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Asymptotic analysis for Korteweg models

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Abstract

This paper deals with a sharp interface limit of the isothermal Navier-Stokes-Korteweg system. The sharp interface limit is performed by matched asymptotic expansions of the fields in powers of the interface width ε. These expansions are considered in the interfacial region (inner expansions) and in the bulk (outer expansion) and are matched order by order. Particularly we consider the first orders of the corresponding inner equations obtained by a change of coordinates in an interfacial layer. For a specific scaling we establish solvability criteria for these inner equations and recover the results within the general setting of jump conditions for sharp interface models.

1 Introduction

Phase transitions in single substance flows are usually described by two kinds of models: the sharp interface model and the diffuse phase field model. The conventional and physically more intuitive approach is the sharp interface model. In the sharp interface approach, interfaces separating the coexisting phases or structural domains are modeled as hypersurfaces at which certain quantities such as the density or the pressure suffer jump conditions. Local quantities at the interface are then determined from the boundary conditions or are calculated from the driving force for interfacial motion. The structure of possible interfacial conditions is quite well-founded from the view point of thermodynamics. However from the numerical point of view the sharp interface approach involves the explicit tracking of the interface and becomes often numerically impractical for complicated microstructures. Because of these disadvantages, the phase field approach has emerged as a powerful method during the last twenty years. A phase field model represents a microstructure, both the compositional domains and the interfaces, as a whole. The interface between different phases is described by a small transition region, where an order parameter, representing the phases, changes its state smoothly. The microstructural evolution is modeled by a system of partial differential equations. The phase field model contains the corresponding sharp interface description as a particular limit, i.e. if the interfacial thickness tends to zero. An overview about a large class of phase field models and their sharp interface limits can be found in [2]. For phase field models it is of overall interest to validate them, by investigating the sharp interface limit in appropriate scaling regimes.

In this work we consider a sharp interface limit of the isothermal Navier-Stokes-Korteweg phase field model for a particular scaling. The sharp interface limit is performed by matched asymptotic expansions of the fields of these models in powers of the interface width ϵ. These expansions are considered in the interfacial region (inner expansion) and in the bulk (outer expansion), and are matched order by order. This results in partial differential equations for the diffuse field and a series of boundary conditions at the interface from which we achieve jump conditions in the sharp interface limit.

We consider the case where the Mach number is of order $\mathcal{O}(1)$ and viscosity and capillarity are both of order $\mathcal{O}(\varepsilon^2)$. Our particular scaling of the viscosity leads to a no-entropy-dissipation kinetic relation in the leading order $\mathcal{O}(1)$. The resulting sharp interface model is the isothermal Euler model. In the first order $\mathcal{O}(\varepsilon)$ we obtain a Young-Laplace law and a non-zero entropy dissipation, which is determined by the viscosity. The fact that the surface tension is of order $\mathcal{O}(\varepsilon)$ is in agreement with the results obtained in [19], [18], [9] for equilibria. A different kinetic relation was considered in [5], there surface tension and entropy dissipation are of order $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ respectively. There are also other physically meaningful scalings, for instance the case where the viscosity is of order $\mathcal{O}(\varepsilon)$, which implies entropy dissipation already in the leading order [14], [3], [21]. Another possibility is to look at a low Mach number limit linked to the sharp interface limit, which can be found in [13], and in [15], [16], [20] for the one phase case.

The paper is organized as follows: We start with describing the Navier-Stokes-Korteweg phase field model in Section 2. The corresponding sharp interface setting is introduced in Section 3. By non-dimensionalization we introduce the smallness parameter ε with physical interpretation, see Section 4. In Section 5 we introduce the asymptotic expansions and provide formulas for the geometric properties of the interface in two dimensions. Then we state our main results in Section 6, which are proven in Sections 7 and 8 respectively. In Section 9 we investigate the kinetic relation comprised in the jump conditions derived in Section 6. We show that the state on one side of the phase boundary already determines the interfacial velocity and the state on the other side of the phase boundary. Furthermore this property does not follow from the jump conditions for mass and momentum alone.

2 The Navier-Stokes-Korteweg phase field model

We consider an isothermal, compressible fluid with density $\rho \in (0,b)$ and velocity $\mathbf{u} \in \mathbb{R}^d$, that is capable to undergo phase transitions. We assume that the stress tensor can be additively decomposed according to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{NS} + \boldsymbol{\sigma}_K,$$

where $\boldsymbol{\sigma}_{NS}$ denotes the classical Navier-Stokes stress and $\boldsymbol{\sigma}_K$ is called Korteweg stress that takes care of possible phase transitions. The Navier-Stokes stress has the form

$$\boldsymbol{\sigma}_{NS} := \lambda(\mathrm{div}\mathbf{u})I + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \tag{2.1}$$

where λ is the bulk viscosity and μ is the shear viscosity, which satisfy $\mu > 0, \lambda + \frac{2\mu}{d} \geq 0$, where d is the space dimension. The Korteweg tensor is given by

$$\boldsymbol{\sigma}_K := \left(\rho\gamma\Delta\rho + \frac{1}{2}\gamma\,|\nabla\rho|^2\right) I - (\gamma\nabla\rho\otimes\nabla\rho)\,, \tag{2.2}$$

where γ is a constant which models capillarity effects. The evolution of the model is described by the isothermal Navier-Stokes-Korteweg equations

$$\left.\begin{aligned} \rho_t + \mathrm{div}(\rho\mathbf{u}) &= 0, && (2.3)\\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) &= \mathrm{div}\boldsymbol{\sigma}_{NS} + \mathrm{div}\boldsymbol{\sigma}_K\,, && (2.4)\end{aligned}\right\} \quad \text{in } \mathbb{R}^d \times (0,T).$$

We assume that the local part of the pressure $p = p(\rho)$ is a non-monotone function of the density given by a van-der-Waals law. From the density the phase can be directly derived, see Figure 1. The corresponding Helmholtz free energy density function $W(\rho)$ is related to the pressure by

$$p(\rho) = \rho W'(\rho) - W(\rho) \tag{2.5}$$

and has the following properties:

- $W \in C^2((0,b),[0,\infty))$,
- $\exists a_1, a_2 \in (0,b) : W'' > 0$ in $(0,a_1)\cup(a_2,b), W'' < 0$ in (a_1,a_2),

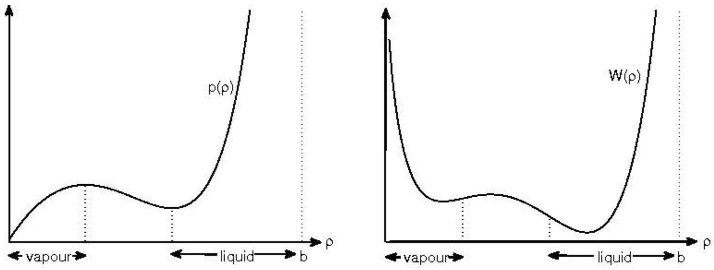

Figure 1: Van-der-Waals pressure and free energy density function

- $\lim_{\rho \to b} W(\rho) = \infty$.

We will denote the Gibbs free energy by g, which is given by $g(\rho) = W'(\rho)$.
Smooth solutions of (2.3), (2.4) identically satisfy the entropy inequality, which is given by

$$\begin{aligned} &\left(W(\rho) + \frac{\rho}{2}|\mathbf{u}|^2 + \frac{\gamma}{2}|\nabla\rho|^2\right)_t + \operatorname{div}\left(\left(W(\rho) + \frac{\rho}{2}|\mathbf{u}|^2 + \frac{\gamma}{2}|\nabla\rho|^2\right)\mathbf{u}\right) \\ &+\operatorname{div}\left((p(\rho)I - \boldsymbol{\sigma}_{NS} - \boldsymbol{\sigma}_K)\,\mathbf{u} + \gamma\rho\nabla\rho(\nabla\cdot\mathbf{u})\right) \\ &= -\boldsymbol{\sigma}_{NS} : (\nabla\mathbf{u}) \leq 0. \end{aligned} \tag{2.6}$$

An alternative form of the inequality is derived in [1], [9]. The existence of classical solutions of (2.3),(2.4) was studied in [12], [17], while weak solutions were investigated in [10], [7].

3 The Euler-Korteweg sharp interface model

Now we describe the phase transition by means of a sharp interface model. In this paper we treat the case where the capillarity γ and the viscosities λ and μ are of the same small order. The corresponding sharp interface model is described by the Euler-Korteweg system. In this context the conservation laws of mass and momentum read in the bulk phases $\Omega^-(t) := \{\mathbf{x} \in \mathbb{R}^d : \rho(t,\mathbf{x}) \in (0, a_1)\}$ and $\Omega^+(t) := \{\mathbf{x} \in \mathbb{R}^d : \rho(t,\mathbf{x}) \in (a_2, b)\}$:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{3.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = 0\,. \tag{3.2}$$

The interface between two adjacent phases $\Omega^\pm$ is described by a moving hypersurface $\Gamma(t)$, i.e. a *sharp interface* of zero thickness. Across $\Gamma(t)$ the bulk quantities may have discontinuities. We are particularly interested in the case that the interface itself is equipped with mass, momentum, energy and entropy. The jump conditions for these quantities rely on the conservation laws for mass, momentum and energy and on the entropy inequality across the interface Γ. To state these jump condition we will use the following notation for some quantity ψ having a jump at the interface. By $\psi^\pm$ we denote the limit at the interface from $\Omega^\pm$ and we abbreviate

$$[\![\psi]\!] := \psi^+ - \psi^-, \qquad \{\psi\} := \frac{\psi^+ + \psi^-}{2}.$$

For more details of the following equations we refer to [6].

The general forms of conservation of mass and momentum read:

$$[\![\rho(u_\nu - w_\nu)]\!] = -\frac{\partial \rho_\Gamma}{\partial t} - (\mathrm{div}_\Gamma(w_t) - \kappa w_\nu)\rho_\Gamma, \tag{3.3}$$

$$[\![\rho(u_\nu - w_\nu)(\mathbf{u} - \mathbf{w}) + p\boldsymbol{\nu}]\!] = -\rho_\Gamma \frac{\partial \mathbf{w}}{\partial t} + \mathrm{div}_\Gamma(\boldsymbol{\sigma}_\Gamma)\,. \tag{3.4}$$

The newly introduced quantities are the normal component of the fluid velocity u_ν, the interfacial mass density ρ_Γ, the interfacial velocity $\mathbf{w}$ which may be decomposed into normal- and tangential speed, i.e. $\mathbf{w} = w_t\mathbf{t} + w_\nu\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\Gamma \in \mathbb{R}^{d\times(d-1)}$ is the surface stress tensor. Each surface point is equipped with a tangential matrix $\mathbf{t}$ and a normal vector $\boldsymbol{\nu}$ and κ denotes the sum of the principal curvatures.

We consider exclusively isothermal processes at the temperature T_0, and these are guaranteed by corresponding heat fluxes $\boldsymbol{q}$, $\boldsymbol{q}_\Gamma$ in the bulk and on the interface. The conservation of energy serves to eliminate the heat fluxes in the entropy inequality, which hereafter becomes the relevant inequality for the isothermal Euler-Korteweg model.

The general form of conservation of energy across the interfaces reads:

$$\left[\!\!\left[\rho(u_\nu - w_\nu)(e + \frac{1}{2}|\mathbf{u}|^2) + \boldsymbol{q}\cdot\boldsymbol{\nu} + u_\nu p\right]\!\!\right] =$$
$$-\frac{\partial \rho_\Gamma(e_\Gamma + \frac{1}{2}|\mathbf{w}|^2)}{\partial t} - (\mathrm{div}_\Gamma(w_t) - \kappa w_\nu)\rho_\Gamma(e_\Gamma + \frac{1}{2}|\mathbf{w}|^2) - \mathrm{div}_\Gamma(\boldsymbol{q}_\Gamma - \mathbf{w}^{\mathbf{T}}\,\boldsymbol{\sigma}_{\boldsymbol{\Gamma}})\,. \tag{3.5}$$

The internal energy densities of bulk and interface are denoted by ρe and $\rho_\Gamma e_\Gamma$, respectively. Finally we give the entropy inequality across the interface. In [1], [9] it is shown that the sharp interface version of the Navier-Stokes-Korteweg phase field model has entropy fluxes in the bulk and at the interface that are given by $\boldsymbol{q}/T_0$ and $\boldsymbol{q}_\Gamma/T_0$, respectively. In this case the entropy inequality across the interface reads

$$\left[\!\!\left[\rho(u_\nu - w_\nu)s + \frac{\boldsymbol{q}\cdot\boldsymbol{\nu}}{T_0}\right]\!\!\right] \geq -\frac{\partial \rho_\Gamma s_\Gamma}{\partial t} - (\mathrm{div}_\Gamma(w_t) - \kappa w_\nu)\rho_\Gamma s_\Gamma - \mathrm{div}_\Gamma(\frac{\boldsymbol{q}_\Gamma}{T_0})\,, \tag{3.6}$$

where ρs is the entropy density of the bulk, and the entropy density of the interface is denoted by $\rho_\Gamma s_\Gamma$.

Next we multiply the momentum balance (3.4) by $\mathbf{w}$ and subtract it from the energy law (3.5). The result is simplified by means of the jump condition for the mass, leading to the general form of the jump condition for the internal energy

$$\left[\!\!\left[\rho(u_\nu - w_\nu)\left(e + \frac{p}{\rho} + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2\right) + \boldsymbol{q}\cdot\boldsymbol{\nu}\right]\!\!\right] =$$
$$-\frac{\partial \rho_\Gamma e_\Gamma}{\partial t} - (\mathrm{div}_\Gamma(w_t) - \kappa w_\nu)\rho_\Gamma e_\Gamma - \mathrm{div}_\Gamma(\boldsymbol{q}_\Gamma) + \nabla_\Gamma(\mathbf{w}) : \boldsymbol{\sigma}_{\boldsymbol{\Gamma}}^{\mathbf{T}}\,. \tag{3.7}$$

We are interested here in a special case where the constitutive law for the surface stress vector $\boldsymbol{\sigma}_\Gamma$ has a simple structure:

$$\boldsymbol{\sigma}_\Gamma = \gamma_\Gamma \frac{\mathbf{t}}{|\mathbf{t}|^2}\,. \tag{3.8}$$

The quantity γ_Γ is called surface tension. In this case the jump condition for the internal energy assumes the special form

$$\left[\!\!\left[\rho(u_\nu - w_\nu)\left(e + \frac{p}{\rho} + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2\right) + \boldsymbol{q}\cdot\boldsymbol{\nu}\right]\!\!\right] = -\frac{\partial \rho_\Gamma e_\Gamma}{\partial t} - (\rho_\Gamma e_\Gamma - \gamma_\Gamma)(\mathrm{div}_\Gamma(w_t) - \kappa w_\nu)\,. \tag{3.9}$$

Finally we multiply the entropy inequality by T_0 and subtract it from the jump condition for the internal energy. There follows the interfacial inequality for the Helmholtz free energy $W_\Gamma = \rho_\Gamma e_\Gamma - T_0 \rho_\Gamma s_\Gamma$:

$$\begin{aligned}\frac{\partial W_\Gamma}{\partial t} &- (\gamma_\Gamma - W_\Gamma)(\mathrm{div}_\Gamma(w_t) - \kappa w_\nu) \\ &+ \{\rho(u_\nu - w_\nu)\} \left[\!\!\left[W'(\rho) + \frac{1}{2}|\mathbf{u}-\mathbf{w}|^2 \right]\!\!\right] + [\![\rho(u_\nu - w_\nu)]\!] \left\{ W'(\rho) + \frac{1}{2}|\mathbf{u}-\mathbf{w}|^2 \right\} \le 0. \quad (3.10)\end{aligned}$$

Among the objectives of this study is to deduce the Euler-Korteweg sharp interface model from the Navier-Stokes-Korteweg phase field model. Note that the sharp interface model must fit into the setting given here, i.e. we have to recover (3.3), (3.4) and (3.10).

