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Parameterizations of sub-attractors in hyperbolic balance laws

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Abstract

This article investigates the properties of the global attractor of hyperbolic balance laws on the circle, given by :

$$u_t + f(u)_x = g(u). \tag{H}$$

The new tool of sub-attractors is introduced. They contain all solutions on the global attractor up to a given number of zeros. The article proves finite dimensionality of all sub-attractors, provides a full parameterization of all sub-attractors and derives a system of ODEs for the embedding parameters that describes the full PDE dynamics on the sub-attractor.

1 Introduction

Existence of global attractors has been proven for many partial differential equations, however in most cases few is known exceeding existence and bounds on the dimension of the attractor. An exception from this rule are hyperbolic balance laws with dissipative source term:

$$u_t(x,t) + [f(u(x,t))]_x = g(u(x,t)). \tag{H}$$

Despite the fact that the global attractor of (H) is infinite dimensional, a lot is known about the structure of the attractor and the connecting properties of rotating waves.

We consider equation (H) for $x \in S^1$ with $S^1 := \mathbb{R}/(2\pi\mathbb{Z})$ which is equivalent to imposing periodic boundary conditions on a domain of length 2π. By a scaling argument all results remain true for the situation of periodic boundary conditions in a domain of size L for any bounded and fixed $L \in \mathbb{R}$. u is a function mapping from $S^1 \times \mathbb{R} \to \mathbb{R}$. The non-linearities f, g map from $\mathbb{R} \to \mathbb{R}$. Furthermore we require the following hypotheses:

(H1) f is C^2 and strictly convex ($\exists \gamma \in \mathbb{R}\ s.t.\ f'' > \gamma > 0$).

(H2) g is C^1 and dissipative, i.e. there exists a constant $M > 0$ such that

$$ug(u) < M \tag{1}$$

for all $|u| > M$.

(H3) g has finitely many zeros at $u_1 < u_2 < \dots u_n$. All zeros are simple. (H2) implies that n is odd.

(H1) - (H3) guarantee the existence of a global attractor (see next section). One of the remaining questions regarding this attractor is its dynamic description. This article closes this gap for all solutions on the attractor with arbitrary but finite zero set. For our description we introduce sub-attractors for hyperbolic balance laws which will turn out to be of finite dimension. This approach allows us to overcome several difficulties arising from the infinite dimensionality of the full global attractor and from solutions with infinite (countable and uncountable) zero set.

In addition the sub-attractors show some striking similarities with the analogously defined sub-attractors of the parabolically regularized version of equation (H)

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u)$$

for small viscosity ε. This relation is explored thoroughly in [3].

The article is organized as follows. The second section reviews what is known about global attractors and the so called connection problem. It provides the necessary background on hyperbolic balance laws for this article. In the third section the notion of sub-attractors is introduced. Section four then formulates and proves the main result of the paper: a parameterization of all sub-attractors, their finite dimensionality and their dynamics. A section with examples follows. The paper concludes in section six with a brief discussion of the results.

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2 Global attractors and the connection problem

We will present the tools and methods used in the proofs of the article and then discuss key results concerning global attractors of scalar hyperbolic balance laws. The initial value problem (Cauchy problem) of (H) can be solved by the method of characteristics. The classical solution $u(x,t)$ to a initial condition $u(x,0) =: u_0(x)$ is given by:

$$u(\chi(t),t) := \underline{\mathrm{v}}(t)$$

where $\underline{v}, \chi$ are curves that solve the following ODE:

$$\begin{array}{rcl} \chi'(t) & = & f'(\underline{v}) \\ \underline{v}'(t) & = & g(\underline{v}) \\ \chi(0) & = & x_0 \\ \underline{v}(0) & = & u_0(x_0) \end{array} \tag{2}$$

for all $x_0 \in S^1$. Classical solutions in general only exist for finite time. To overcome this difficulty we work with weak solutions of equation (H).

In the weak framework solutions are in general not unique. To overcome this obstacle a additional entropy condition can be imposed, that singles out a unique weak solution. This idea derives from the physical entropy in thermodynamics. Entropy conditions for hyperbolic balance laws were first considered by Volpert [12] and Kruzhkov [8]. We follow their approach and define an *entropy* or *admissible solution* of the hyperbolic balance law (H) in the following way:

Definition 2.1 *We call $u \in BV([0,\infty) \times S^1, \mathbb{R})$ an entropy or admissible solution of equation* (H) *to the initial condition $u_0(x)$*

- *if $u(x,0) = u_0(x)$;*
- *if it solves equation* (H) *in the weak sense:*

$$\int_{S^1 \times \mathbb{R}^+} [u\varphi_t + f(u)\varphi_x - g(u)\varphi] dx dt = 0 \tag{3}$$

  for all $\varphi \in C_0^1(S^1 \times \mathbb{R}^+, \mathbb{R})$;
- *and if the entropy condition*

$$u(x+,t) \leq u(x-,t) \tag{4}$$

  holds for all $t > 0$.

Here $u(x+,t)$ defines the right hand, $u(x-,t)$ the left hand limit of u in x at time t and $BV([0,\infty) \times S^1, \mathbb{R})$ denotes the space of functions with bounded variation mapping from $[0,\infty) \times S^1$ to $\mathbb{R}$.

Volpert [12] and later, and for more general initial conditions (L^∞), Kruzhkov [8] were able to prove the following result on the existence of solutions:

Proposition 2.2 *If (H1) holds, then the Cauchy problem of equation* (H) *possesses a unique entropy solution u with the property $u : (0,\infty) \to L^1$ is continuous in time and $u(\cdot,t) \in BV(S^1)$ for all times $t > 0$.*

Equation (H) together with (4) therefore defines a semiflow on $BV(S^1,\mathbb{R})$. We denote that semiflow by

$$\begin{array}{rccl}\Phi: & BV\times\mathbb{R}^+ & \to & BV \\ & u_0,t & \mapsto & \Phi(u_0,t) := u(\cdot,t)\end{array}$$

where $u(\cdot,t)$ is the unique entropy solution to the initial condition u_0 at time t.

For the weak framework Dafermos introduced generalized characteristics [1]:

Definition 2.3 *A Lipschitz curve $x=\chi(t)$, defined on the interval $[a,b]\subset\mathbb{R}$ is called a generalized characteristic associated with the solution u of* (H) *if it satisfies the inequality*

$$\dot\chi \in [f'(u(\chi+,t)), f'(u(\chi-,t))] \tag{5}$$

for almost all $t\in[a,b]$.

Generalized characteristics coincide with classical characteristics $\chi(t)$ defined in (2), wherever the solution is differentiable. Filippov was able to show in [6] that there is at least one forward and one backward characteristic through any point $(x,t)\in S^1\times\mathbb{R}^+$. Equation (5) suggest that there is a lot of freedom in computing forward characteristics. That this is in fact not the case is shown by a proposition found in [6]:

Proposition 2.4 *Let $\chi:[a,b]\to\mathbb{R}$ be a generalized characteristic. Then the following holds for almost all $t\in[a,b]$:*

$$\dot\chi(t) = \begin{cases} f'(u(\chi(t)\pm,t)) & \text{if} \quad u(\chi(t)-,t) = u(\chi(t)+,t) \\ \frac{f(u(\chi(t)+,t))-f(u(\chi(t)-,t))}{u(\chi(t)+,t)-u(\chi(t)-,t)} & \text{if} \quad u(\chi(t)-,t) > u(\chi(t)+,t) \end{cases}.$$

Hence, $\dot\chi(t)$ is uniquely defined even at the position of shocks. If the solution $u(x,t)$ possesses a shock at position x_0 then the shock speed is given by the Rankine-Hugoniot condition for shock speeds

$$c_{shock} = \frac{f(u(x_0+))-f(u(x_0-))}{u(x_0+)-u(x_0-)}. \tag{6}$$

To distinguish between generalized characteristics and the characteristics of classical solutions the notion of genuine characteristics is important:

Definition 2.5 *A characteristic on the interval $[a,b]$ is called genuine, if*

$$u(\chi(t)-,t) = u(\chi(t)+,t)\, \textit{for almost all}\, t\in[a,b].$$

The set of backward characteristics through a point $(\bar x,\bar t)$ spans a funnel between the

- *minimal backward characteristic* $\chi^-(t;\bar x,\bar t)$ and the
- *maximal backward characteristic* $\chi^+(t;\bar x,\bar t)$.

