$ towards a measure-valued process describing a single atom of mass one which fluctuates in macroscopic time according to an α-stable process.

What else can be expected concerning the nature of states of X^0? Here we have the following conjectures about fixed time state properties.

Conjecture 6 (**Superprocess with Neveu's branching mechanism**). Fix $X_0^0 = \mu \in \mathcal{M}_{\mathrm{f}}$ and $t > 0$.

- **(a) (Absolute continuity):** In all dimensions, with probability one, the measure $X_t^0(\mathrm{d}x)$ is absolutely continuous.
- **(b) (Dichotomy of density functions):** If $d = 1$ and $\alpha > 1$, with probability one there exists a continuous version, say $\widetilde{X}_t^0$, of the density function X_t^0 of the measures $X_t^0(\mathrm{d}x)$. On the other hand, if $d > 1$ or $\alpha \leq 1$, we have local unboundedness of X_t^0.

For the remaining statements, suppose $d = 1$ and $\alpha > 1$.

- **(c) (Optimal local Hölder continuity):** Write $\eta_{\mathrm{c}}^0 := \alpha - 1$. Then the continuous density $\widetilde{X}_t^0$ is locally Hölder continuous of every index $\eta < \eta_{\mathrm{c}}^0$. Moreover, η_{c}^0 is optimal.
- **(d) (Lipschitz continuity at a given point):** Fix $x_0 \in \mathsf{R}$. Then a.s. $\widetilde{X}_t^0$ is Lipschitz continuous at x_0. ◇

Note that all these conjectures are based on our results above together with Conjecture 4, by letting formally $\beta \downarrow 0$.

By the way, as in the earlier case of (α, d, β)-superprocesses with $\beta > 0$, the verification of existence of density functions should be done along the lines of proving existence of mild fundamental solutions to the log-Laplace equation (15). Such solutions should exist despite the non-local Lipschitz property in the branching mechanism.

4. Main tools to get the Hölder statements

Clearly, a standard procedure to get an optimal Hölder index of continuity is via Kolmogorov's criterion by using "high" moments. This, for instance, can be done in the case of one-dimensional continuous super-Brownian motion ($\alpha = 2$, $d = 1 = \beta$) to show local Hölder continuity in the space variable of any index smaller than $\frac{1}{2}$, see the estimates in the proof of Corollary 3.4 in Walsh [24]. But in our $\beta < 1$ case, "high" moments do not exist, and it turns out that we cannot use this method for all possible range of parameters α, β. Hence we have to go deeply into the jump structure of the superprocess to obtain the needed estimates.

Actually, the starting point for all of our Hölder proofs is the well-known martingale decomposition of the (α, d, β)-superprocess X, valid for any α, d, β; see, e.g., [5, Lemma 1.6]:

For all sufficiently smooth bounded test functions $\varphi \geq 0$ on R^d and $t \geq 0$,

$$\langle X_t, \varphi\rangle \;=\; \langle \mu, \varphi\rangle + \int_0^t \mathrm{d}s\; \langle X_s, \boldsymbol{\Delta}_\alpha \varphi\rangle + M_t(\varphi) + a\, I_t(\varphi) \tag{16}$$

with discontinuous martingale

$$t \;\mapsto\; M_t(\varphi) \;:=\; \int_{(0,t]\times \mathsf{R}^d \times \mathsf{R}_+} \tilde{N}\big(\mathrm{d}(s,x,r)\big)\, r\, \varphi(x) \tag{17}$$

and increasing process

$$t \;\mapsto\; I_t(\varphi) \;:=\; \int_0^t \mathrm{d}s\; \langle X_s, \varphi\rangle. \tag{18}$$

Here $\tilde{N} := N - \hat{N}$, where $N\big(\mathrm{d}(s,x,r)\big)$ is a random measure on $(0,\infty)\times\mathsf{R}^d\times(0,\infty)$ describing all the jumps $r\delta_x$ of X at times s at sites x of size r, which are the only discontinuities of the process X. Moreover,

$$\hat{N}\big(\mathrm{d}(s,x,r)\big) \;=\; \varrho\; \mathrm{d}s\, X_s(\mathrm{d}x)\, r^{-2-\beta}\mathrm{d}r \tag{19}$$

is the compensator of N, where $\varrho := b(1+\beta)\beta/\Gamma(1-\beta)$ with Γ denoting the Gamma function.

