Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Existence of weak solutions for Cahn-Hilliard systems coupled with elasticity and damage

Christian Heinemann, Christiane Kraus

submitted: June 11, 2010

Weierstraß-Institut
für Angewandte Analysis und Stochastik
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: christian.heinemann@wias-berlin.de
christiane.kraus@wias-berlin.de

No. 1520
Berlin 2010

2000 *Mathematics Subject Classification.* 35K85, 49J40, 74C10, 82C26, 35J50 35K35, 35K55 .

Key words and phrases. Cahn-Hilliard systems, phase separation, damage, elliptic-parabolic systems, energetic solution, weak solution, doubly nonlinear differential inclusions, existence results, rate-dependent systems.

This project is supported by the DFG Research Center "Mathematics for Key Technologies" Matheon in Berlin.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

A typical phase field approach for describing phase separation and coarsening phenomena in alloys is the Cahn-Hilliard model. This model has been generalized to the so-called Cahn-Larché system by combining it with elasticity to capture non-neglecting deformation phenomena, which occur during phase separation and coarsening processes in the material. In order to account for damage effects, we extend the existing framework of Cahn-Hilliard and Cahn-Larché systems by incorporating an internal damage variable of local character. This damage variable allows to model the effect that damage of a material point is influenced by its local surrounding. The damage process is described by a unidirectional rate-dependent evolution inclusion for the internal variable. For the introduced Cahn-Larché systems coupled with rate-dependent damage processes, we establish a suitable notion of weak solutions and prove existence of weak solutions.

1 Introduction

Due to the ongoing miniaturization in the area of micro-electronics the demands on strength and lifetime of the materials used is considerably rising, while the structural size is continuously being reduced. Materials, which enable the functionality of technical products, change the microstructure over time. Phase separation and coarsening phenomena take place and the complete failure of electronic devices like motherboards or mobile phones often results from micro–cracks in solder joints.

Solder joints, for instance, are essential components in electronic devices since they form the electrical and the mechanical bond between electronic components like micro–chips and the circuit–board. The Figures 1 and 2 illustrate the typical morphology in the interior of solder materials. At high temperatures, one homogeneous phase consisting of different components of the alloy is energetically favourable. If the temperature is decreased below a critical value a fine microstructure of two or more phases (different compositions of the components of the material) arises on a very short time scale. The formation of microstructures, also called phase separation or spinodal decomposition, take place to reduce the bulk chemical free energy. Then coarsening phenomena occur, which are mainly driven by decreasing interfacial energy. Due to the misfit of the crystal lattices, the different heat expansion coefficients and the different elastic moduli of the components, very high mechanical stresses occur preferably at the interfaces of the phases. These stress concentrations initiate the nucleation of micro–cracks, whose propagation can finally lead to the failure of the whole electronic device.

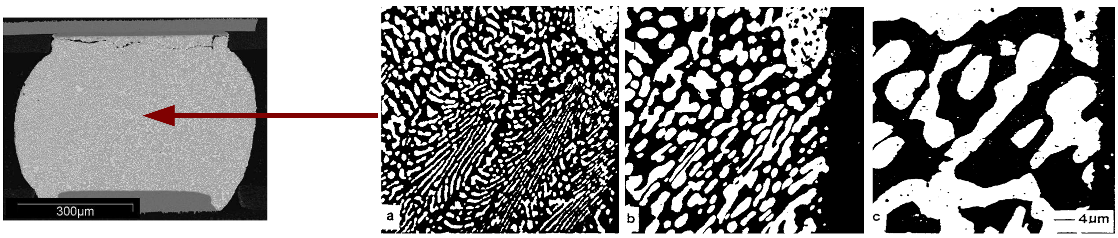

Figure 1: Left: Solder ball and micro–structural coarsening in eutectic Sn–Pb; Right: a) directly after solidification, b) after 3 hours, and c) after 300 hours [HCW91];

The knowledge of the mechanisms inducing phase separation, coarsening and damage phenomena is of great importance for technological applications. A uniform distribution of the original materials is aimed to guarantee evenly distributed material properties of the sample. For instance, mechanical properties, such as the strength and the stability of the material, depend on how finely regions of the original materials are mixed. The control of the evolution of the microstructure and therefore of the lifetime of materials relies on

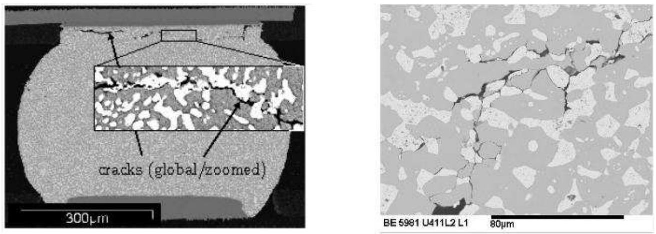

Figure 2: Initiation and propagation of cracks along the phase boundary [FBFD06].

the ability to understand phase separation, coarsening and damage processes. This shows the importance of developing reliable mathematical models to describe such effects.

In the mathematical literature, coarsening and damage processes are treated in general separately. Phase separation and coarsening phenomena are usually described by phase–field models of Cahn-Hilliard type. The evolution is modeled by a parabolic diffusion equation for the phase fractions. To include elastic effects, resulting from stresses caused by different elastic properties of the phases, Cahn-Hilliard systems are coupled with an elliptic equation, describing the quasi-static balance of forces. Such coupled Cahn-Hilliard systems with elasticity are also called Cahn-Larché systems. Since in general the mobility, stiffness and surface tension coefficients depend on the phases (see for instance [BDM07] and [BDDM07] for the explicite structure deduced by the embedded atom method), the mathematical analysis of the coupled problem is very complex. Existence results were derived for special cases in [Gar00, CMP00, BP05] (constant mobility, stiffness and surface tension coefficients), in [BCD$^+$02] (concentration dependent mobility, two space dimensions) and in [PZ08] in an abstract measure-valued setting (concentration dependent mobility and surface tension tensors). For numerical results and simulations we refer [Wei01, Mer05, BM10].

Damage models for elastic materials have been analytically investigated for the last ten years. In the simplest case, the damage variable is a scalar function and describes the local accumulation of damage in the body. The damage process is typically modeled as a unidirectional evolution, which means that damage can increase, but not decrease. Based on the model developed in [FN96], the damage evolution is described by an equation of balance for forces which is coupled with a unidirectional parabolic [BSS05, FK09, Gia05] or rate–independent [MR06, MRZ10] evolution inclusion for the damage variable. The models studied in [FK09, MR06, Gia05] also include the effect that the applied forces have to pass over a threshold before the damage starts to increase.

In this work, we introduce a mathematical model describing both phenomena, phase separation/coarsening and damage processes, in a *unifying* model. We focus on the analytical modeling on the meso– and macroscale. To this end, we couple phase–field models of Cahn-Larché type with damage models. The evolution system consists of an equation of balance for forces which is coupled with a parabolic evolution equation for the phase fractions and a unidirectional evolution inclusion for the damage variable. The evolution inclusion also comprises the phenomenon that a threshold for the loads has to be passed before the damage process increases.

The main aim of the present work is to show existence of weak solutions of the introduced model for rate-dependent damage processes. A crucial step has been to establish a suitable notion of weak solutions. We first study the model with regularization terms and prove existence of weak solutions for the regularized model based on a time–incremental minimization problem with constraints due to the unidirectionality of the damage. The regularization allows us to prove an energy inequality which occurs in the weak notion of our coupled system. The major task has been to prove convergence of the time incremental solutions for the regularized model when the discretization fineness tends to zero. In this context, several approximation results have been established to handle the damage evolution inclusion and the unidirectionality of damage processes. More precisely, the internal variable z, describing damage effects, is bounded with values in $[0, 1]$ and monotonically decreasing with respect to the time variable. The main results are stated in Sections 4.1 and 4.2, see Theorems 4.4 and 4.6.

To the best of our knowledge, phase separation processes coupled with damage are not studied yet in the mathematical literature. However, promising simulations were carried out in the context of phase field models of Cahn-Hilliard and Cahn-Larché type with damage, see [USG07, GUaMM$^+$07].

The paper is organized as follows: We start with introducing a phase field model of Cahn-Larché type coupled with damage, cf. Section 2. Then we state some assumptions for this model, see Section 3. In Section 4, we establish a suitable notation for weak formulations of solutions for the introduced model and a regularized version of the model and state the main results. Section 5.2 is devoted to the existence proof for the regularized Cahn-Larché system coupled with damage. Finally, we pass to the limit in the regularized version, which shows the existence of weak solutions of the original model, see Section 5.3.

2 Model

We consider a material of two components occupying a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$. The state of the system at a fixed time point is specified by a triple $q = (u, c, z)$. The displacement field $u : \Omega \to \mathbb{R}^3$ determines the current position $x + u(x)$ of an undeformed material point x. Throughout this paper, we will work with the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, which is an adequate assumption only when small strains occur in the material. However, this assumption is justified for phase-separation processes in alloys since the deformation usually has a small gradient. The function $c : \Omega \to \mathbb{R}$ is a phase field variable describing a scaled concentration difference of the two components. To account for damage effects, we choose an isotropic damage variable $z : \Omega \to \mathbb{R}$, which models the reduction of the effective volume of the material due to void nucleation, growth, and coalescence. The damage process is modeled unidirectional, i.e. damage may only increase. Self-healing processes in the material are forbidden. No damage at a material point $x \in \Omega$ is described by $z(x) = 1$, whereas $z(x) = 0$ stands for a completely damaged material point $x \in \Omega$. We require that even a damaged material can store a small amount of elastic energy. Plastic effects are not considered in our model.

2.1 Energies and evolutionary equations

Here, we qualify our model formally and postpone a rigorous treatment to Section 4. The presented model is based on two functionals, i.e. a generalized Ginzburg-Landau free energy functional $\mathcal{E}$ and a damage dissipation potential $\mathcal{R}$. The free energy density φ of the system is given by

$$\varphi(e, c, \nabla c, z, \nabla z) := \frac{\gamma}{2}|\nabla c|^2 + \frac{\delta}{p}|\nabla z|^p + W_{\rm ch}(c) + W_{\rm el}(e, c, z), \qquad \gamma, \delta > 0, \tag{1}$$

where the gradient terms penalize spatial changes of the variables c and z, $W_{\rm ch}$ denotes the chemical energy density and $W_{\rm el}$ is the elastically stored energy density accounting for elastic deformations and damage effects. For simplicity of notation, we set $\gamma = \delta = 1$.

The *chemical free energy density* $W_{\rm ch}$ has usually the form of a double well potential for a two phase system. For a rigorous treatment, we need the assumptions (A1)-(A6), see Section 3. Hence, in particular, classical ansatzes such as

$$W_{\rm ch} = (1 - c^2)^2$$

fit in our framework.

The *elastically stored energy density* $\hat{W}_{\rm el}$ due to stresses and strains, which occur in the material, is typically of quadratic form, i.e.

$$\hat{W}_{\rm el}(c, e) = \frac{1}{2}\big(e - e^*(c)\big) : \mathbb{C}(c)\big(e - e^*(c)\big). \tag{2}$$

Here, $e^*(c)$ denotes the *eigenstrain*, which is usually linear in c, and $\mathbb{C}(c) \in \mathcal{L}(\mathbb{R}^{n\times n}_{\rm sym})$ is a fourth order stiffness tensor, which is symmetric and positive definite. If the stiffness tensor does not depend on the concentration, i. e. $\mathbb{C}(c) = \mathbb{C}$, we refer to *homogeneous* elasticity.

To incorporate the effect of damage on the elastic response of the material, $\hat{W}_{\mathrm{el}}$ is replaced by

$$W_{\mathrm{el}} = (\Phi(z) + \tilde{\eta})\,\hat{W}_{\mathrm{el}}, \tag{3}$$

where $\Phi : [0,1] \to \mathbb{R}_+$ is a continuous and monotonically increasing function with $\Phi(0) = 0$ and $\tilde{\eta} > 0$ is a small value. The small value $\tilde{\eta} > 0$ in (3) is introduced for analytical reasons, see for instance (A1).

Rigorous results in the present work are obtained under certain growth conditions for the elastic energy density W_{el}, see Section 3. These conditions are, for instance, satisfied for W_{el} as in (3) in the case of homogeneous elasticity.

The overall free energy $\mathcal{E}$ of Ginzburg-Landau type has the following structure:

$$\begin{aligned} \mathcal{E}(u,c,z) &:= \tilde{\mathcal{E}}(u,c,z) + \int_\Omega I_{[0,\infty)}(z)\,\mathrm{d}x, \\ \tilde{\mathcal{E}}(u,c,z) &:= \int_\Omega \varphi(e(u), c, \nabla c, z, \nabla z)\,\mathrm{d}x. \end{aligned} \tag{4}$$

Here, $I_{[0,\infty)}$ signifies the indicator function of the subset $[0,\infty) \subseteq \mathbb{R}$, i.e. $I_{[0,\infty)}(x) = 0$ for $x \in [0,\infty)$ and $I_{[0,\infty)}(x) = \infty$ for $x < 0$. We assume that the energy dissipation for the damage process is triggered by a dissipation potential $\mathcal{R}$ of the form

$$\begin{aligned} \mathcal{R}(\dot{z}) &:= \tilde{\mathcal{R}}(\dot{z}) + \int_\Omega I_{(-\infty,0]}(\dot{z})\,\mathrm{d}x, \\ \tilde{\mathcal{R}}(\dot{z}) &:= \int_\Omega -\alpha\dot{z} + \frac{1}{2}\beta\dot{z}^2\,\mathrm{d}x \text{ for } \alpha > 0 \text{ and } \beta > 0. \end{aligned} \tag{5}$$

Due to $\beta > 0$, the dissipation potential is referred to as *rate-dependent*. In the case $\beta = 0$, which is not considered in this work, $\mathcal{R}$ is called *rate-independent*. We refer for rate-independent processes to [EM06, MT99, MR06, MRZ10, Rou10] and in particular to [Mie05] for a survey.

The governing evolutionary equations for a system state $q = (u,c,z)$ can be expressed by virtue of the functionals (4) and (5). The evolution is driven by the following elliptic-parabolic system of differential equations and differential inclusion:

$$\begin{aligned} &\textit{Diffusion:} && \partial_t c = \Delta\mu(u,c,z), && \text{(6a)} \\ &\textit{Mechanical equilibrium:} && \mathrm{div}(\sigma(e(u),c,z)) = 0, && \text{(6b)} \\ &\textit{Damage evolution:} && 0 \in \partial_z\mathcal{E}(u,c,z) + \partial_{\dot{z}}\mathcal{R}(\partial_t z), && \text{(6c)} \end{aligned}$$

where $\sigma = \sigma(e,c,z) := \partial_e\varphi(e,c,\nabla c,z,\nabla z)$ denotes the Cauchy stress tensor and μ is the chemical potential given by $\mu = \mu(u,c,z) := \partial_c\varphi(e,c,\nabla c,z,\nabla z) - \mathrm{div}(\partial_{\nabla c}\varphi(e,c,\nabla c,z,\nabla z))$. Equation (6a) is a fourth order quasi-linear parabolic equation of Cahn-Hilliard type and describes phase separation processes for the concentration c while the elliptic equation (6b) constitutes a quasi-static equilibrium for u. This means physically that we neglect kinetic energies and instead assume that mechanical equilibrium is attained at any time. The doubly nonlinear differential inclusion (6c) specifies the flow rule of the damage profile according to the constraints $0 \le z \le 1$ and $\partial_t z \le 0$ (in space and time). The inclusion (6c) has to be read in terms of generalized sub-differentials.

We choose Dirichlet conditions for the displacements u on a subset Γ of the boundary $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$. Let $b : [0,T] \times \Gamma \to \mathbb{R}^n$ be a function which prescribes the displacements on Γ for a fixed chosen time interval $[0,T]$. The imposed boundary and initial conditions and constraints are as follows:

$$\begin{aligned} &\textit{Boundary displacements:} && u(t) = b(t) \text{ on } \Gamma \text{ for all } t \in [0,T], && \text{(IBC1)} \\ &\textit{Initial concentration:} && c(0) = c^0 \text{ in } \Omega, && \text{(IBC2)} \\ &\textit{Initial damage:} && 0 \le z(0) = z^0 \le 1 \text{ in } \Omega, && \text{(IBC3)} \\ &\textit{Damage constraints:} && 0 \le z \le 1 \text{ and } \partial_t z \le 0 \text{ in } \Omega_T. && \text{(IBC4)} \end{aligned}$$

Moreover, we use homogeneous Neumann boundary conditions for the remaining variables on (parts of) the boundary:

$$\sigma \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma, \tag{IBC5}$$

$$\nabla \mu(t) \cdot \nu = 0 \text{ on } \partial\Omega, \tag{IBC6}$$

$$\nabla c(t) \cdot \nu = 0 \text{ on } \partial\Omega, \tag{IBC7}$$

$$\nabla z(t) \cdot \nu = 0 \text{ on } \partial\Omega, \tag{IBC8}$$

where ν stands for the outer unit normal to $\partial\Omega$.

We like to mention that mass conservation of the system follows from the diffusion equation (6a) and (IBC6), i.e.

$$\int_\Omega c(t) - c^0 \,\mathrm{d}x = 0 \text{ for all } t \in [0,T].$$

3 Assumptions and Notation

In the following, we collect all assumptions and constants which are used for a rigorous analysis in the subsequent sections.

(i) *Setting.* $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $n \in \{1,2,3\}$, $p > n$, $\beta > 0$, $W_{\mathrm{el}} \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}_+)$, $W_{\mathrm{ch}} \in C^1(\mathbb{R}; \mathbb{R}_+)$, $W_{\mathrm{el}}(e,c,z) = W_{\mathrm{el}}(e^t,c,z)$ for all $e \in \mathbb{R}^{n\times n}$ and $c, z \in \mathbb{R}$. Furthermore, $C > 0$ always denotes a constant, which may vary from estimate to estimate, and $[0,T]$ is the time interval of interest.