The additional properties of characteristics that are of importance for us are summarized in the next propositions. For proofs we refer to Dafermos' article [1].

Proposition 2.6 *Let $(\bar{x},\bar{t}) \in S^1 \times \mathbb{R}$ be arbitrary. Then the minimal backward characteristic $\chi^-(t;\bar{x},\bar{t})$ and the maximal backward characteristic $\chi^+(t;\bar{x},\bar{t})$ are genuine.*

Proposition 2.7 *Genuine characteristics intersect only at their end points; backward characteristics do not intersect in particular.*

We direct our attention to the existence of global attractors for equation (H). Fan and Hale [5] have settled the existence question for hyperbolic balance laws:

Proposition 2.8 *Assume (H1), (H2) and (H3) hold. Then*

$$\mathscr{A} := \left\{ u_0 \in BV(S^1,\mathbb{R}) : \Phi(u_0,t) \, exists\ for\ all\, t \in \mathbb{R} \, and\ is\ bounded \right\} \tag{7}$$

is the global attractor of (H) *in $L^p(S^1)$, for any $p \in [1,\infty]$, i.e. it is invariant and attracts bounded sets in $L^p(S^1)$.*

This settles the existence of $\mathscr{A}$. We turn to the structure of the global attractor.

Several authors proved Poincaré Bendixson type results for the scalar balance laws. See for example Fan and Hale [4], Sinestrari [11] or Lyberopoulos [9]:

Proposition 2.9 *For $t \to \infty$ any solution of* (H) *tends either to a homogenous solution $u \equiv u_i$ for some $i \in \{1,...n\}$ or it converges to a rotating wave solution*

$$u(x,t) = v(x-ct)$$

where the wave-speed c can only take the values $c = f'(u_{2i})$ for $i \in \{1,\cdots,\frac{n-1}{2}\}$.

For global solutions a theorem similar to 2.9 holds true in backward time. This leads to a description of the global attractor $\mathscr{A}$ as the unification of the homogenous steady states, the frozen and rotating waves and heteroclinic connections between all these objects. A rotating wave is a solution of (H) of the form

$$u(x,t) = v(x-ct)$$

for a profile $v : S^1 \to \mathbb{R}$; c is called the wave speed. If $c = 0$ the wave is called frozen. For the definition of heteroclinic connections we define:

- $\mathscr{E}$ the set of homogenous equilibria of (H);
- $\mathscr{F}$ the set of frozen waves of (H);
- $\mathscr{R}$ the set of rotating waves of (H).

A heteroclinic connection is a solution $u(x,t)$ of (H) that has the property that

$$\begin{aligned} \lim_{t\to+\infty} \quad & u(x,t) \in \mathscr{E} \cup \mathscr{F} \cup \mathscr{R} \\ \lim_{t\to-\infty} \quad & u(x,t) \in \mathscr{E} \cup \mathscr{F} \cup \mathscr{R}. \end{aligned} \tag{8}$$

If we denote the set of heteroclinic connections with $\mathscr{H}$, then the global attractor $\mathscr{A}$ of (H) can be described as

$$\mathscr{A} = \mathscr{E} \cup \mathscr{F} \cup \mathscr{R} \cup \mathscr{H}. \tag{9}$$

Fan and Hale showed in Theorem 3.7 in [5] that if two rotating waves are connected by a heteroclinic orbit, then the waves must have the same velocity. Moreover, if a heteroclinic orbit connects a homogenous equilibrium $u \equiv u_j$ and a rotating wave with speed $f'(u_{2i})$, then $|j-2i| = 1$. It is a consequence of Proposition 1.5 in [5] that all global solutions $u(x,t)$ satisfy

$$u_{2i-1} \leq u(x,t) \leq u_{2i+1}$$

for some $i \in \{1, \dots, \frac{n-1}{2}\}$. This implies that the homogeneous solutions $u \equiv u_{2i\pm1}$ divide the global attractor into separated pieces, connected only at the homogeneous solutions. Hence we can treat all these pieces separately and restrict our analysis to the case where g possesses only three zeros. Without loss of generality we can rewrite assumption (H3) to

$(\widetilde{H}3)$ g has three simple zeros at $u_- < u_0 < u_+$ and $u_0 = 0$.

In our case this implies

$$c = f'(0)$$

due to proposition 2.9. The hyperbolic balance law (H) is homogeneous in x and we can perform a co-ordinate transformation

$$x \mapsto x - f'(0)t$$

which automatically freezes all rotating waves. Hence we can assume without loss of generality

(H4) $f'(0) = 0$.

This assumption fixes our co-ordinate system where all rotating waves have wave speed $c = 0$, hence we have $\mathscr{R} = \emptyset$.

In [10] Sinestrari proved that for any possible wave speed $c = f'(u_0)$ and for any closed set $Z \subset S^1$ there exists a unique rotating wave u_Z with the property

$$Z = \{y \in S^1 : u_Z(y) = 0\}.$$

The uniqueness automatically proves that these are all waves and hence $\mathscr{F}$ is fully described. For the connection question we introduce the map $\mathscr{Z}(\cdot)$ that assigns each function $u : S^1 \to \mathbb{R}$ its zero set:

$$\mathscr{Z}(u(\cdot)) := \{x \in S^1; u(x) = 0\}. \tag{10}$$

In addition we define the zero-number

$$z(u) := \sharp \mathscr{Z}(u),$$

if $\mathscr{Z}(u)$ is uncountable we define $z(u) := \infty$.

Härterich was able to prove the following three theorems A, B and C in [7] which settle the connection question:

Theorem 2.10 (Theorem A) *For any rotating wave $u_{-\infty}$ there exist heteroclinic orbits which connect $u_{-\infty}$ to the homogenous states $u \equiv u_-$ and $u \equiv u_+$.*

Theorem 2.11 (Theorem B) *For any rotating wave $u_{+\infty}$ there exist (several) heteroclinic orbits that connect the spatially homogenous solution $u \equiv 0$ to $u_{+\infty}$.*

Theorem 2.12 (Theorem C) *Suppose that for two rotating waves $u_{-\infty}$ and $u_{+\infty}$ the condition $\mathscr{Z}(u_\infty) \subset \mathscr{Z}(u_{-\infty})$ holds. Then there is a heteroclinic solution that approaches $u_{\pm\infty}$ as the time t tends to $\pm\infty$.*

3 Sub attractors $\mathscr{A}_n$ of order n

One of the main obstacles in the description of the global attractor of (H) is the huge number of stationary solutions due to Sinestrary's result. This results in an infinite dimensionality of the attractor. To overcome this obstacle we introduce the notion of sub-attractors in this section. The underlying idea is to only consider solutions with bounded zero number and to define the sub-attractors in a way such that they remain invariant as sets under the semi flow of the equation. This allows us to get rid of all solutions with infinite or uncountable zero set.

Definition 3.1 *Let $n = 2\alpha$ for $\alpha \in \mathbb{N}$. Then we define:*

- $\mathscr{E}_n := \{u \equiv u_+, u \equiv u_-\}$;
- $\mathscr{F}_n := \{u \in \mathscr{F}; z(u) \leq \alpha\}$;
- $\mathscr{H}_n := \{u \in \mathscr{H}; \lim_{t\to\pm\infty} \in \mathscr{E}_n \cup \mathscr{F}_n\}$.