4 Non-dimensionalization

We introduce the following reference quantities x_r, t_r, u_r, ρ_r, p_r, μ_r and γ_r such that

$$\mathbf{x} = x_r \mathbf{x}^*, \quad t = t_r t^*, \quad \mathbf{u} = u_r \mathbf{u}^*, \quad \rho = \rho_r \rho^*, \quad p = p_r p^*, \quad \lambda = \lambda^* \mu_r, \quad \mu = \mu^* \mu_r, \quad \gamma = \gamma_r \gamma^*.$$

Then we may rewrite the Navier-Stokes-Korteweg equations (2.3), (2.4) as

$$\begin{aligned}\frac{x_r}{u_r t_r}\rho^*_{t^*} + \mathrm{div}^*(\rho^*\mathbf{u}^*) &= 0,\\ \frac{t_r u_r}{x_r}(\rho^*\mathbf{u}^*)_{t^*} + \frac{t_r^2 u_r^2}{x_r^2}\mathrm{div}^*(\rho^*\mathbf{u}^*\otimes\mathbf{u}^*) + \frac{t_r^2 u_r^2}{x_r^2 M^2}\nabla^* p^* &= \frac{t_r^2 u_r^2}{x_r^2 \mathrm{Re}}\mathrm{div}^*\boldsymbol{\sigma}^*_{NS} + C\gamma^*\mathrm{div}^*\boldsymbol{\sigma}^*_K,\end{aligned}$$

where M denotes the Mach number M= $u_r\sqrt{\frac{\rho_r}{p_r}}$, Re= $\frac{\rho_r}{\mu_r}u_r x_r$ is the Reynolds number and $C = \frac{t_r^2 \rho_r \gamma_r}{x_r^4}$ the capillarity number.
There are several possibilities to non-dimensionalize the NSK-system. In the sequel we choose $u_r = \frac{x_r}{t_r}$ and consider for a small parameter $\varepsilon > 0$ the following scaling:

$$M = 1, \quad \tfrac{1}{\mathrm{Re}} = \varepsilon^2, \quad C = \varepsilon^2. \tag{4.1}$$

This leads to

$$\begin{aligned}\rho^*_{t^*} + \mathrm{div}^*(\rho^*\mathbf{u}^*) &= 0, &(4.2)\\ (\rho^*\mathbf{u}^*)_{t^*} + \mathrm{div}^*(\rho^*\mathbf{u}^*\otimes\mathbf{u}^*) + \nabla^* p^* &= \varepsilon^2\mathrm{div}^*\boldsymbol{\sigma}^*_{NS} + \varepsilon^2\gamma^*\mathrm{div}^*\boldsymbol{\sigma}^*_K. &(4.3)\end{aligned}$$

We will call this scaling the capillarity regime. We remark that (4.2),(4.3) is equivalent to

$$\begin{aligned}\rho^*_{t^*} + \mathrm{div}^*(\rho^*\mathbf{u}^*) &= 0, &(4.4)\\ \mathbf{u}^*_{t^*} + (\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^* + \nabla^* g(\rho^*) &= \varepsilon^2\frac{1}{\rho^*}\mathrm{div}^*\boldsymbol{\sigma}^*_{NS} + \varepsilon^2\nabla^*\Delta^*\rho^*. &(4.5)\end{aligned}$$

There are further choices that lead to physically meaningful limits. More details can be found in the introduction.

For simplification we omit the symbol * in the forthcoming considerations. Letting the parameter $\varepsilon \to 0$ leads to jump conditions of the corresponding sharp interface model, which we will deduce in Section 7. Before that, we need some prerequisites for the asymptotic analysis.

5 Inner and outer expansions; matching conditions

We consider the two dimensional case d=2. The position of the phase boundary in the sharp interface limit is described by a function $\mathbf{r}(t,s)$, where s is some coordinate parametrising the interface. We can calculate normal- and tangent vectors as well as the velocity of the interface from $\mathbf{r}$.

The tangent vector pointing in counterclockwise direction is given by

$$\mathbf{t}(t,s) = \left(\frac{\partial r^1}{\partial s}(t,s), \frac{\partial r^2}{\partial s}(t,s)\right)^T,$$

were r^1 and r^2 denote the components of the vector $\mathbf{r}$ in Cartesian coordinates. The inner unit normal to the interface is given by

$$\boldsymbol{\nu}(t,s) = \frac{1}{|\mathbf{t}(t,s)|}\left(-\frac{\partial r^2}{\partial s}(t,s), \frac{\partial r^1}{\partial s}(t,s)\right)^T.$$

The mean curvature, which in two space dimensions coincides with the sum of the principal curvatures, is defined by

$$\kappa := \frac{r_s^1 r_{ss}^2 - r_{ss}^1 r_s^2}{((r_s^1)^2 + (r_s^2)^2)^{\frac{3}{2}}}. \tag{5.1}$$

Next we define the interface velocity and its decomposition into tangential and normal components by

$$\mathbf{w} := \frac{\partial \mathbf{r}}{\partial t} = w_t \mathbf{t} + w_\nu \boldsymbol{\nu}. \tag{5.2}$$

We observe that

$$\left(\frac{t^j}{|\mathbf{t}|}\right)_s = \kappa|\mathbf{t}|\nu^j \quad \text{and} \quad (\nu^j)_s = -t^j\kappa, \quad (j = 1,2) \tag{5.3}$$

and

$$(\nu^j)_\xi \nu^j = 0, \qquad \left(\frac{t^j}{|\mathbf{t}|}\right)_\xi \frac{t^j}{|\mathbf{t}|} = 0, \qquad \left(\frac{t^j}{|\mathbf{t}|}\right)_\xi \nu^j = -\frac{t^j}{|\mathbf{t}|}(\nu^j)_\xi, \tag{5.4}$$

where the index ξ in the expressions denotes the time or tangential derivative. Note that in equation (5.4) and all subsequent calculations we sum over all indices occurring twice.

For some generic quantity ψ the surface divergence in one dimension is defined by

$$\mathrm{div}_\Gamma(\psi) = \frac{1}{|\mathbf{t}|}\frac{\partial(|\mathbf{t}|\psi)}{\partial s}. \tag{5.5}$$

5.1 Outer setting

We assume the existence of expansions in ε for the density and velocity in the bulk phases:

$$\mathbf{u}(t,x^1,x^2;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}_i(t,x^1,x^2), \quad \rho(t,x^1,x^2;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \rho_i(t,x^1,x^2). \tag{5.6}$$

Inserting these expansions into the NSK equations leads to the following equations in the first two orders: In the $\mathcal{O}(\varepsilon^0)$ order:

$$\rho_{0,t} + \operatorname{div}(\rho_0 \mathbf{u}_0) = 0, \tag{5.7}$$
$$(\mathbf{u}_0)_t + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \nabla g(\rho_0) = 0. \tag{5.8}$$

In the $\mathcal{O}(\varepsilon^1)$ order:

$$\rho_{1,t} + \operatorname{div}(\rho_1 \mathbf{u}_0 + \rho_0 \mathbf{u}_1) = 0, \tag{5.9}$$
$$(\mathbf{u}_1)_t + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_0 + \nabla(g'(\rho_0)\rho_1) = 0. \tag{5.10}$$

The conditions at the interface are obtained via matching.

5.2 Inner setting

We assume that the interface Γ_ε defined as

$$\Gamma_\varepsilon(t) = \{(x^1, x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t, x^1, x^2) = \rho_*\} \tag{5.11}$$

is a C^2-hypersurface. We like to mention that this has not to be the case in general for all times t. The value ρ_* is some point in the elliptic region. The interface Γ_ε is explicitly described by $\mathbf{r}_\varepsilon$, the inner unit normal $\boldsymbol{\nu}_\varepsilon$ and the tangential vector $\mathbf{t}_\varepsilon$ respectively. A point in the neighborhood of the interface is represented by

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}(\tau, s, z) = \mathbf{r}_\varepsilon(\tau, s) + \varepsilon z \boldsymbol{\nu}_\varepsilon(\tau, s), \tag{5.12}$$

where z denotes the distance from the interface in normal direction. The reader may note that we could have used t instead of τ in (5.12), but it will become clear in the subsequent analysis why we want rename the time variable in inner coordinates. The small parameter ε is introduced to zoom in the interfacial region. The representation (5.12) can be used to change variables from $(x^1, x^2) \leftrightarrow (s, z)$ and without changing the time variable.

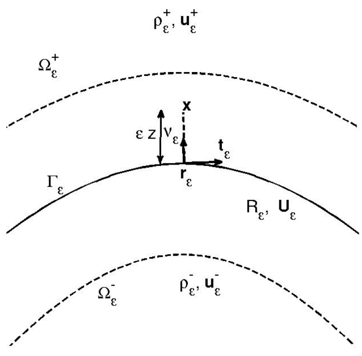

Figure 2: Boundary layer region

We suppose a scalar or a Cartesian component of a vector ψ is defined in inner and outer coordinates, i.e. $\psi(t, x^1, x^2) = \Psi(\tau, s, z)$. In the following we denote quantities in inner coordinates by capital letters in particular R is the density in inner coordinates. The partial derivatives transform as follows:

$$\begin{pmatrix} \frac{\partial\psi}{\partial x^1} \\ \frac{\partial\psi}{\partial x^2} \\ \frac{\partial\psi}{\partial t} \end{pmatrix} = \begin{pmatrix} (1+\varepsilon z\kappa)\frac{1}{|\mathbf{t}_\varepsilon|^2}t^1_\varepsilon & \varepsilon^{-1}\nu^1_\varepsilon & 0 \\ (1+\varepsilon z\kappa)\frac{1}{|\mathbf{t}_\varepsilon|^2}t^2_\varepsilon & \varepsilon^{-1}\nu^2_\varepsilon & 0 \\ -(1+\varepsilon z\kappa)((w_t)_\varepsilon - \varepsilon z\frac{1}{|\mathbf{t}_\varepsilon|^2}t^i_\varepsilon(\nu^i_\varepsilon)_\tau) & -\varepsilon^{-1}(w_\nu)_\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial\Psi}{\partial s} \\ \frac{\partial\Psi}{\partial z} \\ \frac{\partial\Psi}{\partial \tau} \end{pmatrix} + \mathcal{O}(\varepsilon^2), \quad (5.13)$$

where $(w_t)_\varepsilon = w^i_\varepsilon \frac{t^i_\varepsilon}{|\mathbf{t}_\varepsilon|^2}$ and $(w_\nu)_\varepsilon = w^i_\varepsilon \nu^i_\varepsilon$.

Accordingly to the outer expansion (5.6) we assume the existence of inner expansions in ε. In particular we assume

$$\mathbf{U}(\tau, s, z; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{U}_i(\tau, s, z), \quad R(\tau, s, z; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i R_i(\tau, s, z), \quad (5.14)$$

where $\mathbf{U}_i, R_i$ do not depend on ε for all $i \in \mathbb{N}_0$. Furthermore we suppose expansions of the geometric quantity

$$\mathbf{r}(\tau, s; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{r}_i(\tau, s), \quad (5.15)$$

which induces $\varepsilon-$expansions of $\boldsymbol{\nu}_\varepsilon, \mathbf{t}_\varepsilon$ and $\mathbf{w}_\varepsilon$.

Particularly we get the following expansions, up to terms of order $\mathcal{O}(\varepsilon^2)$:

$$\begin{aligned}
(w_\nu)_\varepsilon &= \underbrace{\nu^i_0 w^i_0}_{=:w_{\nu 0}} + \varepsilon \underbrace{(\nu^i_1 w^i_0 + \nu^i_0 w^i_1)}_{=:w_{\nu 1}} + \mathcal{O}(\varepsilon^2), & (5.16)\\
(U_\nu)_\varepsilon &= \underbrace{\nu^i_0 U^i_0}_{=:U_{\nu 0}} + \varepsilon \underbrace{(\nu^i_1 U^i_0 + \nu^i_0 U^i_1)}_{=:U_{\nu 1}} + \mathcal{O}(\varepsilon^2), & (5.17)\\
(U_t)_\varepsilon &= \underbrace{\frac{t^i_0 U^i_0}{|\mathbf{t}_0|^2}}_{=:U_{t0}} + \varepsilon \underbrace{\left(\frac{t^i_1 U^i_0}{|\mathbf{t}_0|^2} - 2\frac{t^i_0 U^i_0 t^j_0 t^j_1}{|\mathbf{t}_0|^4} + \frac{t^i_0 U^i_1}{|\mathbf{t}_0|^2}\right)}_{=:U_{t1}} + \mathcal{O}(\varepsilon^2), & (5.18)\\
(w_t)_\varepsilon &= \underbrace{\frac{t^i_0 w^i_0}{|\mathbf{t}_0|^2}}_{=:w_{t0}} + \varepsilon \underbrace{\left(\frac{t^i_1 w^i_0}{|\mathbf{t}_0|^2} - 2\frac{t^i_0 w^i_0 t^j_0 t^j_1}{|\mathbf{t}_0|^4} + \frac{t^i_0 w^i_1}{|\mathbf{t}_0|^2}\right)}_{=:w_{t1}} + \mathcal{O}(\varepsilon^2), & (5.19)\\
J_\varepsilon &:= R_\varepsilon((U_\nu)_\varepsilon - (w_\nu)_\varepsilon) & (5.20)\\
&= \underbrace{R_0(U_{\nu 0} - w_{\nu 0})}_{=:J_0} + \varepsilon \underbrace{(R_1(U_{\nu 0} - w_{\nu 0}) + R_0(U_{\nu 1} - w_{\nu 1}))}_{=:J_1} + \mathcal{O}(\varepsilon^2). & (5.21)
\end{aligned}$$

With respect to the expansions of the geometric quantities we observe

$$1 = |\boldsymbol{\nu}_\varepsilon|^2 = \nu^i_0 \nu^i_0 + 2\varepsilon \nu^i_0 \nu^i_1 + \mathcal{O}(\varepsilon^2),$$

which implies

$$\nu^i_0 \nu^i_0 = 1, \quad \nu^i_0 \nu^i_1 = 0. \quad (5.22)$$

Furthermore we have

$$0 = t^i_\varepsilon \nu^i_\varepsilon = t^i_0 \nu^i_0 + \varepsilon(t^i_1 \nu^i_0 + t^i_0 \nu^i_1) + \mathcal{O}(\varepsilon^2),$$

which implies

$$t_0^i \nu_0^i = 1, \quad \nu_0^i t_1^i + \nu_1^i t_0^i = 0. \tag{5.23}$$

Additionally an easy calculation shows

$$(\nu_0^i)_s = -\kappa_0 t_0^i, \quad \left(\frac{t_0^i}{|\mathbf{t}_0|}\right)_s = \kappa_0 |t_0| \nu_0^i. \tag{5.24}$$

Using (5.12), (5.14) and (5.15) we get the inner equations by comparing coefficients of different powers of ε.