(ii) *Convexity and growth assumptions.* The function W_{el} is assumed to satisfy for some constants $\eta > 0$ and $C > 0$ the following estimates:

$$\eta|e_1 - e_2|^2 \le (\partial_e W_{\mathrm{el}}(e_1,c,z) - \partial_e W_{\mathrm{el}}(e_2,c,z)) : (e_1 - e_2), \tag{A1}$$

$$W_{\mathrm{el}}(e,c,z) \le C(|e|^2 + |c|^2 + 1), \tag{A2}$$

$$|\partial_e W_{\mathrm{el}}(e_1 + e_2, c, z)| \le C(W_{\mathrm{el}}(e_1,c,z) + |e_2| + 1), \tag{A3}$$

$$|\partial_c W_{\mathrm{el}}(e,c,z)| \le C(|e| + |c|^2 + 1), \tag{A4}$$

$$|\partial_z W_{\mathrm{el}}(e,c,z)| \le C(|e|^2 + |c|^2 + 1) \tag{A5}$$

for arbitrary $c \in \mathbb{R}$, $z \in [0,1]$ and symmetric $e, e_1, e_2 \in \mathbb{R}^{n\times n}$.

The chemical energy density function W_{ch} satisfies

$$|\partial_c W_{\mathrm{ch}}(c)| \le \hat{C}(|c|^{2^\star/2} + 1) \tag{A6}$$

for some constant $\hat{C} > 0$. For dimension $n = 3$, the constant $2^\star$ denotes the Sobolev critical exponent given by $\frac{2n}{n-2}$. In the two dimensional case $n = 2$, the constant $2^\star$ can be an arbitrary positive real number and in one space dimension (A6) can be dropped.

(iii) *Boundary displacements.* We assume that Γ is a $\mathcal{H}^{n-1}$-measurable subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$ and that the boundary displacement $b : [0,T] \times \Gamma \to \mathbb{R}^n$ may be extended by $\hat{b} \in W^{1,1}([0,T]; W^{1,\infty}(\Omega; \mathbb{R}^n))$ such that $b(t)|_\Gamma = \hat{b}(t)|_\Gamma$ in the sense of traces for a.e. $t \in [0,T]$. In the following, we write b instead of $\hat{b}$.

Remark 3.1 *Conditions* (A1), (A2) *and* (A3) *imply the following estimates*

$$|\partial_e W_{\mathrm{el}}(e,c,z)| \le C(|e| + |c|^2 + 1), \tag{11a}$$

$$\eta|e|^2 - C(|c|^4 + 1) \le W_{\mathrm{el}}(e,c,z) \tag{11b}$$

for some appropriate constants $\eta > 0$ and $C > 0$, cf. [Gar00, Section 3.2] for (11b).

We introduce some auxiliary spaces to shorten the notation for the construction of solution curves of the evolutionary problem. First of all, we define the trajectory space $\mathcal{Q}$ for the limit problem (6a)-(6c) as

$$\mathcal{Q} := \left\{ q = (u,c,z) \quad \text{with} \quad \begin{array}{c} u \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \\ c \in L^\infty([0,T]; H^1(\Omega)) \cap H^1([0,T], (H^1(\Omega))^\star), \\ z \in L^\infty([0,T]; W^{1,p}(\Omega)) \cap H^1([0,T]; L^2(\Omega)) \end{array} \right\}.$$

Based on $\mathcal{Q}$, the set of admissible functions of the viscous problem (see Section 4) is

$$\mathcal{Q}^{\mathrm{v}} := \big\{ q = (u,c,z) \in \mathcal{Q} \,|\, c \in H^1([0,T]; L^2(\Omega)) \text{ and } u \in L^\infty([0,T]; W^{1,4}(\Omega;\mathbb{R}^n)) \big\}.$$

It will be convenient for the variational formulation to define Sobolev spaces with functions taking only non-negative and non-positive values, respectively, and Sobolev spaces consisting of functions with vanishing traces on the boundary Γ:

$$\begin{aligned} &W^{1,r}_+(\Omega) := \big\{ \zeta \in W^{1,r}(\Omega) \,\big|\, \zeta \geq 0 \text{ a.e. in } \Omega \big\}, \\ &W^{1,r}_-(\Omega) := \big\{ \zeta \in W^{1,r}(\Omega) \,\big|\, \zeta \leq 0 \text{ a.e. in } \Omega \big\}, \\ &W^{1,r}_\Gamma(\Omega;\mathbb{R}^n) := \big\{ \zeta \in W^{1,r}(\Omega;\mathbb{R}^n) \,\big|\, \zeta|_\Gamma = 0 \text{ in the sense of traces} \big\} \end{aligned}$$

for $r \in [1,\infty]$. In this context, $I_{W^{1,r}_\pm(\Omega)} : W^{1,r}(\Omega) \to \{0,\infty\}$ denote the indicator functions given by

$$I_{W^{1,r}_\pm(\Omega)}(\zeta) := \begin{cases} 0, & \text{if } \zeta \in W^{1,r}_\pm(\Omega), \\ \infty, & \text{else.} \end{cases}$$

Since Cahn-Hilliard systems can be expressed as H^{-1}-gradient flows, we introduce the following spaces in order to apply the direct method in the time-discrete version (see Section 5):

$$\begin{aligned} V_0 &:= \left\{ \zeta \in H^1(\Omega) \,\Big|\, \int_\Omega \zeta \,\mathrm{d}x = 0 \right\}, \\ \tilde{V}_0 &:= \left\{ \zeta \in (H^1(\Omega))^* \,\Big|\, \langle \zeta, \mathbf{1} \rangle_{(H^1)^* \times H^1} = 0 \right\}. \end{aligned}$$

This permits us to define the operator $(-\Delta)^{-1} : \tilde{V}_0 \to V_0$ as the inverse of the operator $-\Delta : V_0 \to \tilde{V}_0$, $u \mapsto \langle \nabla u, \nabla \cdot \rangle_{L^2(\Omega)}$. The space $\tilde{V}_0$ will be endowed with the scalar product $\langle u, v \rangle_{\tilde{V}_0} := \langle \nabla(-\Delta)^{-1}u, \nabla(-\Delta)^{-1}v \rangle_{L^2(\Omega)}$.

We end this section by introducing some notation which is frequently used for some approximation features in this paper. The expression $B_R(K)$ denotes the open neighborhood with width $R > 0$ of a subset $K \subseteq \mathbb{R}^n$. Whenever we consider the zero set of a function $\zeta \in W^{1,p}(\Omega)$ for $p > n$ abbreviated in the following by $\{\zeta = 0\}$ we mean $\{x \in \overline{\Omega} \,|\, \zeta(x) = 0\}$ by taking the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ into account. We adapt the convention that for two given functions $\zeta, \xi \in L^1([0,T]; W^{1,p}(\Omega))$ the inclusion $\{\zeta = 0\} \supseteq \{\xi = 0\}$ is an abbreviation for $\{\zeta(t) = 0\} \supseteq \{\xi(t) = 0\}$ for a.e. $t \in [0,T]$.

4 Weak formulation and existence theorems

Existence results for multi-phase Cahn-Larché systems without considering damage phase fields are shown in [Gar00] provided that the chemical energy density W_{ch} can be decomposed into $W^1_{\mathrm{ch}} + W^2_{\mathrm{ch}}$ with convex W^1_{ch} and linear growth behavior of $\partial_c W^2_{\mathrm{ch}}$ (see [Gar00, Section 3.2] for a detailed explanation). Logarithmic free energies W_{ch} are also studied in [Gar00] as well as in [Gar05b]. Further variants of Cahn-Larché systems are investigated in [CMP00], [BP05], [BCD+02] and [Gar05a].

Purely mechanical systems with rate-independent damage processes are analytically considered and reviewed for instance in [MR06] and [MRZ10]. The rate-independence enables the concept of the so-called *global energetic solutions* (see Remark 4.2 (i)) to such systems.

Coupling rate-independent systems with other (rate-dependent) processes (such as with inertial or thermal effects) may lead, however, to serious mathematical difficulties as pointed out in [Rou10].

In our situation where the Cahn-Larché system is coupled with rate-dependent damage, we will treat our model problem analytically by a regularization method that gives better regularity property for c and integrability for u in the first instance. A passage to the limit will finally give us solutions to the original problem. In doing so, the notion of a weak solution consists of variational equalities and inequalities as well as an energy estimate, inspired by the concept of energetic solutions in the framework of rate-independent systems.

4.1 Regularization

The regularization, we want to consider here, is achieved by adding the term $\varepsilon\Delta\partial_t c$ to the Cahn-Hilliard equation (referred to as viscous Cahn-Hilliard equation [BP05]) and the 4-Laplacian $\varepsilon\mathrm{div}(|\nabla u|^2\nabla u)$ to the quasi-static equilibrium equation of the model problem. The classical formulation of the regularized problem for $\varepsilon > 0$ now reads as

$$\partial_t c = \Delta(-\Delta c + \partial_c W_{\mathrm{ch}}(c) + \partial_c W_{\mathrm{el}}(e(u), c, z) + \varepsilon\partial_t c), \tag{12a}$$

$$\mathrm{div}(\sigma(e(u), c, z)) + \varepsilon\mathrm{div}(|\nabla u|^2\nabla u) = 0, \tag{12b}$$

$$0 \in \partial_z \mathcal{E}_\varepsilon(u, c, z) + \partial_{\dot z}\mathcal{R}(\partial_t z) \tag{12c}$$

with the regularized energies

$$\mathcal{E}_\varepsilon(u, c, z) := \mathcal{E}(u, c, z) + \varepsilon\int_\Omega \frac{1}{4}|\nabla u|^4\,\mathrm{d}x,$$

$$\tilde{\mathcal{E}}_\varepsilon(u, c, z) := \tilde{\mathcal{E}}(u, c, z) + \varepsilon\int_\Omega \frac{1}{4}|\nabla u|^4\,\mathrm{d}x.$$

In the following, we motivate a formulation of weak solutions of the system (12a)-(12b) admissible for curves $q = (u, c, z) \in \mathcal{Q}^{\mathrm{v}}$. For every $t \in [0, T]$, equation (12a) can be translated with the boundary conditions in a weak formulation as follows:

$$\int_\Omega (\partial_t c(t))\zeta\,\mathrm{d}x = -\int_\Omega \nabla\mu(t)\cdot\nabla\zeta\,\mathrm{d}x \tag{13}$$

for all $\zeta \in H^1(\Omega)$ and

$$\int_\Omega \mu(t)\zeta\,\mathrm{d}x = \int_\Omega \nabla c(t)\cdot\nabla\zeta + \partial_c W_{\mathrm{ch}}(c(t))\zeta + \partial_c W_{\mathrm{el}}(e(u(t)), c(t), z(t))\zeta + \varepsilon(\partial_t c(t))\zeta\,\mathrm{d}x \tag{14}$$

for all $\zeta \in H^1(\Omega)$. In the same spirit, we rewrite (12b) as

$$\int_\Omega \partial_e W_{\mathrm{el}}(e(u(t)), c(t), z(t)) : e(\zeta) + \varepsilon|\nabla u(t)|^2\nabla u(t) : \nabla\zeta\,\mathrm{d}x = 0 \tag{15}$$

for all $\zeta \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$ by using the symmetry condition

$$\partial_e W_{\mathrm{el}}(e, c, z) = (\partial_e W_{\mathrm{el}}(e, c, z))^t \quad \text{for } e \in \mathbb{R}^{n\times n}_{\mathrm{sym}},\ c, z \in \mathbb{R},$$

following from the assumptions in Section 3 (i). The differential inclusion (12c) is equivalent to

$$0 = \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(u(t), c(t), z(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)) + s(t)$$

with some $r(t) \in \partial I_{W^{1,p}_+(\Omega)}(z(t))$ and $s(t) \in \partial I_{W^{1,p}_-(\Omega)}(\partial_t z(t))$ (see (4) and (5) for the definitions of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{R}}$). This can be expressed to the following system of variational inequalities:

$$I_{W^{1,p}_-(\Omega)}(\partial_t z(t)) - \left\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \zeta - \partial_t z(t)\right\rangle \le I_{W^{1,p}_-(\Omega)}(\zeta) \quad \text{for } \zeta \in W^{1,p}(\Omega),$$

$$I_{W^{1,p}_+(\Omega)}(z(t)) + \langle r(t), \zeta - z(t)\rangle \le I_{W^{1,p}_+(\Omega)}(\zeta) \quad \text{for } \zeta \in W^{1,p}(\Omega).$$

Here, $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $(W^{1,p}(\Omega))^\star$ and $W^{1,p}(\Omega)$. This system is, in turn, equivalent to the inequality system

$$z(t) \ge 0 \text{ and } \partial_t z(t) \le 0, \tag{16a}$$

$$-\Big\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \partial_t z(t)\Big\rangle \ge 0, \tag{16b}$$

$$\Big\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \zeta\Big\rangle \ge 0 \quad \text{for } \zeta \in W^{1,p}_-(\Omega), \tag{16c}$$

$$\langle r(t), \zeta - z(t)\rangle \le 0 \quad \text{for } \zeta \in W^{1,p}_+(\Omega). \tag{16d}$$

Due to the lack of regularity of q, (16b) cannot be justified rigorously. To overcome this difficulty, we use a formal calculation originating from energetic formulations introduced in [MT99].

Proposition 4.1 (Energetic characterization) *Let $q \in \mathcal{Q}^{\mathrm{v}} \cap C^2(\Omega_T; \mathbb{R}^n \times \mathbb{R} \times \mathbb{R})$ be a smooth solution of* (13)-(15) *with* (IBC1)-(IBC8)*. Then the following two conditions are equivalent:*

(i) (16b) *with $r(t) \in \partial I_{W^{1,p}_+(\Omega)}(z(t))$ for all $t \in [0,T]$,*

(ii) for all $0 \le t_1 \le t_2 \le T$:

$$\begin{aligned}
&\mathcal{E}_\varepsilon(q(t_2)) + \int_{t_1}^{t_2} \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle\,\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla\mu|^2 + \varepsilon|\partial_t c|^2\,\mathrm{d}x\mathrm{d}s - \mathcal{E}_\varepsilon(q(t_1))\\
&\le \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{17}$$

Proof. We first show for all $t \in [0,T]$:

$$\langle r, \partial_t z(t)\rangle = 0 \text{ for all } r \in \partial I_{W^{1,p}_+(\Omega)}(z(t)). \tag{18}$$

The inequality $0 \le \langle r, \partial_t z(t)\rangle$ follows directly from (16d) by putting $\zeta = z(t) - \partial_t z(t)$. The '$\ge$' - part can be shown by an approximation argument. Applying Lemma 5.1 with $f_M = z(t)$ and $f = z(t)$ and $\zeta = -\partial_t z(t)$, we obtain a sequence $\{\zeta_M\} \subseteq W^{1,p}_+(\Omega)$ and constants $\nu_M > 0$ such that $-\zeta_M \to \partial_t z(t)$ in $W^{1,p}(\Omega)$ as $M \to \infty$ and $0 \le z(t) - \nu_M\zeta_M$ a.e. in Ω for all $M \in \mathbb{N}$. Testing (16d) with $\zeta = z(t) - \nu_M\zeta_M$ shows $\langle r, -\zeta_M\rangle \le 0$. Passing to $M \to \infty$ gives $\langle r, \partial_t z(t)\rangle \le 0$.

To $(ii) \Rightarrow (i)$: We remark that (14) and (15) can be written in the following form:

$$\int_\Omega \mu(t)\zeta_1 - \varepsilon(\partial_t c(t))\zeta_1\,\mathrm{d}x = \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q(t)), \zeta_1\rangle, \tag{19a}$$

$$\langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t)), \zeta_2\rangle = 0, \tag{19b}$$

for all $t \in [0,T]$, all $\zeta_1 \in H^1(\Omega)$ and all $\zeta_2 \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$.

Let $t_0 \in [0,T)$. It follows

$$\begin{aligned}
&\frac{\mathcal{E}_\varepsilon(q(t_0+h)) - \mathcal{E}_\varepsilon(q(t_0))}{h} + \fint_{t_0}^{t_0+h} \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle\,\mathrm{d}t + \fint_{t_0}^{t_0+h}\int_\Omega |\nabla\mu|^2 + \varepsilon|\partial_t c|^2\,\mathrm{d}x\mathrm{d}t\\
&\le \fint_{t_0}^{t_0+h}\int_\Omega \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t + \varepsilon\fint_{t_0}^{t_0+h}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Letting $h \searrow 0$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_\varepsilon(q(t_0)) + \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t_0)), \partial_t z(t_0)\rangle + \int_\Omega |\nabla\mu(t_0)|^2 + \varepsilon|\partial_t c(t_0)|^2\,\mathrm{d}x\\
&\qquad\leq \int_\Omega \partial_e W_{\mathrm{el}}(e(u(t_0)), c(t_0), z(t_0)) : e(\partial_t b(t_0))\mathrm{d}x + \varepsilon\int_\Omega |\nabla u(t_0)|^2\nabla u(t_0) : \nabla\partial_t b(t_0)\,\mathrm{d}x\\
&\qquad= \langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t b(t_0)\rangle.
\end{aligned}$$

Using the chain rule and (13)-(15) yield

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_\varepsilon(q(t_0)) &= \underbrace{\langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t u(t_0)\rangle}_{\text{apply (19b)}} + \underbrace{\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t c(t_0)\rangle}_{\text{apply (19a) and (13)}} + \langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t z(t_0)\rangle\\
&= \langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t b(t_0)\rangle + \int_\Omega -|\nabla\mu(t_0)|^2 - \varepsilon|\partial_t c(t_0)|^2\,\mathrm{d}x + \langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t z(t_0)\rangle.
\end{aligned}$$

In consequence, property (i) follows together with (18). The case $t_0 = T$ can be derived similarly by considering the difference quotient of t_0 and $t_0 - h$.