Then we define the sub-attractor of order n of the hyperbolic balance law (H) *by*

$$\mathscr{A}_n := \mathscr{E}_n \cup \mathscr{F}_n \cup \mathscr{H}_n. \tag{11}$$

We first prove the following lemma:

Lemma 3.2 *(i) Let $\mathscr{A}_n$ and $\mathscr{A}_m$ be defined as above for some $m,n \in \mathbb{N}$, then*

$$\mathscr{A}_n \subset \mathscr{A}_m \Leftrightarrow n < m$$

(ii) We have the following alternative description for $\mathscr{A}_n$:

$$\mathscr{A}_n = \overline{\{W^u(\mathscr{F}_n)\}}$$

(iii) $\mathscr{A}_n$ is invariant under the semi-flow Φ generated by equation (H)*.*

Proof. (i) is obvious by the definition of $\mathscr{A}_n$, (ii) directly follows through

$$\begin{aligned} \mathscr{A}_n &= \bigcup_{\beta=1}^{\alpha} \left\{W^u(u^0); u^0 \in \mathscr{F}, z(u^0) = \beta\right\} \cup \mathscr{F}_n \cup \mathscr{E}_n \\ &= \bigcup_{\beta=1}^{\alpha} \overline{\{W^u(u^0); u^0 \in \mathscr{F}, z(u^0) = \beta\}} \\ &= \overline{\{W^u(\mathscr{F}_n)\}}, \end{aligned} \tag{12}$$

(iii) is a direct consequence of the invariance of $\mathscr{E}_n$ and $\mathscr{F}_n$.

□

At a first glance it seems strange to denote the sub-attractors by $\mathscr{A}_n$ and not $\mathscr{A}_\alpha$. However one of the results in the following section will be $\dim \mathscr{A}_n = n$, which justifies the notation.

4 Parameterizations for $\mathscr{A}_n$

We now turn to the question of parameterizing the sub-atractors $\mathscr{A}_n$. We follow an idea introduced by Härterich in [7]. In Section 4 Härterich presents an example of one heteroclinic connection between two defined states for Burgers equation ($f(u) = \frac{u^2}{2}$). The key idea is that the connection consits of stationary profiles which are separated by shocks. The solutions on the connections only change if at all close to the shocks. This idea guides the path towards the parameterization of the sub attractors by the position of the stationary profiles on the one hand and the positions of the shocks on the other. A key step in the proof is to show that this approach covers all heteroclinic connections.

We begin with the definition of the stationary profiles.
Let $\phi(x)$ be the unique solution of the following equation:

$$\begin{aligned} v_x &= \frac{g(v)}{f'(v)} \\ v(0) &= 0. \end{aligned} \tag{13}$$

Then $\phi(x)$ exists for all $x \in \mathbb{R}$ and

$$\lim_{x\to-\infty}\phi(x) = u_- \qquad \lim_{x\to\infty}\phi(x) = u_+.$$

Let $n = 2\alpha$ for some $\alpha \in \mathbb{N}$ be given. Then we choose a sequence of α zeros $0 < x_1 < x_2 < \cdots < x_\alpha < 2\pi$. Due to Sinestrari [10] there exists a unique frozen wave $v(x)$ with

$$\mathscr{Z}(v) = \{x_1, \cdots, x_\alpha\}.$$

Without loss of generality we assume $x_1 = 0$. All other cases can be generated by a shift. Note that for every solution of equation (H) it is true that between two zeros there must be a shock and between two shocks with sign changing left- and right-hand states there must be a zero, see [2]. It is in particular true for v_α. Hence there is a unique sequence of shocks $\hat{y}_1, \ldots, \hat{y}_\alpha$ with

$$0 = x_1 < \hat{y}_1 < x_2 < \hat{y}_2 < \cdots < \hat{y}_{\alpha-1} < x_\alpha < \hat{y}_\alpha < 2\pi$$

such that v is given by

$$v = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, \hat{y}_i] \\ \phi(x - x_{i+1}) & \text{for} \quad x \in [\hat{y}_i, x_{i+1}] \end{cases} . \tag{14}$$

For convenience let us define the notation

$$\{\mathbf{x}_\alpha\} := \{x_1, \cdots, x_\alpha\}$$

and denote the unique frozen wave with zero set $\mathbf{x}_\alpha$ by $v_{\mathbf{x}_\alpha}$.

We now define the solution $u_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}}$ with α shocks located between the zeros $\{x_1, \cdots, x_\alpha\}$ that consists piecewise of shifted copies of $\phi(x)$. In general $u_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}}$ is not stationary. Let $0 \le x_1 \le y_1 < x_2 \le \cdots < x_\alpha \le y_\alpha < 2\pi$ then we define

$$u_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, y_i] \\ \phi(x - x_{i+1}) & \text{for} \quad x \in (y_i, x_{i+1}] \end{cases} \tag{15}$$

for $i = 1, \ldots, \alpha$.

Finally let us define the general solution $\tilde{u}_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}}$ with α or less shocks that consists piecewise of shifted copies of $\phi(x)$ where all shocks have sign changing left and right states.
Let $0 \le \tilde{y}_1 \le \tilde{y}_2 \le \cdots \le \tilde{y}_\alpha < 2\pi$ then we define if $\tilde{y}_i < \tilde{y}_{i+1}$

$$\tilde{u}_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, \tilde{y}_i] \\ \phi(x - x_{i+1}) & \text{for} \quad x \in (\tilde{y}_i, x_{i+1}] \end{cases} , \tag{16}$$

and if $\tilde{y}_i = \tilde{y}_{i+1} = \cdots = \tilde{y}_{i+m}$

$$\tilde{u}_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, \tilde{y}_i] \\ \phi(x - x_{i+m+1}) & \text{for} \quad x \in (\tilde{y}_{i+m}, x_{i+m+1}] \end{cases} . \tag{17}$$

Then the two sets $A_{\{\mathbf{x}_\alpha\}}$ and $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ of all these solutions with fixed $\{x_1,\cdots,x_\alpha\}=\{\mathbf{x}_\alpha\}$ are given by

$$A_{\{\mathbf{x}_\alpha\}} := \{u_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}}; 0 \le x_1 \le y_1 < x_2 \le \cdots < x_\alpha \le y_\alpha < 2\pi\} \tag{18}$$

and

$$\tilde{A}_{\{\mathbf{x}_\alpha\}} := \{\tilde{u}_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}}; 0 \le y_1 \le \cdots \le y_\alpha < 2\pi\}. \tag{19}$$

Then we have the following lemma:

Lemma 4.1 *Let $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ be given with $\mathbf{x}_\beta \subset \mathbf{x}_\alpha$. Then we have*

(i) $v_{\mathbf{x}_\alpha} \in A_{\{\mathbf{x}_\alpha\}} \subset \tilde{A}_{\{\mathbf{x}_\alpha\}}$

(ii) $\tilde{A}_{\{\mathbf{x}_\beta\}} \subset \tilde{A}_{\{\mathbf{x}_\alpha\}}$

(iii) There is no $u \in \tilde{A}_{\{\mathbf{x}_\alpha\}}$ with more than α shocks.

Proof. We only proof (iii). We first argue for two zeros: Assume that the solution has a zero located at $x_1=0$ and another zero at x_2. We explicitly construct the set of all admissible solutions $u(x)$ that consist piecewise of shifted copies of $\phi(x-x_i-2\pi k_j)$ for some $k_j \in \mathbb{Z}$ and $i \in \{1,2\}$; with the additional property that $u(x_1=0)=0$ and show that in fact $j=2$ necessarily.

Let us denote all shock positions by $0 < y_1 < \cdots < y_j \le 2\pi$. Due to the fact that between zeros there has to be shock we obtain $j \ge 2$. Let us define the sequence of stationary profiles

$$..., \phi(x+2\pi), \phi(x+x_2), \phi(x), \phi(x-x_2), \phi(x-2\pi), \phi(x-x_2-2\pi), ... \tag{20}$$

Because $u(0)=0$ we start at $x=0$ with $u(x)=\phi(x)$ locally. At each of the shocks y_i the solution jumps to a profile to the right in the above sequence due to the entropy condition (4). However we have to end with $u(x)=\phi(x-2\pi)$ locally at x close to 2π. Hence $j \le 2$ and therefore $j=2$.