5.3 Matching conditions

Inner- and outer quantities are matched by the usual procedure, see [8] for details. For convenience of the reader we sketch the arguments. Let ψ and Ψ be any function possessing expansions like (5.6) and (5.14). Near the interfacial layer we formally equate the two expansions

$$\Psi(\tau, s, z; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \Psi_i(\tau, s, z) = \sum_{i=0}^{\infty} \varepsilon^i \psi_i(t, \mathbf{r}_\varepsilon(\tau, s) + \varepsilon z \boldsymbol{\nu}_\varepsilon(\tau, s)) = \psi(t, x^1, x^2; \varepsilon). \tag{5.25}$$

Then we expand the right-hand side in a Taylor series yielding

$$\sum_{i=0}^{\infty} \varepsilon^i \Psi_i(\tau, s, z) = \sum_{i=0}^{N} \varepsilon^i P_i^{\pm}(\tau, s, z) + \varepsilon^{N+1} R_{N+1}^{\pm}, \tag{5.26}$$

where

$$P_i^{\pm} = \frac{1}{i!} \frac{d^i}{d\varepsilon^i} \psi(t, \mathbf{r}_\varepsilon(\tau, s) + \varepsilon z \boldsymbol{\nu}_\varepsilon(\tau, s)) \Big|_{\varepsilon=0} \tag{5.27}$$

and $R_{N+1}^{\pm}$ is a Lagrange remainder evaluated at some small $\hat{\varepsilon}$. We observe that $P_i^{\pm}$ is a polynomial of order i in z with coefficients depending on $\mathbf{r}$ and its derivatives and on (x^1, x^2)-derivatives of ψ. We will assume that all the (x^1, x^2)-derivatives of ψ up to order $N+1$ are bounded in a neighborhood of the interface. The polynomials P_i^+, P_i^- are valid for $z > 0$ and $z < 0$ respectively. Now letting $z \to \pm\infty$ with ε coupled to z such that $\varepsilon z^{N+1} \to 0$ but otherwise arbitrary. Then the remainder term in (5.26) has higher order than all the preceding terms. This fact leads to the identification of the inner quantities with the Taylor polynomials for $z \to \infty$, i.e.

$$\Psi_i(\tau, s, z) = P_i^{\pm}(\tau, s, z) + o(1) \text{ for } z \to \pm\infty. \tag{5.28}$$

Applying this to the first two orders of the asymptotic expansions shows

$$\Psi_0(\tau, s, z) \to \quad \psi_0^{\pm}(\tau, \mathbf{r}_0(\tau, s)) \qquad z \to \pm\infty, \tag{5.29}$$

$$\Psi_1(\tau, s, z) \to \quad \psi_1^{\pm}(\tau, \mathbf{r}_0(\tau, s)) + \left(\tfrac{\partial \psi_0}{\partial x^j}\right)^{\pm} (\tau, \mathbf{r}_0(\tau, s))(r_1^j(\tau, s) + \nu_0^j(\tau, s) z) \qquad z \to \pm\infty. \tag{5.30}$$

By differentiating both sides in (5.25) with respect to s, z or τ we can derive further matching conditions:

$$\Psi_{0,\tau} \to \quad \left(\tfrac{\partial \psi_0}{\partial x^j}\right)^{\pm} (\tau, \mathbf{r}_0(\tau, s)) w_0^j(\tau, s) + \left(\tfrac{\partial \psi_0}{\partial t}\right)^{\pm} (\tau, \mathbf{r}_0(\tau, s)) \qquad z \to \pm\infty, \tag{5.31}$$

$$\Psi_{0,s} \to \quad \left(\tfrac{\partial \psi_0}{\partial x^j}\right)^{\pm} (\tau, \mathbf{r}_0(\tau, s)) t_0^j(\tau, s) \qquad z \to \pm\infty, \tag{5.32}$$

$$\Psi_{0,z} \to \quad 0 \qquad z \to \pm\infty, \tag{5.33}$$

$$\Psi_{1,z} \to \quad \left(\tfrac{\partial \psi_0}{\partial x^j}\right)^{\pm} (\tau, \mathbf{r}_0(\tau, s)) \nu_0^j(\tau, s) \qquad z \to \pm\infty. \tag{5.34}$$

We will assume that all these limits are attained superlinearly fast.

When we define

$$
\begin{aligned}
\psi_0^+(s,\tau) &:= \lim_{z>0, z\to 0} \psi_0(\tau, \mathbf{r}_0(s,\tau) + z\boldsymbol{\nu}_0(s,\tau)), && (5.35)\\
\psi_0^-(s,\tau) &:= \lim_{z<0, z\to 0} \psi_0(\tau, \mathbf{r}_0(s,\tau) + z\boldsymbol{\nu}_0(s,\tau)), && (5.36)
\end{aligned}
$$

we can rewrite (5.31) and (5.32) as

$$
\begin{aligned}
\Psi_{0,\tau} \to &\quad (\psi_0^{\pm})_\tau \quad z \to \pm\infty, && (5.37)\\
\Psi_{0,s} \to &\quad (\psi_0^{\pm})_s \quad z \to \pm\infty. && (5.38)
\end{aligned}
$$

6 Main results

In our considerations we assume that for all relevant quantities there exists an asymptotic expansion in ε. In particular we consider solutions of the structure given below:

Definition 1 *Let $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ be a classical solution of* (4.4),(4.5) *existing in the time interval* $[0,T)$ *with inner and outer expansions as in* (5.6) *and* (5.14).

In addition let

$$
\begin{aligned}
\Omega_\varepsilon^-(t) &:= \{(x^1,x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t,x^1,x^2) < \rho_*\},\\
\Omega_\varepsilon^+(t) &:= \{(x^1,x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t,x^1,x^2) > \rho_*\},\\
\Gamma_\varepsilon(t) &:= \{(x^1,x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t,x^1,x^2) = \rho_*\},
\end{aligned}
$$

for $t \in [0,T)$.

When Γ_ε is a $C^1([0,T), C^2(\mathbb{R}^2))$-hypersurface with an asymptotic ε-expansion as in (5.15) *we call $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1)$ an* **outer solution** *if*

$$
\begin{aligned}
\rho_0, \rho_1 &\in C^1([0,T), C^0(\bar\Omega_0^{\pm}(t))) \cap C^0([0,T), C^1(\bar\Omega_0^{\pm}(t))), && (6.1)\\
\mathbf{u}_0, \mathbf{u}_1 &\in C^1([0,T), C^0(\bar\Omega_0^{\pm}(t))) \cap C^0([0,T), C^1(\bar\Omega_0^{\pm}(t))) && (6.2)
\end{aligned}
$$

and $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1)$ satisfies

$$
\begin{aligned}
\rho_{0,t} + \operatorname{div}(\rho_0 \mathbf{u}_0) &= 0, && (6.3)\\
(\mathbf{u}_0)_t + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \nabla g(\rho_0) &= 0, && (6.4)\\
\rho_{1,t} + \operatorname{div}(\rho_1 \mathbf{u}_0 + \rho_0 \mathbf{u}_1) &= 0, && (6.5)\\
(\mathbf{u}_1)_t + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_0 + \nabla(g'(\rho_0)\rho_1) &= 0. && (6.6)
\end{aligned}
$$

When Γ_ε is a $C^1([0,T), C^2(\Omega))$-hypersurface with an asymptotic ε-expansion as in (5.15) *we call $(R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ an* **inner solution** *if*

$$
\begin{aligned}
R_0, R_1 &\in C^1(I \times [0,T), C^3(\mathbb{R})), && (6.7)\\
\mathbf{U}_0, \mathbf{U}_1 &\in C^1(I \times [0,T), C^1(\mathbb{R})) && (6.8)\\
& && (6.9)
\end{aligned}
$$

and $(R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ *satisfies*

$$
\begin{aligned}
-w_{\nu 0} R_{0,z} + (U_{\nu 0} R_0)_z &= 0, \quad (6.10)\\
-w_{\nu 0}(U_0^j)_z + U_{\nu 0}(U_0^j)_z + \nu_0^j g(R_0)_z - \nu_0^j \gamma R_{0,zzz} &= 0, \quad (6.11)\\
-w_{\nu 1} R_{0,z} - w_{\nu 0} R_{1,z} - w_{t0} R_{0,s} + R_{0,\tau} + \frac{t_0^i}{|\mathbf{t}_0|^2}(R_0 U_0^i)_s + (R_1 U_{\nu 0} + R_0 U_{\nu 1})_z &= 0, \quad (6.12)
\end{aligned}
$$

$$
\begin{aligned}
& -w_{\nu 1}(U_0^j)_z - w_{\nu 0}(U_1^j)_z + (U_0^j)_\tau - w_{t0}(U_0^j)_s + U_{\nu 1}\left(U_0^j\right)_z + U_{\nu 0}\left(U_1^j\right)_z \qquad (6.13)\\
& + U_{t0}(U_0^j)_s + g(R_0)_z \nu_1^j + (g'(R_0)R_1)_z \nu_0^j + g(R_0)_s \frac{t_0^j}{|\mathbf{t}_0|^2} - (\lambda + 2\mu)\,\nu_0^j \frac{1}{R_0}(U_{\nu 0})_{zz}\\
& = \mu t_0^j \frac{1}{R_0}(U_{t0})_{zz} + \gamma \nu_0^j R_{1,zzz} + \gamma \nu_1^j R_{0,zzz} - \gamma \kappa_0 \nu_0^j R_{0,zz} + \gamma \frac{t_0^j}{|\mathbf{t}_0|^2} R_{0,szz}.
\end{aligned}
$$

We call $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ *a* **matching solution***, when* $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1)$ *is an outer solution,* $(R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ *is an inner solution and both are linked by the matching conditions* (5.29)-(5.34)*. We call a matching solution* **admissible***, if the mass flux* $\rho_0(u_{\nu 0} - w_{\nu 0}) \neq 0$.

Remark 6.1
Equations (6.3)-(6.6) *were motivated in Section 5.1. To derive* (6.10)-(6.13) *we perform the coordinate change* (5.12) *in* (4.4) *and* (4.5)*, which changes the partial derivatives according to* (5.13)*. Then we insert the expansions* (5.14) *and compare coefficients of different powers of* ε.

Remark 6.2
The admissibility condition $\rho_0(u_{\nu 0} - w_{\nu 0}) \neq 0$ *is due to two reasons. We think that the case of real phase transitions with non-zero mass flux over the interface is more interesting. Furthermore the subsequent analysis does not cover the case* $\rho_0(u_{\nu 0} - w_{\nu 0}) = 0$.

Now we are well-prepared to state our main results.

Theorem 6.3
If $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ *is an admissible matching solution defined as in Definition 1, then the following jump conditions up to order* $\mathcal{O}(\varepsilon^2)$ *are fulfilled:*

$$
\begin{aligned}
[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!] &= -\varepsilon \frac{\partial \rho_\Gamma}{\partial \tau} - \varepsilon(\mathrm{div}_\Gamma(w_{t0}) - \kappa_0 w_{\nu 0})\rho_\Gamma, \qquad (6.14)\\
[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\boldsymbol{\nu}_\varepsilon]\!] &= -\varepsilon \rho_\Gamma \frac{\partial \mathbf{w}_0}{\partial \tau} + \varepsilon \mathrm{div}_\Gamma(\boldsymbol{\sigma}_\Gamma), \qquad (6.15)\\
\left[\!\!\left[\frac{1}{2}\left(\frac{j_\varepsilon}{\rho_\varepsilon}\right)^2 + g(\rho_\varepsilon)\right]\!\!\right] &= -\varepsilon \int_0^\infty \nu_0^j \left(U_0^j - (u_0^j)^+\right)_\tau dz \qquad (6.16)\\
&\quad -\varepsilon \int_{-\infty}^0 \nu_0^j \left(U_0^j - (u_0^j)^-\right)_\tau dz\\
&\quad -\varepsilon(\lambda + 2\mu) j_0 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0}\right)_z\right)^2 dz + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

where

$$\rho_\Gamma \;:=\; \int_0^\infty (R_0-\rho_0^+)\,dz - \int_{-\infty}^0 (R_0-\rho_0^-)\,dz + \mathcal{O}(\varepsilon), \tag{6.17}$$

$$\boldsymbol{\sigma}_\Gamma \;:=\; \sigma_\Gamma^j = \left(\int_0^\infty \left(\left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^+} + \gamma R_{0,z}^2\right)\right) dz + \int_{-\infty}^0 \ldots\, dz\right) \frac{t_0^j}{|\mathbf{t}_0|^2} + \mathcal{O}(\varepsilon). \tag{6.18}$$

Remark 6.4
We want to emphasize that (6.14) *and* (6.15) *recover* (3.3) *and* (3.4) *respectively and we will show in Theorem 6.6 that* (6.14) - (6.16) *are in agreement with* (3.10).

Remark 6.5
The zeroth orders of (6.14) - (6.16) *imply that the states* $\rho_0^\pm$ *are the Maxwell points of the free energy density function*

$$W_{j_0}(\rho) := W(\rho) - \frac{j_0^2}{2\rho}.$$

Theorem 6.6
If $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ *is an admissible matching solution defined as in Definition 1, then for* ε *sufficiently small the jump conditions derived in Theorem 6.3 imply the inequality*

$$\begin{aligned} 0 \;\geq\; & -\varepsilon(\lambda+2\mu) j_0^2 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0}\right)_z\right)^2 dz + \mathcal{O}(\varepsilon^2) \qquad (6.19)\\ =\; & \varepsilon \frac{\partial W_\Gamma}{\partial \tau} - \varepsilon\left(\gamma_\Gamma - W_\Gamma\right)\left(\mathrm{div}_\Gamma((w_t)_\varepsilon) - \kappa_\varepsilon (w_\nu)_\varepsilon\right) + [\![j_\varepsilon]\!] \left\{ W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right\} \\ & + \{j_\varepsilon\} \left[\!\!\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right]\!\!\right]. \end{aligned}$$

Recall that the surface tension γ_Γ *is related to the surface stress tensor by* $(3.8)_2$. *The interface mass density* ρ_Γ *and the surface stress tensor are as in* (6.17), (6.18) *and the Helmholtz surface energy density* W_Γ *is given by*

$$\begin{aligned} W_\Gamma \;=\; & \int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{1}{2}\frac{j_0^2}{R_0} - \frac{1}{2}\frac{j_0^2}{\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2\right) dz \qquad (6.20)\\ & + \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{1}{2}\frac{j_0^2}{R_0} - \frac{1}{2}\frac{j_0^2}{\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2\right) dz. \end{aligned}$$

The inequality (6.19) *is identical to the entropy inequality* (3.10). *Thus the jump conditions derived in Theorem 6.3 are compatible to the second law of thermodynamics.*

Remark 6.7
The inequality (6.19) *can be obtained by straightforward but cumbersome calculations using asymptotic analysis to the entropy inequality in the bulk (see* (2.6)*), which is derived in [9].*

7 Proof of Theorem 6.3

In the following lemma we will establish the zeroth order of (6.14), (6.15) and (6.16).

Lemma 7.1
If $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ is an admissible matching solution defined as in Definition 1, then the following zeroth order jump conditions are satisfied:

$$
\begin{aligned}
[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!]_0 &= 0, &(7.1)\\
[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\boldsymbol{\nu}_\varepsilon]\!]_0 &= 0, &(7.2)\\
[\![(u_t)_\varepsilon]\!]_0 &= 0, &(7.3)\\
\left[\!\!\left[\frac{1}{2}\left(\frac{j_\varepsilon}{\rho_\varepsilon}\right)^2 + g(\rho_\varepsilon)\right]\!\!\right]_0 &= 0. &(7.4)
\end{aligned}
$$

Proof:
The mass flux over the interface is given by $j := \rho(u_\nu - w_\nu)$. Hence equation (6.10) implies $J_{0,z} = 0$ and therefore we get the jump condition (7.1), i.e.

$$[\![j]\!]_0 = 0. \tag{7.5}$$

Multiplying (6.11) with the zeroth order of the tangent vector $\mathbf{t}_0$ we obtain

$$\frac{J_0}{R_0}(U_{t0})_z = 0. \tag{7.6}$$

Hence, the leading order of the tangent velocity U_{t0} is constant over the interface. Furthermore to solve (6.10) and (6.11) it is sufficient to substitute $U_{\nu 0} = \frac{J_0}{R_0} - w_{\nu 0}$ in (6.11) and to solve (6.11) times $\boldsymbol{\nu}_0$ which gives

$$\frac{J_0}{R_0}\left(\frac{J_0}{R_0}\right)_z + g(R_0)_z = \gamma R_{0,zzz}. \tag{7.7}$$

By Proposition 1.2 in the work of Benzoni-Gavage et. al. [4] there exist values $\rho_0^\pm$ satisfying certain jump conditions which we will state in (7.9) and (7.12) below. Given these states $\rho_0^\pm$ there exists a solution R_0 of (7.7) attaining $\rho_0^\pm$ as boundary values.

In the sequel we will show that (7.9) and (7.12) are necessary conditions for boundary values of solutions of (7.7). From (7.7) we infer

$$\frac{1}{2}\left(\frac{J_0}{R_0}\right)^2 + g(R_0) = \gamma R_{0,zz} + c_1 \tag{7.8}$$

for some $c_1 \in \mathbb{R}$. Using the matching conditions (5.29) and (5.33) this implies

$$c_1 = \frac{1}{2}\left(\frac{j_0}{\rho_0^-}\right)^2 + g(\rho_0^-) = \frac{1}{2}\left(\frac{j_0}{\rho_0^+}\right)^2 + g(\rho_0^+). \tag{7.9}$$

Multiplying (7.7) by R_0 we get

$$J_0\left(\frac{J_0}{R_0}\right)_z + p(R_0)_z = \gamma R_0 R_{0,zzz}, \tag{7.10}$$

which implies

$$\frac{J_0^2}{R_0} + p(R_0) = \gamma \left(R_0 R_{0,zz} - \frac{1}{2}(R_{0,z})^2 \right) + c_2 \tag{7.11}$$

for some c_2 in $\mathbb{R}$. Applying (5.29) and (5.33) to (7.11) we find

$$c_2 = \frac{j_0^2}{\rho_0^-} + p(\rho_0^-) = \frac{j_0^2}{\rho_0^+} + p(\rho_0^+). \tag{7.12}$$

■

Remark 7.2
In Lemma 7.1 we have seen that the tangential velocity is constant over the interface and up to now we have not chosen a specific parameterization of the interface. So we are free to impose

$$U_0^i = w_{t0} t_0^i + U_{\nu 0} \nu_0^i, \quad \text{i.e. } U_{t0} = w_{t0}. \tag{7.13}$$

We want to point out that the admissibility condition from Definition 1 is crucial to establish the equality of the tangent velocities of the fluid and the interface.