To *(i)* $\Rightarrow$ *(ii)* : This implication follows from the relation $\mathcal{E}_\varepsilon(q(t_2)) - \mathcal{E}_\varepsilon(q(t_1)) = \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_\varepsilon(q(t))\,\mathrm{d}t$ as well as the equations (13)-(15) and (18). ■

Remark 4.2

(i) *In the rate-independent case $\beta = 0$ and for convex $\mathcal{E}_\varepsilon$ with respect to z, condition* (16c) *can be characterized by a stability condition which reads as*

$$\mathcal{E}_\varepsilon(u(t), c(t), z(t)) \leq \mathcal{E}_\varepsilon(u(t), c(t), \zeta) + \mathcal{R}(\zeta - z(t)) \tag{20}$$

for all $t \in [0,T]$ and all test-functions $\zeta \in W_+^{1,p}(\Omega)$. Thereby, (17) *and* (20) *give an equivalent description of the differential inclusion* (12c) *for smooth solutions. This concept of solutions is referred to as global energetic solutions and was introduced in [MT99]. We emphasize that the damage variable z in the rate-independent case $\beta = 0$ is a function of bounded variation and is allowed to exhibit jumps. For a comprehensive introduction, we refer to [AFP00]. To tackle rate-dependent systems and non-convexity of $\mathcal{E}_\varepsilon$ with respect to z, we can not use formulation* (20) *(cf. [MRS09, MRZ10]).*

(ii) *For smooth solutions q, satisfying* (13)-(15)*, the energy inequality* (17) *and the variational inequality* (16c)*, we even obtain the following energy balance:*

$$\begin{aligned}
&\mathcal{E}_\varepsilon(q(t_2)) + \int_{t_1}^{t_2}\langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle\,\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla\mu|^2 + \varepsilon|\partial_t c|^2\,\mathrm{d}x\mathrm{d}s\\
&\qquad= \mathcal{E}_\varepsilon(q(t_1)) + \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s
\end{aligned}$$

for all $0 \leq t_1 \leq t_2 \leq T$.

This motivates the definition of a solution in the following sense:

Definition 4.3 (Weak solution - viscous problem) *A triple $q = (u, c, z) \in \mathcal{Q}^{\mathrm{v}}$ with $c(0) = c^0$, $z(0) = z^0$, $z \geq 0$ and $\partial_t z \leq 0$ a.e. in Ω_T is called a weak solution of the viscous system* (12a)-(12c) *with initial-boundary data and constraints* (IBC1)-(IBC8) *if it satisfies the following conditions:*

(i) for all $\zeta \in L^2([0,T];H^1(\Omega))$

$$\int_{\Omega_T} (\partial_t c)\zeta \,\mathrm{d}x\mathrm{d}t = -\int_{\Omega_T} \nabla\mu\cdot\nabla\zeta \,\mathrm{d}x\mathrm{d}t, \tag{21}$$

where $\mu \in L^2([0,T];H^1(\Omega))$ *satisfies for all* $\zeta \in L^2([0,T];H^1(\Omega))$

$$\int_{\Omega_T} \mu\zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla c\cdot\nabla\zeta + \partial_c W_{\mathrm{ch}}(c)\zeta + \partial_c W_{\mathrm{el}}(e(u),c,z)\zeta + \varepsilon(\partial_t c)\zeta \,\mathrm{d}x\mathrm{d}t, \tag{22}$$

(ii) for all $\zeta \in L^4([0,T];W^{1,4}_\Gamma(\Omega;\mathbb{R}^n))$

$$\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\zeta) + \varepsilon|\nabla u|^2\nabla u : \nabla\zeta \,\mathrm{d}x\mathrm{d}t = 0, \tag{23}$$

(iii) for all $\zeta \in L^p([0,T];W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$

$$0 \le \int_{\Omega_T} |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + (\partial_z W_{\mathrm{el}}(e(u),c,z) - \alpha + \beta(\partial_t z))\zeta \,\mathrm{d}x\mathrm{d}t + \int_0^T \langle r(t),\zeta(t)\rangle \,\mathrm{d}t, \tag{24}$$

where $r \in L^1(\Omega_T) \subset L^1\big([0,T];(W^{1,p}(\Omega))^*\big)$ *satisfies for all* $\zeta \in W^{1,p}_+(\Omega)$ *and for a.e.* $t \in [0,T]$

$$\langle r(t),\zeta - z(t)\rangle \le 0, \tag{25}$$

(iv) for a.e. $0 \le t_1 \le t_2 \le T$

$$\begin{aligned}\mathcal{E}_\varepsilon(q(t_2)) &+ \int_\Omega \alpha(z(t_1)-z(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z|^2 \,\mathrm{d}x\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla\mu|^2 + \varepsilon|\partial_t c|^2 \,\mathrm{d}x\mathrm{d}s \\ &\le \mathcal{E}_\varepsilon(q(t_1)) + \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b \,\mathrm{d}x\mathrm{d}s. \end{aligned}\tag{26}$$

Theorem 4.4 (Existence theorem - viscous problem) *Let the assumptions in Section 3 be satisfied and let* $c^0 \in H^1(\Omega)$*,* $z^0 \in W^{1,p}(\Omega)$ *with* $0 \le z^0 \le 1$ *a.e. in* Ω *and a viscosity factor* $\varepsilon \in (0,1]$ *be given. Then there exists a weak solution* $q \in \mathcal{Q}^{\mathrm{v}}$ *of the viscous system* (12a)-(12c) *in the sense of Definition 4.3. In addition:*

$$r = -\chi_{\{z=0\}}[\partial_z W_{\mathrm{el}}(e(u),c,z)]^+, \tag{27}$$

where $[\cdot]^+$ *is defined by* $\max\{0,\cdot\}$.

4.2 Limit problem

Our main aim in this work is to establish an existence result for the system (12a)-(12c) with vanishing ε-terms, i.e. with $\varepsilon = 0$. In the same fashion as in Section 4.1 we introduce a weak notion of (6a)-(6c) as follows.

Definition 4.5 (Weak solution - limit problem) *A triple* $q = (u,c,z) \in \mathcal{Q}$ *with* $z(0) = z^0$*,* $z \ge 0$ *and* $\partial_t z \le 0$ *a.e. in* Ω_T *is called a weak solution of the system* (6a)-(6c) *with boundary and initial conditions* (IBC1)-(IBC8) *if it satisfies the following conditions:*

(i) for all $\zeta \in L^2([0,T];H^1(\Omega))$ *with* $\partial_t\zeta \in L^2(\Omega_T)$ *and* $\zeta(T) = 0$

$$\int_{\Omega_T} (c-c^0)\partial_t\zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla\mu\cdot\nabla\zeta \,\mathrm{d}x\mathrm{d}t,$$

where $\mu \in L^2([0,T]; H^1(\Omega))$ *satisfies for all* $\zeta \in L^2([0,T]; H^1(\Omega))$

$$\int_{\Omega_T} \mu\zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla c \cdot \nabla\zeta + \partial_c W_{\mathrm{ch}}(c)\zeta + \partial_c W_{\mathrm{el}}(e(u), c, z)\zeta \,\mathrm{d}x\mathrm{d}t,$$

(ii) for all $\zeta \in L^2([0,T]; H^1_\Gamma(\Omega; \mathbb{R}^n))$

$$\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\zeta) \,\mathrm{d}x\mathrm{d}t = 0,$$

(iii) for all $\zeta \in L^p([0,T]; W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$

$$0 \leq \int_{\Omega_T} |\nabla z|^{p-2}\nabla z \cdot \nabla\zeta + \partial_z W_{\mathrm{el}}(e(u), c, z)\zeta - \alpha\zeta + \beta(\partial_t z)\zeta \,\mathrm{d}x\mathrm{d}t + \int_0^T \langle r(t), \zeta(t)\rangle \,\mathrm{d}t,$$

where $r \in L^1(\Omega_T)$ *satisfies for all* $\zeta \in W^{1,p}_+(\Omega)$ *and for a.e.* $t \in [0,T]$

$$\langle r(t), \zeta - z(t)\rangle \leq 0,$$

(iv) for a.e. $0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned}\mathcal{E}(q(t_2)) + \int_\Omega \alpha(z(t_1) - z(t_2)) \,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z|^2 \,\mathrm{d}x\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla\mu|^2 \,\mathrm{d}x\mathrm{d}s \\ \leq \mathcal{E}(q(t_1)) + \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\partial_t b)\mathrm{d}x \,\mathrm{d}s.\end{aligned}$$

Theorem 4.6 (Existence theorem - limit problem) *Let the assumptions in Section 3 be satisfied and let* $c^0 \in H^1(\Omega)$*,* $z^0 \in W^{1,p}(\Omega)$ *with* $0 \leq z^0 \leq 1$ *a.e. in* Ω *be given. Then there exists a weak solution* $q \in \mathcal{Q}$ *of the system* (6a)-(6c) *in the sense of Definition 4.5.*

5 Proof of the existence theorems

5.1 Preliminaries

The proof of Theorem 4.4 is based on recursive functional minimization that comes from an implicit Euler scheme of the system (12a)-(12c) with respect to the time variable. To obtain from the time-discrete model the time-continuous model (12a)-(12c), we need some preliminary results on approximation schemes for test-functions, which will be presented in this section.

Lemma 5.1 (Approximation of test-functions) *Let* $p > n$ *and* $f, \zeta \in W^{1,p}_+(\Omega)$ *with* $\{\zeta = 0\} \supseteq \{f = 0\}$*. Furthermore, let* $\{f_M\}_{M\in\mathbb{N}} \subseteq W^{1,p}_+(\Omega)$ *be a sequence with* $f_M \rightharpoonup f$ *in* $W^{1,p}(\Omega)$ *as* $M \to \infty$*. Then, there exist a sequence* $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq W^{1,p}_+(\Omega)$ *and constants* $\nu_M > 0$*,* $M \in \mathbb{N}$*, such that*

(i) $\zeta_M \to \zeta$ *in* $W^{1,p}(\Omega)$ *as* $M \to \infty$*,*

(ii) $\zeta_M \leq \zeta$ *a.e. in* Ω *for all* $M \in \mathbb{N}$*,*

(iii) $\nu_M\zeta_M \leq f_M$ *a.e. in* Ω *for all* $M \in \mathbb{N}$*.*

Proof. Without loss of generality we may assume $\zeta \not\equiv 0$ on $\overline{\Omega}$.

Let $\{\delta_k\}$ be a sequence with $\delta_k \searrow 0$ as $k \to \infty$ and $\delta_k > 0$. Define for every $k \in \mathbb{N}$ the approximation function $\tilde{\zeta}_k \in W^{1,p}_+(\Omega)$ as

$$\tilde{\zeta}_k := [\zeta - \delta_k]^+,$$

where $[\cdot]^+$ stands for $\max\{0, \cdot\}$. Let $0 < \alpha < 1 - \frac{n}{p}$ be a fixed constant. Then $\tilde{\zeta}_k \in C^{0,\alpha}(\overline{\Omega})$ due to $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$. Furthermore, set the constant R_k, $k \in \mathbb{N}$, to

$$R_k := \left(\delta_k / \|\zeta\|_{C^{0,\alpha}(\overline{\Omega})}\right)^{1/\alpha} > 0.$$

It follows $\{\tilde{\zeta}_k = 0\} \supseteq \overline{\Omega} \cap B_{R_k}(\{\zeta = 0\}) \supseteq \overline{\Omega} \cap B_{R_k}(\{f = 0\})$. Without loss of generality we may assume $\overline{\Omega} \setminus B_{R_k}(\{f = 0\}) \neq \emptyset$ for all $k \in \mathbb{N}$. Furthermore, there exists a strictly increasing sequence $\{M_k\} \subseteq \mathbb{N}$ such that we find for all $k \in \mathbb{N}$:

$$f_M \geq \eta_k/2 \text{ a.e. on } \overline{\Omega} \setminus B_{R_k}(\{f = 0\}) \text{ for all } M \geq M_k$$

with $\eta_k := \inf\{f(x) \,|\, x \in \overline{\Omega} \setminus B_{R_k}(\{f = 0\})\} > 0$, $k \in \mathbb{N}$, (note that $f_M \to f$ in $C^{0,\alpha}(\overline{\Omega})$ as $M \to \infty$). This implies $\tilde{\nu}_k \tilde{\zeta}_k \leq f_M$ a.e. on $\overline{\Omega}$ for all $M \geq M_k$ by setting $\tilde{\nu}_k := \eta_k / (2\|\zeta\|_{L^\infty(\Omega)}) > 0$. The claim follows with $\zeta_M := 0$ and $\nu_k := 1$ for $M \in \{1, \ldots, M_1 - 1\}$ and $\zeta_M := \tilde{\zeta}_{\delta_k}$ and $\nu_M := \tilde{\nu}_k$ for each $M \in \{M_k, \ldots, M_{k+1} - 1\}$, $k \in \mathbb{N}$. ■

Lemma 5.2 (Approximation of time-dependent test-functions) *Let $p > n$, $q \geq 1$ and $f, \zeta \in L^q([0,T]; W^{1,p}_+(\Omega))$ with $\{\zeta = 0\} \supseteq \{f = 0\}$. Furthermore, let $\{f_M\}_{M \in \mathbb{N}} \subseteq L^q([0,T]; W^{1,p}_+(\Omega))$ be a sequence with $f_M(t) \rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ as $M \to \infty$ for a.e. $t \in [0,T]$. Then, there exist a sequence $\{\zeta_M\}_{M \in \mathbb{N}} \subseteq L^q([0,T]; W^{1,p}_+(\Omega))$ and constants $\nu_{M,t} > 0$ such that*

(i) $\zeta_M \to \zeta$ in $L^q([0,T]; W^{1,p}(\Omega))$ as $M \to \infty$,

(ii) $\zeta_M \leq \zeta$ a.e. in Ω_T for all $M \in \mathbb{N}$ (in particular $\{\zeta_M = 0\} \supseteq \{\zeta = 0\}$),

(iii) $\nu_{M,t}\zeta_M(t) \leq f_M(t)$ a.e. in Ω for a.e. $t \in [0,T]$ and for all $M \in \mathbb{N}$.

If, in addition, $\zeta \leq f$ a.e. in Ω_T then condition (iii) can be refined to

(iii)' $\zeta_M \leq f_M$ a.e. in Ω_T for all $M \in \mathbb{N}$.

Proof. Let $\{\delta_k\}$ with $\delta_k \searrow 0$ as $k \to \infty$ and $\delta_k > 0$ be a sequence and $0 < \alpha < 1 - \frac{n}{p}$ be a fixed constant. We construct the approximation functions $\zeta_M \in L^q([0,T]; W^{1,p}_+(\Omega))$, $M \in \mathbb{N}$, as follows:

$$\zeta_M(t) := \sum_{k=1}^{M} \chi_{A^k_M}(t)[\zeta(t) - \delta_k]^+, \tag{28}$$

where $\chi_{A^k_M} : [0,T] \to \{0,1\}$ is defined as the characteristic function of the measurable set A^k_M given by

$$A^k_M := \begin{cases} P^k_M \setminus \left(\bigcup_{i=k+1}^{M} P^i_M\right) & \text{if } k < M, \\ P^M_M & \text{if } k = M, \end{cases}$$

with

$$P^k_M := \Big\{t \in [0,T] \,\Big|\, \overline{\Omega} \setminus B_{R_k(t)}(\{f(t) = 0\}) \neq \emptyset$$

$$\text{and } f_M(t) \geq \eta_k(t)/2 \text{ a.e. on } \overline{\Omega} \setminus B_{R_k(t)}(\{f(t)=0\})\Big\}, \tag{29}$$

where the functions $R_k, \eta_k : [0,T] \to \mathbb{R}^+$ are defined by

$$R_k(t) = \Big(\delta_k / \|\zeta(t)\|_{C^{0,\alpha}(\overline{\Omega})}\Big)^{1/\alpha},$$
$$\eta_k(t) = \inf\{f(t,x) \,|\, x \in \overline{\Omega} \setminus B_{R_k(t)}(\{f(t)=0\})\}.$$

Here, we use the convention $R_k(t) := \infty$ for $\zeta(t) \equiv 0$. Note that A_M^k, $1 \leq k \leq M$, are pairwise disjoint by construction.

Consider a $t \in [0,T]$ with $f_M(t) \rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ and $\zeta(t) \not\equiv 0$ with $\{\zeta(t)=0\} \supseteq \{f(t)=0\}$. Let $K \in \mathbb{N}$ be arbitrary but large enough such that $\overline{\Omega} \setminus B_{R_K(t)}(\{f(t)=0\}) \neq \emptyset$ holds. It follows the existence of an $\tilde{M} \geq K$ with $t \in P_M^K$ for all $M \geq \tilde{M}$. Therefore, for each $M \geq \tilde{M}$ exists a $k \geq K$ such that $t \in A_M^k$, i.e. $\zeta_M(t) = [\zeta(t) - \delta_k]^+$. Thus $\zeta_M(t) \to \zeta(t)$ in $W^{1,p}(\Omega)$ as $K \to \infty$. Lebesgue's convergence theorem shows (i). Property (ii) follows immediately from (28). It remains to show (iii). Let $M \in \mathbb{N}$ be arbitrary. If $\zeta_M(t) \equiv 0$ we set $\nu_{M,t} = 1$. Otherwise we find a unique $1 \leq k \leq M$ with $t \in A_M^k$ and $\zeta_M(t) = [\zeta(t) - \delta_k]^+$. This, in turn, implies the existence of a $\nu_{M,t} > 0$ with $\nu_{M,t}\zeta_M \leq f_M$ (see proof of Lemma 5.1).