We now state the main theorem:

Theorem 4.2 *Let $n=2\alpha$ and $\alpha \in \mathbb{N}$. Then the following is true:*

a) The local unstable manifold $W^u_{loc}(v_{\{\mathbf{x}_\alpha\}})$ of $v_{\{\mathbf{x}_\alpha\}}$ is given by $A_{\{\mathbf{x}_\alpha\}}$ defined in equation (18)*:*

$$W^u_{loc}(v_{\{\mathbf{x}_\alpha\}}) = A_{\{\mathbf{x}_\alpha\}} \tag{21}$$

where $v_{\{\mathbf{x}_\alpha\}}$ is the unique frozen wave of equation (H) *with zeros at $x_1,\ldots,x_\alpha$.*

b) The global unstable manifold $W^u(v_{\{\mathbf{x}_\alpha\}})$ of $v_{\{\mathbf{x}_\alpha\}}$ is then given by

$$W^u(v_{\{\mathbf{x}_\alpha\}}) = \left\{\Phi(u,t); u \in A_{\{\mathbf{x}_\alpha\}}, t \in \mathbb{R}^+\right\} \tag{22}$$

where Φ denotes the semiflow in $BV(S^1,\mathbb{R})$ generated by equation (H)*.*

c) The dynamics on $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ *defined in equation* (19) *can be described by the following equation for the shock parameters* y_j*:*

$$\dot{y}_j(t) = \frac{f(\phi(y_j - x_j)) - f(\phi(y_j - x_{j+1})}{\phi(y_j - x_j) - \phi(y_j - x_{j+1})}. \tag{23}$$

d) The dimension of the sub-attractors $\mathscr{A}_n$ *of order* n *is given by*

$$\dim \mathscr{A}_n = n.$$

e) Let v_1 *be a frozen wave of equation* (H) *with*

$$z(v_1) = 1.$$

Then there exist unique heteroclinic connections $\tilde{u}(\cdot,t)$ *and* $\hat{u}(\cdot,t)$ *with*

$$\lim_{t\to-\infty} \tilde{u}(\cdot,t) = \lim_{t\to-\infty} \hat{u}(\cdot,t) = v_1$$
$$\lim_{t\to\infty} \tilde{u}(\cdot,t) \equiv u_+$$
$$\lim_{t\to\infty} \hat{u}(\cdot,t) \equiv u_-.$$

f) Let $0 \le x_1 < x_2 < \cdots < x_\alpha < 2\pi$ *and let* v_1 *and* v_2 *be frozen waves of equation* (H) *with the property*

$$\mathscr{Z}(v_1) = \{x_1, \ldots, x_\alpha\}$$

and

$$\mathscr{Z}(v_2) = \{x_{k_1}, \ldots, x_{k_\beta}\}$$

with $k_{i+1} - k_i \in \{0,1\}$ *for all* $1 \le i \le \beta - 1$ *where we have set* $\beta + 1 = \alpha$*. Then there exists an up to shifts in time unique heteroclinic connection* $u(x,t)$ *with the property*

$$\lim_{t\to-\infty} u(\cdot,t) = v_1(\cdot)$$
$$\lim_{t\to\infty} u(\cdot,t) = v_2(\cdot).$$

The proof of the theorem will use the overflowing invariance of the sets $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ and $A_{\{\mathbf{x}_\alpha\}}$, which we prove first:

Lemma 4.3 *Let* $\{\mathbf{x}_\alpha\} := \{x_1, \ldots, x_\alpha\}$ *with* $0 \le x_1 < \cdots < x_\alpha < 2\pi$ *be given.*

(i) The set $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ *is overflowing invariant under the semiflow of equation* (H)*. Overflowing means that if a solution* $u \in \tilde{A}_{\{\mathbf{x}_\alpha\}}$ *leaves* $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ *at time* $t = \tilde{t}$*, hence we have* $\Phi(u,t) \in \tilde{A}_{\{\mathbf{x}_\alpha\}}$ *for* $t < \tilde{t}$ *and* $\Phi(u,t) \notin \tilde{A}_{\{\mathbf{x}_\alpha\}}$ *for* $t > \tilde{t}$ *then either* $y_1 = x_1$ *or* $y_\alpha = 2\pi$ *in* $u_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}} := \Phi(u,\tilde{t})$*.*

(ii) The set $A_{\{\mathbf{x}_\alpha\}}$ *is overflowing invariant under the semiflow of equation (H).*

Proof. Let $u(x,0) \in \tilde{A}_{\{\mathbf{x}_\alpha\}}$ such that $u(x,0) = u_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}}$ with $y_1 > 0$ and $y_\alpha < 2\pi$.

Local forward invariance of $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ follows from the fact that the profiles ϕ that define $u_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}}$ are stationary. Hence $u(x,t)$ is stationary except near the points y_j, and so we only have to prove invariance locally at the shock points. We only investigate the shock located at y_1, the argument works for any other shock equivalently.

Let therefore $u(x,0)$ be given by

$$u(x,0) = \begin{cases} \phi(x) & \text{for} \quad y_1 - \delta x \leq y_1 \\ \phi(x-x_2) & \text{for} \quad y_1 + \delta x > y_1 \end{cases} \tag{24}$$

for some $\delta > 0$. At y_1 there is a unique forward characteristic $\chi(t)$ on which the shock evolves. The other characteristics in a neighbourhood of y_1 necessarily point towards $\chi(t)$ for $t > 0$. Hence for $x \notin [y_1 - \delta, y_1 + \delta]$ the solution $u(x,t)$ is stationary and given by $\phi(x)$ for $x \leq \chi(t)$ and by $\phi(x-x_2)$ for $x > \chi(t)$. See the Figure 4 for illustration. $\chi(t)$ is uniquely determined by the differential equation:

$$\dot{\chi}(t) = \frac{f(\phi(\chi(t))) - f(\phi((\chi(t)-x_2)))}{\phi(\chi(t)) - \phi((\chi(t)-x_2))} \tag{25}$$
$$\chi(0) = y_1.$$

The slope of $\chi(t)$ is bounded from above and hence, if t is sufficiently small we have obtained local forward invariance of the shock.

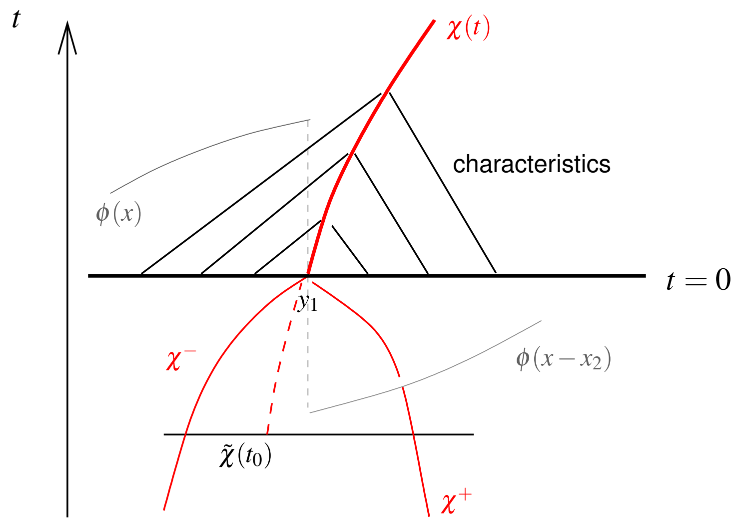

Figure 1: Illustration for the proof of Lemma 4.3.

For the backward invariance we observe that for $t < 0$ a minimal characteristic $\chi^-(t)$ and a maximal backward characteristic $\chi^+(t)$ emanate from y_1. For the area between χ^- and χ^+ there are in principle many possibilities to define the solution such that we obtain $u(x,t)$ for $t \geq 0$ (there is no backward uniqueness!). For backward invariance it is however enough if we can find one $u(x,t) \in \tilde{A}_{\{\mathbf{x}_\alpha\}}$ for $t < 0$.

Let now $t_0 < 0$ be sufficiently small. Then we define

$$\tilde{u}(x,t_0) := \begin{cases} \phi(x) & \text{for} \quad x \in [\chi^-(t_0), \tilde{\chi}(t_0)] \\ \phi(x - x_2) & \text{for} \quad x \in (\tilde{\chi}(t_0), \chi^+(t_0)] \end{cases}$$

for some $\tilde{\chi}(t_0) \in [\chi^-(t_0), \chi^+(t_0)]$. Local backward invariance follows if we can prove that there is one $\tilde{\chi}(t_0)$ such that if we solve equation 25 with initial condition $\tilde{\chi}(t_0)$ we obtain

$$\tilde{\chi}(0) = \chi(0) = y_1.$$

If we assume $\tilde{\chi}(t_0) = \chi^-(t_0)$, then monotonicity of ϕ and convexity of f imply $\tilde{\chi}(0) < y_1$; if we assume on the other hand $\tilde{\chi}(t_0) = \chi^+(t_0)$, then the same argument yields $\tilde{\chi}(0) > y_1$. The intermediate value theorem yields the existence of a $\tilde{y} \in (\chi^-, \chi^+)$ such that $\tilde{\chi}(t)$ with $\tilde{\chi}(t_0) := \tilde{y}$ has the desired property. Due to the convexity of f and the monotonicity of ϕ the $\tilde{y}$ is even unique. Hence backward invariance follows.