Recall that under our assumption $d < \frac{\alpha}{\beta}$, for fixed $t > 0$, the random measure $X_t(\mathrm{d}x)$ is absolutely continuous a.s. From the Green function representation related to (16), see, e.g., [5, (1.9)] we obtain the following representation of a version of the density function of $X_t(\mathrm{d}x)$ (see, e.g., [5, (1.12)]):

$$\begin{aligned} X_t(x) \;&=\; \mu * p^\alpha_t(x) \;+\; \int_{(0,t]\times\mathsf{R}^d} M\big(\mathrm{d}(s,y)\big)\, p^\alpha_{t-s}(x-y) \\ &+\; a\int_{(0,t]\times\mathsf{R}^d} I\big(\mathrm{d}(s,y)\big)\, p^\alpha_{t-s}(x-y) =: \; Z^1_t(x) + Z^2_t(x) + Z^3_t(x), \quad x \in \mathsf{R}^d, \end{aligned} \tag{20}$$

with notation in the obvious correspondence. Here $M\big(\mathrm{d}(s,y)\big)$ is the martingale measure related to (17) and $I\big(\mathrm{d}(s,y)\big)$ the random measure related to (18).

It is easy to see that the deterministic function Z^1_t is locally Lipschitz continuous. It is also not difficult to show that Z^3_t is a.s. locally Lipschitz, and hence we fix our attention on the main term Z^2_t involving the martingale measure M. Note that Z^2_t is the most difficult term to analyze. Here the starting point is, that the random increment $Z^2_t(x_1) - Z^2_t(x_2)$ can be represented as the difference of the values of two spectrally positive $(1+\beta)$-stable processes L^1, L^2 at some random times T_+,

T_-, respectively. Recall that per definition L is a *spectrally positive* $(1+\beta)$*-stable process*, if it is an R-valued time-homogeneous process with independent increments and with Laplace transform given by

$$\mathbf{E}\,\mathrm{e}^{-\lambda L(t)} \,=\, \mathrm{e}^{t\lambda^{1+\beta}}, \quad \lambda, t \geq 0. \tag{21}$$

Consequently, there is a representation

$$Z_t^2(x_1) - Z_t^2(x_2) = L^1(T_+) - L^2(T_-). \tag{22}$$

Here the random times $T_\pm$ are given by

$$T_\pm := \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y) \left(p^\alpha_{t-s}(x_1-y) - p^\alpha_{t-s}(x_2-y)\right)_\pm^{1+\beta} \tag{23}$$

with $\pm$ referring to the positive and negative parts.

It follows from (22), that the Hölder continuity can be destroyed by "big" values of the processes L^1 and L^2. Now, it is known from the standard theory of spectrally positive stable processes that "big" values are due to "big" positive jumps. Thus, to prove the Hölder continuity, one needs to control all the jumps of the processes L^1, L^2 by time $T_\pm$. More precisely, we show in the proof that there are no jumps, which can destroy the Hölder continuity of order η smaller than the critical index η_{c} or $\bar{\eta}_{\mathrm{c}}$.

The more complicated parts are the optimality proofs for the indexes. To prove the optimality of η_{c} we are going to show that there exists a sequence of "big" jumps of X that occur close to time t in the considered domain U in Theorem 2(b), and these jumps indeed destroy the local Hölder continuity of index η_{c}. But the existence of such a sequence is not sufficient for the proof of optimality. We need additionally to show that the influence of "big" jumps of one of the stable processes L^1, L^2 cannot be compensated by "big" jumps of the other one.

To prove the optimality of $\bar{\eta}_{\mathrm{c}}$, additionally the "big" jumps have to be found in the vicinity of the fixed x_0. Moreover, values of "big" jumps in Theorem 3(b) are of a smaller order than those in Theorem 2(b), because it is more likely to have "big" jumps in a domain than in a vicinity of a fixed point. This creates additional technical difficulties in the proof of Theorem 3(b).

Now, we would like to explain a bit the occurrence of the critical value $\eta_{\mathrm{c}} = \frac{\alpha}{1+\beta} - 1$ for local Hölder continuity in Theorem 2 and we will skip the discussion on $\bar{\eta}_{\mathrm{c}}$ which goes along similar lines.