Lemma 7.3
Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution and let the operator

$$L : W^{3,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R}) \to L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times L^1(\mathbb{R})$$

be given by

$$L \begin{pmatrix} A \\ B \\ C \end{pmatrix} := \begin{pmatrix} \left(A(U_{\nu 0} - w_{\nu 0}) + R_0 B\right)_z \\ \left((U_{\nu 0} - w_{\nu 0})B + g'(R_0)A - \gamma A_{zz}\right)_z \\ (U_{\nu 0} - w_{\nu 0})C_z \end{pmatrix}, \tag{7.14}$$

then

$$L \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{7.15}$$

has a solution if and only if

$$\int_{-\infty}^{\infty} f_1 \, dz = 0, \tag{7.16}$$

$$\int_{-\infty}^{\infty} f_2 \, dz = 0, \tag{7.17}$$

$$\int_{-\infty}^{\infty} R_0 f_3 \, dz = 0, \tag{7.18}$$

$$\int_{-\infty}^{\infty} \frac{j_0}{R_0} f_1 + R_0 f_2 \, dz = 0, \tag{7.19}$$

where $f_1, f_2, f_3 \in L^1(\mathbb{R})$.

Proof:
By the Fredholm alternative theorem the system (7.15) is solvable if and only if the right hand side satisfies

$$\int_{-\infty}^{\infty} f \cdot \tilde{A} + g \cdot \tilde{B} + h \cdot \tilde{C} \, dz = 0$$

for every solution $(\tilde{A}, \tilde{B}, \tilde{C})^T$ of the homogeneous problem for the adjoint operator. The adjoint operator

$$L^* : L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \to W^{-3,1}(\mathbb{R}) \times W^{-1,1}(\mathbb{R}) \times W^{-1,1}(\mathbb{R})$$

is given by

$$L^* \begin{pmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{pmatrix} := \begin{pmatrix} (U_{\nu 0} - w_{\nu 0})\tilde{A}_z + g'(R_0)\tilde{B}_z - \gamma \tilde{B}_{zzz} \\ R_0 \tilde{A}_z + (U_{\nu 0} - w_{\nu 0})\tilde{B}_z \\ \Big((U_{\nu 0} - w_{\nu 0})\tilde{C} \Big)_z \end{pmatrix}. \tag{7.20}$$

In order to determine solvability criteria for (7.15) we have to find all $(\tilde{A}, \tilde{B}, \tilde{C})^T \in (L^\infty(\mathbb{R}))^3$ satisfying

$$L^*(\tilde{A}, \tilde{B}, \tilde{C}) = 0. \tag{7.21}$$

Considering the homogeneous problem (7.21) it is well known (cf. [11] for example) that all distributional solutions are already classical solutions. There are five linearly independent solutions of (7.21) in $C^3(\mathbb{R}) \times C^1(\mathbb{R}) \times C^1(\mathbb{R})$. We have to determine whether they are elements of $(L^\infty(\mathbb{R}))^3$.

The homogeneous problem decouples into one problem for $\tilde{A}$ and $\tilde{B}$ and one for $\tilde{C}$. The latter immediately implies due to (5.20) that $\tilde{C}$ has to be some multiple of R_0. The equations for $\tilde{A}$ and $\tilde{B}$ are more involved. They amount to

$$\begin{aligned} \frac{j_0}{R_0}\tilde{A}_z + g'(R_0)\tilde{B}_z - \gamma \tilde{B}_{zzz} &= 0, && (7.22) \\ R_0 \tilde{A}_z + (U_{\nu 0} - w_{\nu 0})\tilde{B}_z &= 0. && (7.23) \end{aligned}$$

We can solve (7.23) for $\tilde{A}_z$, which yields

$$\tilde{A}_z = -\frac{j_0}{R_0^2}\tilde{B}_z. \tag{7.24}$$

Inserting (7.24) in (7.22) we obtain

$$-\frac{j_0^2}{R_0^3}\tilde{B}_z + g'(R_0)\tilde{B}_z - \gamma \tilde{B}_{zzz} = 0. \tag{7.25}$$

For simplification of notation we introduce a new energy density function and a corresponding Gibbs free energy function

$$\tilde{W}(R_0) := W(R_0) - \frac{1}{2}\frac{j_0^2}{R_0}, \qquad \tilde{g}(R_0) := \tilde{W}'(R_0) = g(R_0) + \frac{1}{2}\frac{j_0^2}{R_0^2}. \tag{7.26}$$

Then (7.25) becomes

$$\tilde{g}'(R_0)\tilde{B}_z - \gamma \tilde{B}_{zzz} = 0. \tag{7.27}$$

We observe that each constant function and $\tilde{B} = R_0$ are solutions of (7.27). So we only have to find one more linearly independent solution of (7.27). When we define $\tilde{D} := \tilde{B}_z$ and $\tilde{E} = \tilde{D}_z$, we get from (7.27)

$$\begin{pmatrix} \tilde{D} \\ \tilde{E} \end{pmatrix}_z = \begin{pmatrix} 0 & 1 \\ \frac{1}{\gamma}\tilde{g}'(R_0) & 0 \end{pmatrix} \begin{pmatrix} \tilde{D} \\ \tilde{E} \end{pmatrix}. \tag{7.28}$$

As $\tilde{B} = R_0$ is a solution of (7.27), $(R_{0,z}, R_{0,zz})^T$ is a solution of (7.28). Due to the d'Alembert reduction principle we make the following ansatz:

$$\tilde{D}(z) = \varphi(z)R_{0,z}(z), \tag{7.29}$$
$$\tilde{E}(z) = \varphi(z)R_{0,zz}(z) + \psi(z), \tag{7.30}$$

where the functions φ and ψ are to be determined. Inserting (7.29), (7.30) in (7.28) we get

$$\varphi_z(z)R_{0,z}(z) + \varphi(z)R_{0,zz}(z) = \varphi(z)R_{0,zz} + \psi(z), \tag{7.31}$$
$$\varphi_z(z)R_{0,zz}(z) + \varphi(z)R_{0,zzz}(z) + \psi_z(z) = \frac{1}{\gamma}\tilde{g}'(R_0)\varphi(z)R_{0,z}(z). \tag{7.32}$$

We can solve (7.31) with respect to φ_z, which gives

$$\varphi_z = \frac{\psi}{R_{0,z}}. \tag{7.33}$$

Using (7.7) and (7.26) in (7.32) we obtain

$$\psi_z(z) = -\varphi_z(z)R_{0,zz}(z) \overset{(7.33)}{=} -\psi(z)\frac{R_{0,zz}(z)}{R_{0,z}(z)}. \tag{7.34}$$

From this we find

$$\psi = \frac{k}{R_{0,z}}$$

for $k \in \mathbb{R}$ and hence by (7.33)

$$\varphi_z = \frac{k}{(R_{0,z})^2}. \tag{7.35}$$

We only need one solution F of (7.27) which is linearly independent of the constant and R_0. Thus we consider the special case

$$\varphi(z) = \int_0^z \frac{1}{(R_{0,z}(\tilde{z}))^2}\, d\tilde{z}.$$

Then F is given as a primitive function of $\varphi R_{0,z}$. We will show $F \notin L^\infty(\mathbb{R})$. To do so it is sufficient to show

$$|\varphi(z)R_{0,z}(z)| \to \infty \quad \text{for } |z| \to \infty, \tag{7.36}$$

because it directly implies

$$|F(z)| \to \infty \quad \text{for } |z| \to \infty.$$

To verify (7.36) we rely on the fact

$$R_{0,z}(z), R_{0,zz}(z) \to 0 \quad \text{for } |z| \to \infty,$$

which is prescribed by the matching conditions. Given $\varepsilon > 0$ there exists $K > 0$ such that

$$|R_{0,z}(z)| < \varepsilon,\ |R_{0,zz}(z)| < \varepsilon \quad \text{for } |z| > K.$$

We consider some pair $(z, \bar{z})$ with $|\bar{z}| > K + 1$, $|z| > K$ and $z\bar{z} > 0$. Then we have

$$|R_{0,z}(z) - R_{0,z}(\bar{z})| = \left| \int_{\bar{z}}^{z} R_{0,zz}(\tilde{z})\, d\tilde{z} \right| \le |z - \bar{z}|\varepsilon. \tag{7.37}$$

Therefore

$$|R_{0,z}(z)| \le 2|R_{0,z}(\bar{z})| \quad \text{for all } |z - \bar{z}| \le \frac{|R_{0,z}(\bar{z})|}{\varepsilon}\ (< 1).$$

Next we define $z^* := \bar{z} - \operatorname{sgn}(\bar{z})\frac{|R_{0,z}(\bar{z})|}{\varepsilon}$ and find

$$|\varphi(\bar{z})| \ge \left| \int_{z^*}^{\bar{z}} \left(\frac{1}{R_{0,z}(\tilde{z})} \right)^2 d\tilde{z} \right| = |\bar{z} - z^*| \frac{1}{4} \left| \frac{1}{R_{0,z}(\bar{z})} \right|^2, \tag{7.38}$$

which implies

$$|\varphi(\bar{z}) R_{0,z}(\bar{z})| \ge \frac{1}{4} |\bar{z} - z^*| \left| \frac{1}{R_{0,z}(\bar{z})} \right| = \frac{1}{4\varepsilon}. \tag{7.39}$$

So for every $\varepsilon > 0$ we find some $K > 0$ such that

$$|\varphi(\bar{z}) R_{0,z}(\bar{z})| \ge \frac{1}{4\varepsilon} \quad \text{for all } |\bar{z}| > K + 1, \tag{7.40}$$

which proves (7.36). Hence there are only two linearly independent solutions of (7.27) in $L^\infty(\mathbb{R})$.

Using (7.24) the general solution of the homogeneous problem (7.21) in $(L^\infty(\mathbb{R}))^3$ has the form

$$\begin{aligned} \tilde{A} &= k_1 + k_2 \frac{j_0}{R_0}, \\ \tilde{B} &= k_3 + k_2 R_0, \\ \tilde{C} &= k_4 R_0, \end{aligned} \tag{7.41}$$

for coefficients $k_1, k_2, k_3, k_4 \in \mathbb{R}$. This gives rise to the solvabilty criteria

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} f_1\, dz = 0, && \int_{-\infty}^{\infty} f_2\, dz = 0, \\ &\textstyle\int_{-\infty}^{\infty} R_0 f_3\, dz = 0, && \int_{-\infty}^{\infty} \frac{j_0}{R_0} f_1 + R_0 f_2\, dz = 0. \end{aligned}$$

■

Lemma 7.4
Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ *be an admissible matching solution and let* $\Phi, \Psi_\nu, \Psi_t \in C^\infty(\mathbb{R})$ *satisfying*

$$\Phi(z) = \rho_1^\pm + \left(\frac{\partial \rho_0}{\partial x^k} \right)^\pm (r_1^k + z\nu_0^k), \tag{7.42}$$

$$\Psi_\nu(z) = (u_0^j)^\pm \nu_1^j + (u_1^j)^\pm \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \nu_0^j (r_1^k + z\nu_0^k), \tag{7.43}$$

$$\Psi_t(z) = (u_1^j)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2} (r_1^k + z\nu_0^k) + (u_0^j)^\pm \left(\frac{t_1^j}{|\mathbf{t}_0|^2} - 2\frac{t_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4} \right), \tag{7.44}$$

for $z > 1$ and $z < -1$ respectively. Then equation (6.12), (6.13) *can be equivalently written as*

$$L\begin{pmatrix} \tilde{R}_1 \\ \tilde{U}_{\nu 1} \\ \tilde{U}_{t1} \end{pmatrix} = \begin{pmatrix} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] \\ f_2[R_0, \mathbf{U}_0, \Phi, \Psi] \\ f_3[R_0, \mathbf{U}_0, \Phi, \Psi] \end{pmatrix}, \tag{7.45}$$

where L is the operator from Lemma 7.3, $\tilde{R}_1 = R_1 - \Phi, \tilde{U}_{\nu 1} = R_1 - \Psi_\nu, \tilde{U}_{t1} = U_{t1} - \Psi_t$ and

$$\begin{aligned} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] := &-\frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{\nu 0} \\ &- (\Phi(U_{\nu 0} - w_{\nu 0}) + R_0(\Psi_\nu - w_{\nu 1}))_z\,, \end{aligned} \tag{7.46}$$

$$\begin{aligned} f_2[R_0, \mathbf{U}_0, \Phi, \Psi] := &-\nu_0^j (U_0^j)_\tau + (\lambda + 2\mu)\frac{1}{R_0}(U_{\nu 0})_{zz} - \gamma\kappa_0 R_{0,zz} \\ &- \left((U_{\nu 0} - w_{\nu 0})(\Psi_\nu - w_{\nu 1}) + g'(R_0)\Phi - \gamma\Phi_{zz}\right)_z\,, \end{aligned} \tag{7.47}$$

and

$$f_3[R_0, \mathbf{U}_0, \Phi, \Psi] := -\left(\frac{t_0^j}{|\mathbf{t}_0|^2}\right)(U_0^j)_\tau - \frac{1}{|\mathbf{t}_0|^2}\left(g(R_0) - \gamma R_{0,zz}\right)_s - (U_{\nu 0} - w_{\nu 0})(\Psi_t)_z. \tag{7.48}$$

Proof:
We will not consider (6.13) directly but separate it into a normal and a tangential part. The normal part is given by (6.13) times ν_0^j plus (6.10) times ν_1^j, i.e.

$$\begin{aligned} &-w_{\nu 1}(U_{\nu 0})_z - w_{\nu 0}(U_{\nu 1})_z + \nu_0^j(U_0^j)_\tau + U_{\nu 1}(U_{\nu 0})_z + U_{\nu 0}(U_{\nu 1})_z + (g'(R_0)R_1)_z \\ &\qquad = \frac{1}{R_0}(\lambda + 2\mu)(U_{\nu 0})_{zz} + \gamma R_{1,zzz} - \kappa_0\gamma R_{0,zz}. \end{aligned} \tag{7.49}$$

Rearranging the terms in (7.49) gives

$$\begin{aligned} &\left((U_{\nu 0} - w_{\nu 0})(U_{\nu 1} - w_{\nu 1}) + g'(R_0)R_1\right)_z \\ &\qquad = -\nu_0^j(U_0^j)_\tau + \frac{1}{R_0}(\lambda + 2\mu)(U_{\nu 0})_{zz} + \gamma R_{1,zzz} - \kappa_0\gamma R_{0,zz}. \end{aligned} \tag{7.50}$$

The tangential part of (6.13) is given by (6.13) times $\frac{t_0^j}{|\mathbf{t}_0|^2}$ plus (6.10) times $\frac{t_1^j}{|\mathbf{t}_0|^2} - 2\frac{t_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4}$, i.e.

$$\begin{aligned} &-w_{\nu 1}(U_{t0})_z - w_{\nu 0}\left(\frac{t_0^j U_1^j}{|\mathbf{t}_0|^2}\right)_z + \frac{t_0^j}{|\mathbf{t}_0|^2}(U_0^j)_\tau + U_{\nu 1}(U_{t0})_z U_{\nu 0}\left(\frac{t_0^j U_1^j}{|\mathbf{t}_0|^2}\right)_z \\ &+\frac{t_0^j\nu_1^j}{|\mathbf{t}_0|^2}g(R_0)_z + \frac{1}{|\mathbf{t}_0|^2}g(R_0)_s + (U_{\nu 0} - w_{\nu 0})\left(\frac{t_1^j U_0^j}{|\mathbf{t}_0|^2} - 2\frac{t_0^j U_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4}\right)_z + \frac{t_1^j\nu_0^j}{|\mathbf{t}_0^2|}g(R_0)_z \\ &= \mu\frac{1}{R_0}(U_{t0})_{zz} + \gamma\frac{\nu_1^j t_0^j}{|\mathbf{t}_0|^2}R_{0,zzz} + \gamma\frac{1}{|\mathbf{t}_0|^2}R_{0,szz} + \gamma\frac{t_1^j\nu_0^i}{|t_0|^2}R_{0,zzz}. \end{aligned} \tag{7.51}$$