In the case $\zeta \leq f$, we use instead of (29) the set:

$$P_M^k := \Big\{t \in [0,T] \,\Big|\, \|f_M(t) - f(t)\|_{C^0(\overline{\Omega})} \leq \delta_k\Big\}.$$

With a similar argumentation, $\{\zeta_M\}$ fulfills (i), (ii) and (iii)'. ■

Lemma 5.3 *Let $f \in L^p(\Omega;\mathbb{R}^n)$, $g \in L^p(\Omega)$ and $z \in W^{1,p}(\Omega)$ with $f \cdot \nabla z \geq 0$ and $\{f=0\} \supseteq \{z=0\}$ a.e.. Furthermore, we assume that*

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \geq 0 \quad \text{for all } \zeta \in W_-^{1,p}(\Omega) \text{ with } \{\zeta=0\} \supseteq \{z=0\}.$$

Then

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \geq \int_{\{z=0\}} [g]^+\zeta \,\mathrm{d}x \quad \text{for all } \zeta \in W_-^{1,p}(\Omega).$$

Proof. We assume $z \not\equiv 0$ on Ω. Let $\zeta \in W_-^{1,p}(\Omega)$ be a test-function. For $\delta > 0$ small enough such that $\overline{\Omega} \setminus B_\delta(\{z=0\}) \neq \emptyset$, we define

$$\zeta_\delta := \max\big\{\zeta, -z\|\zeta\|_{L^\infty} C_\delta^{-1}\big\}$$

with the constant

$$C_\delta := \inf\big\{z(x) \,|\, x \in \overline{\Omega} \setminus B_\delta(\{z=0\})\big\} > 0.$$

We consider the following partition of $\overline{\Omega}$:

$$\overline{\Omega} = \Sigma_1 \cup \Sigma_2^{\leq} \cup \Sigma_2^{>}$$

with

$$\Sigma_1 := \overline{\Omega} \setminus B_\delta(\{z=0\}),$$
$$\Sigma_2^{\leq} := \overline{\Omega} \cap B_\delta(\{z=0\}) \cap \{\zeta \leq -z\|\zeta\|_{L^\infty} C_\delta^{-1}\},$$
$$\Sigma_2^{>} := \overline{\Omega} \cap B_\delta(\{z=0\}) \cap \{\zeta > -z\|\zeta\|_{L^\infty} C_\delta^{-1}\}.$$

By construction, the sequence $\{\zeta_\delta\}_{\delta\in(0,1]}$ satisfies

$$\zeta_\delta(x) = \begin{cases} \zeta(x), & \text{if } x \in \Sigma_1 \cup \Sigma_2^{>}, \\ -z(x)\|\zeta\|_{L^\infty} C_\delta^{-1}, & \text{if } x \in \Sigma_2^{\leq}. \end{cases}$$

In particular, $\zeta_\delta = 0$ on $\{z = 0\}$ for every $\delta \in (0,1]$ and $\zeta_\delta \overset{\star}{\rightharpoonup} \zeta$ in $L^\infty(\{z > 0\})$ as $\delta \searrow 0$. By using the assumptions, we estimate

$$\begin{aligned}
&\int_\Omega f\cdot\nabla\zeta + g\zeta \,\mathrm{d}x - \int_{\{z=0\}} [g]^+\zeta\,\mathrm{d}x \\
&\quad= \int_\Omega f\cdot\nabla(\zeta-\zeta_\delta) + g(\zeta-\zeta_\delta)\,\mathrm{d}x - \int_{\{z=0\}}[g]^+\zeta\,\mathrm{d}x + \underbrace{\int_\Omega f\cdot\nabla\zeta_\delta + g\zeta_\delta\,\mathrm{d}x}_{\geq 0} \\
&\quad\geq \int_\Omega f\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad= \underbrace{\int_{\Sigma_1} f\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x}_{=0} + \int_{\Sigma_2^{\leq}} f\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x + \underbrace{\int_{\Sigma_2^{>}} f\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x}_{=0} \\
&\qquad + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad= \|\zeta\|_{L^\infty} C_\delta^{-1}\int_{\Sigma_2^{\leq}} \underbrace{f\cdot\nabla z}_{\geq 0}\,\mathrm{d}x + \underbrace{\int_{\Sigma_2^{\leq}} f\cdot\nabla\zeta\,\mathrm{d}x}_{=\int_{\Sigma_2^{\leq}\setminus\{z=0\}} f\cdot\nabla\zeta\,\mathrm{d}x} + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad\geq \int_{\Sigma_2^{\leq}\setminus\{z=0\}} f\cdot\nabla\zeta\,\mathrm{d}x + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x.
\end{aligned}$$

The terms on the right hand side converge to 0 as $\delta \searrow 0$. ■

5.2 Viscous case

This section is aimed to prove Theorem 4.4. The initial displacement u_ε^0 is chosen to be a minimizer of the functional $u \mapsto \mathcal{E}_\varepsilon(u, c^0, z^0)$ defined on the space $W^{1,4}(\Omega)$ with the constraint $u|_\Gamma = b(0)|_\Gamma$ (the existence proof is based on direct methods in the calculus of variations - see the proof of Lemma 5.4 below). We now apply an implicit Euler scheme of the system (12a)-(12c). The discretization fineness is given by $\tau := \frac{T}{M}$, where $M \in \mathbb{N}$. We set $q_{M,\varepsilon}^0 := (u_{M,\varepsilon}^0, c_{M,\varepsilon}^0, z_{M,\varepsilon}^0) := (u_\varepsilon^0, c^0, z^0)$ and construct $q_{M,\varepsilon}^m$ for $m \in \{1,\dots,M\}$ recursively by considering the functional

$$\mathbb{E}_{M,\varepsilon}^m(u,c,z) := \tilde{\mathcal{E}}_\varepsilon(u,c,z) + \tilde{\mathcal{R}}\left(\frac{z - z_{M,\varepsilon}^{m-1}}{\tau}\right)\tau + \frac{1}{2\tau}\|c - c_{M,\varepsilon}^{m-1}\|_{\tilde{V}_0}^2 + \frac{\varepsilon}{2\tau}\|c - c_{M,\varepsilon}^{m-1}\|_{L^2(\Omega)}^2.$$

The set of admissible states for $\mathbb{E}_{M,\varepsilon}^m$ is

$$\begin{aligned}
\mathcal{Q}_{M,\varepsilon}^m := \Big\{& q = (u,c,z) \in W^{1,4}(\Omega;\mathbb{R}^n) \times H^1(\Omega) \times W^{1,p}(\Omega) \\
&\text{with } u|_\Gamma = b(m\tau)|_\Gamma,\ \int_\Omega c - c^0\,\mathrm{d}x = 0 \text{ and } 0 \leq z \leq z_{M,\varepsilon}^{m-1} \text{ a.e. in } \Omega\Big\}.
\end{aligned}$$

A minimization problem for the functional $\mathbb{E}^m_{M,\varepsilon}(u,c,z)=\mathbb{E}^m_{M,\varepsilon}(u,c)=\int_\Omega \frac{1}{2}|\nabla c|^2+W_{\rm ch}(c)+W_{\rm el}(e(u),c)\,\mathrm{d}x+\frac{1}{2\tau}\|c-c^{m-1}_{M,\varepsilon}\|^2_{\rm L}$ containing a weighted $(H^1(\Omega,\mathbb{R}^n))^\star$-scalar product $\langle\cdot,\cdot\rangle_{\rm L}$ has been considered in [Gar00]. However, due to the additional internal variable z, the passage to $M\to\infty$ becomes much more involved.

In the following, we will omit the ε-dependence in the notation since $\varepsilon\in(0,1]$ is fixed until Section 5.3.

Lemma 5.4 *The functional $\mathbb{E}^m_M$ has a minimizer $q^m_M=(u^m_M,c^m_M,z^m_M)\in\mathcal{Q}^m_M$.*

Proof. The existence is shown by direct methods in the calculus of variations. We can immediately see that $\mathcal{Q}^m_M$ is closed with respect to the weak topology in $W^{1,4}(\Omega;\mathbb{R}^n)\times H^1(\Omega)\times W^{1,p}(\Omega)$. Furthermore, we need to show coercivity and sequentially weakly lower semi-continuity of $\mathbb{E}^m_M$ defined on $\mathcal{Q}^m_M$.

(i) *Coercivity.* We have the estimate

$$\mathbb{E}^m_M(q)\geq\frac{1}{2}\|\nabla c\|^2_{L^2(\Omega)}+\frac{1}{p}\|\nabla z\|^p_{L^p(\Omega)}+\frac{\varepsilon}{4}\|\nabla u\|^4_{L^4(\Omega)}.$$

Therefore, given a sequence $\{q_k\}_{k\in\mathbb{N}}$ in $\mathcal{Q}^m_M$ with the boundedness property $\mathbb{E}^m_M(q_k)<C$ for all $k\in\mathbb{N}$, we obtain the boundedness of u_k in $W^{1,4}(\Omega)$ by Poincaré's inequality (u_k has fixed boundary data on Γ), the boundedness of c_k in $H^1(\Omega)$ by Poincaré's inequality ($\int_\Omega c_k\,\mathrm{d}x$ is conserved) and the boundedness of z_k in $W^{1,p}(\Omega)$ by also considering the restriction $0\leq z_k\leq 1$ a.e. in Ω.

(ii) *Sequentially weakly lower semi-continuity.* All terms in $\mathbb{E}^m_M$ except $\int_\Omega W_{\rm ch}(c)\,\mathrm{d}x$ and $\int_\Omega W_{\rm el}(e(u),c,z)\,\mathrm{d}x$ are convex and continuous and therefore sequentially weakly l.s.c.. Now let $(u_k,c_k,z_k)\rightharpoonup(u,c,z)$ be a weakly converging sequence in $\mathcal{Q}^m_M$. In particular, $z_k\to z$ in $L^p(\Omega)$, $z_k\to z$ a.e. in Ω and $c_k\to c$ in $L^r(\Omega)$ as $k\to\infty$ for all $1\leq r<2^\star$ and $c_k\to c$ a.e. in Ω for a subsequence. Lebesgue's generalized convergence theorem yields $\int_\Omega W_{\rm ch}(c_k)\,\mathrm{d}x\to\int_\Omega W_{\rm ch}(c)\,\mathrm{d}x$ using (A6). The remaining term can be treated by employing the uniform convexity of $W_{\rm el}(\cdot,c,z)$ (see (A1)):

$$\begin{aligned}&\int_\Omega W_{\rm el}(e(u_k),c_k,z_k)-W_{\rm el}(e(u),c,z)\,\mathrm{d}x\\&\quad=\int_\Omega W_{\rm el}(e(u),c_k,z_k)-W_{\rm el}(e(u),c,z)\,\mathrm{d}x+\int_\Omega W_{\rm el}(e(u_k),c_k,z_k)-W_{\rm el}(e(u),c_k,z_k)\,\mathrm{d}x\\&\quad\geq\underbrace{\int_\Omega W_{\rm el}(e(u),c_k,z_k)-W_{\rm el}(e(u),c,z)\,\mathrm{d}x}_{\to 0\text{ by Lebesgue's gen. conv. theorem and (A2)}}+\int_\Omega\partial_e W_{\rm el}(e(u),c_k,z_k)(e(u_k)-e(u))\,\mathrm{d}x.\end{aligned}$$

The second term also converges to 0 because of $\partial_e W_{\rm el}(e(u),c_k,z_k)\to\partial_e W_{\rm el}(e(u),c,z)$ in $L^2(\Omega)$ (by Lebesgue's generalized convergence theorem and (11a)) and $e(u_k)-e(u)\rightharpoonup 0$ in $L^2(\Omega)$.

Thus there exists $q^m_M=(u^m_M,c^m_M,z^m_M)\in\mathcal{Q}^m_M$ such that $\mathbb{E}^m_M(q^m_M)=\inf_{q\in\mathcal{Q}^m_M}\mathbb{E}^m_M(q)$. ■

The minimizers q^m_M for $m\in\{0,\ldots,M\}$ are used to construct approximate solutions q_M and $\hat{q}_M$ to our viscous problem by a piecewise constant and linear interpolation in time, respectively. More precisely,

$$\begin{aligned}q_M(t)&:=q^m_M,\\\hat{q}_M(t)&:=\beta q^m_M+(1-\beta)q^{m-1}_M\end{aligned}$$

with $t\in((m-1)\tau,m\tau]$ and $\beta=\frac{t-(m-1)\tau}{\tau}$. The retarded function q^-_M is set to

$$q^-_M(t):=\begin{cases}q_M(t-\tau), & \text{if } t\in[\tau,T],\\ q^0_\varepsilon, & \text{if } t\in[0,\tau).\end{cases}$$

The functions b_M and b_M^- are analogously defined adapting the notation $b_M^m := b(m\tau)$. Furthermore, the discrete chemical potential is given by (note that $\partial_t \hat{c}_M(t) \in V_0$)

$$\mu_M(t) := -(-\Delta)^{-1} (\partial_t \hat{c}_M(t)) + \lambda_M(t) \tag{30}$$

with the Lagrange multiplier λ_M originating from mass conservation:

$$\lambda_M(t) := \fint_\Omega \partial_c W_{\mathrm{ch}}(c_M(t)) + \partial_c W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\,\mathrm{d}x. \tag{31}$$

The discretization of the time variable t will be expressed by the functions

$$\begin{aligned} d_M(t) &:= \min\{m\tau \,|\, m \in \mathbb{N}_0 \text{ and } m\tau \geq t\}, \\ d_M^-(t) &:= \min\{(m-1)\tau \,|\, m \in \mathbb{N}_0 \text{ and } m\tau \geq t\}. \end{aligned}$$

The following lemma clarifies why the functions q_M, q_M^- and $\hat{q}_M$ are approximate solutions to our problem.

Lemma 5.5 (Euler-Lagrange equations and energy estimate) *The tuples q_M, q_M^- and $\hat{q}_M$ satisfy the following properties:*

(i) for all $t \in (0,T)$ and all $\zeta \in H^1(\Omega)$

$$\int_\Omega (\partial_t \hat{c}_M(t))\zeta\,\mathrm{d}x = -\int_\Omega \nabla \mu_M(t) \cdot \nabla \zeta\,\mathrm{d}x, \tag{32}$$

(ii) for all $t \in (0,T)$ and all $\zeta \in H^1(\Omega)$

$$\begin{aligned} \int_\Omega \mu_M(t)\zeta\,\mathrm{d}x = &\int_\Omega \nabla c_M(t) \cdot \nabla \zeta + \partial_c W_{\mathrm{ch}}(c_M(t))\zeta\,\mathrm{d}x \\ &+ \int_\Omega \partial_c W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\zeta + \varepsilon(\partial_t \hat{c}_M(t))\zeta\,\mathrm{d}x, \end{aligned} \tag{33}$$

(iii) for all $t \in [0,T]$ and for all $\zeta \in W_\Gamma^{1,4}(\Omega;\mathbb{R}^n)$

$$0 = \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t)) : e(\zeta) + \varepsilon |\nabla u_M(t)|^2 \nabla u_M(t) : \nabla \zeta\,\mathrm{d}x, \tag{34}$$

(iv) for all $t \in (0,T)$ and all $\zeta \in W^{1,p}(\Omega)$ such that there exists a constant $\nu > 0$ with $0 \leq \nu\zeta + z_M(t) \leq z_M^-(t)$ a.e. in Ω

$$0 \leq \int_\Omega |\nabla z_M(t)|^{p-2} \nabla z_M(t) \cdot \nabla \zeta + \partial_z W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\zeta - \alpha\zeta + \beta(\partial_t \hat{z}_M(t))\zeta\,\mathrm{d}x, \tag{35}$$

(v) for all $t \in [0,T]$

$$\begin{aligned} &\mathcal{E}_\varepsilon(q_M(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t \hat{z}_M)\,\mathrm{d}s + \int_0^{d_M(t)} \int_\Omega \frac{\varepsilon}{2}|\partial_t \hat{c}_M|^2 + \frac{1}{2}|\nabla \mu_M|^2\,\mathrm{d}x\mathrm{d}s \\ &\quad \leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + \int_0^{d_M(t)} \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\ &\qquad + \varepsilon \int_0^{d_M(t)} \int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) : \nabla \partial_t b\,\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{36}$$

Proof. Using Lebesgue's generalized convergence theorem, the mean value theorem of differentiability and growth conditions (11a), (A4)-(A6), we obtain the variational derivatives of $\tilde{\mathcal{E}}_\varepsilon$ with respect to u, c and z:

$$\langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q),\zeta\rangle = \int_\Omega \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\zeta) + \varepsilon|\nabla u|^2\nabla u : \nabla\zeta \,\mathrm{d}x \text{ for } \zeta\in W^{1,4}(\Omega;\mathbb{R}^n), \tag{37a}$$

$$\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q),\zeta\rangle = \int_\Omega \nabla c\cdot\nabla\zeta + \partial_c W_{\mathrm{ch}}(c)\zeta + \partial_c W_{\mathrm{el}}(e(u),c,z)\zeta \,\mathrm{d}x \text{ for } \zeta\in H^1(\Omega), \tag{37b}$$

$$\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q),\zeta\rangle = \int_\Omega |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + \partial_z W_{\mathrm{el}}(e(u),c,z)\zeta \,\mathrm{d}x \text{ for } \zeta\in W^{1,p}(\Omega). \tag{37c}$$

To (i)-(v):

(i) This follows from (30).

(ii) q_M^m fulfills $\langle \mathrm{d}_c\mathbb{E}_M^m(q_M^m),\zeta_1\rangle = 0$ for all $\zeta_1\in V_0$ and all $m\in\{1,\ldots,M\}$. Therefore,

$$0 = \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta_1\rangle + \langle\partial_t\hat{c}_M(t),\zeta_1\rangle_{\tilde{V}_0} + \varepsilon\langle\partial_t\hat{c}_M(t),\zeta_1\rangle_{L^2(\Omega)}.$$

On the one hand, definition (30) implies

$$\begin{aligned}\langle\partial_t\hat{c}_M(t),\zeta_1\rangle_{\tilde{V}_0} &= \langle(-\Delta)^{-1}\left(\partial_t\hat{c}_M(t)\right),\zeta_1\rangle_{L^2(\Omega)}\\ &= \langle -\mu_M(t)+\lambda_M(t),\zeta_1\rangle_{L^2(\Omega)}\\ &= -\langle\mu_M(t),\zeta_1\rangle_{L^2(\Omega)}\end{aligned}$$

and consequently

$$0 = \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta_1\rangle - \langle\mu_M(t),\zeta_1\rangle_{L^2(\Omega)} + \varepsilon\langle\partial_t\hat{c}_M(t),\zeta_1\rangle_{L^2(\Omega)} \quad \text{for all } \zeta_1\in V_0. \tag{38}$$

On the other hand, definitions (30) and (31) yield for $\zeta_2\equiv\tilde{C}$ with constant $\tilde{C}\in\mathbb{R}$:

$$\begin{aligned}&\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta_2\rangle - \langle\mu_M(t),\zeta_2\rangle_{L^2(\Omega)} + \varepsilon\langle\partial_t\hat{c}_M(t),\zeta_2\rangle_{L^2(\Omega)}\\ &\quad = \tilde{C}\mathcal{L}^n(\Omega)\lambda_M(t) + \underbrace{\langle(-\Delta)^{-1}\left(\partial_t\hat{c}_M(t)\right),\zeta_2\rangle_{L^2(\Omega)}}_{=0} - \underbrace{\langle\lambda_M(t),\zeta_2\rangle_{L^2(\Omega)}}_{\tilde{C}\mathcal{L}^n(\Omega)\lambda_M(t)} + 0\\ &\quad = 0.\end{aligned} \tag{39}$$

Setting $\zeta_1 = \zeta - \fint\zeta$ and $\zeta_2 = \fint\zeta$, inserting (37b) into (38) and (39), and adding (38) to (39) shows finally (ii) (cf. [Gar00, Lemma 3.2]).