Our first observation is that, up to an probability error of $\varepsilon \in (0, \frac{1}{1+\beta})$, all the jumps $\Delta M(s,y)$ of the martingale measure $M\big(\mathrm{d}(s,y)\big)$ at times $s < t$ are bounded by $(t-s)^{\frac{1}{1+\beta}-\varepsilon}$, that is

$$\mathbf{P}\Big(\Delta M(s,y) \leq c\,(t-s)^{\frac{1}{1+\beta}-\varepsilon} \text{ for all } s<t \text{ and } y \in \mathsf{R}\Big) \,\geq\, 1-\varepsilon, \tag{24}$$

see [5, Lemma 2.14]. However, it follows from (20) and (22) that the jumps ΔL^1 and ΔL^2 of L^1 and L^2, respectively, generated by the jumps $\Delta M(s,y)$ do not exceed

$$\Delta M(s,y) \sup_{y\in\mathsf{R}} \left|p^\alpha_{t-s}(x_1-y) - p^\alpha_{t-s}(x_2-y)\right|. \tag{25}$$

Hence, from (24) and an estimate for α-stable kernels (see [5, Lemma 2.1]) we obtain the following bound

$$c\,(t-s)^{\frac{1}{1+\beta}-\varepsilon}\sup_{y\in\mathsf{R}}\left|p^{\alpha}_{t-s}(x_1-y)-p^{\alpha}_{t-s}(x_2-y)\right| \tag{26}$$

$$\leq c\,(t-s)^{\frac{1}{1+\beta}-\varepsilon}\,\frac{|x_1-x_2|^{\delta}}{(t-s)^{\delta/\alpha+1/\alpha}}\,,\quad 0<\delta\leq 1,$$

for the jumps ΔL^1 and ΔL^2. If now $\delta=\frac{\alpha}{1+\beta}-1-\alpha\varepsilon=\eta_{\mathrm{c}}-\alpha\varepsilon$ (for sufficiently small ε), then

$$\Delta L^1,\Delta L^2\;\leq\; c\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\varepsilon}. \tag{27}$$

But if the jumps of a spectrally positive stable process are not "big", then the process values cannot be "big" as well. Consequently,

$$\mathbf{P}\Big(L^1,L^2\leq c\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\varepsilon}\Big)\;\geq\;1-\varepsilon. \tag{28}$$

In view of (22), the latter implies

$$\mathbf{P}\Big(\left|Z^2_t(x_1)-Z^2_t(x_2)\right|\leq c\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\varepsilon}\;\;\forall x_1,x_2\in K\Big)\;\geq\;1-\varepsilon \tag{29}$$

(with K a compact), which gives the Hölder continuity of Z^2_t of any exponent smaller than η_{c}.

To show the optimality of η_{c} we first prove that there exists a sequence (s_n,y_n,r_n) such that

$$s_n\uparrow t,\;\;y_n\in(-1,1),\;\;\Delta M(s_n,y_n)=r_n\geq(t-s_n)^{\frac{1}{1+\beta}}\,\log\frac{1}{t-s_n}\,. \tag{30}$$

Using again a representation as in (22), with corresponding spectrally positive stable processes and random times indexed by n, we have

$$Z^2_t(y_n)-Z^2_t\big(y_n+(t-s_n)^{1/\alpha}\big)\;=\;L^1_n(T_{n,+})-L^2_n(T_{n,-}). \tag{31}$$

One can see, that the jump of L^1_n generated by $\Delta M(s_n,y_n)$ is bounded from below by

$$r_n\Big(p^{\alpha}_{t-s_n}(0)-p^{\alpha}_{t-s_n}\big((t-s_n)^{1/\alpha}\big)\Big)\;\geq\;\big(p^a_1(0)-p^a_1(1)\big)\,(t-s_n)^{\frac{1}{1+\beta}-\frac{1}{\alpha}}\,\log\frac{1}{t-s_n}\,.$$

Now "big" jumps of a spectrally positive stable process lead to "big" values of the process, that is

$$L^1_n(T_{n,+})\geq c\,(t-s_n)^{\frac{1}{1+\beta}-\frac{1}{\alpha}}\,\log\frac{1}{t-s_n}\,. \tag{32}$$

Since the probability of having another "big" jump is small, one has

$$L^2_n(T_{n,-})\leq c\,(t-s_n)^{\frac{1}{1+\beta}-\frac{1}{\alpha}}. \tag{33}$$

As a result we have

$$\frac{Z^2_t(y_n)-Z^2_t\big(y_n+(t-s_n)^{1/\alpha}\big)}{\big((t-s_n)^{1/\alpha}\big)^{\eta_{\mathrm{c}}}}\;=\;\frac{L^1_n(T_{n,+})-L^2_n(T_{n,-})}{\big((t-s_n)^{1/\alpha}\big)^{\eta_{\mathrm{c}}}}$$

$$\geq\;c\,\frac{(t-s_n)^{\frac{1}{1+\beta}-\frac{1}{\alpha}}\,\log\frac{1}{t-s_n}}{(t-s_n)^{\frac{1}{1+\beta}-\frac{1}{\alpha}}}\;=\;c\log\frac{1}{t-s_n}\;\underset{n\uparrow\infty}{\longrightarrow}\infty. \tag{34}$$

In other words, Z^2_t is not Hölder continuous of index η_{c}.