Using (7.13) and (5.23) this is equivalent to

$$(U_{\nu 0} - w_{\nu 0})(U_{t1})_z + \frac{t_0^j}{|\mathbf{t}_0|^2}(U_0^j)_\tau + \frac{1}{|\mathbf{t}_0|^2}g(R_0)_s = \gamma\frac{1}{|\mathbf{t}_0|^2}R_{0,szz}. \tag{7.52}$$

By the matching conditions, $R_1, U_{\nu 1}, U_{t1}$ must have the following asymptotic behavior for $z \to \pm\infty$:

$$R_1(z) \quad\to\quad \rho_1^\pm + \left(\frac{\partial \rho_0}{\partial x^k}\right)^\pm (r_1^k + z\nu_0^k), \tag{7.53}$$

$$U_{\nu 1}(z) \quad\to\quad (u_0^j)^\pm \nu_1^j + (u_1^j)^\pm \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm \nu_0^j (r_1^k + z\nu_0^k), \tag{7.54}$$

$$U_{t1}(z) \quad\to\quad (u_1^j)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2}(r_1^k + z\nu_0^k) + (u_0^j)^\pm \left(\frac{t_1^j}{|\mathbf{t}_0|^2} - 2\frac{t_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4}\right). \tag{7.55}$$

Hence, solving (6.12),(6.13) is equivalent to solving

$$\begin{aligned}\left(\tilde{R}_1(U_{\nu 0} - w_{\nu 0}) + R_0 \tilde{U}_{\nu 1}\right)_z &= -\frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{\nu 0} \\ &\quad - \left(\Phi(U_{\nu 0} - w_{\nu 0}) + R_0(\Psi_\nu - w_{\nu 1})\right)_z ,\end{aligned}$$

$$\begin{aligned}\left((U_{\nu 0} - w_{\nu 0})\tilde{U}_{\nu 1} + g'(R_0)\tilde{R}_1 - \gamma \tilde{R}_{1,zz}\right)_z &= -\nu_0^j (U_0^j)_\tau + (\lambda + 2\mu)\,\frac{1}{R_0}(U_{\nu 0})_{zz} - \gamma\kappa R_{0,zz} \\ &\quad - \left((U_{\nu 0} - w_{\nu 0})(\Psi_\nu - w_{\nu 1}) + g'(R_0)\Phi - \gamma \Phi_{zz}\right)_z\end{aligned}$$

and

$$(U_{\nu 0} - w_{\nu 0})(\tilde{U}_{t1})_z \quad=\quad -\left(\frac{t_0^j}{|\mathbf{t}_0|^2}\right)(U_0^j)_\tau - \frac{1}{|\mathbf{t}_0|^2}\left(g(R_0) - \gamma R_{0,zz}\right)_s - (U_{\nu 0} - w_{\nu 0})(\Psi_t)_z,$$

with the new boundary conditions

$$\tilde{R}_1(z), \tilde{U}_{\nu 1}(z), \tilde{U}_{t1}(z) \to 0 \quad \text{for } z \to \infty. \tag{7.56}$$

■

In the following we deduce the first order jump conditions in Theorem 6.3 by exploiting the solvabilty conditions from Lemma 7.3.

7.1 $\mathcal{O}(1)$ -order terms

7.1.1 Mass balance

Relying on Lemma 7.3 and Lemma 7.4 we can determine the first order of the jump of the mass flux over the interface.

Lemma 7.5
Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution then

$$[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!]_1 = -\frac{\partial \rho_\Gamma}{\partial \tau} - (div_\Gamma(w_{t0}) - \kappa_0 w_{\nu 0})\rho_\Gamma, \tag{7.57}$$

where

$$\rho_\Gamma = \int_0^\infty (R_0 - \rho_0^+)\, dz - \int_{-\infty}^0 (R_0 - \rho_0^-)\, dz + \mathcal{O}(\varepsilon). \tag{7.58}$$

Proof:
From Lemma 7.3 and Lemma 7.4 we know

$$\int_{-\infty}^{\infty} f_1[R_0, \mathbf{U}_0, \Phi, \Psi]\, dz = 0,$$

where

$$\begin{aligned} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] \;=\; & -\frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{\nu 0} \\ & - \left(\Phi(U_{\nu 0} - w_{\nu 0}) + R_0(\Psi_\nu - w_{\nu 1})\right)_z, \end{aligned}$$

when $f_1 \in L^1(\mathbb{R})$. We start with showing that $f_1 \in L^1(\mathbb{R})$.

We observe that

$$(\rho_{0,t})^{\pm} + \frac{t_0^i}{|\mathbf{t}_0|^2}\left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j}\right)^{\pm} t_0^j + \nu_0^i \left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j}\right)^{\pm} \nu_0^j = 0, \tag{7.59}$$

because $\rho_0, \mathbf{u}_0$ satisfy the mass conservation equation in the bulk. Utilizing $w_{\nu 0}\nu_0^i + w_{t0}t_0^i = w_0^i$ equation (7.59) is equivalent to

$$\begin{aligned} & -w_{\nu 0}\left(\frac{\partial \rho_0}{\partial x^i}\right)^{\pm} \nu_0^i - w_{t0}\left(\frac{\partial \rho_0}{\partial x^i}\right)^{\pm} t_0^i + (\rho_{0,t})^{\pm} \\ & + \left(\frac{\partial \rho_0}{\partial x^i}\right)^{\pm} w_0^i + \frac{t_0^i}{|\mathbf{t}_0|^2}\left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j}\right)^{\pm} t_0^j + \nu_0^i\left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j}\right)^{\pm} \nu_0^j = 0. \end{aligned} \tag{7.60}$$

Using the different forms of the matching conditions implies

$$-w_{\nu 0}\left(\frac{\partial \rho_0}{\partial x^i}\right)^{\pm} \nu_0^i - w_{t0}\left(\rho_0^{\pm}\right)_s + (\rho_0^{\pm})_\tau + \frac{t_0^i}{|\mathbf{t}_0|}(\rho_0^{\pm}(u_0^i)^{\pm})_s + \nu_0^i\left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j}\right)^{\pm} \nu_0^j = 0. \tag{7.61}$$

Due to (5.24) and (7.13) we obtain

$$(u_{\nu 0}^{\pm} - w_{\nu 0})\left(\frac{\partial \rho_0}{\partial x^i}\right)^{\pm} \nu_0^i + \rho_0^{\pm}\left(\frac{\partial u_0^i}{\partial x^j}\right)^{\pm} \nu_0^i \nu_0^j + \frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|)_s \rho_0^{\pm} + (\rho_0^{\pm})_\tau - \kappa_0 \rho_0^{\pm} u_{\nu 0}^{\pm} = 0. \tag{7.62}$$

Keeping in mind the definition of $f_1[R_0, \mathbf{U}_0, \Phi, \Psi]$ (see (7.46)), equation (7.62) implies due to (5.31), (5.32) and (5.34)

$$f_1[R_0(z), \mathbf{U}_0(z), \Phi(z), \Psi(z)] \to 0 \quad \text{for } |z| \to \infty \tag{7.63}$$

superlinearly and hence $f_1[R_0, \mathbf{U}_0, \Phi, \Psi] \in L^1(\mathbb{R})$.

Thus we can write (7.16) as

$$\begin{aligned} 0 \;=\; \lim_{a\to\infty} \Big[& \left(\frac{\partial \rho_0}{\partial x^j}\right)^{+} (\nu_0^j a + r_1^j)\left(u_{\nu 0}^{+} - w_{\nu 0}\right) - \left(\frac{\partial \rho_0}{\partial x^j}\right)^{-} (r_1^j - \nu_0^j a)\left(u_{\nu 0}^{-} - w_{\nu 0}\right) \\ & + \rho_1^{+}(u_{\nu 0}^{+} - w_{\nu 0}) - \rho_1^{-}(u_{\nu 0}^{-} - w_{\nu 0}) + \rho_0^{+}\left(\frac{\partial u_0^i}{\partial x^j}\right)^{+} \nu_0^i(\nu_0^j a + r_1^j) - \rho_0^{-}\left(\frac{\partial u_0^i}{\partial x^j}\right)^{-} \nu_0^i(r_1^j - \nu^j a) \\ & - w_{\nu 1}\rho_0^{+} + w_{\nu 1}\rho_0^{-} + \rho_0^{+}(u_0^j)^{+}\nu_1^j + \rho_0^{+}(u_1^j)^{+}\nu_0^j - \rho_0^{-}(u_0^j)^{-}\nu_1^j - \rho_0^{-}(u_1^j)^{-}\nu_0^j \\ & - \int_{-a}^{a} \left(-\frac{1}{|\mathbf{t}_0|}\left(w_{t0}|\mathbf{t}_0|\right)_s R_0 - R_{0,\tau} + \kappa_0 R_0 U_{\nu 0}\right) dz \Big]. \end{aligned}$$

Now we would like to decompose the limit into several parts, which have to converge. This can be achieved by subtracting a times $(7.62)^{\pm}$ before letting $a \to \infty$, i.e.

$$\begin{aligned}
0 \;=\; & \lim_{a\to\infty} \Big[\left(\frac{\partial \rho_0}{\partial x^j}\right)^{+} r_1^j \left(u_{\nu 0}^{+} - w_{\nu 0}\right) - \left(\frac{\partial \rho_0}{\partial x^j}\right)^{-} r_1^j \left(u_{\nu 0}^{-} - w_{\nu 0}\right) \\
& + \rho_1^{+}(u_{\nu 0}^{+} - w_{\nu 0}) - \rho_1^{-}(u_{\nu 0}^{-} - w_{\nu 0}) + \rho_0^{+} \left(\frac{\partial u_0^i}{\partial x^j}\right)^{+} \nu_0^i r_1^j - \rho_0^{-} \left(\frac{\partial u_0^i}{\partial x^j}\right)^{-} \nu_0^i r_1^j \\
& - w_{\nu 1}\rho_0^{+} + w_{\nu 1}\rho_0^{-} + \rho_0^{+}(u_0^j)^{+}\nu_1^j + \rho_0^{+}(u_1^j)^{+}\nu_0^j - \rho_0^{-}(u_0^j)^{-}\nu_1^j - \rho_0^{-}(u_1^j)^{-}\nu_0^j \\
& - \int_0^a -\frac{1}{|\mathbf{t}_0|}\left(w_{t0}|\mathbf{t}_0|\right)_s (R_0 - \rho_0^{+}) - (R_0 - \rho_0^{+})_\tau + \kappa_0 (R_0 U_{\nu 0} - \rho_0^{+} u_{\nu 0}^{+})\, dz - \int_{-a}^0 \ldots\, dz \Big].
\end{aligned}$$

Inserting (10.2) in (7.64) and using

$$R_0 U_{\nu 0} - \rho_0^{+} u_{\nu 0}^{+} = R_0 w_{\nu 0} - \rho_0^{+} w_{\nu 0},$$

we obtain

$$\begin{aligned}
& [\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!]_1 \qquad (7.64) \\
& = \int_0^\infty \left(-R_{0,\tau} + (\rho_0^{+})_\tau\right) dz + \int_{-\infty}^0 \ldots \\
& + \int_0^\infty \left(-\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s R_0 + \frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s \rho_0^{+} + \kappa_0 R_0 w_{\nu 0} - \kappa_0 \rho_0^{+} w_{\nu 0} \right) dz + \int_{-\infty}^0 \ldots .
\end{aligned}$$

In view of (3.3) we can identify the surface mass density as

$$\rho_\Gamma := \int_0^\infty (R_0 - \rho_0^{+})\, dz + \int_{-\infty}^0 (R_0 - \rho_0^{-})\, dz + \mathcal{O}(\varepsilon^2), \qquad (7.65)$$

which finishes the proof. ■

7.1.2 Gibbs free energy

We deduce the first order of the jump of the Gibbs free energy over the interface. The proof is based on Lemma 7.3 and Lemma 7.4.

Lemma 7.6
Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution. Then

$$\begin{aligned}
\left[\!\!\left[\frac{1}{2}\left(\frac{j_\varepsilon}{\rho_\varepsilon}\right)^2 + g(\rho_\varepsilon) \right]\!\!\right]_1 \;=\; & -\int_0^\infty \nu_0^j \left(U_0^j - (u_0^j)^{+}\right)_\tau dz \qquad (7.66) \\
& - \int_{-\infty}^0 \nu_0^j \left(U_0^j - (u_0^j)^{-}\right)_\tau dz \\
& - (\lambda + 2\mu) j_0 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0}\right)_z\right)^2 dz. \qquad (7.67)
\end{aligned}$$

Proof:
From Lemma 7.3 and Lemma 7.4 we know

$$\int_{-\infty}^{\infty} f_2[R_0, \mathbf{U}_0, \Phi, \Psi]\, dz = 0,$$

where

$$\begin{aligned} f_2[R_0, \mathbf{U}_0, \Phi, \Psi] \;=\; & -\nu_0^j (U_0^j)_\tau + (\lambda + 2\mu)\, \frac{1}{R_0} (U_{\nu 0})_{zz} - \gamma \kappa_0 R_{0,zz} \\ & - \big((U_{\nu 0} - w_{\nu 0})(\Psi_\nu - w_{\nu 1}) + g'(R_0)\Phi - \gamma \Phi_{zz} \big)_z\,, \end{aligned}$$

when $f_2 \in L^1(\mathbb{R})$. We start with showing that $f_2 \in L^1(\mathbb{R})$.

As the outer quantities ρ_0, $\mathbf{u}_0$ satisfy (4.5), we have

$$\nu_0^j (u_{0,t}^j)^\pm + (u_0^k)^\pm \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \nu_0^j + g'(\rho_0^\pm) \left(\frac{\partial \rho_0}{\partial x^j} \right)^\pm \nu_0^j = 0. \tag{7.68}$$

Due to the matching conditions (5.33), (5.34), (5.37) and the assumptions (7.42), (7.43) and (7.44) on Φ, Ψ_ν and Ψ_t we get

$$\begin{aligned} f_2[R_0(z), \mathbf{U}_0(z), \Phi(z), \Psi(z)] \to \; & -(u_{\nu 0}^\pm - w_{\nu 0}) \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \nu_0^j \nu_0^k - g'(\rho_0^\pm) \left(\frac{\partial \rho_0}{\partial x^k} \right)^\pm \nu_0^k \\ & -\nu_0^j \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm w_0^k - \nu_0^j (u_{0,t}^j)^\pm. \end{aligned} \tag{7.69}$$

Inserting (7.68) in (7.69) we get

$$f_2[R_0(z), \mathbf{U}_0(z), \Phi(z), \Psi(z)] \to \left(-u_{\nu 0}\nu_0^k + (u_0^k)^\pm + w_{\nu 0}\nu_0^k - w_0^k \right) \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \nu_0^k = 0 \quad \text{for } |z| \to \infty, \tag{7.70}$$

where the convergence is obtained superlinear, whence $f_2[R_0, \mathbf{U}_0, \Phi, \Psi] \in L^1(\mathbb{R})$.