(iii) This property follows from (37a) and $0 = \langle \mathrm{d}_u\mathbb{E}_M^m(q_M^m),\zeta\rangle = \langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q_M^m),\zeta\rangle$ for all $\zeta\in W_\Gamma^{1,4}(\Omega;\mathbb{R}^n)$.

(iv) By construction, z_M^m minimizes $\mathbb{E}_M^m(u_M^m,c_M^m,\cdot)$ in the space $W^{1,p}(\Omega)$ with the constraints $0\le z$ and $z-z_M^{m-1}\le 0$ a.e. in Ω. This implies

$$-\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q_M^m),\zeta-z_M^m\rangle - \left\langle \mathrm{d}_{\dot{z}}\tilde{\mathcal{R}}\left(\frac{z_M^m-z_M^{m-1}}{\tau}\right),\zeta-z_M^m\right\rangle_{L^2(\Omega)} \le 0 \tag{40}$$

for all $\zeta\in W^{1,p}(\Omega)$ with $0\le\zeta\le z_M^{m-1}$ a.e. in Ω. Now, let the functions $\zeta\in W^{1,p}(\Omega)$ and $\nu>0$ with $0\le\nu\zeta+z_M(t)\le z_M^-(t)$ a.e. in Ω be given. Since $\nu>0$, we obtain from (40):

$$-\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta(t)\rangle - \langle \mathrm{d}_{\dot{z}}\tilde{\mathcal{R}}\left(\partial_t\hat{z}_M(t)\right),\zeta(t)\rangle_{L^2(\Omega)} \le 0.$$

This and (37c) gives (iv).

(v) Testing $\mathbb{E}_M^m$ with $q = (u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1})$ and using the chain rule yields:

$$\begin{aligned}
&\mathcal{E}_\varepsilon(q_M^m) + \mathcal{R}\left(\frac{z_M^m - z_M^{m-1}}{\tau}\right)\tau + \frac{1}{2\tau}\|c_M^m - c_M^{m-1}\|_{\tilde{V}_0}^2 + \frac{\varepsilon}{2\tau}\|c_M^m - c_M^{m-1}\|_{L^2(\Omega)}^2 \\
&\quad \le \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1}) \\
&\quad = \mathcal{E}_\varepsilon(q_M^{m-1}) + \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1}) - \mathcal{E}_\varepsilon(q_M^{m-1}) \\
&\quad = \mathcal{E}_\varepsilon(q_M^{m-1}) + \int_{(m-1)\tau}^{m\tau} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\varepsilon(u_M^{m-1} + b(s) - b_M^{m-1}, c_M^{m-1}, z_M^{m-1})\,\mathrm{d}s \\
&\quad = \mathcal{E}_\varepsilon(q_M^{m-1}) \\
&\qquad + \int_{(m-1)\tau}^{m\tau}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^{m-1} + b(s) - b_M^{m-1}), c_M^{m-1}, z_M^{m-1}) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\
&\qquad + \varepsilon\int_{(m-1)\tau}^{m\tau}\int_\Omega |\nabla u_M^{m-1} + \nabla b(s) - \nabla b_M^{m-1}|^2\nabla(u_M^{m-1} + b(s) - b_M^{m-1}) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Summing this inequality for $k = 1, \ldots, m$ one gets:

$$\begin{aligned}
&\mathcal{E}_\varepsilon(q_M^m) + \sum_{k=1}^m \tau\left(\mathcal{R}\left(\frac{z_M^k - z_M^{k-1}}{\tau}\right) + \frac{1}{2}\left\|\frac{c_M^k - c_M^{k-1}}{\tau}\right\|_{\tilde{V}_0}^2 + \frac{\varepsilon}{2}\left\|\frac{c_M^k - c_M^{k-1}}{\tau}\right\|_{L^2(\Omega)}^2\right) \\
&\quad \le \mathcal{E}_\varepsilon(q_\varepsilon^0) + \int_0^{m\tau}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\
&\qquad + \varepsilon\int_0^{m\tau}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Because of $\left\|\frac{c_M^k - c_M^{k-1}}{\tau}\right\|_{\tilde{V}_0}^2 = \|\nabla\mu_M^k\|_{L^2(\Omega)}^2$ by (30), above estimate shows (v). ■

The discrete energy inequality (36) gives rise to a-priori estimates for the approximate solutions.

Lemma 5.6 (Energy boundedness) *There exists a constant $C > 0$ independent of M, t and ε such that*

$$\mathcal{E}_\varepsilon(q_M(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t \hat{z}_M)\,\mathrm{d}s + \int_0^{d_M(t)}\int_\Omega \frac{\varepsilon}{2}|\partial_t\hat{c}_M|^2 + \frac{1}{2}|\nabla\mu_M|^2\,\mathrm{d}x\mathrm{d}s \le C(\mathcal{E}_\varepsilon(q_\varepsilon^0) + 1).$$

Proof. Exploiting (A3) yields the estimate ($C > 0$ denotes a context-dependent constant independent of M, t and ε):

$$\begin{aligned}
&\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^-(s) + b(s) - b_M^-(s)), c_M^-(s), z_M^-(s)) : e(\partial_t b(s))\,\mathrm{d}x \\
&\qquad \le C\|\nabla\partial_t b(s)\|_{L^\infty(\Omega)}\int_\Omega W_{\mathrm{el}}(e(u_M^-(s)), c_M^-(s), z_M^-(s)) + |e(b(s) - b_M^-(s))| + 1\,\mathrm{d}x. \qquad (41)
\end{aligned}$$

In addition,

$$\begin{aligned}
&\int_\Omega |\nabla u_M^-(s) + \nabla b(s) - \nabla b_M^-(s)|^2\nabla(u_M^-(s) + b(s) - b_M^-(s)) : \nabla\partial_t b(s)\,\mathrm{d}x \\
&\qquad \le C\|\nabla\partial_t b(s)\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_M^-(s)|^3 + |\nabla(b(s) - b_M^-(s))|^3\,\mathrm{d}x. \qquad (42)
\end{aligned}$$

To simplify the notation, we define the function:

$$\gamma(t) := \begin{cases} \mathcal{E}_\varepsilon(q_M(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t\hat{z}_M)\,\mathrm{d}s + \int_0^{d_M(t)}\int_\Omega \frac{\varepsilon}{2}|\partial_t\hat{c}_M|^2 + \frac{1}{2}|\nabla\mu_M|^2\,\mathrm{d}x\mathrm{d}s, & \text{if } t \in [0,T], \\ \mathcal{E}_\varepsilon(q_\varepsilon^0), & \text{if } t \in [-\tau, 0). \end{cases}$$

Using (41) and (42), the discrete energy inequality (36) can be estimated as follows:

$$
\begin{aligned}
\gamma(t) &\leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + C\int_0^{d_M(t)} \|\nabla\partial_t b(s)\|_{L^\infty(\Omega)}\mathcal{E}_\varepsilon(q_M^-(s))\,\mathrm{d}s \\
&\quad + C\big\|\nabla\partial_t b\big\|_{L^1([0,T];L^\infty(\Omega))}\big\||\nabla(b-b_M^-)|^3 + |e(b-b_M^-)| + 1\big\|_{L^\infty([0,T];L^1(\Omega))} \\
&\leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + C\int_{-\tau}^{d_M^-(t)} \|\nabla\partial_t b(s+\tau)\|_{L^\infty(\Omega)}\mathcal{E}_\varepsilon(q_M(s))\,\mathrm{d}s + C \\
&\leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + C\int_{-\tau}^{t} \|\nabla\partial_t b(s+\tau)\|_{L^\infty(\Omega)}\gamma(s)\,\mathrm{d}s + C.
\end{aligned}
$$

Gronwall's inequality shows for all $t \in [0,T]$

$$
\begin{aligned}
\gamma(t) &\leq C + \mathcal{E}_\varepsilon(q_\varepsilon^0) + C\int_{-\tau}^{t} (C + \mathcal{E}_\varepsilon(q_\varepsilon^0))\|\nabla\partial_t b(s+\tau)\|_{L^\infty(\Omega)} \exp\left(\int_s^t \|\nabla\partial_t b(l+\tau)\|_{L^\infty(\Omega)}\,\mathrm{d}l\right)\mathrm{d}s \\
&\leq C(\mathcal{E}_\varepsilon(q_\varepsilon^0) + 1).
\end{aligned}
$$

■

Corollary 5.7 (A-priori estimates) *There exists a constant $C > 0$ independent of M such that*

(i) $\|u_M\|_{L^\infty([0,T];W^{1,4}(\Omega;\mathbb{R}^n))} \leq C$,

(ii) $\|c_M\|_{L^\infty([0,T];H^1(\Omega))} \leq C$,

(iii) $\|z_M\|_{L^\infty([0,T];W^{1,p}(\Omega))} \leq C$,

(iv) $\|\partial_t \hat{c}_M\|_{L^2(\Omega_T)} \leq C$,

(v) $\|\partial_t \hat{z}_M\|_{L^2(\Omega_T)} \leq C$,

(vi) $\|\mu_M\|_{L^2([0,T];H^1(\Omega))} \leq C$

for all $M \in \mathbb{N}$.

Proof. We use Lemma 5.6. The boundedness of $\{\nabla(u_M(t) - b_M(t))\}$ in $L^4(\Omega;\mathbb{R}^n)$ and $u_M(t) - b_M(t) \in H^1_\Gamma(\Omega;\mathbb{R}^n)$ yield (i) by Poincaré's inequality. The boundedness of $\{\nabla c_M(t)\}$ in $L^2(\Omega)$ and mass conservation imply (ii) by Poincaré's inequality. The boundedness of $\{\nabla z_M(t)\}$ in $L^p(\Omega)$ and $0 \leq z_M(t) \leq 1$ a.e. in Ω for all M and all $t \in [0,T]$ show (iii). The properties (iv) and (v) follow immediately from Lemma 5.6. The boundedness of $\{\nabla\mu_M\}$ in $L^2(\Omega_T)$ and $\{\int_\Omega \mu_M(t)\,\mathrm{d}x\}$ with respect to M and t show (vi) by Poincaré's inequality. Indeed, $\{\int_\Omega \mu_M(t)\,\mathrm{d}x\}$ is bounded with respect to M and t because of (33) and (32) tested with $\zeta \equiv 1$. ■

Due to the a-priori estimates we can select weakly (weakly-$\star$) convergent subsequences (see Lemma 5.8). Furthermore, exploiting the Euler-Lagrange equations of the approximate solutions, we even attain strong convergence properties (see Lemma 5.9 and Lemma 5.11).

Lemma 5.8 (Weak convergence of the approximate solutions) *There exists a subsequence $\{M_k\}$ and elements $(u,c,z) = q \in \mathcal{Q}^{\mathrm{v}}$ and $\mu \in L^2([0,T];H^1(\Omega))$ with $c(0) = c^0$, $z(0) = z^0$, $0 \leq z \leq 1$ and $\partial_t z \leq 0$ a.e. in Ω_T such that the following properties are satisfied:*

(i) $z_{M_k}, z_{M_k}^- \stackrel{\star}{\rightharpoonup} z$ *in* $L^\infty([0,T];W^{1,p}(\Omega))$,
$z_{M_k}(t), z_{M_k}^-(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$ *a.e.* t,
$z_{M_k}, z_{M_k}^- \to z$ *a.e. in* Ω_T *and*
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$,

(ii) $u_{M_k} \stackrel{\star}{\rightharpoonup} u$ *in* $L^\infty([0,T];W^{1,4}(\Omega))$,

(iii) $c_{M_k}, c_{M_k}^- \stackrel{\star}{\rightharpoonup} c$ *in* $L^\infty([0,T];H^1(\Omega))$,
$c_{M_k}(t), c_{M_k}^-(t) \rightharpoonup c(t)$ *in* $H^1(\Omega)$ *a.e.* t,
$c_{M_k}, c_{M_k}^- \to c$ *a.e. in* Ω_T *and*
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1([0,T];L^2(\Omega))$,

(iv) $\mu_{M_k} \rightharpoonup \mu$ *in* $L^2([0,T];H^1(\Omega))$

as $k \to \infty$.

Proof. To simplify notation we omit the index k in the proof.

(iii) Since $\hat{c}_M$ is bounded in $L^2([0,T];H^1(\Omega))$ and $\partial_t\hat{c}_M$ is bounded in $L^2(\Omega_T)$ we obtain $\hat{c}_M \to \hat{c}$ in $L^2(\Omega_T)$ as $M \to \infty$ for a subsequence by a compactness result from J. P. Aubin and J. L. Lions (see [Sim86]). Therefore, we can extract a subsequence such that $\hat{c}_M(t) \to \hat{c}(t)$ in $L^2(\Omega)$ for a.e. $t \in [0,T]$ and $\hat{c}_M \to \hat{c}$ a.e. on Ω_T. We denote this subsequence also with $\{\hat{c}_M\}$. The boundedness of $\{\hat{c}_M(t)\}_{M\in\mathbb{N}}$ in $H^1(\Omega)$ even shows $\hat{c}_M(t) \rightharpoonup \hat{c}(t)$ in $H^1(\Omega)$ for a.e. $t \in [0,T]$. In addition, the boundedness of $\{\hat{c}_M\}$ in $L^\infty([0,T];H^1(\Omega))$ shows $\hat{c}_M \overset{\star}{\rightharpoonup} \hat{c}$ in $L^\infty([0,T];H^1(\Omega))$. Furthermore, we obtain from the boundedness of $\{\partial_t\hat{c}_M\}$ in $L^2(\Omega_T)$ for every $t \in [0,T]$:

$$\begin{aligned}\|c_M(t) - \hat{c}_M(t)\|_{L^1(\Omega)} &= \|\hat{c}_M(d_M(t)) - \hat{c}_M(t)\|_{L^1(\Omega)}\\ &\le \int_t^{d_M(t)} \|\partial_t\hat{c}_M(s)\|_{L^1(\Omega)}\,\mathrm{d}s\\ &\le C(d_M(t)-t)^{1/2}\|\partial_t\hat{c}_M\|_{L^2(\Omega_T)} \to 0 \text{ as } M\to\infty.\end{aligned}$$

Lebesgue's convergence theorem yields $\|c_M - \hat{c}_M\|_{L^1(\Omega_T)} \to 0$ as $M \to \infty$. Analogously, we obtain $\|c_M - c_M^-\|_{L^1(\Omega_T)} \to 0$ as $M \to \infty$. Thus, the convergence properties for $\hat{c}_M$ also holds for c_M and c_M^- with the same limit $c = c^- = \hat{c}$ a.e. . The boundedness of $\{\hat{c}_M\}$ in $H^1([0,T];L^2(\Omega))$ shows $\hat{c}_M \rightharpoonup c$ in $H^1([0,T];L^2(\Omega))$ for a subsequence.

(i) We obtain the convergence properties for $\{z_M\}$ with the same argumentation as in (iii). Note that the limit function is also monotonically decreasing with respect to t.

(ii) This property follows from the boundedness of $\{u_M\}$ in $L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$.

(iv) This property follows from the boundedness of $\{\mu_M\}$ in $L^2([0,T];H^1(\Omega))$. ■

In the sequel, we take advantage from the elementary inequality (x, y are elements of an inner product space X with scalar product $\langle\cdot,\cdot\rangle$)

$$C_{\mathrm{uc}}\|x-y\|^q \le \big\langle\big(\|x\|^{q-2}x - \|y\|^{q-2}y\big), x-y\big\rangle \tag{43}$$

for a constant $C_{\mathrm{uc}} > 0$ depending on X and $q \ge 2$. To see this, (43) is equivalent to

$$C_{\mathrm{uc}} \le \big\langle b, \|a+b\|^{q-2}(a+b) - \|a\|^{q-2}a\big\rangle \text{ for all } a,b\in X, \|b\|=1$$

by introducing the variables $a := x/\|x-y\|$ and $b := (x-y)/\|x-y\|$ for $x \ne y$. This is equivalent to

$$C_{\mathrm{uc}} \le \|a+b\|^{q-2} + \langle b,a\rangle\left(\|a+b\|^{q-2} - \|a\|^{q-2}\right) \text{ for all } a,b\in X, \|b\|=1. \tag{44}$$

Now the equivalence $\|a+b\| \le \|a\| \;\Leftrightarrow\; \langle a,b\rangle \le -\frac{1}{2}\|b\|^2$ gives the estimate:

$$\begin{aligned}\|a+b\|^{q-2} + \langle b,a\rangle\left(\|a+b\|^{q-2} - \|a\|^{q-2}\right) &\ge \|a+b\|^{q-2} + \frac{1}{2}\|b\|^2(\|a\|^{q-2} - \|a+b\|^{q-2})\\ &= \frac{1}{2}\|a+b\|^{q-2} + \frac{1}{2}\|a\|^{q-2}\end{aligned}$$

Since $\|b\| = 1$, the right hand side is bounded from below by a positive constant and therefore (44) follows.