References

[1] D.A. Dawson. Infinitely divisible random measures and superprocesses. In *Stochastic analysis and related topics (Silivri, 1990)*, volume 31 of *Progr. Probab.*, pages 1–129. Birkhäuser Boston, Boston, MA, 1992.

[2] A. Dembo, Y. Peres, J. Rosen, and O. Zeitouni. Thick points for planar Brownian motion and the Erdös-Taylor conjecture on random walk. *Acta Math.*, 186:239–270, 2001.

[3] A. Durand. Singularity sets of Lévy processes. *Probab. Theory Related Fields*, 143(3-4):517–544, 2009.

[4] K. Fleischmann. Critical behavior of some measure-valued processes. *Math. Nachr.*, 135:131–147, 1988.

[5] K. Fleischmann, L. Mytnik, and V. Wachtel. Optimal local Hölder index for density states of superprocesses with $(1+\beta)$-branching mechanism. *Ann. Probab.*, 38(3):1180–1220, 2010.

[6] K. Fleischmann, L. Mytnik, and V. Wachtel. Hölder index at a given point for density states of super-α-stable motion of index $1+\beta$. arXiv:0901.2315v2 [math.PR], 2010.

[7] K. Fleischmann and A. Sturm. A super-stable motion with infinite mean branching. *Ann. Inst. Henri Poincaré Probab. Statist.*, 40(5):513–537, 2004.

[8] K. Fleischmann and V. Wachtel. Large scale localization of a spatial version of Neveu's branching process. *Stoch. Proc. Appl.*, 116(7):983–1011, 2006.

[9] X. Hu and S.J. Taylor. Multifractal structure of a general subordinator. *Stochastic Process. Appl.*, 88:245–258, 2000.

[10] I. Iscoe. A weighted occupation time for a class of measure-valued critical branching Brownian motions. *Probab. Theory Related Fields*, 71:85–116, 1986.

[11] I. Iscoe. Ergodic theory and a local occupation time for measure-valued critical branching Brownian motion. *Stochastics*, 18:197–243, 1986.

[12] S. Jaffard. The multifractal nature of Lévy processes. *Probab. Theory Related Fields*, 114:207–227, 1999.

[13] S. Jaffard. On Lacunary wavelet series. *Ann. Appl. Probab.*, 10(1):313–329, 2000.

[14] S. Jaffard. Wavelet techniques in multifractal analysis. *Proc. Symp. Pure Math.*, 72(2):91–151, 2004.

[15] S. Jaffard and Y. Meyer. Wavelet methods for pointwise regularity and local oscillations of functions. *Memb. Am. Math. Soc.*, 123:587, 1996.

[16] A. Klenke and P. Mörters. The multifractal spectrum of Brownian intersection local time. *Ann. Probab.*, 33:1255–1301, 2005.

[17] N. Konno and T. Shiga. Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Related Fields*, 79:201–225, 1988.

[18] J.-F. Le Gall and E.A. Perkins. The Hausdorff measure of the support of two-dimensional super-Brownian motion. *Ann. Probab.*, 23(4):1719–1747, 1995.

[19] P. Mörters and N.R. Shieh. On the multifractal spectrum of the branching measure on a Galton-Watson tree. *J. Appl. Probab.*, 41:1223–1229, 2004.

[20] L. Mytnik. Stochastic partial differential equation driven by stable noise. *Probab. Theory Related Fields*, 123:157–201, 2002.

[21] L. Mytnik and E. Perkins. Regularity and irregularity of $(1+\beta)$-stable super-Brownian motion. *Ann. Probab.*, 31(3):1413–1440, 2003.

[22] E.A. Perkins and S.J. Taylor. The multifractal structure of super-Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(1):97–138, 1998.

[23] M. Reimers. One-dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Related Fields*, 81:319–340, 1989.

[24] J.B. Walsh. An introduction to stochastic partial differential equations. volume 1180 of *Lecture Notes Math.*, pages 265–439. École d'Été de Probabilités de Saint-Flour XIV–1984, Springer-Verlag Berlin, 1986.