Due to the matching conditions and (7.42), (7.43) the solvability condition (7.17) is equivalent to

$$\begin{aligned} 0 \;=\; \lim_{a\to\infty} \Big[& (u_{\nu 0}^+ - w_{\nu 0}) w_{\nu 1} - (u_{\nu 0}^- - w_{\nu 0}) w_{\nu 1} \\ & -(u_{\nu 0}^+ - w_{\nu 0}) \left((u_0^j)^+ \nu_1^j + (u_1^j)^+ \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ \nu_0^j (r_1^k + \nu_0^k a) \right) \\ & +(u_{\nu 0}^- - w_{\nu 0}) \left((u_0^j)^- \nu_1^j + (u_1^j)^- \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- \nu_0^j (r_1^k - \nu_0^k a) \right) \\ & -g'(\rho_0^+)\rho_1^+ - g'(\rho_0^+) \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ (r_1^k + \nu_0^k a) + g'(\rho_0^-)\rho_1^- \\ & +g'(\rho_0^-) \left(\frac{\partial \rho_0}{\partial x^k} \right)^- (r_1^k - \nu_0^k a) - \int_{-a}^{a} \nu_0^j (U_0^j)_\tau\, dz + (\lambda + 2\mu) \int_{-a}^{a} \frac{1}{R_0} (U_{\nu 0})_{zz}\, dz \Big]. \end{aligned} \tag{7.71}$$

Now we subtract a times $(7.68)^+$ and a times $(7.68)^-$ before letting $a \to \infty$ and obtain

$$
\begin{aligned}
0 \;=\; & \lim_{a\to\infty} \Big[(u_{\nu 0}^+ - w_{\nu 0}) w_{\nu 1} - (u_{\nu 0}^- - w_{\nu 0}) w_{\nu 1} \\
& -(u_{\nu 0}^+ - w_{\nu 0}) \left((u_0^j)^+ \nu_1^j + (u_1^j)^+ \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k}\right)^+ \nu_0^j r_1^k \right) \\
& +(u_{\nu 0}^- - w_{\nu 0}) \left((u_0^j)^- \nu_1^j + (u_1^j)^- \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k}\right)^- \nu_0^j r_1^k \right) \\
& -g'(\rho_0^+)\rho_1^+ - g'(\rho_0^+) \left(\frac{\partial \rho_0}{\partial x^k}\right)^+ r_1^k + g'(\rho_0^-)\rho_1^- + g'(\rho_0^-) \left(\frac{\partial \rho_0}{\partial x^k}\right)^- r_1^k \\
& - \int_0^a \nu_0^j (U_0^j - (u_0^j)^+)_\tau \, dz - \int_{-a}^0 \ldots \, dz + (\lambda + 2\mu) \int_{-a}^a \frac{1}{R_0} (U_{\nu 0})_{zz} \, dz \Big].
\end{aligned}
\tag{7.72}
$$

Using (10.5) equation (7.72) means

$$
\begin{aligned}
\left[\!\!\left[\frac{1}{2}((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)^2 + g(\rho_\varepsilon) \right]\!\!\right]_1 \;=\; & -\int_0^\infty \nu_0^j \left((U_0^j)_\tau(z) - ((u_0^j)^+)_\tau \right) dz \\
& - \int_{-\infty}^0 \nu_0^j \left((U_0^j)_\tau(z) - ((u_0^j)^-)_\tau \right) dz \\
& + \int_{-\infty}^\infty (\lambda + 2\mu) \frac{1}{R_0(z)} (U_{\nu 0})_{zz}(z) \, dz.
\end{aligned}
\tag{7.73}
$$

Inserting $U_{\nu 0} = \frac{J_0}{R_0} + w_\nu$ in (7.73) yields

$$
\begin{aligned}
\left[\!\!\left[\frac{1}{2}((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)^2 + g(\rho_\varepsilon) \right]\!\!\right]_1 \;=\; & -\int_0^\infty \nu_0^j \left(U_0^j(z) - (u_0^j)^+ \right)_\tau dz \\
& - \int_{-\infty}^0 \nu_0^j \left(U_0^j(z) - (u_0^j)^- \right)_\tau dz \\
& - \int_{-\infty}^\infty (\lambda + 2\mu) \, J_0 \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz.
\end{aligned}
\tag{7.74}
$$

■

7.1.3 Momentum balance

Using the last two solvability conditions from Lemma 7.3 together with Lemma 7.4, we will determine the first order of the pressure jump over the interface.

Lemma 7.7
Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution. Then

$$
[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\boldsymbol{\nu}_\varepsilon]\!]_1 \;=\; -\rho_\Gamma \frac{\partial \mathbf{w}_0}{\partial \tau} + div_\Gamma(\boldsymbol{\sigma}_\Gamma), \tag{7.75}
$$

where

$$\rho_\Gamma \;=\; \int_0^\infty (R_0-\rho_0^+)\,dz - \int_{-\infty}^0 (R_0-\rho_0^-)\,dz + \mathcal{O}(\varepsilon), \tag{7.76}$$

$$\sigma_\Gamma^j \;=\; \left(\int_0^\infty \left(p(\rho_0^+)-p(R_0)-\frac{\gamma}{2}(R_{0,z})^2\right)dz + \int_{-\infty}^0 \dots\right)\frac{t_0^j}{|\mathbf{t}_0|^2} + \mathcal{O}(\varepsilon). \tag{7.77}$$

Proof:
From Lemma 7.3 and Lemma 7.4 we have the solvabilty criteria

$$\int_{-\infty}^\infty R_0 f_3\,dz = 0, \qquad \int_{-\infty}^\infty \frac{j_0}{R_0} f_1 + R_0 f_2\,dz = 0,$$

where

$$\begin{aligned}
f_1[R_0,\mathbf{U}_0,\Phi,\Psi] &= -\frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{\nu 0}\\
&\quad - \left(\Phi(U_{\nu 0}-w_{\nu 0}) + R_0(\Psi_\nu - w_{\nu 1})\right)_z,\\
f_2[R_0,\mathbf{U}_0,\Phi,\Psi] &= -\nu_0^j(U_0^j)_\tau + (\lambda+2\mu)\,\frac{1}{R_0}(U_{\nu 0})_{zz} - \gamma\kappa_0 R_{0,zz}\\
&\quad - \left((U_{\nu 0}-w_{\nu 0})(\Psi_\nu - w_{\nu 1}) + g'(R_0)\Phi - \gamma\Phi_{zz}\right)_z,\\
f_3[R_0,\mathbf{U}_0,\Phi,\Psi] &= -\left(\frac{t_0^j}{|\mathbf{t}_0|^2}\right)(U_0^j)_\tau - \frac{1}{|\mathbf{t}_0|^2}\left(g(R_0)-\gamma R_{0,zz}\right)_s - (U_{\nu 0}-w_{\nu 0})(\Psi_t)_z,
\end{aligned}$$

when $f_1, f_2, f_3 \in L^1(\mathbb{R})$. From the proofs of Lemma 7.5 and Lemma 7.6 we already know that $f_1, f_2 \in L^1(\mathbb{R})$. It remains to show that $f_3 \in L^1(\mathbb{R})$.

By the matching conditions and (7.44) we get

$$f_3 \to -\frac{t_0^j}{|\mathbf{t}_0|^2}\left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm w_0^k - \frac{t_0^j}{|\mathbf{t}_0|^2}\left(\frac{\partial u_0^j}{\partial t}\right)^\pm - \frac{t_0^k}{|\mathbf{t}_0|^2}\left(\frac{\partial g(\rho_0)}{\partial x^k}\right)^\pm - (u_{\nu 0}^\pm - w_{\nu 0}) - \frac{t_0^j}{|\mathbf{t}_0|^2}\left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm \nu_0^k. \tag{7.78}$$

On the other hand, $\rho_0, \mathbf{u}_0$ satisfy (4.5) in the bulk, such that

$$\frac{t_0^j}{|\mathbf{t}_0|^2}\left(\left(\frac{\partial u_0^j}{\partial t}\right)^\pm + (u_0^k)^\pm\left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm + \left(\frac{\partial g(\rho_0)}{\partial x^j}\right)^\pm\right) = 0. \tag{7.79}$$

Inserting (7.79) in (7.78) we get

$$f_3 \to -\frac{t_0^j}{|\mathbf{t}_0|^2}\left((u_0^k)^\pm - w_0^k - (u_{\nu 0}^\pm - w_{\nu 0})\nu_0^k\right)\left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm = 0. \tag{7.80}$$

To obtain a solvability criterion, which has the form of the Young Laplace law, we will deal with the solvabilty criteria (7.18) and (7.19) simultaneously. As $\boldsymbol{\nu}_0$ and $\mathbf{t}_0$ are linearly independent equations (7.18) and (7.19) are equivalent to

$$\int_{-\infty}^\infty R_0 f_3[R_0,\mathbf{U}_0,\Phi,\Psi]t_0^i + \frac{j_0}{R_0}f_1[R_0,\mathbf{U}_0,\Phi,\Psi]\nu_0^i + R_0 f_2[R_0,\mathbf{U}_0,\Phi,\Psi]\nu_0^i\,dz = 0. \tag{7.81}$$

As f_1, f_2, f_3 go to zero superlinearly for $|z| \to \infty$, we have

$$\lim_{z\to\pm\infty} R_0 f_3[R_0, \mathbf{U}_0, \Phi, \Psi]t^i + \frac{j_0}{R_0} f_1[R_0, \mathbf{U}_0, \Phi, \Psi]\nu^i + R_0 f_2[R_0, \mathbf{U}_0, \Phi, \Psi]\nu^i = 0. \tag{7.82}$$

Inserting (7.46) - (7.48) in (7.81) yields

$$\begin{aligned} 0 = \int_{-\infty}^{\infty} \Big[& \frac{J_0}{R_0}\nu_0^i \left(-\frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|)_s R_0 - R_{0,\tau} + \kappa_0 R_0 U_{\nu 0} - \left(\Phi \frac{J_0}{R_0} + R_0(\Psi_\nu - w_{\nu 1}) \right)_z \right) \\ & + R_0 \nu_0^i \left(-\nu_0^j (U_0^j)_\tau + (\lambda + 2\mu)\frac{1}{R_0}(U_{\nu 0})_{zz} - \gamma \kappa_0 R_{0,zz} \right) \\ & - R_0 \nu_0^i \left(\frac{J_0}{R_0}(\Psi_\nu - w_{\nu 1}) + g'(R_0)\Phi - \gamma \Phi_{zz} \right)_z \\ & - R_0 t_0^i \left(\frac{t_0^j}{|\mathbf{t}_0|^2}(U_0^j)_\tau + \frac{1}{|\mathbf{t}_0|^2}(g(R_0) - \gamma R_{0,zz})_s + \frac{J_0}{R_0}(\Psi_t)_z \right) \Big] \, dz. \end{aligned} \tag{7.83}$$

The terms containing $w_{\nu 1}$ cancel out and using integration by parts, (7.83) becomes

$$\begin{aligned} 0 = \int_{-\infty}^{\infty} \Big[& - J_0 \nu_0^i \frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|)_s - \frac{J_0}{R_0}\nu_0^i R_{0,\tau} + \kappa_0 \nu_0^i J_0 U_{\nu 0} \\ & - \nu_0^i \left(J_0 \Psi_\nu + \frac{J_0^2}{R_0^2}\Phi + J_0 \Psi_\nu + p'(R_0)\Phi - \gamma(\Phi_{zz} R_0 - \Phi_z R_{0,z} + \Phi R_{0,zz}) \right)_z \\ & + \frac{J_0}{R_0}\left(\frac{J_0}{R_0}\right)_z \nu_0^i \Phi + \left(\frac{J_0}{R_0}\right)_z \nu_0^i R_0 \Psi_\nu - R_0 (U_0^i)_\tau + (\lambda + 2\mu)(U_{\nu 0})_{zz}\nu_0^i \\ & - \gamma \kappa_0 \nu_0^i R_0 R_{0,zz} + \nu_0^i R_{0,z}\frac{J_0}{R_0}\Psi_\nu + g(R_0)_z \Phi \nu_0^i - \gamma R_{0,zzz}\Phi \nu_0^i \\ & - \frac{t_0^i}{|\mathbf{t}_0|^2} p(R_0)_s + \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,szz} - J_0 t_0^i (\Psi_t)_z \Big] \, dz. \end{aligned} \tag{7.84}$$

We observe that (6.10) and (7.13) imply

$$\frac{J_0}{R_0}\nu_0^i R_{0,\tau} + R_0 (U_0^i)_\tau = (J_0 \nu_0^i)_\tau + R_0 (w_0^i)_\tau \quad \text{and} \quad \left(\frac{J_0}{R_0}\right)_z R_0 + \left(\frac{J_0}{R_0}\right) R_{0,z} = 0.$$

Using this (7.84) yields

$$\begin{aligned} 0 = \int_{-\infty}^{\infty} \Big[& - J_0 \nu_0^i \frac{1}{|\mathbf{t}_0|}(w_{t0}|\mathbf{t}_0|)_s - (J_0 \nu_0^i)_\tau - R_0 (w_0^i)_\tau + \kappa_0 \nu_0^i J_0 U_{\nu 0} \\ & - \nu_0^i \left(2 J_0 \Psi_\nu + \frac{J_0^2}{R_0^2}\Phi + p'(R_0)\Phi - \gamma(\Phi_{zz} R_0 - \Phi_z R_{0,z} + \Phi R_{0,zz}) \right)_z \\ & + (\lambda + 2\mu)(U_{\nu 0})_{zz}\nu_0^i - \gamma \kappa_0 \nu_0^i R_0 R_{0,zz} - \frac{t_0^i}{|\mathbf{t}_0|^2} p(R_0)_s + \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,szz} - J_0 t_0^i (\Psi_t)_z \\ & + \Phi \nu_0^i \left(\frac{J_0}{R_0}\left(\frac{J_0}{R_0}\right)_z + g(R_0)_z - \gamma R_{0,zzz} \right) \Big] \, dz. \end{aligned} \tag{7.85}$$

We remark that the last line of (7.85) vanishes due to (6.11).

Now we will consider some of the terms from (7.85) separately. We start with multiplying (6.11) with R_0 and subtracting (6.10) times w_0^j. We obtain

$$\Big(R_0U_{\nu 0}(U_0^j - w_0^j)\Big)_z - \underbrace{\Big(R_0w_{\nu 0}(U_0^j - w_0^j)\Big)_z}_{=0} = -\nu_0^j p(R_0)_z + \nu_0^j\gamma R_0R_{0,zzz}. \tag{7.86}$$

So there exists D_1^i independent of z such that

$$\nu_0^i j_0 U_{\nu 0} = -\nu_0^i p(R_0) + \gamma\nu_0^i R_0R_{0,zz} - \gamma\nu_0^i\frac{1}{2}(R_{0,z})^2 + D_1^i. \tag{7.87}$$

With (5.3) we compute

$$\begin{aligned} p(R_0)_s\frac{t_0^j}{|\mathbf{t}_0|^2} &= \frac{1}{|\mathbf{t}_0|}\left(p(R_0)\frac{t_0^j}{|\mathbf{t}_0|}\right)_s - p(R_0)\frac{1}{|\mathbf{t}_0|}\left(\frac{t_0^j}{|\mathbf{t}_0|}\right)_s \\ &= \frac{1}{|\mathbf{t}_0|}\left(\frac{t_0^j}{|\mathbf{t}_0|}p(R_0)\right)_s - \nu_0^j\kappa_0 p(R_0). \end{aligned} \tag{7.88}$$

Additionally we observe that due to (6.10)

$$D_2^j := (R_0(U_0^j - w_0^j))_\tau = (j_0\nu_0^j)_\tau \tag{7.89}$$

and

$$D_3^j := \frac{1}{|\mathbf{t}_0|}(|\mathbf{t}_0|w_{t0})_s R_0(U_0^j - w_0^j) = \frac{1}{|\mathbf{t}_0|}(|\mathbf{t}_0|w_{t0})_s j_0\nu_0^j \tag{7.90}$$

are independent of z. Now we insert (7.87) and (7.88) in (7.85) and we end up with

$$\begin{aligned} 0 = \int_{-\infty}^{\infty}\Big[&- D_3^i - D_2^i - R_0(w_0^i)_\tau - \kappa_0\gamma\nu_0^i\frac{1}{2}(R_{0,z})^2 + \kappa_0 D_1^i \\ &-\left(2J_0\Psi_\nu\nu_0^i + J_0\Psi_t t_0^i + \nu_0^i\frac{J_0^2}{R_0^2}\Phi + p'(R_0)\Phi\nu_0^i - \gamma\nu_0^i\left(\Phi_{1,zz}R_0 - \Phi_{1,z}R_{0,z} + \Phi_1 R_{0,zz}\right)\right)_z \\ &+(\lambda+2\mu)(U_{\nu 0})_{zz}\nu_0^i - \frac{1}{|\mathbf{t}|_0}\left(\frac{t_0^i}{|\mathbf{t}_0|}p(R_0)\right)_s + \gamma\frac{t_0^i}{|\mathbf{t}_0|^2}R_0R_{0,szz}\Big]\,dz. \end{aligned} \tag{7.91}$$

By (7.82) the integrand in (7.91) goes to zero for $|z| \to \infty$, i.e.

$$\begin{aligned} 0 \;=\; &-D_2^i - D_3^i + \kappa_0 D_1^i - \rho_0^\pm(w_0^i)_\tau - 2j_0\left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm \nu_0^i\nu_0^j\nu_0^k - j_0\left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm\frac{t_0^it_0^j}{|\mathbf{t}_0|^2}\nu_0^k \\ &-\frac{j_0^2}{(\rho_0^\pm)^2}\nu_0^i\left(\frac{\partial\rho_0}{\partial x^k}\right)^\pm\nu_0^k - p'(\rho_0^\pm)\nu_0^i\left(\frac{\partial\rho_0}{\partial x^k}\right)^\pm\nu_0^k - \frac{1}{|\mathbf{t}_0|}\left(p(\rho_0^\pm)\frac{t_0^i}{|\mathbf{t}_0|}\right)_s. \end{aligned} \tag{7.92}$$