Although $\tilde{y}$ is unique the backward solution is not unique in $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ in general, due to the possibility of shock splittings in backward time direction. However if we assume that no shock splitting occurs we even obtain uniqueness of the backward solution in $\tilde{A}_{\{\mathbf{x}_\alpha\}}$.

For the overflowing property we assume $u(x,0) \in \tilde{A}_{\{\mathbf{x}_\alpha\}}$ with $y_1 = 0$. Then the forward characteristic $\chi(t)$ in $x_1 = y_1 = 0$ is given by the equation

$$\chi(t) = \frac{-f(\phi(y_\alpha - 2\pi)}{-\phi(y_\alpha - 2\pi)} < 0$$

for $t \in [0, \delta_1)$, δ_1 positive and small and $\chi(0) = 2\pi$. Thus, after identification of 0 and 2π we obtain that the solution is locally given by

$$\begin{array}{cll} \phi(x - x_2) & \text{for} & 0 < x < y_2 \\ \phi(x - x_2 - 2\pi) & \text{for} & \chi(t) < x < 2\pi \\ \phi(x - 2\pi) & \text{for} & y_\alpha < x < \chi(t). \end{array}$$

This proves the overflowing property of $\tilde{A}_{\{\mathbf{x}_\alpha\}}$, because the above solution is not in $\tilde{A}_{\{\mathbf{x}_\alpha\}}$ due to the fact that there are only $\alpha - 1$ zeros but α shocks, one of which has the same sign at the left and right states. This proves (i).

Due to the fact that $A_{\{\mathbf{x}_\alpha\}} \subset \tilde{A}_{\{\mathbf{x}_\alpha\}}$ we conclude invariance of $A_{\{\mathbf{x}_\alpha\}}$ by virtue of the same construction. The overflowing property works just as for $\tilde{A}_{\{\mathbf{x}_\alpha\}}$, here the boundary is given by the condition $y_j = x_j$ or $y_j = x_{j+1}$ for some $j \in \{1, \ldots, \alpha\}$. □

Corollary 4.4 *For every $u(x,0) \in A_{\{\mathbf{x}_\alpha\}}$ there is a unique backward orbit in $A_{\{\mathbf{x}_\alpha\}}$.*

Proof. From the proof of the previous lemma we deduce that it is sufficient to show that shocks in u cannot split in backward time. By construction any solution in $A_{\{\mathbf{x}_\alpha\}}$ has exactly α zeros and α shocks due to Lemma 4.1 (iii) shock splitting cannot occur. □

Proof of Theorem 4.2:
We have already proven part **c)**. Equation (25) yields exactly equation (23) if we replace $\chi(t)\pm$ by the y_j. Hence we can integrate solutions along the (invariant) manifold $A_{\{\mathbf{x}_\alpha\}}$ by using equation (23) for every y_j $(1 \le j \le n)$. Note that y_j and y_{j+1} can meet. Thus the y_j are only lipschitz in t not C^1.

For **a)** we prove that all solutions $u(\cdot,0) \in A_{\{\mathbf{x}_\alpha\}}$ converge in backward time to $v_{\{\mathbf{x}_\alpha\}}$, this shows

$$A_{\{\mathbf{x}_\alpha\}} \subset W^u(v_{\{\mathbf{x}_\alpha\}}). \tag{26}$$

Then we show maximality of $A_{\{\mathbf{x}_\alpha\}}$ by proving that all solutions $u(\cdot,t)$ converging to $v_{\{\mathbf{x}_\alpha\}}$ in backward time are contained in $A_{\{\mathbf{x}_\alpha\}}$ for sufficiently small $t<0$ which proves

$$W^u_{loc}(v_{\{\mathbf{x}_\alpha\}}) \subset A_{\{\mathbf{x}_\alpha\}}. \tag{27}$$

This yields **a)** for appropriately chosen local neighborhood in $W^u(v_{\{\mathbf{x}_\alpha\}})$.

The first part is a consequence of Lemma 4.3 and the convexity of f: Let $u(\cdot,0) \in A_{\{\mathbf{x}_\alpha\}}$. Because of the overflowing invariance and backward uniqueness (Corollary 4.4) we conclude

$$u(\cdot,t) \in A_{\{\mathbf{x}_\alpha\}}$$

for all $t<0$. In addition

$$\lim_{t\to-\infty} u(\cdot,t) \in \mathscr{F} \cup \mathscr{E}$$

due to proposition 2.9 $v_{\{\mathbf{x}_\alpha\}}$ is the only frozen wave in $A_{\{\mathbf{x}_\alpha\}}$ and hence

$$A_{\{\mathbf{x}_\alpha\}} \cap \mathscr{E} \cup \mathscr{F} = \{v_{\{\mathbf{x}_\alpha\}}\}.$$

This yields equation (26).
For the other direction we argue indirectly. Assume there exists $\tilde{u}(x,t)$ with

$$\lim_{t\to-\infty} \tilde{u}(x,t) = v_{\{\mathbf{x}_\alpha\}} \text{ and} \tag{28}$$

$$\tilde{u}(x,t) \notin A_{\{\mathbf{x}_\alpha\}} \text{ for all } t<0 \tag{29}$$

then for sufficiently small $\tilde{t}<0$ there exists $\tilde{x} \in S^1$ such that for all $1 \le j \le \alpha+1$

$$\tilde{u}(\tilde{x},\tilde{t}) \neq \phi(\tilde{x}-x_j), \tag{30}$$

where we have set $x_{\alpha+1} = 2\pi$.
Due to the fact that $\tilde{u}$ connects to $v_{\{\mathbf{x}_\alpha\}}$ we can always choose $(\tilde{x},\tilde{t})$ such that $\tilde{u}(\tilde{x},\tilde{t})$ is smaller than the maximum and larger than the minimum of the stationary solution with one zero.
We now construct a contradiction by proving that $\lim_{t\to-\infty}\tilde{u}(\cdot,t)$ has a zero x_s not coinciding with one of the $x_1,\dots,x_\alpha$ and therefore $\tilde{u}$ cannot converge to $v_{\{\mathbf{x}_\alpha\}}$ in backward time. We use a stationary solution u_s coinciding with $\tilde{u}(\cdot,0)$ at $\tilde{x}$ to calculate explicitly the backward characteristic of $\tilde{u}$ emanating from $(\tilde{x},\tilde{t})$.

From equation (30) we deduce, that there is a stationary solution $u_s \in \mathscr{F}$ with the following properties:

$$\begin{aligned} u_s(\tilde{x}) &= \tilde{u}(\tilde{x},\tilde{t}) \\ \mathscr{Z}(u_s) &= \{x_s\} \end{aligned}$$

where $x_s \notin \{x_1,\ldots,x_\alpha\}$. We investigate the (genuine!) backward characteristic $(\chi(t),\underline{\mathrm{v}}(t))$ with

$$\begin{aligned} \chi(\tilde{t}) &= \tilde{x} \\ \underline{\mathrm{v}}(\tilde{t}) &= u_s(\tilde{x},\tilde{t}) = \tilde{u}(\tilde{x},\tilde{t}) \end{aligned}$$

Because u_s is stationary, the characteristic has the property that

$$\lim_{t\to-\infty}\chi(t) = x_s$$

and

$$\lim_{t\to-\infty}\underline{\mathrm{v}}(t) = 0$$

From this we deduce

$$\Rightarrow \lim_{t\to-\infty} u_s(\chi(t),t) = u_s(x_s,\cdot) = 0,$$

this implies

$$\lim_{t\to-\infty} u(x_s,t) = \lim_{t\to-\infty} u_s(x_s,t) = 0$$

hence

$$\lim_{t\to-\infty}\tilde{u}(\chi(t),t) = v_{\{x_\alpha\}}(x_s) = 0.$$

This contradicts $x_s \notin \{x_1,\ldots,x_\alpha\}$ and maximality of $A_{\{\mathbf{x}_\alpha\}}$ is proved.

b) follows from the fact that due to unique forward solvability we obtain the global unstable manifold by using the semiflow to forward–solve the local unstable manifold. $A_{\{\mathbf{x}_\alpha\}} \subset \mathscr{A}$ ensures boundedness of the forward iteration, hence equation (22) follows.