Lemma 5.9 *There exists a subsequence $\{M_k\}$ such that $u_{M_k}, u_{M_k}^- \to u$ in $L^4([0,T];W^{1,4}(\Omega;\mathbb{R}^n))$ as $k \to \infty$.*

Proof. We omit the index k in the proof.
Applying (A1), taking inequality (43) for $q = 4$ into account and considering (34) with the test-function $\zeta = u_M(t) - u(t) - b_M(t) + b(t)$, we get

$$\eta\|e(u_M) - e(u)\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})} + \varepsilon C_{\mathrm{uc}}\|\nabla u_M - \nabla u\|^4_{L^4(\Omega_T;\mathbb{R}^{n\times n})}$$

$$
\begin{aligned}
&\leq \int_{\Omega_T} (\partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) - \partial_e W_{\mathrm{el}}(e(u), c_M, z_M)) : (e(u_M) - e(u))\,\mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon \int_{\Omega_T} (|\nabla u_M|^2 \nabla u_M - |\nabla u|^2 \nabla u) : (\nabla u_M - \nabla u)\,\mathrm{d}x\mathrm{d}t \\
&= \underbrace{\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) : e(\zeta) + \varepsilon |\nabla u_M|^2 \nabla u_M : \nabla \zeta\,\mathrm{d}x\mathrm{d}t}_{=0 \text{ by } (34)} \\
&\quad + \underbrace{\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) : (e(b_M) - e(b))\,\mathrm{d}x\mathrm{d}t}_{(\star)} + \underbrace{\varepsilon \int_{\Omega_T} |\nabla u_M|^2 \nabla u_M : (\nabla b_M - \nabla b)\,\mathrm{d}x\mathrm{d}t}_{(\star\star)} \\
&\quad - \underbrace{\int_{\Omega_T} (\partial_e W_{\mathrm{el}}(e(u), c_M, z_M) : (e(u_M) - e(u))\,\mathrm{d}x\mathrm{d}t}_{(\star\star\star)} - \underbrace{\varepsilon \int_{\Omega_T} |\nabla u|^2 \nabla u : (\nabla u_M - \nabla u)\,\mathrm{d}x\mathrm{d}t}_{(\star\star\star\star)}\,. \qquad (45)
\end{aligned}
$$

Since $\partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M)$ is bounded in $L^2(\Omega_T; \mathbb{R}^{n\times n})$ (by (11a) and Corollary 5.7) as well as $e(b_M) \to e(b)$ in $L^2(\Omega_T; \mathbb{R}^{n\times n})$, we obtain $(\star) \to 0$ as $M \to \infty$. The boundedness of $|\nabla u_M|^2 \nabla u_M$ in $L^{4/3}(\Omega_T; \mathbb{R}^{n\times n})$ by Corollary 5.7 and $\nabla b_M \to \nabla b$ in $L^4(\Omega_T; \mathbb{R}^{n\times n})$ lead to $(\star\star) \to 0$. We also have $\partial_e W_{\mathrm{el}}(e(u), c_M, z_M) \to \partial_e W_{\mathrm{el}}(e(u), c, z)$ in $L^2(\Omega_T; \mathbb{R}^{n\times n})$ by (11a) and Lebesgue's generalized convergence theorem. Furthermore, $e(u_M) \rightharpoonup e(u)$ in $L^2(\Omega_T; \mathbb{R}^n \times \mathbb{R}^n)$ by Lemma 5.8. This gives $(\star\star\star) \to 0$. Since $\nabla u_M \rightharpoonup \nabla u$ in $L^4(\Omega_T; \mathbb{R}^n)$ by Lemma 5.8, we obtain $(\star\star\star\star) \to 0$. Therefore, (45) implies $e(u_M) \to e(u)$ in $L^2(\Omega_T; \mathbb{R}^{n\times n})$ and $\nabla u_M \to \nabla u$ in $L^4(\Omega_T; \mathbb{R}^{n\times n})$ as $M \to \infty$. Poincaré's inequality finally shows $u_M \to u$ in $L^4([0,T]; W^{1,4}(\Omega; \mathbb{R}^n))$. Now, we choose a subsequence such that $u_M(t) \to u(t)$ in $W^{1,4}(\Omega; \mathbb{R}^n)$ for a.e. $t \in [0,T]$ and $u_M \to u$ a.e. in Ω_T. We also denote this subsequence with $\{u_M\}$.

Analogously, we obtain a $u^- \in L^4([0,T]; W^{1,4}(\Omega))$ satisfying $u_M^- \to u^-$ with the same convergence properties. We will show $u = u^-$ a.e. . Consider (34) for $q_M(t)$ and for $q_M^-(t)$:

$$
0 = \int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) : e(\zeta) + \varepsilon |\nabla u_M|^2 \nabla u_M : \nabla \zeta\,\mathrm{d}x\mathrm{d}t, \qquad (46a)
$$

$$
0 = \int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-) : e(\zeta) + \varepsilon |\nabla u_M^-|^2 \nabla u_M^- : \nabla \zeta\,\mathrm{d}x\mathrm{d}t. \qquad (46b)
$$

We choose the test-function $\zeta(t) = u_M(t) - u_M^-(t) - b_M(t) + b_M^-(t) \in W_\Gamma^{1,4}(\Omega)$. An estimate similar to (45) gives:

$$
\begin{aligned}
&\eta \|e(u_M) - e(u_M^-)\|^2_{L^2(\Omega_T)} + \varepsilon C_{\mathrm{ineq}}^{-1} \|\nabla u_M - \nabla u_M^-\|^4_{L^4(\Omega_T)} \\
&\quad \leq \int_{\Omega_T} (\partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) - \partial_e W_{\mathrm{el}}(e(u_M^-), c_M, z_M)) : (e(u_M) - e(u_M^-))\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \varepsilon \int_{\Omega_T} (|\nabla u_M|^2 \nabla u_M - |\nabla u_M^-|^2 \nabla u_M^-) : (\nabla u_M - \nabla u_M^-)\,\mathrm{d}x\mathrm{d}t \\
&\quad = \underbrace{\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) : e(\zeta) + \varepsilon |\nabla u_M|^2 \nabla u_M : \nabla \zeta\,\mathrm{d}x\mathrm{d}t}_{=0 \text{ by } (46a)} \\
&\qquad - \underbrace{\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-) : e(\zeta) + \varepsilon |\nabla u_M^-|^2 \nabla u_M^- : \nabla \zeta\,\mathrm{d}x\mathrm{d}t}_{=0 \text{ by } (46b)} \\
&\qquad + \int_{\Omega_T} (\partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-) - \partial_e W_{\mathrm{el}}(e(u_M^-), c_M, z_M)) : (e(u_M) - e(u_M^-))\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \int_{\Omega_T} (\partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) - \partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-)) : (e(b_M) - e(b_M^-))\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$+\varepsilon\int_{\Omega_T}(|\nabla u_M|^2\nabla u_M-|\nabla u_M^-|^2\nabla u_M^-):(\nabla b_M-\nabla b_M^-)\,\mathrm{d}x\mathrm{d}t.$$

Observe that $\partial_e W_{\mathrm{el}}(e(u_M^-),c_M^-,z_M^-)-\partial_e W_{\mathrm{el}}(e(u_M^-),c_M,z_M)\to 0$ in $L^2(\Omega_T)$ by Lebesgue's generalized convergence theorem (using growth condition (11a), Lemma 5.8 and convergence properties of u_M and u_M^-) as well as $e(b_M)-e(b_M^-)\to 0$ in $L^2(\Omega_T;\mathbb{R}^{n\times n})$ and $\nabla b_M-\nabla b_M^-\to 0$ in $L^4(\Omega_T;\mathbb{R}^{n\times n})$. Hence, each term on the right hand side converges to 0 as $M\to\infty$ ■

Lemma 5.10 *There exists a subsequence $\{M_k\}$ such that $c_{M_k},c_{M_k}^-\to c$ in $L^2([0,T];H^1(\Omega))$ as $k\to\infty$.*

Proof. We omit the index k in the proof.
Lemma 5.8 implies $c_M(t)\to c(t)$ in $L^{2^\star/2+1}(\Omega)$ for a.e. $t\in[0,T]$. Using Corollary 5.7 and Lebesgue's convergence theorem, we get $c_M\to c$ in $L^{2^\star/2+1}(\Omega_T)$. Next, we test (33) with $\zeta=c_M(t)$ and integrate from $t=0$ to $t=T$. Then we use Lebesgue's generalized convergence theorem, growth conditions (A4) and (A6) as well as Lemma 5.8 to obtain

$$\int_{\Omega_T}|\nabla c_M|^2\,\mathrm{d}x\mathrm{d}t\to-\int_{\Omega_T}\partial_c W_{\mathrm{ch}}(c)c+\partial_c W_{\mathrm{el}}(e(u),c,z)c+\varepsilon(\partial_t c)c-\mu c\,\mathrm{d}x\mathrm{d}t$$

as $M\to\infty$. On the other hand, we test (33) with $c(t)$ and integrate from $t=0$ to $t=T$. Note that $c\in L^{2^\star}(\Omega_T)$ and $\partial_c W_{\mathrm{ch}}(c_M)\to\partial_c W_{\mathrm{ch}}(c)$ in $L^{2^\star/(2^\star-1)}(\Omega_T)$ as $M\to\infty$ by Lebesgue's generalized convergence theorem. Hence, we derive for $M\to\infty$:

$$\int_{\Omega_T}|\nabla c|^2\,\mathrm{d}x\mathrm{d}t=-\int_{\Omega_T}\partial_c W_{\mathrm{ch}}(c)c+\partial_c W_{\mathrm{el}}(e(u),c,z)c+\varepsilon(\partial_t c)c-\mu c\,\mathrm{d}x\mathrm{d}t.$$

Therefore, $c_M\to c$ in $L^2([0,T];H^1(\Omega))$ as $M\to\infty$. The convergence $\|c_M\|_{L^2([0,T];H^1(\Omega))}\to\|c\|_{L^2([0,T];H^1(\Omega))}$ implies $\|c_M^-\|_{L^2([0,T];H^1(\Omega))}\to\|c\|_{L^2([0,T];H^1(\Omega))}$. We also have $c_M^-\rightharpoonup c$ in $L^2([0,T];H^1(\Omega))$ (by Lemma 5.8 (ii)) and consequently $c_M^-\to c$ in $L^2([0,T];H^1(\Omega))$ as $M\to\infty$.

■

Note that in connection with Corollary 5.7 we even get for each $q\geq 1$

$$c_M,c_M^-\to c\text{ in }L^q([0,T];H^1(\Omega))$$

for a subsequence as $M\to\infty$.

Lemma 5.11 *There exists a subsequence $\{M_k\}$ such that $z_{M_k},z_{M_k}^-\to z$ in $L^p([0,T];W^{1,p}(\Omega))$ as $k\to\infty$.*

Proof. To simplify notation we omit the index k in the proof.
Applying Lemma 5.2 with $f=\zeta=z$ and $f_M=z_M^-$ gives a sequence of approximations $\{\zeta_M\}_{M\in\mathbb{N}}\subseteq L^p([0,T];W_+^{1,p}(\Omega))\cap L^\infty(\Omega_T)$ with the properties (note that we have $z_M^-(t)\rightharpoonup z(t)$ in $W^{1,p}(\Omega)$ for a.e. $t\in[0,T]$ by Lemma 5.8):

$$\zeta_M\to z\text{ in }L^p([0,T];W^{1,p}(\Omega))\text{ as }M\to\infty \tag{47a}$$

$$0\leq\zeta_M\leq z_M^-\text{ a.e. on }\Omega_T\text{ for all }M\in\mathbb{N}. \tag{47b}$$

We test (35) with $\zeta=\zeta_M(t)-z_M(t)$ for $\nu=1$ (possible due to (47b)), integrate from $t=0$ to $t=T$ and use (43) to obtain the following estimate:

$$\begin{aligned}C_{\mathrm{uc}}&\int_{\Omega_T}|\nabla z_M-\nabla z|^p\,\mathrm{d}x\mathrm{d}t\\&\leq\int_{\Omega_T}(|\nabla z_M|^{p-2}\nabla z_M-|\nabla z|^{p-2}\nabla z)\cdot\nabla(z_M-z)\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$\le \int_{\Omega_T} |\nabla z_M|^{p-2}\nabla z_M \cdot \nabla(z_M - \zeta_M)\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{\Omega_T} |\nabla z_M|^{p-2}\nabla z_M \cdot \nabla(\zeta_M - z) - |\nabla z|^{p-2}\nabla z \cdot \nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t$$
$$\le \int_{\Omega_T} (\partial_z W_{\mathrm{el}}(e(u_M), c_M, z_M) - \alpha + \beta\partial_t \hat{z}_M)(\zeta_M - z_M)\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{\Omega_T} |\nabla z_M|^{p-2}\nabla z_M \cdot \nabla(\zeta_M - z) - |\nabla z|^{p-2}\nabla z \cdot \nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t$$
$$\le \underbrace{\|\partial_z W_{\mathrm{el}}(e(u_M), c_M, z_M) - \alpha + \beta\partial_t \hat{z}_M\|_{L^2(\Omega_T)}}_{\text{bounded by (A5) and Cor. 5.7}} \|\zeta_M - z_M\|_{L^2(\Omega_T)}$$
$$+ \underbrace{\|\nabla z_M\|^{p-1}_{L^p(\Omega_T)}}_{\text{bounded by Cor. 5.7}} \|\nabla\zeta_M - \nabla z\|_{L^p(\Omega_T)} - \int_{\Omega_T} |\nabla z|^{p-2}\nabla z \cdot \nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t.$$

Observe that $\nabla\zeta_M - \nabla z \to 0$ in $L^p(\Omega_T;\mathbb{R}^n)$ and $\zeta_M - z_M \to 0$ in $L^2(\Omega_T)$ (by property (47a) and by Lemma 5.8) as well as $\nabla z_M - \nabla z \rightharpoonup 0$ in $L^p(\Omega_T;\mathbb{R}^n)$ by Lemma 5.8. Using these properties, each term on the right hand side converges to 0 as $M \to \infty$.

We also obtain $\|z_M^-\|_{L^p([0,T];W^{1,p}(\Omega))} \to \|z\|_{L^p([0,T];W^{1,p}(\Omega))}$ from $\|z_M\|_{L^p([0,T];W^{1,p}(\Omega))} \to \|z\|_{L^p([0,T];W^{1,p}(\Omega))}$. Because of $z_M^- \rightharpoonup z$ in $L^p([0,T];W^{1,p}(\Omega))$ (by Lemma 5.8 (i)) we even have $z_M^- \to z$ in $L^p([0,T];W^{1,p}(\Omega))$ as $M \to \infty$. ■

In conclusion, Corollary 5.7, Lemma 5.8, Lemma 5.9, Lemma 5.10 and Lemma 5.11 imply the following convergence properties:

Corollary 5.12 *There exists subsequence $\{M_k\}$ and an element $(u,c,z) = q \in \mathcal{Q}^{\mathrm{v}}$ with $c(0) = c^0$ and $z(0) = z^0$ such that*

(i) $z_{M_k}, z_{M_k}^- \to z$ *in* $L^p([0,T];W^{1,p}(\Omega))$,
$z_{M_k}(t), z_{M_k}^-(t) \to z(t)$ *in* $W^{1,p}(\Omega)$ *a.e.* t,
$z_{M_k}, z_{M_k}^- \to z$ *a.e. in* Ω_T *and*
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$

(ii) $c_{M_k}, c_{M_k}^- \to c$ *in* $L^{2^\star}([0,T];H^1(\Omega))$,
$c_{M_k}(t), c_{M_k}^-(t) \to c(t)$ *in* $H^1(\Omega)$ *a.e.* t,
$c_{M_k}, c_{M_k}^- \to c$ *a.e. in* Ω_T *and*
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1([0,T];L^2(\Omega))$

(iii) $u_{M_k}, u_{M_k}^- \to u$ *in* $L^4([0,T];W^{1,4}(\Omega;\mathbb{R}^n))$,
$u_{M_k}(t), u_{M_k}^-(t) \to u(t)$ *in* $W^{1,4}(\Omega;\mathbb{R}^n)$ *a.e.* t,
$u_{M_k}, u_{M_k}^- \to u$ *a.e. in* Ω_T

(iv) $\mu_{M_k} \rightharpoonup \mu$ *in* $L^2([0,T];H^1(\Omega))$

(v) $\partial_c W_{\mathrm{ch}}(c_{M_k}) \to \partial_c W_{\mathrm{ch}}(c)$ *in* $L^2(\Omega_T)$

as $k \to \infty$.

The above convergence properties allow us to establish an energy estimate, which is in an asymptotic sense stronger than the one in Lemma 5.5 (v). We emphasize that (36) has in comparison with (48) no factor 1/2 in front of the terms $\beta|\partial_t \hat{z}_M|^2$, $\varepsilon|\partial_t \hat{c}_M|^2$ and $|\nabla\mu_M|^2$.

Lemma 5.13 (Precise energy inequality) *For every* $0 \le t_1 < t_2 \le T$*:*

$$\mathcal{E}_\varepsilon(q_M(t_2)) + \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega -\alpha\partial_t \hat{z}_M + \beta|\partial_t \hat{z}_M|^2 + \varepsilon|\partial_t \hat{c}_M|^2 + |\nabla\mu_M|^2\,\mathrm{d}x\mathrm{d}s - \mathcal{E}_\varepsilon(q_M^-(t_1))$$
$$\le \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s$$
$$+ \varepsilon \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s + \kappa_M \tag{48}$$

with $\kappa_M \to 0$ as $M \to \infty$.