Using the matching conditions and (7.42)-(7.44) equation (7.91) is equivalent to

$$
\begin{aligned}
0 \;=\; & \lim_{a\to\infty}\Big[-2aD_2^i - 2aD_3^i + 2a\kappa_0 D_1^i - \int_{-a}^{a} R_0 (w_0^i)_\tau + \frac{\gamma}{2}\kappa_0 \nu_0^i (R_{0,z})^2\, dz \\
& -2j_0\nu_0^i \left((u_0^j)^+ \nu_1^j + (u_1^j)^+ \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k}\right)^+ \nu_0^j (r_1^k + a\nu_0^k)\right) \\
& +2j_0\nu_0^i \left((u_0^j)^- \nu_1^j + (u_1^j)^- \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k}\right)^- \nu_0^j (r_1^k - a\nu_0^k)\right) \\
& -j_0 t_0^i \left((u_1^j)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k}\right)^+ \frac{t_0^j}{|\mathbf{t}_0|^2}(r_1^k + a\nu_0^k) + (u_0^k)^+ \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2\frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4}\right)\right) \\
& +j_0 t_0^i \left((u_1^j)^- \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k}\right)^- \frac{t_0^j}{|\mathbf{t}_0|^2}(r_1^k - a\nu_0^k) + (u_0^k)^- \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2\frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4}\right)\right) \\
& -\nu_0^i \left(\frac{j_0^2}{(\rho_0^+)^2} + p'(\rho_0^+)\right)\left(\rho_1^+ + \left(\frac{\partial \rho_0}{\partial x^k}\right)^+ (r_1^k + a\nu_0^k)\right) \\
& +\nu_0^i \left(\frac{j_0^2}{(\rho_0^-)^2} + p'(\rho_0^-)\right)\left(\rho_1^- + \left(\frac{\partial \rho_0}{\partial x^k}\right)^- (r_1^k - a\nu_0^k)\right) \\
& -\int_{-a}^{a} \frac{1}{|\mathbf{t}_0|}\left(p(R_0)\frac{t_0^i}{|\mathbf{t}_0|}\right)_s - \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,szz}\, dz\Big].
\end{aligned} \tag{7.93}
$$

We subtract a times $(7.92)^+$ and a times $(7.92)^-$ before letting $a \to \infty$ which yields:

$$
\begin{aligned}
0 \;=\; & -\int_0^\infty (R_0 - \rho_0^+)(w_0^i)_\tau + \frac{\gamma}{2}\kappa_0 \nu_0^i (R_{0,z})^2\, dz - \int_{-\infty}^0 \dots\, dz \\
& -2j_0\nu_0^i \left((u_0^j)^+ \nu_1^j + (u_1^j)^+ \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k}\right)^+ \nu_0^j r_1^k\right) \\
& +2j_0\nu_0^i \left((u_0^j)^- \nu_1^j + (u_1^j)^- \nu_0^j + \left(\frac{\partial u_0^j}{\partial x^k}\right)^- \nu_0^j r_1^k\right) \\
& -j_0 t_0^i \left((u_1^j)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k}\right)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} r_1^k + (u_0^k)^+ \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2\frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4}\right)\right) \\
& +j_0 t_0^i \left((u_1^j)^- \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k}\right)^- \frac{t_0^j}{|\mathbf{t}_0|^2} r_1^k + (u_0^k)^- \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2\frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4}\right)\right) \\
& -\nu_0^i \left(\frac{j_0^2}{(\rho_0^+)^2} + p'(\rho_0^+)\right)\left(\rho_1^+ + \left(\frac{\partial \rho_0}{\partial x^k}\right)^+ r_1^k\right) \\
& +\nu_0^i \left(\frac{j_0^2}{(\rho_0^-)^2} + p'(\rho_0^-)\right)\left(\rho_1^- + \left(\frac{\partial \rho_0}{\partial x^k}\right)^- r_1^k\right) \\
& -\int_0^\infty \frac{1}{|\mathbf{t}_0|}\left((p(R_0) - p(\rho_0^+))\frac{t_0^i}{|\mathbf{t}_0|}\right)_s - \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,szz}\, dz - \int_{-\infty}^0 \dots\, dz.
\end{aligned} \tag{7.94}
$$

As explained in the appendix, (7.94) can be written as

$$\begin{aligned}
& [\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\boldsymbol{\nu}_\varepsilon]\!]_1 \qquad (7.95)\\
= \quad & -(w_0^j)_\tau \int_0^\infty (R_0 - \rho_0^+)\, dz - \int_{-\infty}^0 \dots\, dz \\
& - \int_0^\infty \left[\frac{1}{|\mathbf{t}_0|} \left(p(R_0) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s - \frac{1}{|\mathbf{t}_0|} \left(p(\rho_0^+) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s \right] dz - \int_{-\infty}^0 \dots\, dz \\
& -\kappa_0 \gamma \frac{1}{2} \nu_0^j \int_{-\infty}^\infty (R_{0,z})^2\, dz - \gamma \frac{1}{2} \frac{t_0^j}{|\mathbf{t}_0|^2} \int_{-\infty}^\infty ((R_{0,z})^2)_s\, dz
\end{aligned}$$

or by (5.3) equivalently

$$\begin{aligned}
& [\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\boldsymbol{\nu}_\varepsilon]\!]_1 \qquad (7.96)\\
= \quad & -(w_0^j)_\tau \int_0^\infty (R_0 - \rho_0^+)\, dz - \int_{-\infty}^0 \dots\, dz \\
& - \int_0^\infty \left[\frac{1}{|\mathbf{t}_0|} \left(p(R_0) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s - \frac{1}{|\mathbf{t}_0|} \left(p(\rho_0^\pm) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s \right] dz - \int_{-\infty}^0 \dots\, dz \\
& -\gamma \frac{1}{2} \frac{1}{|\mathbf{t}_0|} \int_{-\infty}^\infty \left(\frac{t_0^j}{|\mathbf{t}_0|} (R_{0,z})^2 \right)_s dz.
\end{aligned}$$

Because of (7.11) we have

$$\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^\pm} - \gamma R_0 R_{0,zz} = p(\rho_0^\pm) - p(R_0) - \frac{\gamma}{2} R_{0,z}^2.$$

Inserting this into (7.96) we can, in view of (3.4), identify the surface stress vector $\boldsymbol{\sigma}_\Gamma$ as

$$\boldsymbol{\sigma}_\Gamma = \sigma_\Gamma^j = \left(\int_0^\infty \left(\left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^+} + \gamma R_{0,z}^2 \right) \right) dz + \int_{-\infty}^0 \dots\, dz \right) \frac{t_0^j}{|\mathbf{t}_0|^2} + \mathcal{O}(\varepsilon). \qquad (7.97)$$

■

8 Proof of Theorem 6.6

Now we will establish Theorem 6.6.

Proof:
To prove the theorem we will compute the first non-vanishing order of

$$[\![j_\varepsilon]\!] \left\{ W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right\} + \{j_\varepsilon\} \left[\!\!\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right]\!\!\right] \qquad (8.1)$$

by the jump conditions in Theorem 6.3.

As we see from (6.14) and (6.15) there are no $\mathcal{O}(1)$ contributions in the jump terms in (8.1), so the leading order terms are of $\mathcal{O}(\varepsilon)$. They are given as products of the order $\mathcal{O}(\varepsilon)$-terms in the jumps and the order $\mathcal{O}(1)$-terms in the mean values, i.e. the leading order of (8.1) is given by

$$[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!]_1 \left\{W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2}\right\} + \{\rho_0(u_{\nu 0} - w_{\nu 0})\} \left[\!\!\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2}\right]\!\!\right]_1 . \tag{8.2}$$

By (7.13) we have $u_{t0} = w_{t0}$ such that $u_0^j - w_0^j = (u_{\nu 0} - w_{\nu 0})\nu_0^j$, whence

$$|\mathbf{u}_0 - \mathbf{w}_0|^2 = (u_{\nu 0} - w_\nu 0)^2 \quad \text{and} \quad (u_0^j - w_0^j)u_1^j = (u_{\nu 0} - w_\nu 0)u_{\nu 1}. \tag{8.3}$$

Hence, (8.2) equals

$$[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!]_1 \left\{W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2}\right\} + \{\rho_0(u_{\nu 0} - w_{\nu 0})\} \left[\!\!\left[W'(\rho_\varepsilon) + \frac{|(u_\nu)_\varepsilon - (w_\nu)_\varepsilon|^2}{2}\right]\!\!\right]_1 . \tag{8.4}$$

Now we use (6.14) and (6.15) to get an equality

$$\begin{aligned}
&[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!]_1 \left\{W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2}\right\} \qquad (8.5)\\
&+\{\rho_0(u_{\nu 0} - w_{\nu 0})\} \left[\!\!\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2}\right]\!\!\right]_1\\
&= -j_0 \left(\int_0^\infty \nu_0^j \left(U_0^j - (u_0^j)^+\right)_\tau dz + \int_{-\infty}^0 \nu_0^j \left(U_0^j - (u_0^j)^+\right)_\tau dz\right)\\
&- \left(g(\rho_0^+) + \frac{j_0^2}{2(\rho_0^+)^2}\right) \int_0^\infty (R_0 - \rho_0^+)_\tau \, dz - \left(g(\rho_0^-) + \frac{j_0^2}{2(\rho_0^-)^2}\right) \int_{-\infty}^0 (R_0 - \rho_0^-)_\tau \, dz\\
&- \left(\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s - \kappa_0 w_{\nu 0}\right) \left(g(\rho_0^+) + \frac{j_0^2}{2(\rho_0^+)^2}\right) \int_0^\infty (R_0 - \rho_0^+) \, dz\\
&- \left(\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s - \kappa_0 w_{\nu 0}\right) \left(g(\rho_0^-) + \frac{j_0^2}{2(\rho_0^-)^2}\right) \int_{-\infty}^0 (R_0 - \rho_0^-) \, dz\\
&-(\lambda + 2\mu) j_0^2 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0}\right)_z\right)^2 dz.
\end{aligned}$$

To simplify the notation we define

$$A^\pm := \mp j_0 \int_0^{\pm\infty} \nu_0^j \left(U_0^j - (u_0^j)^\pm\right)_\tau dz \mp \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) \int_0^{\pm\infty} \left(R_0 - \rho_0^\pm\right)_\tau dz, \tag{8.6}$$

$$B^\pm := \mp \left(\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s - \kappa_0 w_{\nu 0}\right) \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) \int_0^{\pm\infty} (R_0 - \rho_0^\pm) \, dz. \tag{8.7}$$

We will first deal with $A^\pm$. Due to (5.4) there exists $c \in \mathbb{R}$, which is independent of z as $\boldsymbol{\nu}_0$ and $\mathbf{t}_0$ are independent of z, such that

$$(\nu_0^j)_\tau = c t_0^j.$$

Therefore we get

$$\int_0^{\pm\infty} \nu_0^j \left(U_0^j - (u_0^j)^\pm\right)_\tau dz = \int_0^{\pm\infty} \left(U_{\nu 0} - (u_{\nu 0})^\pm\right)_\tau dz - \int_0^{\pm\infty} (\nu_0^j)_\tau \left(U_0^j - (u_0^j)^\pm\right) dz \quad (8.8)$$
$$= \int_0^{\pm\infty} \left(U_{\nu 0} - (u_{\nu 0})^\pm\right)_\tau dz - c\int_0^{\pm\infty} \left(t_0^j U_0^j - t_0^j (u_0^j)^\pm\right) dz$$
$$\overset{(7.6)}{=} \int_0^{\pm\infty} \left((U_{\nu 0} - w_{\nu 0})_\tau - \left(u_{\nu 0}^\pm - w_{\nu 0}\right)_\tau\right) dz.$$

Using (7.9) yields

$$\left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) \int_0^{\pm\infty} \left(R_0 - \rho_0^\pm\right)_\tau dz \quad (8.9)$$
$$= \int_0^{\pm\infty} \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) \left(R_0 - \rho_0^\pm\right)_\tau dz$$
$$= \int_0^{\pm\infty} \left(g(R_0) + \frac{j_0^2}{2R_0^2} - \gamma R_{0,zz}\right) R_{0,\tau} - \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) (\rho_0^\pm)_\tau \, dz.$$

Inserting (8.8) and (8.9) in (8.6) implies

$$\begin{aligned} A^\pm &= \mp \int_0^{\pm\infty} \left(j_0 \left(\frac{j_0}{R_0}\right)_\tau - j_0 \left(\frac{j_0}{\rho_0^\pm}\right)_\tau\right) dz \\ &\quad \mp \int_0^{\pm\infty} \left(g(R_0) + \frac{j_0^2}{2R_0^2} - \gamma R_{0,zz}\right) R_{0,\tau} - \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) (\rho_0^\pm)_\tau \, dz \\ &= \mp \int_0^{\pm\infty} \left(W(R_0) - W(\rho_0^\pm) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^\pm}\right)_\tau dz \pm \gamma \int_0^{\pm\infty} R_{0,zz} R_{0,\tau} \, dz. \end{aligned} \quad (8.10)$$

We observe

$$\int_{-\infty}^{\infty} R_{0,zz} R_{0,\tau} = -\int_{-\infty}^{\infty} \left(\frac{1}{2} (R_{0,z})^2\right)_\tau dz,$$

whence (8.10) implies

$$\begin{aligned} A^+ + A^- &= -\int_0^{\infty} \left(W(R_0) - W(\rho_0^+) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2\right)_\tau dz \\ &\quad - \int_{-\infty}^{0} \left(W(R_0) - W(\rho_0^-) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2\right)_\tau dz. \end{aligned} \quad (8.11)$$

This finishes our calculations for $A^\pm$ and we turn to $B^\pm$.

First we observe that due to (7.9)

$$\left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) \int_0^{\pm\infty} \left(R_0 - \rho_0^\pm\right) dz, \quad (8.12)$$
$$= \int_0^{\pm\infty} \left(g(R_0) + \frac{j_0^2}{2R_0^2} - \gamma R_{0,zz}\right) R_0 - \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2}\right) \rho_0^\pm \, dz$$
$$= \int_0^{\pm\infty} \left(p(R_0) + W(R_0) + \frac{j_0^2}{2R_0} - \gamma R_0 R_{0,zz} - p(\rho_0^\pm) - W(\rho_0^\pm) - \frac{j_0^2}{2\rho_0^\pm}\right) dz.$$

From (7.11) and (7.12) we already know

$$p(R_0) - p(\rho_0^\pm) - \gamma R_0 R_{0,zz} = -\frac{j_0^2}{R_0} + \frac{j_0^2}{\rho_0^\pm} - \frac{\gamma}{2} R_{0,z}^2. \tag{8.13}$$

Inserting (8.12) and (8.13) in (8.7) we find

$$\begin{aligned} B^+ \quad &+ \quad B^- \qquad\qquad (8.14)\\ = \quad &-\left(\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s - \kappa_0 w_{\nu 0}\right)\int_0^\infty \left(W(R_0) - W(\rho_0^+) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^+} - \frac{\gamma}{2} R_{0,z}^2\right) dz \\ &-\left(\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s - \kappa_0 w_{\nu 0}\right)\int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^-} - \frac{\gamma}{2} R_{0,z}^2\right) dz. \end{aligned}$$

Plugging (8.11) and (8.14) in (8.5) we get

$$\begin{aligned} &[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)]\!]_1 \left\{W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2}\right\} \qquad\qquad (8.15)\\ &\quad + \{\rho_0((u_{\nu 0} - w_{\nu 0})\} \left[\!\!\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2}\right]\!\!\right]_1 \\ &= \; -\int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2\right)_\tau dz \\ &\quad -\int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2\right)_\tau dz \\ &\quad -\left(\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s - \kappa_0 w_{\nu 0}\right)\int_0^\infty \left(W(R_0) - W(\rho_0^+) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^+} - \frac{\gamma}{2} R_{0,z}^2\right) dz \\ &\quad -\left(\frac{1}{|\mathbf{t}_0|}\left(|\mathbf{t}_0| w_{t0}\right)_s - \kappa_0 w_{\nu 0}\right)\int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^-} - \frac{\gamma}{2} R_{0,z}^2\right) dz \\ &\quad -(\lambda + 2\mu) j_0^2 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0}\right)_z\right)^2 dz. \end{aligned}$$

Recalling equation (3.10) and definitions (6.17), (6.18) we can identify W_Γ as

$$\begin{aligned} W_\Gamma \quad = \quad &\int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{1}{2}\frac{j_0^2}{R_0} - \frac{1}{2}\frac{j_0^2}{\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2\right) dz \\ &+ \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{1}{2}\frac{j_0^2}{R_0} - \frac{1}{2}\frac{j_0^2}{\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2\right) dz. \end{aligned}$$

and obtain the assertion of Theorem 6.6. ■

9 Kinetic relation

We will show that the leading order jump conditions stated in Theorem 6.3 consist of the two conservation laws for mass and momentum and in addition of a kinetic relation. In particular, when the direction of the mass flux is given, the normal velocity of the interface is determined by the state on one side of the interface.