For **d)** we use the fact that

$$\dim\left(W^u_{loc}(v_{\{\mathbf{x}_\alpha\}})\right) = \dim\left(W^u(v_{\{\mathbf{x}_\alpha\}})\right) \tag{31}$$

which is true due to forward uniqueness of solutions.

The sub-attractor of order $n = 2$ consists by definition of all frozen waves with one zero and heteroclinic connections from these waves to $u_\pm$. In other words

$$\mathscr{A}_2 = W^u(\mathscr{F}_2) \cup \mathscr{E}_2$$

For fixed x_1 we have

$$\dim\left(W^u(v_{\{x_1\}})\right) = \dim\left(A_{\{x_1\}}\right) = 1.$$

From the uniqueness of frozen waves with given $x_1 \in S^1$ we deduce

$$\dim \mathscr{A}_2 = 2$$

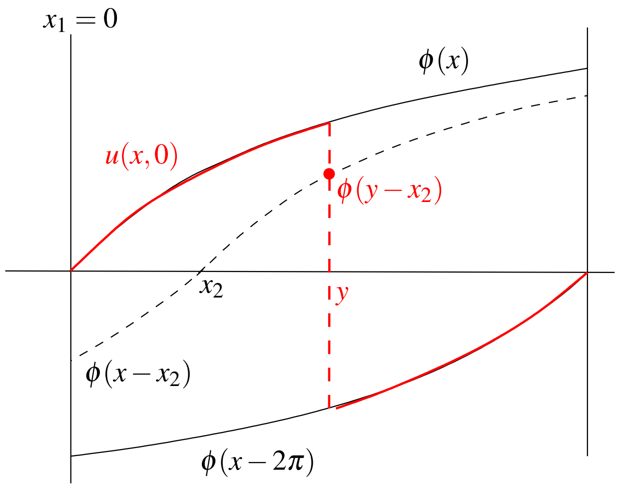

Figure 2: Unique shock-splitting of one shock in backward time in $A_{\{x_1,x_2\}}$.

For $n = 2\alpha > 2$ we use

$$\mathscr{A}_n = \{W^u(u); u \in \mathscr{F}_n\} \cup \mathscr{E}_n. \tag{32}$$

First we prove

$$\dim \{W^u(u); u \in \mathscr{F}, z(u) = \alpha\} = 2\alpha = n.$$

For each fixed set of zeros $\{0 \leq x_1 < \cdots < x_\alpha < 2\pi\}$ we have by part a) of this theorem

$$\dim \left(W^u_{loc}(v_{\{\mathbf{x}_\alpha\}})\right) = \dim \left(A_{\{\mathbf{x}_\alpha\}}\right) = \alpha.$$

Moreover, all frozen waves v with zero-number $z(v) \leq \alpha$ can be parameterized by $(x_1, \ldots x_\alpha) \in \left(S^1\right)^\alpha = \mathbb{T}^\alpha$, hence

$$\dim \mathscr{F}_n = \dim \mathbb{T}^\alpha = \alpha.$$

Putting everything together we obtain by using equation (32)

$$\dim \mathscr{A}_n = \dim W^u_{loc}(\{\mathscr{F}_n\}) = \dim W^u_{loc}(v_{\{\mathbf{x}_\alpha\}}) + \dim \mathbb{T}^\alpha = \alpha + \alpha = n.$$

For **e)** we count dimensions to obtain uniqueness. For $\alpha = 1$ the unstable manifold of v_1 is one dimensional, thus the connection must be unique.

For **f)** we argue in the following way: the condition $k_{i+1} - k_i \in \{0, 1\}$ implies that at most every second zero can vanish, hence we can reduce the proof to the situation where

$$\mathscr{Z}(v_1) = \{0, x_2\}$$

and

$$\mathscr{Z}(v_2) = \{0\}.$$

Let us denote the unique shock position of v_2 by y and the two unique shock positions of v_1 by y_1 and y_2.

It is a consequence of **c)** that in the class of solutions $A_{\{x_1,x_2\}}$ all stationary shocks are unstable. In order to obtain the solution v_2 with only one shock, the two shocks emanating form y_1 and y_2 consequently have to meet at the position y in such a way that the resulting shock is stationary.

We define $t = 0$ as the time at which the two shocks collide. So the question of uniqueness of heteroclinic connections reduces to the question of uniqueness of shock collisions in $A_{\{x_1,x_2\}}$, or in negative time direction the questions of uniqueness of the splitting of shocks at a given position.

Let $u(x,t)$ be the solution where two shocks meet at time $t = 0$ at position $x = y$ then the lower state of the left shock and the upper state of the right shock have to coincide. By construction of $\tilde{A}_{\{x_1,x_2\}}$ it is given by $\phi(y-x_2)$:

$$\lim_{x\searrow y}\lim_{t\nearrow 0} u(x,t) = \lim_{x\nearrow y}\lim_{t\nearrow 0} u(x,t) \overset{!}{=} \phi(y-x_2).$$

See Figure 2 for illustration.

Hence uniqueness of the splitting follows by uniqueness of backward solutions in the case of $u \in A_{\{x_1,x_2\}}$ with two shocks proved in Corollary 4.4 . This proves **e)** and the Theorem is proven.

□

Note that for the situation of Theorem 4.2 e) we can explicitly parameterize the whole heteroclinic connection from v_1 to $u_\pm$. The stationary solution v_1 with $\mathscr{Z}(v_1) = \{x_1\}$ has one unique shock at position y_1. Then using Theorem 4.2 b) and c) we can parameterize the whole connection manifold $W^u(v_1)$ as follows: for any $k \in \mathbb{Z}$ and any $y_1 \in [2k\pi, 2(k+1)\pi)$ we define

$$u^*_{\{x_1,y_1\}}(x) := \begin{cases} \phi(x-x_1+2k\pi) & \text{for} \quad 0 \le x \le y_1 - 2k\pi \\ \phi(x-x_1+2(k-1)\pi) & \text{for} \quad y_1 - 2k\pi < x < 2\pi \end{cases}. \tag{33}$$

Then $W^u(v_1)$ is given by

$$W^u(v_1) := \{u^*_{\{x_1,y_1\}} \in BV(s^1,\mathbb{R}); y_1 \in \mathbb{R}\}. \tag{34}$$

The next section will present a geometric representation of $\mathscr{A}_2 = \overline{W^u(\mathscr{F}_2)}$

Corollary 4.5 *Again let $\alpha \in \mathbb{N}$ and $n = 2\alpha$. Then the set of heteroclinic connections between two frozen waves with zero-number $z \le \alpha$ is completely contained in*

$$\tilde{\mathbf{A}}_{\mathbf{n}} := \left\{\tilde{A}_{\{\mathbf{x}_\alpha\}}; \mathbf{x}_\alpha \in \mathbb{T}^\alpha \; and \; 0 \le x_1 < \cdots < x_\alpha < 2\pi\right\}. \tag{35}$$

Proof Let v_1, v_2 be two frozen waves with

$$\mathscr{Z}(v_1) = \{x_1,\ldots,x_\beta\}$$
$$\mathscr{Z}(v_2) \subset \mathscr{Z}(v_1)$$

for some given $0 \le x_1 < \cdots x_\beta < 2\pi$ and $\beta \le \alpha$. Let $u(x,t)$ denote a heteroclinic connection between v_1 and v_2. Then

$$u(\cdot,t) \in A_{\{\mathbf{x}_\beta\}} \subset \tilde{A}_{\{\mathbf{x}_\beta\}} \subset \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$$

for some $x_{\beta+1},\ldots,x_\alpha$ and t sufficiently small.