Proof. We know $\mathbb{E}_M^m(q_M^m) \leq \mathbb{E}_M^m(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, z_M^m)$. The regularity properties of the functions b, $\hat{c}_M$ and $\hat{z}_M$ ensure that the chain rule can be applied and the following integral terms are well defined:

$$
\begin{aligned}
&\mathcal{E}_\varepsilon(u_M^m, c_M^m, z_M^m) \\
&\quad \leq \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, z_M^m) \\
&\quad = \mathcal{E}_\varepsilon(u_M^{m-1}, c_M^{m-1}, z_M^{m-1}) \\
&\qquad + \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1}) - \mathcal{E}_\varepsilon(u_M^{m-1}, c_M^{m-1}, z_M^{m-1}) \\
&\qquad + \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, z_M^{m-1}) - \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1}) \\
&\qquad + \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, z_M^m) - \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, z_M^{m-1}) \\
&\quad = \mathcal{E}_\varepsilon(u_M^{m-1}, c_M^{m-1}, z_M^{m-1}) \\
&\qquad + \int_{(m-1)\tau}^{m\tau} \langle \mathrm{d}_u \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1} + b(s) - b_M^{m-1}, c_M^{m-1}, z_M^{m-1}), \partial_t b(s)\rangle_{(H^1)^* \times H^1} \,\mathrm{d}s \\
&\qquad + \int_{(m-1)\tau}^{m\tau} \langle \mathrm{d}_c \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, \hat{c}_M(s), z_M^{m-1}), \partial_t \hat{c}_M(s)\rangle_{(H^1)^* \times H^1} \,\mathrm{d}s \\
&\qquad + \int_{(m-1)\tau}^{m\tau} \langle \mathrm{d}_z \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, \hat{z}_M(s)), \partial_t \hat{z}_M(s)\rangle_{(W^{1,p})^* \times W^{1,p}} \,\mathrm{d}s.
\end{aligned}
$$

Summing from $m = \frac{d_M^-(t_1)}{\tau} + 1$ to $\frac{d_M(t_2)}{\tau}$ yields:

$$
\begin{aligned}
&\mathcal{E}_\varepsilon(q_M(t_2)) - \mathcal{E}_\varepsilon(q_M^-(t_1)) \\
&\quad \leq \varepsilon \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega |\nabla(u_M^- + b - b_M^-)|^2 \nabla(u_M^- + b - b_M^-) : \nabla \partial_t b \,\mathrm{d}x\mathrm{d}s \\
&\qquad + \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b) \,\mathrm{d}x\mathrm{d}s \\
&\qquad + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_c W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), \hat{c}_M, z_M^-) \partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s}_{(\star)} \\
&\qquad + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \nabla \hat{c}_M \cdot \nabla \partial_t \hat{c}_M + \partial_c W_{\mathrm{ch}}(\hat{c}_M) \partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s}_{(\star\star)} \\
&\qquad + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_z W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), c_M, \hat{z}_M)\, \partial_t \hat{z}_M + |\nabla \hat{z}_M|^{p-2} \nabla \hat{z}_M \cdot \nabla \partial_t \hat{z}_M \,\mathrm{d}x\mathrm{d}s}_{(\star\star\star)}.
\end{aligned}
\tag{49}
$$

By using convexity of $x \mapsto |x|^p$, we obtain the following elementary inequality

$$
(|\nabla \hat{z}_M(t,x)|^{p-2} \nabla \hat{z}_M(t,x) - |\nabla z_M(t,x)|^{p-2} \nabla z_M(t,x)) \cdot \nabla \partial_t \hat{z}_M(t,x) \leq 0.
$$

This estimate and (35), tested with $\zeta = -\partial_t \hat{z}_M(t)$ for $\nu = \tau$ and integrated from $t = 0$ to $t = T$, lead to the estimate:

$$
\begin{aligned}
(\star\star\star) \leq & -\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega -\alpha\partial_t \hat{z}_M + \beta|\partial_t \hat{z}_M|^2 \,\mathrm{d}x\mathrm{d}s \\
& + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega (\partial_z W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), c_M, \hat{z}_M) - \partial_z W_{\mathrm{el}}(e(u_M), c_M, z_M))\partial_t \hat{z}_M \,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^3}\,.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
(\star) \leq & \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_c W_{\mathrm{el}}(e(u_M), c_M, z_M)\partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s \\
& + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega (\partial_c W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), \hat{c}_M, z_M^-) - \partial_c W_{\mathrm{el}}(e(u_M), c_M, z_M))\partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^1}\,.
\end{aligned}
$$

Using the elementary estimate $(\nabla\hat{c}_M - \nabla c_M)\nabla\partial_t\hat{c}_M \leq 0$, we obtain

$$
\begin{aligned}
(\star\star) \leq & \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \nabla c_M \cdot \nabla\partial_t \hat{c}_M + \partial_c W_{\mathrm{ch}}(c_M)\partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s \\
& + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega (\partial_c W_{\mathrm{ch}}(\hat{c}_M) - \partial_c W_{\mathrm{ch}}(c_M))\partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^2}\,.
\end{aligned}
$$

Hence, applying equations (33) and (32) shows

$$
\begin{aligned}
\int_{d_M^-(t_1)}^{d_M(t_2)} \langle \mathrm{d}_c \tilde{\mathcal{E}}_\varepsilon(q_M), \partial_t \hat{c}_M\rangle_{(H^1)^* \times H^1} \,\mathrm{d}s &= \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \mu_M \partial_t \hat{c}_M - \varepsilon|\partial_t \hat{c}_M|^2 \,\mathrm{d}x\mathrm{d}s \\
&= \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega -|\nabla\mu_M|^2 - \varepsilon|\partial_t \hat{c}_M|^2 \,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Thus,

$$
(\star) + (\star\star) \leq \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega -|\nabla\mu_M|^2 - \varepsilon|\partial_t \hat{c}_M|^2 \,\mathrm{d}x\mathrm{d}s + \kappa_M^1 + \kappa_M^2.
$$

Lebesgue's generalized convergence theorem, the growth conditions (A4), (A5), (A6) and Corollary 5.12 ensure that κ_M^1, κ_M^2 and κ_M^3 converge to 0 as $M \to \infty$. Here, we want to emphasize that we need boundedness of $\partial_t \hat{c}_M$ and $\partial_t \hat{z}_M$ in $L^2(\Omega_T)$ and the convergence $e(u_M) \to e(u)$ in $L^4(\Omega_T)$, which we have only due to the regularization for every fixed $\varepsilon > 0$ as $M \to \infty$ (see Corollary 5.12). To finish the proof, set $\kappa_M := \kappa_M^1 + \kappa_M^2 + \kappa_M^3$. ■

We are now in the position to prove the existence theorem for the viscous case.

Proof of Theorem 4.4. The proof is divided into several steps:

(i) Using growth conditions (A4), (A6), (11a), Corollary 5.12 and Lebesgue's generalized convergence theorem, we can pass to $M \to \infty$ in the time integrated version of the integral equations (32), (33) and (34). This shows (i) and (ii) of Definition 4.3.

(ii) Let $0 \leq t_1 < t_2 \leq T$ be arbitrary. Because of $d_M^-(t_1) \leq t_1 < t_2 \leq d_M(t_2)$, Lemma 5.13 particularly implies

$$\begin{aligned}\mathcal{E}_\varepsilon(q_M(t_2)) + \int_{t_1}^{t_2}\int_\Omega -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2 + \varepsilon|\partial_t\hat{c}_M|^2 + |\nabla\mu_M|^2\,\mathrm{d}x\mathrm{d}t - \mathcal{E}_\varepsilon(q_M^-(t_1))\\ \leq \int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t\\ + \varepsilon\int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t + \kappa_M \qquad (50)\end{aligned}$$

with $\kappa_M \to 0$ as $M \to \infty$. Due to growth condition (A2), (A6), Corollary 5.12 and Lebesgue's generalized convergence theorem we obtain

$$\mathcal{E}_\varepsilon(q_M(t)) \to \mathcal{E}_\varepsilon(q(t)) \text{ and } \mathcal{E}_\varepsilon(q_M^-(t)) \to \mathcal{E}_\varepsilon(q(t)) \qquad (51)$$

as $M \to \infty$ for a.e. $t \in [0,T]$. A sequentially weakly lower semi-continuity argument based on Corollary 5.12 shows:

$$\begin{aligned}\liminf_{M\to\infty}\int_{t_1}^{t_2}\int_\Omega -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2 + \varepsilon|\partial_t\hat{c}_M|^2 + |\nabla\mu_M|^2\,\mathrm{d}x\mathrm{d}t\\ \geq \int_\Omega \alpha(z(t_1) - z(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z|^2 + \varepsilon|\partial_t c|^2 + |\nabla\mu|^2\,\mathrm{d}x\mathrm{d}t. \qquad (52)\end{aligned}$$

Growth condition (11a), Corollary 5.12 and Lebesgue's generalized convergence theorem show:

$$\begin{aligned}\partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M) &\overset{\star}{\rightharpoonup} \partial_e W_{\mathrm{el}}(e(u), c, z) && \text{in } L^\infty([0,T]; L^2(\Omega)),\\ |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) &\overset{\star}{\rightharpoonup} |\nabla u|^2\nabla u && \text{in } L^\infty([0,T]; L^{4/3}(\Omega)).\end{aligned}$$

Since $e(\partial_t b) \in L^1([0,T]; L^2(\Omega))$ and $\nabla\partial_t b \in L^1([0,T]; L^4(\Omega))$ we get:

$$\begin{aligned}&\int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t\\ &\qquad\to \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t,\\ &\int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t\\ &\qquad\to \int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t. \qquad (53)\end{aligned}$$

Now, using (51), (52) and (53) gives (iv) of Definition 4.3 by passing to $M \to \infty$ in (50) for a subsequence.

(iii) Let $\tilde{\zeta} \in L^p([0,T]; W_-^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ be a test-function with $\{\tilde{\zeta} = 0\} \supseteq \{z = 0\}$. Applying Lemma 5.2 with $f = z$ and $f_M = z_M$ and $\zeta = -\tilde{\zeta}$ gives a sequence of approximations $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq L^p([0,T]; W_+^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ with the properties:

$$\zeta_M \to -\tilde{\zeta} \text{ in } L^p([0,T]; W^{1,p}(\Omega)) \text{ as } M \to \infty, \qquad (54a)$$

$$0 \leq \nu_{M,t}\zeta_M(t) \leq z_M(t) \text{ a.e. in } \Omega \text{ for a.e. } t \in [0,T] \text{ and all } M \in \mathbb{N}. \qquad (54b)$$

Let $\tilde{\zeta}_M$ denote the function $-\zeta_M$. Then, (54b) in particular implies $0 \leq \nu_{M,t}\tilde{\zeta}_M(t) + z_M(t) \leq z_M^-(t)$ a.e. in Ω for a.e. $t \in [0,T]$. Now, (35) holds for $\zeta = \tilde{\zeta}_M(t)$. Integration from $t = 0$ to $t = T$ and using growth condition (A5), Corollary 5.12 and Lebesgue's generalized convergence theorem as well as the strong convergence (54a) yield for $M \to \infty$:

$$-\int_{\Omega_T} |\nabla z|^{p-2}\nabla z \cdot \nabla\tilde{\zeta} + \partial_z W_{\mathrm{el}}(e(u), c, z)\tilde{\zeta} - \alpha\tilde{\zeta} + \beta(\partial_t z)\tilde{\zeta}\,\mathrm{d}x\mathrm{d}t \leq 0. \tag{55}$$

(iv) Property (55) implies that

$$-\int_{\Omega} |\nabla z(t)|^{p-2}\nabla z(t) \cdot \nabla\zeta + \big(\partial_z W_{\mathrm{el}}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))\big)\zeta\,\mathrm{d}x \leq 0$$

holds for all $\zeta \in W_-^{1,p}(\Omega)$ with $\{\zeta = 0\} \supseteq \{z(t) = 0\}$ and for a.e. $t \in [0, T]$. Applying Lemma 5.3 with $f = |\nabla z(t)|^{p-2}\nabla z(t)$ and $g = \partial_z W_{\mathrm{el}}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))$ shows

$$\begin{aligned}\int_{\Omega} &|\nabla z(t)|^{p-2}\nabla z(t) \cdot \nabla\zeta + \big(\partial_z W_{\mathrm{el}}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))\big)\zeta\,\mathrm{d}x\\ &\geq \int_{\{z(t)=0\}} [\partial_z W_{\mathrm{el}}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))]^+\zeta\,\mathrm{d}x\\ &\geq \int_{\{z(t)=0\}} [\partial_z W_{\mathrm{el}}(e(u(t)), c(t), z(t))]^+\zeta\,\mathrm{d}x\end{aligned} \tag{56}$$

for all $\zeta \in W_-^{1,p}(\Omega)$. Setting

$$r := -\chi_{\{z=0\}}[\partial_z W_{\mathrm{el}}(e(u), c, z)]^+,$$

we get (24) from (56) by integration from $t = 0$ to $t = T$ and we also have

$$\langle r(t), \zeta - z(t)\rangle = -\int_{\{z(t)=0\}} [\partial_z W_{\mathrm{el}}(e(u(t)), c(t), z(t))]^+(\zeta - z(t))\,\mathrm{d}x \leq 0$$

for any $\zeta \in W_+^{1,p}(\Omega)$ and a.e. $t \in [0, T]$. Therefore, (25) is shown. ■

5.3 Vanishing viscosity: $\varepsilon \searrow 0$

For each $\varepsilon \in (0, 1]$, we denote with $q_\varepsilon = (u_\varepsilon, c_\varepsilon, z_\varepsilon) \in \mathcal{Q}^{\mathrm{v}}$ a viscous solution according to Theorem 4.4. Whenever we refer to the equations and inequalities (21)-(27) of Definition 4.3 the variables $q = (u, c, z)$, μ and r should be replaced by $q_\varepsilon = (u_\varepsilon, c_\varepsilon, z_\varepsilon)$, μ_ε and r_ε. By the use of Lemma 5.14, Lemma 5.15 and Lemma 5.16 below, we identify a suitable subsequence where we can pass to the limit.

Lemma 5.14 (A-priori estimates) *There exists a $C > 0$ independent of $\varepsilon > 0$ such that*

(i) $\|u_\varepsilon\|_{L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))} \leq C$,

(ii) $\varepsilon^{1/4}\|u_\varepsilon\|_{L^\infty([0,T];W^{1,4}(\Omega;\mathbb{R}^n))} \leq C$,

(iii) $\|c_\varepsilon\|_{L^\infty([0,T];H^1(\Omega))} \leq C$,

(iv) $\|z_\varepsilon\|_{L^\infty([0,T];W^{1,p}(\Omega))} \leq C$,

(v) $\|\partial_t c_\varepsilon\|_{L^2([0,T];(H^1(\Omega))^\star)} \leq C$,

(vi) $\varepsilon^{1/2}\|\partial_t c_\varepsilon\|_{L^2(\Omega_T)} \leq C$,

(vii) $\|\partial_t z_\varepsilon\|_{L^2(\Omega_T)} \leq C$,

(viii) $\|\mu_\varepsilon\|_{L^2([0,T];H^1(\Omega))} \leq C$

for all $\varepsilon \in (0, 1]$.

Proof. According to Lemma 5.6, the discretization $q_{M,\varepsilon}$ of q_ε fulfills

$$\mathcal{E}_\varepsilon(q_{M,\varepsilon}(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t \hat{z}_{M,\varepsilon})\,\mathrm{d}s + \int_0^{d_M(t)} \int_\Omega \frac{\varepsilon}{2}|\partial_t \hat{c}_{M,\varepsilon}|^2 + \frac{1}{2}|\nabla \mu_{M,\varepsilon}|^2\,\mathrm{d}x\mathrm{d}s \le C(\mathcal{E}_\varepsilon(q_\varepsilon^0) + 1), \tag{57}$$

where C is independent of M, t, ε. By the minimizing property of q_ε^0, we also obtain $\mathcal{E}_\varepsilon(q_\varepsilon^0) \le \mathcal{E}_\varepsilon(q_1^0) \le \mathcal{E}_1(q_1^0)$ for all $\varepsilon \in (0,1]$. Therefore, the left hand side of (57) is bounded with respect to $M \in \mathbb{N}$, $t \in [0,T]$ and $\varepsilon \in (0,1]$. This leads to the boundedness of

$$\mathcal{E}_\varepsilon(q_\varepsilon(t)) + \int_0^t \mathcal{R}(\partial_t z_\varepsilon)\,\mathrm{d}s + \int_0^t \int_\Omega \frac{\varepsilon}{2}|\partial_t c_\varepsilon|^2 + \frac{1}{2}|\nabla \mu_\varepsilon|^2\,\mathrm{d}x\mathrm{d}s \le C \tag{58}$$

for a.e. $t \in [0,T]$ and for all $\varepsilon \in (0,1]$. We immediately obtain (iv), (vi) and (vii). Due to $\int c_\varepsilon(t)\,\mathrm{d}x = \text{const}$ and the boundedness of $\|\nabla c_\varepsilon(t)\|_{L^2(\Omega)}$, Poincaré's inequality yields (iii). In addition, (ii) follows from Poincaré's inequality. Now, using (58), growth conditions (11b) and Korn's inequality, we attain the desired a-priori estimate (i). Due to (22) and (21) we obtain boundedness of $\int_\Omega \mu_\varepsilon(t)\,\mathrm{d}x$. Since $\|\nabla \mu_\varepsilon(t)\|_{L^2(\Omega_T)}$ is also bounded, Poincaré's inequality yields (viii).

Finally, we know from the boundedness of $\{\nabla \mu_\varepsilon\}$ in $L^2(\Omega_T)$ that $\{\partial_t c_\varepsilon\}$ is also bounded in $L^2([0,T];(H^1(\Omega))^*)$ with respect to ε by using equation (21). Therefore, (v) holds. ■

Lemma 5.15 (Weak convergence of viscous solutions) *There exists a subsequence $\{\varepsilon_k\}$ (which is also denoted by ε) and elements $(u,c,z) = q \in \mathcal{Q}$ and $\mu \in L^2([0,T];H^1(\Omega))$ with $z(0) = z^0$, $0 \le z \le 1$ and $\partial_t z \le 0$ a.e. in Ω_T such that*

(i) $z_\varepsilon \overset{\star}{\rightharpoonup} z$ *in* $L^\infty([0,T];W^{1,p}(\Omega))$,
$z_\varepsilon(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$ *a.e.* t,
$z_\varepsilon \to z$ *a.e. in* Ω_T *and*
$z_\varepsilon \rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$,

(ii) $u_\varepsilon \overset{\star}{\rightharpoonup} u$ *in* $L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$,

(iii) $c_\varepsilon \overset{\star}{\rightharpoonup} c$ *in* $L^\infty([0,T];H^1(\Omega))$,
$c_\varepsilon(t) \rightharpoonup c(t)$ *in* $H^1(\Omega)$ *a.e.* t *and*
$c_\varepsilon \to c$ *a.e. in* Ω_T,

(iv) $\mu_\varepsilon \rightharpoonup \mu$ *in* $L^2([0,T];H^1(\Omega))$

as $\varepsilon \searrow$
0.