Lemma 9.1
Let $\rho_0^+, \rho_0^-, u_{\nu 0}^+, u_{\nu 0}^-, u_{t0}^+, u_{t0}^-, w_{\nu 0}$ satisfy the jump conditions from Lemma 7.1, such that $\rho_0^- \notin [\alpha, \beta]$, where α, β are the Maxwell points of W, and the equal area construction

$$0 = -\int_{\frac{1}{\rho_0^-}}^{\frac{1}{\rho_0^+}} p(\rho_0)\, d\left(\frac{1}{\rho_0}\right) + \{p(\rho_0)\} \left[\!\left[\frac{1}{\rho_0}\right]\!\right] \tag{9.1}$$

is possible. If $\rho_0^-, u_{\nu 0}^-, u_{t0}^-$ and the sign of j_0 are specified, then $\rho_0^+, u_{\nu 0}^+, u_{t0}^+, w_{\nu 0}$ are determined.

Proof:
Due to (7.3) we know u_{t0}^+ equals u_{t0}^-. Using (7.2) we get the following expression for the mass flux

$$j_0^2 = -\frac{[\![p(\rho_0)]\!]}{\left[\!\left[\frac{1}{\rho_0}\right]\!\right]}. \tag{9.2}$$

Inserting (9.2) in (7.4) we find

$$[\![g(\rho_0)]\!] = -\frac{1}{2} j_0^2 \left[\!\left[\left(\frac{1}{\rho_0}\right)^2\right]\!\right] = \frac{1}{2}\, [\![p(\rho_0)]\!]\, \frac{\left(\frac{1}{\rho_0^+}\right)^2 - \left(\frac{1}{\rho_0^-}\right)^2}{\frac{1}{\rho_0^+} - \frac{1}{\rho_0^-}} = [\![p(\rho_0)]\!] \left\{\frac{1}{\rho_0}\right\}. \tag{9.3}$$

This is equivalent to

$$0 = \left[\!\left[g(\rho_0) - \frac{p(\rho_0)}{\rho_0}\right]\!\right] + \{p(\rho_0)\} \left[\!\left[\frac{1}{\rho_0}\right]\!\right]. \tag{9.4}$$

From equation (2.5) we conclude

$$\frac{d}{d\frac{1}{\rho_0}} \left(g(\rho_0) - \frac{p(\rho_0)}{\rho_0}\right) = -p(\rho_0). \tag{9.5}$$

Inserting (9.5) in (9.4) we obtain

$$0 = -\int_{\frac{1}{\rho_0^-}}^{\frac{1}{\rho_0^+}} p(\rho_0)\, d\left(\frac{1}{\rho_0}\right) + \{p(\rho_0)\} \left[\!\left[\frac{1}{\rho_0}\right]\!\right]. \tag{9.6}$$

Note that (9.6) is an equal area rule which determines ρ_0^+. Then (9.2) prescribes j_0 which we can, in turn, use to obtain $w_{\nu 0}$ and $u_{\nu 0}^+$ from (7.1). ■

As shown in [5] the kinetic relation derived in Lemma 9.1 guarantees the solvabilty of the free boundary value problem for $\rho_0, \mathbf{u}_0$. Hence, when we want to determine $\rho_1^+, u_{\nu 1}^+, u_{t1}^+$ and $w_{\nu 1}$ from $\rho_1^-, u_{\nu 1}^-$ and u_{t1}^-, we can assume that we can, at least numerically, calculate ρ_0, $\mathbf{u}_0$ and $\mathbf{w}_0$. Furthermore R_0 and $\mathbf{U}_0$ are traveling wave solutions of the non-viscous problem. Their phase portraits can be found in [4].

For the following lemma we assume that we have solved, at least numerically, the inner and outer problem of zeroth order. Hence, we know the right hand side of the first order jump conditions in Theorem 6.3.

Lemma 9.2
Let ρ_1^+, ρ_1^-, $u_{\nu 1}^+$, $u_{\nu 1}^-$, u_{t1}^+, u_{t1}^-, $w_{\nu 1}$ satisfy the first order of the jump conditions in Theorem 6.3. If ρ_1^-, $u_{\nu 1}^-$, u_{t1}^- are given and $j_0 \neq 0$, then ρ_1^+, $u_{\nu 1}^+$, u_{t1}^+, $w_{\nu 1}$ are determined.

Proof:
We start by determining u_{t1}^+. We multiply (7.75) by $\mathbf{t}_0$ and obtain

$$j_0 u_{t1}^+ = h_1, \tag{9.7}$$

where h_1 comprises terms depending only on ρ_1^-, $u_{\nu 1}^-$, u_{t1}^- and terms of zeroth order and their derivatives. As $j_0 \neq 0$ equation (9.7) prescribes the value of u_{t1}^+. To determine ρ_1^+, $u_{\nu 1}^+$, $w_{\nu 1}$ we consider (7.57), (7.66) and (7.75) times $\boldsymbol{\nu}_0$. We obtain a system of linear equations for ρ_1^+, $u_{\nu 1}^+$, $w_{\nu 1}$ which reads

$$\underbrace{\begin{pmatrix} \frac{j_0}{\rho_0^+} & \rho_0^+ & -\rho_0^+ + \rho_0^- \\ g'(\rho_0^+) & \frac{j_0}{\rho_0^+} & -\frac{j_0}{\rho_0^+} + \frac{j_0}{\rho_0^-} \\ \left(\frac{j_0}{\rho_0^+}\right)^2 + p'(\rho_0^+) & 2j_0 & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} \rho_1^+ \\ u_{\nu 1}^+ \\ w_{\nu 1} \end{pmatrix} = \begin{pmatrix} h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{9.8}$$

where $h_2, \ldots, h_4$ only contain terms depending on ρ_1^-, $u_{\nu 1}^-$, u_{t1}^-, terms of zeroth order and their derivatives and r_1. An easy calculation shows

$$\det A = \frac{j_0}{\rho_0^-}(\rho_0^+ - \rho_0^-)^2 \left(g'(\rho_0^+) - \frac{j_0^2}{(\rho_0^+)^3} \right) = \frac{j_0}{\rho_0^-(\rho_0^+)^3}(\rho_0^+ - \rho_0^-)^2 \left(-\frac{\partial p}{\partial\left(\frac{1}{\rho_0}\right)}(\rho_0^+) + \frac{[\![p(\rho_0)]\!]}{\left[\!\left[\frac{1}{\rho_0}\right]\!\right]} \right).$$

The last expression vanishes if and only if the tangent at ρ_0^+ and the secant connecting ρ_0^+ to ρ_0^- coincide. However this is not possible due to the equal area rule (9.1).

Thus we have a relation $w_{\nu 1}(r_1)$ such that we end up with an ODE to determine the interface position up to the first order. ■

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10 Appendix: List of the ε-expansions of the jump brackets

In this section we provide identities for the first two orders of the jump brackets of mass, Gibbs free energy and pressure.

10.1 Mass flux

$$
\begin{aligned}
[\![\rho_\varepsilon((u_\nu)_\varepsilon-(w_\nu)_\varepsilon)]\!]_0 &= [\rho_0(u_{\nu 0}-w_{\nu 0})] && (10.1)\\
[\![\rho_\varepsilon((u_\nu)_\varepsilon-(w_\nu)_\varepsilon)]\!]_1 &= \left(\rho_1^+ + \left(\frac{\partial\rho_0}{\partial x^k}\right)^+ r_1^k\right)\left(\nu_0^i(u_0^i)^+ - \nu_0^i w_0^i\right) && (10.2)\\
&\quad +\rho_0^+\left(\nu_1^i(u_0^i)^+ + \nu_0^i(u_1^i)^+ + \nu_0^i\left(\frac{\partial u_0^i}{\partial x^k}\right)^+ r_1^k - \nu_1^i w_0^i - \nu_0^i w_1^i\right)\\
&\quad -\left(\rho_1^- + \left(\frac{\partial\rho_0}{\partial x^k}\right)^- r_1^k\right)\left(\nu_0^i(u_0^i)^- - \nu_0^i w_0^i\right)\\
&\quad -\rho_0^-\left(\nu_1^i(u_0^i)^- + \nu_0^i(u_1^i)^- + \nu_0^i\left(\frac{\partial u_0^i}{\partial x^k}\right)^- r_1^k - \nu_1^i w_0^i - \nu_0^i w_1^i\right)
\end{aligned}
$$

10.1.1 Explanation of (10.1) and (10.2)

We have

$$
\begin{aligned}
&[\![\rho_\varepsilon((u_\nu)_\varepsilon-(w_\nu)_\varepsilon)]\!] && (10.3)\\
=\ &\lim_{z\to 0, z>0}\rho_\varepsilon(\mathbf{r}_\varepsilon+z\boldsymbol{\nu}_\varepsilon)\left(\nu_\varepsilon^i u_\varepsilon^i(\mathbf{r}_\varepsilon+z\boldsymbol{\nu}_\varepsilon)-\nu_\varepsilon^i w_\varepsilon^i\right) - \lim_{z\to 0, z<0}\rho_\varepsilon(\mathbf{r}_\varepsilon+z\boldsymbol{\nu}_\varepsilon)\left(\nu_\varepsilon^i u_\varepsilon^i(\mathbf{r}_\varepsilon+z\boldsymbol{\nu}_\varepsilon)-\nu_\varepsilon^i w_\varepsilon^i\right).
\end{aligned}
$$

To determine the first orders of the jump we will look at the $\varepsilon-$expansion

$$
\begin{aligned}
&\rho_\varepsilon(\mathbf{r}_\varepsilon+z\boldsymbol{\nu}_\varepsilon)\left(\nu_\varepsilon^i u_\varepsilon^i(\mathbf{r}_\varepsilon+z\boldsymbol{\nu}_\varepsilon)-\nu_\varepsilon^i w_\varepsilon^i\right)\\
=\ &\rho_0(\mathbf{r}_0+z\boldsymbol{\nu}_0)\left(\nu_0^i u_\varepsilon^i(\mathbf{r}_0+z\boldsymbol{\nu}_0)-\nu_0^i w_0^i\right)\\
&+\varepsilon\Big[\left(\rho_1(r_0+z\nu_0)+\left(\frac{\partial\rho_0}{\partial x^k}\right)(\mathbf{r}_0+z\boldsymbol{\nu}_0)(r_1^k+z\nu_1^k)\right)\left(\nu_0^i u_0^i(\mathbf{r}_0+z\boldsymbol{\nu}_0)-\nu_0^i w_0^i\right)\\
&\quad+\rho_0(\mathbf{r}_0+z\boldsymbol{\nu}_0)\Big(\nu_1^i u_0^i(\mathbf{r}_0+z\boldsymbol{\nu}_0)+\nu_0^i u_1^i(\mathbf{r}_0+z\boldsymbol{\nu}_0^i)+\nu_0^i\left(\frac{\partial u_0^i}{\partial x^k}\right)(\mathbf{r}_0+z\boldsymbol{\nu}_0)(r_1^k+z\nu_1^k)\\
&\quad-\nu_1^i w_0^i-\nu_0^i w_1^i\Big)\Big]+\mathcal{O}(\varepsilon^2)
\end{aligned}
$$

which shows (10.1) and (10.2).

10.2 Balance of Gibbs free energy

$$\left[\!\!\left[\frac{1}{2}((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)^2 + g(\rho_\varepsilon)\right]\!\!\right]_0 = \left[\!\!\left[\frac{1}{2}(u_{\nu 0} - w_{\nu 0})^2 + g(\rho_0)\right]\!\!\right] \tag{10.4}$$

$$\begin{aligned}\left[\!\!\left[\frac{1}{2}((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)^2 + g(\rho_\varepsilon)\right]\!\!\right]_1 &= -(u^+_{\nu 0} - w_{\nu 0})w_{\nu 1} + (u^-_{\nu 0} - w_{\nu 0})w_{\nu 1} \\ &+(u^+_{\nu 0} - w_{\nu 0})\left((u^j_0)^+\nu^j_1 + (u^j_1)^+\nu^j_0 + \left(\frac{\partial u^j_0}{\partial x^k}\right)^+ \nu^j_0 r^k_1\right) \\ &-(u^-_{\nu 0} - w_{\nu 0})\left((u^j_0)^-\nu^j_1 + (u^j_1)^-\nu^j_0 + \left(\frac{\partial u^j_0}{\partial x^k}\right)^- \nu^j_0 r^k_1\right) \\ &+g'(\rho^+_0)\rho^+_1 + g'(\rho^+_0)\left(\frac{\partial \rho_0}{\partial x^k}\right)^+ r^k_1 \\ &-g'(\rho^-_0)\rho^-_1 - g'(\rho^-_0)\left(\frac{\partial \rho_0}{\partial x^k}\right)^- r^k_1\end{aligned} \tag{10.5}$$

The derivation of (10.4) and (10.5) is analogous to the derivation of (10.1) and (10.2).

10.3 Momentum balance

$$[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\boldsymbol{\nu}_\varepsilon]\!]_0 = [\![\rho_0(u_{\nu 0}) - w_{\nu 0})(\mathbf{u}_0 - \mathbf{w}_0) + p(\rho_0)\boldsymbol{\nu}_0]\!] \tag{10.6}$$

$$\begin{aligned}&[\![\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\boldsymbol{\nu}_\varepsilon]\!]_1 \\ &= 2j_0\nu^i_0\left((u^j_0)^+\nu^j_1 + (u^j_1)^+\nu^j_0 + \left(\frac{\partial u^j_0}{\partial x^k}\right)^+ \nu^j_0 r^k_1\right) \\ &-2j_0\nu^i_0\left((u^j_0)^-\nu^j_1 + (u^j_1)^-\nu^j_0 + \left(\frac{\partial u^j_0}{\partial x^k}\right)^- \nu^j_0 r^k_1\right) \\ &+j_0 t^i_0\left((u^j_1)^+\frac{t^j_0}{|\mathbf{t}_0|^2} + \left(\frac{\partial u^j_0}{\partial x^k}\right)^+ \frac{t^j_0}{|\mathbf{t}_0|^2} r^k_1 + (u^k_0)^+\left(\frac{t^k_1}{|\mathbf{t}_0|^2} - 2\frac{t^k_0 t^j_0 t^j_1}{|\mathbf{t}_0|^4}\right)\right) \\ &-j_0 t^i_0\left((u^j_1)^-\frac{t^j_0}{|\mathbf{t}_0|^2} + \left(\frac{\partial u^j_0}{\partial x^k}\right)^- \frac{t^j_0}{|\mathbf{t}_0|^2} r^k_1 + (u^k_0)^-\left(\frac{t^k_1}{|\mathbf{t}_0|^2} - 2\frac{t^k_0 t^j_0 t^j_1}{|\mathbf{t}_0|^4}\right)\right) \\ &+\nu^i_0\left(\frac{j^2_0}{(\rho^+_0)^2} + p'(\rho^+_0)\right)\left(\rho^+_1 + \left(\frac{\partial \rho_0}{\partial x^k}\right)^+ r^k_1\right) \\ &-\nu^i_0\left(\frac{j^2_0}{(\rho^-_0)^2} + p'(\rho^-_0)\right)\left(\rho^-_1 + \left(\frac{\partial \rho_0}{\partial x^k}\right)^- r^k_1\right)\end{aligned} \tag{10.7}$$

The derivation of (10.6) and (10.7) is analogous to the derivation of (10.1) and (10.2).

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