Now assume $u(\cdot,\tilde{t}) \notin \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$ for some $\tilde{t} \in \mathbb{R}$. Then we conclude

$$u(\cdot,t) \notin \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$$

for all $t > \tilde{t}$ due to the overflowing property of $\tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$.

This contradicts

$$\lim_{t\to\infty} u(\cdot,t) = v_2$$

because

$$v_2 \in \mathring{\tilde{A}}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}.$$

Here $\mathring{\tilde{A}}$ denotes the interior of $\tilde{A}$ in the topology of the manifold $\tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$.

$\square$

In the next section a geometric representation of the sub-attractors of order $n = 2$ and $n = 4$ will be given.

5 Examples

5.1 The sub-attractor $\mathscr{A}_2$

According to the definition the sub-attractor $\mathscr{A}_2$ consists of all frozen waves with zero-number $z = 1$, the two stable homogeneous equilibria $u \equiv u_\pm$ and all heteroclinic connections between these objects. The frozen waves can be represented as an S^1.

Due to Theorem A (2.10) all frozen waves are connected to $u(x) \equiv u_\pm$. Theorem C (2.10) states that these are all heteroclinic connections in $\mathscr{A}_2$ and Theorem 4.2 e) yields uniqueness of these heteroclinics. Equation (33) provides together with equation (34) an explicit parameterization of these connections. Hence we can define an explicit embedding

$$\begin{aligned} \Sigma_2 : S^1 \times \mathbb{R} &\to BV(S^1,\mathbb{R}) \\ (x_1,y_1) &\mapsto \Sigma_2(x_1,y_1) := u_{\{x_1,y_1\}} \end{aligned}$$

where $u^*_{\{x_1,y_1\}}$ is defined in equation (33). The flow on $graph(\Sigma_2)$ can be computed explicitly and is given by equation (23).

By a stereographic projection we can map $graph(\Sigma_2)$ onto the surface of a ball, thus obtaining a representation of $\mathscr{A}_2$ as an S^2, shown in Figure 3.

The three diagrams on the right in Figure 3 show schematically how the shape of these solutions evolves on the S^2 along a heteroclinic connection.

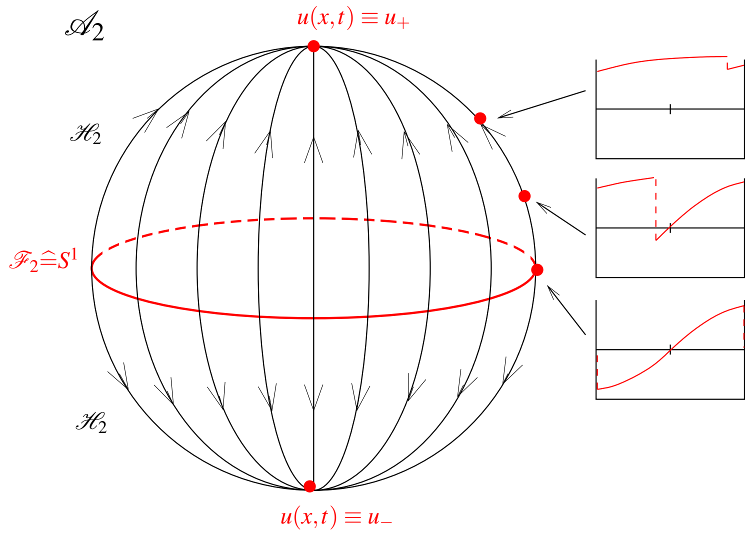

Figure 3: Geometric representation of the sub attractor $\mathscr{A}_2$.

5.2 The sub-attractor $\mathscr{A}_4$

Following the definition of $\mathscr{A}_4 := \mathscr{E}_4 \cup \mathscr{F}_4 \cup \mathscr{H}_4$ we will first classify all homogeneous equilibria and frozen waves. Due to Sinestrati the frozen waves can be uniquely parameterized by the position of their zeros x_1, x_2 hence form a two-torus:

$$\mathscr{F}_4 = \mathbb{T}^2 := S^1 \times S^1,$$

and again $\mathscr{E}_4 = \{u_-, u_+\}$.

Each element of this torus has a heteroclinic connection to the homogeneous equilibria $u \equiv u_\pm$. This can be depicted by a spindle with a quadratic horizontal section and $u_\pm$ located at the top and bottom. See panel a) in Figure 4. The heteroclinic connections are drawn with arrows. The edges of the quadratic horizontal section have to be identified in order to obtain the torus. The sub-attractor $\mathscr{A}_2$ is contained in this picture as well and is depicted in green. Figure 3 is obtained after identification of the two opposite corners on the torus $\mathscr{F}_4$. The spindle is completely filled with heteroclinics starting in $\mathscr{F}_4$ and ending at $u \equiv u_-$ or $u \equiv u_+$ respectively.

The more interesting part of $\mathscr{A}_4$ is the part of the attractor that consists of all heteroclinic connections between $\mathscr{F}_4$ and $\mathscr{F}_2$. Theorem C (2.12) yields that every frozen wave $\tilde{u}$ with zero-number $z(\tilde{u}) = 2$ is connected to two waves $\tilde{u}_a, \tilde{u}_b$ with zero-numbers $z(\tilde{u}_{a,b}) = 1$, Theorem 4.2 f) yields uniqueness of these connections.

Hence every point on the torus of frozen waves $\mathscr{F}_4 \backslash \mathscr{F}_2$ has two heteroclinic connections to two points on the diagonal curve on that torus representing $\mathscr{F}_2$. This is shown in panel b) in Figure 4, where we have parameterized the torus by the zeros (x_1, x_2) given as the horizontal and vertical axes. Some heteroclinics are shown as arrows for illustration. The lines are vertical if the zero x_1 persists, horizontal if the zero x_2 persists. Two heteroclinics emerge from every point.

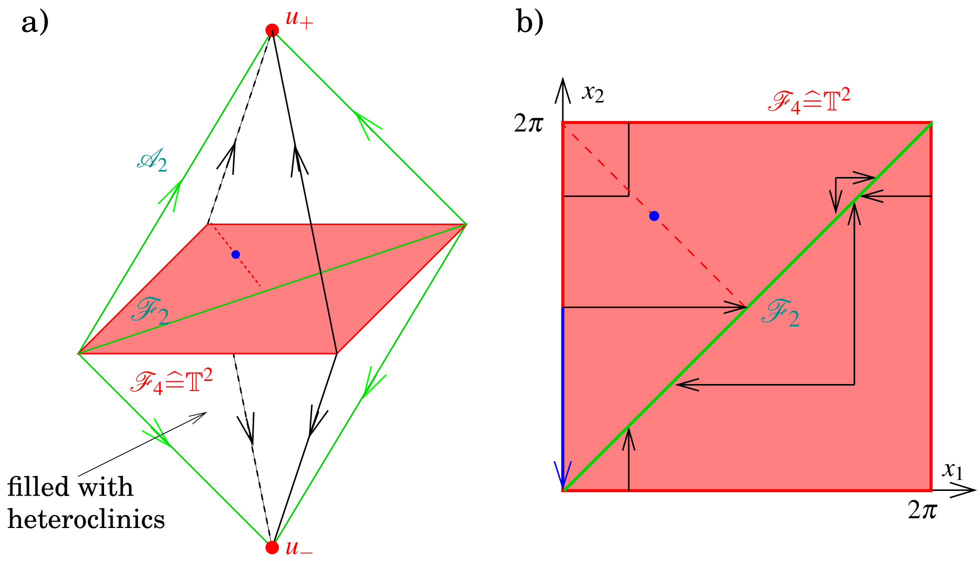

Figure 4: Heteroclinic connections in $\mathscr{A}_4$ with targets $u \equiv u_\pm$.

Equations (16) and (17) provide an explicit parameterization of these connections.

To show the complete connection picture it is convenient to use another representation that divides out the S^1 symmetry. This representation is shown in Figure 5.

To understand the Figure it is best to start with the vertical vertical line. This line represents $\mathscr{F}_4/S^1$: the manifold that contains all frozen waves with zero-number $z = 2$ after having divided out the S^1 symmetry. The center point on this line is the π-periodic frozen wave with equidistant zeros.