Proof.

(i) This property follows from the boundedness of $\{z_\varepsilon\}$ in $L^\infty([0,T];W^{1,p}(\Omega))$ and in $H^1([0,T];L^2(\Omega))$ (see proof of Lemma 5.14). The function z obtained in this way is monotonically decreasing with respect to t, i.e. $\partial_t z \le 0$ a.e. in Ω_T.

(ii) This property follows from the boundedness of $\{u_\varepsilon\}$ in $L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$.

(iii) Properties (iii) and (v) of Lemma 5.14 show that c_ε converges strongly to an element c in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$ for a subsequence by a compactness result due to J. P. Aubin and J. L. Lions (see [Sim86]). This allows us to extract a further subsequence such that $c_\varepsilon(t) \to c(t)$ in $L^2(\Omega)$ for a.e. $t \in [0,T]$. Taking also the boundedness of $\{c_\varepsilon\}$ in $L^\infty([0,T];H^1(\Omega))$ into account, we obtain a subsequence with $c_\varepsilon(t) \rightharpoonup c(t)$ in $H^1(\Omega)$ for a.e. $t \in [0,T]$ and $c_\varepsilon \to c$ a.e. in Ω_T as well as $c_\varepsilon \overset{\star}{\rightharpoonup} c$ in $L^\infty([0,T];H^1(\Omega))$.

(iv) This property follows from the boundedness of $\{\mu_\varepsilon\}$ in $L^2([0,T];H^1(\Omega))$. ■

Lemma 5.16 (Strong convergence of viscous solutions) *The following convergence properties are satisfied for a subsequence $\varepsilon \searrow 0$:*

(i) $u_\varepsilon \to u$ *in* $L^2([0,T];H^1(\Omega;\mathbb{R}^n))$,

(ii) $c_\varepsilon \to c$ *in* $L^2([0,T];H^1(\Omega))$,

(iii) $z_\varepsilon \to z$ *in* $L^p([0,T];W^{1,p}(\Omega))$.

Proof.

(i) We consider an approximation sequence $\{\tilde{u}_\delta\}_{\delta\in(0,1]} \subseteq L^4([0,T];W^{1,4}(\Omega))$ with

$$\tilde{u}_\delta \to u \text{ in } L^2([0,T];H^1(\Omega)) \text{ as } \delta \searrow 0, \tag{59a}$$

$$\tilde{u}_\delta - b \in L^4([0,T];W^{1,4}_\Gamma(\Omega)) \text{ for all } \delta > 0. \tag{59b}$$

Since ε and δ are independent, we consider a sequence $\{\delta_\varepsilon\}_{\varepsilon\in(0,1]}$ with

$$\varepsilon^{1/4}\|\nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)} \to 0 \text{ and } \delta_\varepsilon \searrow 0 \text{ as } \varepsilon \searrow 0. \tag{60}$$

Testing (23) with $\zeta = u_\varepsilon - \tilde{u}_{\delta_\varepsilon}$ (possible due to (59b)), applying the uniform monotonicity of $\partial_e W_{\mathrm{el}}$ (assumption (A1)) and (43) for $p = 4$ (compare with the calculation performed in (45)) gives

$$\begin{aligned}
&\frac{\eta}{2}\|e(u_\varepsilon) - e(u)\|^2_{L^2(\Omega_T)}\\
&\quad\le \eta\|e(u) - e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)} + \eta\|e(u_\varepsilon) - e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)} + \varepsilon C_{\mathrm{uc}}\|\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon}\|^4_{L^4(\Omega_T)}\\
&\quad\le \eta\|e(u) - e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)}\\
&\qquad+ \int_{\Omega_T}(\partial_e W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) - \partial_e W_{\mathrm{el}}(e(\tilde{u}_{\delta_\varepsilon}),c_\varepsilon,z_\varepsilon)) : (e(u_\varepsilon) - e(\tilde{u}_{\delta_\varepsilon}))\,\mathrm{d}x\mathrm{d}t\\
&\qquad+ \varepsilon\int_{\Omega_T}(|\nabla u_\varepsilon|^2\nabla u_\varepsilon - |\nabla\tilde{u}_{\delta_\varepsilon}|^2\nabla\tilde{u}_{\delta_\varepsilon}) : (\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon})\,\mathrm{d}x\mathrm{d}t\\
&\quad= \eta\|e(u) - e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)}\\
&\qquad+ \underbrace{\int_{\Omega_T}\partial_e W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) : (e(u_\varepsilon) - e(\tilde{u}_{\delta_\varepsilon})) + \varepsilon|\nabla u_\varepsilon|^2\nabla u_\varepsilon : (\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon})\,\mathrm{d}x\mathrm{d}t}_{=0 \text{ by } (23)}\\
&\qquad- \int_{\Omega_T}\partial_e W_{\mathrm{el}}(e(\tilde{u}_{\delta_\varepsilon}),c_\varepsilon,z_\varepsilon) : (e(u_\varepsilon) - e(\tilde{u}_{\delta_\varepsilon}))\,\mathrm{d}x\mathrm{d}t\\
&\qquad\underbrace{- \varepsilon\int_{\Omega_T}|\nabla\tilde{u}_{\delta_\varepsilon}|^2\nabla\tilde{u}_{\delta_\varepsilon} : (\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon})\,\mathrm{d}x\mathrm{d}t}_{(\star)}.
\end{aligned} \tag{61}$$

Finally,

$$\begin{aligned}
|(\star)| &\le \varepsilon\|\nabla\tilde{u}_{\delta_\varepsilon}\|^3_{L^4(\Omega_T)}\|\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)}\\
&\le \Big(\underbrace{\varepsilon^{1/4}\|\nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)}}_{\to 0 \text{ as } \varepsilon\searrow 0 \text{ by } (60)}\Big)^3\Big(\underbrace{\varepsilon^{1/4}\|\nabla u_\varepsilon\|_{L^4(\Omega_T)}}_{\le C \text{ by Lemma 5.14}} + \underbrace{\varepsilon^{1/4}\|\nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)}}_{\to 0 \text{ as } \varepsilon\searrow 0 \text{ by } (60)}\Big).
\end{aligned}$$

From growth condition (11a), Lemma 5.15 and Lebesgue's generalized convergence theorem, we obtain

$$\partial_e W_{\mathrm{el}}(e(\tilde{u}_{\delta_\varepsilon}),c_\varepsilon,z_\varepsilon) \to \partial_e W_{\mathrm{el}}(e(u),c,z) \text{ in } L^2(\Omega_T)$$

for a subsequence $\varepsilon \searrow 0$. By $u_\varepsilon \overset{\star}{\rightharpoonup} u$ in $L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$ for a subsequence $\varepsilon \searrow 0$ (Lemma 5.15 (iii)) as well as (59a), we also have

$$e(u_\varepsilon) - e(\tilde{u}_{\delta_\varepsilon}) \rightharpoonup 0 \text{ in } L^2(\Omega_T)$$

as $\varepsilon \searrow 0$ for a subsequence. Therefore, every term on the right hand side of (61) converges to 0 as $\varepsilon \searrow 0$ for a subsequence. This shows $u_\varepsilon \to u$ in $L^2([0,T];H^1(\Omega;\mathbb{R}^n))$ as $\varepsilon \searrow 0$ for a subsequence by Korn's inequality.

(ii) Testing (22) with c_ε and c and passing to $\varepsilon \searrow 0$ for a subsequence eventually shows strong convergence $c_\varepsilon \to c$ in $L^2([0,T];H^1(\Omega))$ (see the argumentation in Lemma 5.10 and notice that $\int_{\Omega_T} \varepsilon(\partial_t c_\varepsilon)c_\varepsilon \,\mathrm{d}x\mathrm{d}t \leq \varepsilon\|\partial_t c_\varepsilon\|_{L^2(\Omega_T)}\|c_\varepsilon\|_{L^2(\Omega_T)} \to 0$ as $\varepsilon \searrow 0$).

(iii) According to Lemma 5.2 with $f = \zeta = z$ and $f_M = z_{\varepsilon_M}$ (here we choose $\varepsilon_M = 1/M$) we find an approximation sequence $\{\zeta_{\varepsilon_k}\} \subseteq L^p([0,T];W^{1,p}_+(\Omega)) \cap L^\infty(\Omega_T)$ with $\varepsilon_k \searrow 0$ and the properties:

$$\zeta_{\varepsilon_k} \to z \text{ in } L^p([0,T];W^{1,p}(\Omega)) \text{ as } k \to \infty, \tag{62a}$$

$$0 \leq \zeta_{\varepsilon_k} \leq z_{\varepsilon_k} \text{ a.e. in } \Omega_T \text{ for all } k \in \mathbb{N}. \tag{62b}$$

We denote the subsequences also with $\{z_\varepsilon\}$ and $\{\zeta_\varepsilon\}$, respectively. The desired property $z_\varepsilon \to z$ in $L^p([0,T];W^{1,p}(\Omega))$ as $\varepsilon \searrow 0$ follows with the same estimate as in the proof of Lemma 5.11 by using the uniform convexity of $x \mapsto |x|^p$ and the integral inequality (24) with $\zeta := \zeta_\varepsilon - z_\varepsilon$ (note that $\langle r_\varepsilon, \zeta_\varepsilon - z_\varepsilon\rangle = 0$ holds by (27) and (62b)). Indeed, we obtain

$$\begin{aligned} &C_{\mathrm{ineq}}^{-1}\int_{\Omega_T} |\nabla z_\varepsilon - \nabla z|^p \,\mathrm{d}x\mathrm{d}t \\ &\leq \underbrace{\|\partial_z W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) - \alpha + \beta\partial_t z_\varepsilon\|_{L^2([0,T];L^1(\Omega))}}_{\text{bounded}}\underbrace{\|\zeta_\varepsilon - z_\varepsilon\|_{L^2([0,T];L^\infty(\Omega))}}_{\to 0} \\ &\quad + \underbrace{\|\nabla z_\varepsilon\|_{L^p(\Omega_T)}^{p-1}}_{\text{bounded}}\underbrace{\|\nabla\zeta_\varepsilon - \nabla z\|_{L^p(\Omega_T)}}_{\to 0} - \underbrace{\int_{\Omega_T} |\nabla z|^{p-2}\nabla z\cdot\nabla(z_\varepsilon - z)\,\mathrm{d}x\mathrm{d}t}_{\to 0} \end{aligned}$$

as $\varepsilon \searrow 0$ for a subsequence. Here, we have used $z_\varepsilon \to z$ and $\zeta_\varepsilon \to z$ in $L^2([0,T];L^\infty(\Omega))$ as $\varepsilon \searrow 0$ for a subsequence due to Lemma 5.15 and the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. ■

Corollary 5.17 *The following convergence properties are fulfilled:*

(i) $z_\varepsilon \to z$ *in* $L^p([0,T];W^{1,p}(\Omega))$,
$z_\varepsilon(t) \to z(t)$ *in* $W^{1,p}(\Omega)$ *a.e.* t,
$z_\varepsilon \to z$ *a.e. in* Ω_T *and*
$z_\varepsilon \rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$,

(ii) $c_\varepsilon \to c$ *in* $L^{2^\star}([0,T];H^1(\Omega))$,
$c_\varepsilon(t) \to c(t)$ *in* $H^1(\Omega)$ *a.e.* t *and*
$c_\varepsilon \to c$ *a.e. in* Ω_T,

(iii) $u_\varepsilon \to u$ *in* $L^2([0,T];H^1(\Omega;\mathbb{R}^n))$,
$u_\varepsilon(t) \to u(t)$ *in* $H^1(\Omega;\mathbb{R}^n)$ *a.e.* t *and*
$u_\varepsilon \to u$ *a.e. in* Ω_T,

(iv) $\mu_\varepsilon \rightharpoonup \mu$ *in* $L^2([0,T];H^1(\Omega))$,

(v) $\partial_c W_{\mathrm{ch}}(c_\varepsilon) \to \partial_c W_{\mathrm{ch}}(c)$ *in* $L^2(\Omega_T)$

as $\varepsilon \searrow 0$ *for a subsequence.*

Now we are well prepared to prove the main result of this work.

Proof of Theorem 4.6. We can pass to $\varepsilon \searrow 0$ in (22) and (23) by the already known convergence features (see Corollary 5.17) noticing that $\int_{\Omega_T} \varepsilon|\nabla u_\varepsilon|^2\nabla u_\varepsilon : \nabla\zeta \,\mathrm{d}x\mathrm{d}t$ and $\int_{\Omega_T} \varepsilon(\partial_t c_\varepsilon)\zeta \,\mathrm{d}x\mathrm{d}t$ converge to 0 as $\varepsilon \searrow 0$. We get

$$\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\zeta)\,\mathrm{d}x\mathrm{d}t = 0 \tag{63}$$

for all $\zeta \in L^4([0,T];W^{1,4}_\Gamma(\Omega;\mathbb{R}^n))$. A density argument shows that (63) also holds for all $\zeta \in L^2([0,T];H^1_\Gamma(\Omega;\mathbb{R}^n))$. Writing (21) in the form

$$\int_{\Omega_T} (c_\varepsilon - c^0)\partial_t\zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla\mu_\varepsilon\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t,$$

by only allowing test-functions $\zeta \in L^2([0,T];H^1(\Omega))$ with $\partial_t\zeta \in L^2(\Omega_T)$ and $\zeta(T)=0$, we can also pass to $\varepsilon \searrow 0$ by using Corollary 5.17.

To obtain a limit equation in (24) and (25), observe that

$$\begin{aligned} [\partial_z W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon)]^+ &\to [\partial_z W_{\mathrm{el}}(e(u),c,z)]^+ && \text{in } L^1(\Omega_T),\\ \chi_{\{z_\varepsilon=0\}} &\overset{\star}{\rightharpoonup} \chi, && \text{in } L^\infty(\Omega_T)\end{aligned}$$

for a subsequence $\varepsilon\searrow 0$ and an element $\chi\in L^\infty(\Omega_T)$. Setting $r := -\chi[\partial_z W_{\mathrm{el}}(e(u),c,z)]^+$ and keeping (27) into account, we find for all $\zeta\in L^\infty(\Omega_T)$:

$$\int_{\Omega_T} r_\varepsilon\zeta \,\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} r\zeta\,\mathrm{d}x\mathrm{d}t \tag{64}$$

for a subsequence $\varepsilon\searrow 0$. Thus, we can also pass to $\varepsilon\searrow 0$ for a subsequence in (24) by using Lebesgue's generalized convergence theorem, growth condition (A5), Corollary 5.17 and (64). Let $\xi\in L^\infty([0,T])$ with $\xi\geq 0$ a.e. on $[0,T]$ be a further test-function. Then, (25) and (27) imply

$$\begin{aligned} 0 \geq \int_0^T\left(\int_\Omega r_\varepsilon(t)(\zeta - z_\varepsilon(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t &= \int_{\Omega_T} r_\varepsilon(\zeta-z_\varepsilon)\xi\,\mathrm{d}x\mathrm{d}t\\ \to \int_{\Omega_T} r(\zeta - z)\xi\,\mathrm{d}x\mathrm{d}t &= \int_0^T\left(\int_\Omega r(t)(\zeta-z(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t.\end{aligned}$$

This shows $\int_\Omega r(t)(\zeta - z(t))\,\mathrm{d}x \leq 0$ for a.e. $t\in[0,T]$.

It remains to show that (26) also yields to a limit inequality. First observe that (26) implies:

$$\begin{aligned} &\mathcal{E}_\varepsilon(q_\varepsilon(t_2)) + \int_\Omega \alpha(z_\varepsilon(t_1)-z_\varepsilon(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z_\varepsilon|^2 + |\nabla\mu_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t - \mathcal{E}_\varepsilon(q_\varepsilon(t_1))\\ &\quad\leq \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon):e(\partial_t b)\,\mathrm{d}x\mathrm{d}t + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u_\varepsilon|^2\nabla u_\varepsilon : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t. \end{aligned}\tag{65}$$

To proceed, we need to prove $\varepsilon\int_\Omega|\nabla u_\varepsilon(t)|^4\,\mathrm{d}x \to 0$ as $\varepsilon\searrow 0$ for a.e. $t\in[0,T]$. Indeed, testing (23) with $\zeta := u_\varepsilon - b$ gives

$$\varepsilon\int_{\Omega_T}|\nabla u_\varepsilon|^4\,\mathrm{d}x\mathrm{d}t = \varepsilon\int_{\Omega_T}|\nabla u_\varepsilon|^2\nabla u_\varepsilon:\nabla b\,\mathrm{d}x\mathrm{d}t - \int_{\Omega_T}\partial_e W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon):e(u_\varepsilon - b)\,\mathrm{d}x\mathrm{d}t.$$

We immediately see that the first term converges to 0 as $\varepsilon\searrow 0$. The second term also converges to 0 because of $\int_{\Omega_T}\partial_e W_{\mathrm{el}}(e(u),c,z):e(u-b)\,\mathrm{d}x\mathrm{d}t = 0$ (equation (63)). This, together with Corollary 5.17, proves $\mathcal{E}_\varepsilon(q_\varepsilon(t))\to\mathcal{E}(q(t))$ for a.e. $t\in[0,T]$. In conclusion, we can pass to $\varepsilon\searrow 0$ in (65) for a.e. $0\leq t_1 < t_2\leq T$ by Corollary 5.17 together with Lebesgue's generalized convergence theorem, growth condition (A2), (11a) and (A6) as well as by a sequentially weakly lower semi-continuity argument for $\int_\Omega\beta|\partial_t z_\varepsilon|^2\,\mathrm{d}x$ and for $\int_\Omega|\nabla\mu_\varepsilon|^2\,\mathrm{d}x$. ■

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