The coordinates on the vertical manifold are given by the distance between the two zeros x_1 and x_2. On the bottom the distance is zero, in the middle at the dot it is π and then it goes to zero again towards the top. x_1 and x_2 change in such a way that the two shocks always remain in the same position (for Burgers equation this means due to symmetries that $\frac{x_1+x_2}{2} = \pi$). Three of the solution profiles in Figure 5 show how the solutions evolve along the vertical manifold. This manifold is also included in panels a) and b) of Figure 4 as a dashed line with a dot on the torus $\mathbb{T}^2$.

Each of the frozen waves has two connections to frozen waves with $z = 1$, one connection where the zero at x_1 persists and one where the one at x_2 persists. These are represented by the black arrows connecting to the circle representing $\mathscr{F}_2$ ($=$ frozen waves with one zero). To the left x_1 persists and to the right x_2 persist, this induces coordinates on the circle of frozen waves with zero-number $z = 1$. The eight remaining solution profiles in Figure 5 indicate how solutions evolve along the circle. A clockwise rotation along the S^1 in the figure corresponds to a shift of the solution to the right.

Now we are ready to include the S^1 symmetry in the figure that was divided out before. To do this we just have to rotate the whole figure along a circle in transverse direction attached to the dot representing the wave with two equidistant zeros. We obtain a filled torus where we have a figure similar to the one in Figure 5 in every slice.

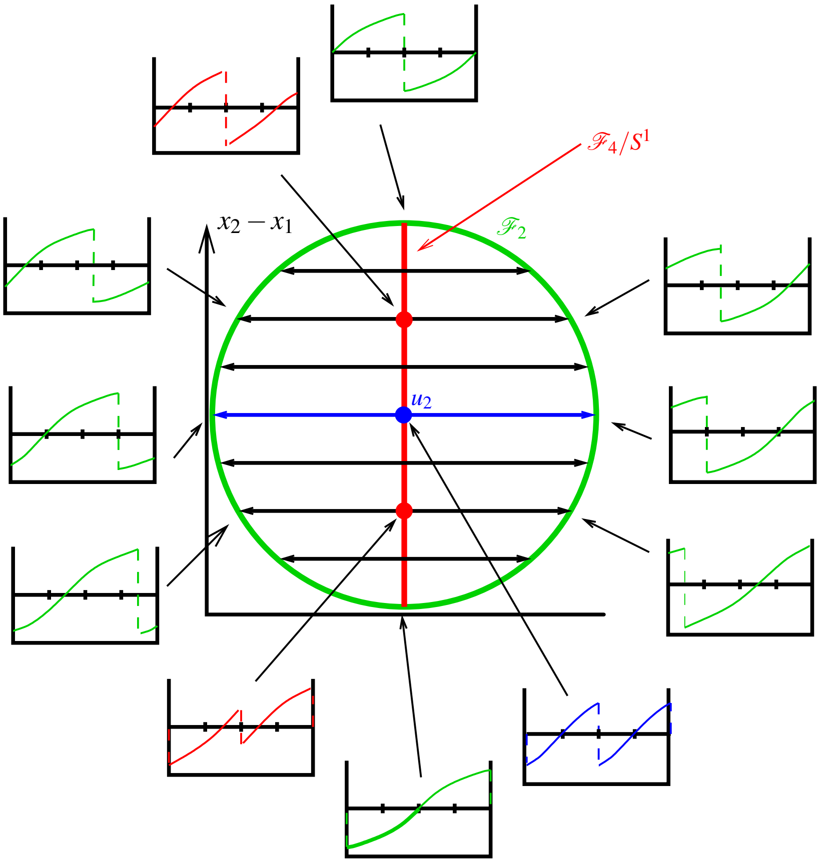

Figure 5: Heteroclinic connections in $\mathscr{A}_4$ from frozen waves with zero-number $z = 2$ to waves with zero-number $z = 1$. The S^1 symmetry is divided out.

Inside the torus the vertical line and the heteroclinic connections rotate once around the center point with higher symmetry and therefore form a spiral. Figure 6 shows a geometric representation of this. We have plotted half of the torus. The thick halfcircular line corresponds to the frozen waves in $\mathscr{A}_4$ with higher symmetry (equidistant zeros). The heteroclinics are shown only in the beginning and the end. They rotate with the vertical manifold. There is a colour gradient included to illustrate the rotation of the heteroclinics. Note that the there is no rotation on the torus' surface.
To obtain the full picture we have to identify all points on the surface of the torus with the S^1 labeled with $\mathscr{F}_2$, hence retract the torus surface to the S^1 without rotating it!

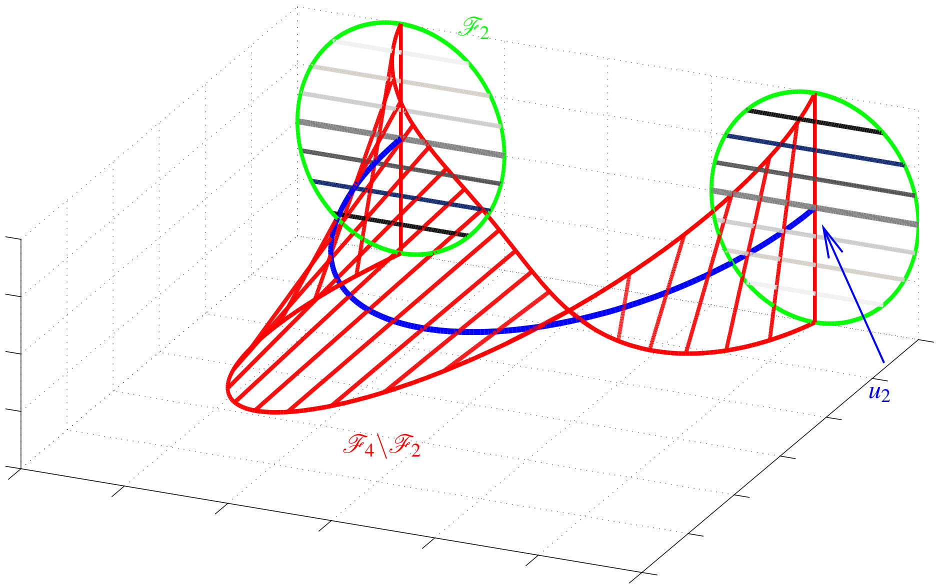

Figure 6: Torus representing $W^u(\mathscr{F}_4) \,\overline{\pitchfork}\, W^s(\mathscr{F}_2)$.

6 Conclusions and Discussion

Building on earlier results of Fan and Hale [5], Sinestrari [10] and Härterich [7] and others this article closes on of the last remaining gaps in the full dynamic description of the global attractor of hyperbolic balance laws. The introduction of finite dimensional sub-attractors in Section 3 allowed us to overcome difficulties coming from the infinite dimensional nature of the global attractor.

Theorem 4.2 and Corollary 4.5 provide explicit parameterizations of all finite dimensional sub-attractors of the global attractor and allow a geometric interpretation of the results as given in the examples section.

A remaining question concerns the uniqueness of heteroclinic connections in situations where the assumption in Theorem 4.2 f) is violated. It is unclear whether convexity of f and monotonicity of the profiles ϕ is enough to guarantee uniqueness of heteroclinic connections in case more than two shocks meet to form a stationary shock.

In addition the question remains how to describe the remaining part of the global attractor. In principle I believe that heteroclinic connections emanating from waves with infinite zero set can be treated analogously. A uniform explicit parameterization covering the whole attractor seems to be difficult, due to the infinite dimensional nature of the global attractor.

Moreover I believe that the introduced sub-attractors are a suitable tool to investigate the relation between global solutions of the hyperbolic balance law with global solutions of its parabolically regularized version the viscous balance laws given by

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u). \tag{36}$$

This relation is explored in [3]. Due to the non persistence result of the author in [3] the relation of global solutions of the hyperbolic and parabolic equation is more complicated than one might expect, however the sub-attractors help facilitating the description of that relation.

Finally the explicit results on the structure of the connections between waves with finite zero number in this article open an alternative door for the description of heteroclinic connections in the parabolically regularized equation (36) other than by proving invariant manifold results by spectral methods, which despite serious efforts by many people in the last decades is still an unsolved problem.

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