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New connections between finite element formulations of the Navier-Stokes equations

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Abstract

We show the velocity solutions to the convective, skew-symmetric, and rotational Galerkin finite element formulations of the Navier-Stokes equations are identical if Scott-Vogelius elements are used, and thus all three formulations will the same pointwise divergence free solution velocity. A connection is then established between the formulations for grad-div stabilized Taylor-Hood elements: under mild restrictions, the formulations' velocity solutions converge to each other (and to the Scott-Vogelius solution) as the stabilization parameter tends to infinity. Thus the benefits of using Scott-Vogelius elements can be obtained with the less expensive Taylor-Hood elements, and moreover the benefits of all the formulations can be retained if the rotational formulation is used. Numerical examples are provided that confirm the theory.

1 Introduction

We consider finite element formulations of the Navier-Stokes equations (NSE), and for simplicity we restrict to the equilibrium case with homogeneous Dirichlet boundary conditions on a polygonal or polyhedral domain Ω,

$$\begin{aligned} u\cdot\nabla u+\nabla p-\nu\Delta u &= f \text{ in } \Omega, &(1.1)\\ \nabla\cdot u &= 0 \text{ in } \Omega, &(1.2)\\ u &= 0 \text{ on } \partial\Omega, &(1.3)\end{aligned}$$

where u is velocity, and p is the zero mean pressure. The results herein are easily extendable to the time-dependent case and more general boundary conditions.

We prove that when computing solutions to the Galerkin finite element method for (1.1)-(1.3) using Scott-Vogelius (SV) elements (see below for a detailed description), the velocity solutions obtained by convective, skew-symmetric, and rotational (1.8)-(1.9) formulations are identical. For general LBB stable elements, these formulations can give very different answers [10], and thus this result shows that if one formulation is attractive for a particular problem (e.g. type of flow, use of particular preconditioners, conservation properties, etc.), then using SV elements can alleviate adverse consequences due to the chosen formulation. This equivalence result is then used to prove our second result, which establishes a new connection between the formulations for grad-div stabilized Taylor-Hood (TH) elements. We prove that under mild restrictions, as the stabilization parameter tends to infinity, the velocity solutions of the formulations all converge to *the* SV solution, and thus to each other.

The difference in accuracy between different discrete NSE formulations is well documented, and is not restricted to finite element formulations. In [5, 6], Horiuti and Itami found using finite difference methods that discretization errors in near wall regions can be worse with rotational form, causing overall accuracy to suffer. Zang, in [16], found in spectral methods, rotational form usage can lead to greater aliasing errors, although for (P_k, P_{k-2}) spectral elements, although Wilhelm and Kleiser found that instabilities can occur if the rotational form is *not* used. In the finite element context, even though matrix properties of the rotational form can allow for better preconditioning of the resulting linear systems [1, 2], its use of Bernoulli pressure can result in larger error in both pressure and velocity solutions [10]. The results herein show how, in a particular (although not restrictive) setting, the best features of each of the formulations can be retained, even using the less expensive TH elements.

Denote the $L^2(\Omega)$ inner product by $(\cdot,\cdot)$, and with LBB stable finite dimensional spaces $(X_h, Q_h) \subset (H_0^1(\Omega), L_0^2(\Omega))$. Listed below are the three common finite element formulations of (1.1)-(1.3): Find $(u_h, p_h) \in (X_h, Q_h)$ satisfying $\forall (v_h, q_h) \in (X_h, Q_h)$,

Convective Form:

$$(u_h \cdot \nabla u_h, v_h) - (p_h, \nabla \cdot v_h) + \nu(\nabla u_h, \nabla v_h) = (f, v_h) \tag{1.4}$$
$$(\nabla \cdot u_h, q_h) = 0 \tag{1.5}$$

Skew-symmetric Form:

$$(u_h \cdot \nabla u_h, v_h) + \frac{1}{2}((\nabla \cdot u_h)u_h, v_h) - (p_h, \nabla \cdot v_h) + \nu(\nabla u_h, \nabla v_h) = (f, v_h) \tag{1.6}$$
$$(\nabla \cdot u_h, q_h) = 0 \tag{1.7}$$

Rotational Form:

$$((\nabla \times u_h) \times u_h, v_h) - (p_h, \nabla \cdot v_h) + \nu(\nabla u_h, \nabla v_h) = (f, v_h) \tag{1.8}$$
$$(\nabla \cdot u_h, q_h) = 0 \tag{1.9}$$

Each of the formulations arises from a consistent variational formulation of (1.1)-(1.2), followed by the usual Galerkin finite element discretization procedure. Skew symmetry adds an additional term to the convective formulation to enforce unconditional stability, and rotational form arises from the identity

$$u \cdot \nabla u = (\nabla \times u) \times u + \frac{1}{2}\nabla |u|^2 \tag{1.10}$$

being applied at the continuous level, then forming the Bernoulli pressure $P = p + \frac{1}{2}|u|^2$. Thus the pressure p_h in (1.9) approximates P, but in (1.7) and (1.5) it approximates p.

1.1 Scott-Vogelius elements

Scott-Vogelius elements have recently become interesting for approximating NSE solutions due to results of Zhang [17] and Burman and Linke [3]. Zhang showed that the

$((P_k)^d, P_{k-1}^{disc})$ element pair is LBB stable with optimal approximation properties provided $k \geq d$ and the mesh is constructed as a barycenter refinement of a quasi-uniform mesh (a mild restriction). With SV elements, the weak enforcement of mass conservation provides *pointwise* mass conservation. This is because $\nabla \cdot X_h \subset Q_h$, and thus one can choose the special test function $q_h = \nabla \cdot u_h$ (in (1.5),(1.7), or (1.9), resulting in

$$\|\nabla \cdot u_h\| = 0 \quad \Longrightarrow \quad \nabla \cdot u_h = 0.$$

In general, e.g for TH elements, such a choice of test functions is not possible. We note a drawback of these elements is the use of a discontinuous pressure space, which leads to significantly larger linear systems.

Work in [3, 12, 4] has shown that using the SV element pair can provide excellent results for approximating NSE solutions, as well as providing pointwise mass conservation. Moreover, it can be shown that with SV elements, the velocity error is independent of the pressure error, whereas for most element choices the velocity error can be scaled (at least) by $Re * pressureError$ [9, 10].

Since only in the case of $k \geq d$ and the mesh condition are SV elements known to be LBB stable with optimal approximation properties, all references to SV elements will assume these restrictions are met.

2 A equivalence theorem for the formulations with SV elements

We now prove the equivalence of the velocities of the three formulations.

Theorem 2.1. *Suppose SV elements are used to solve the problems* (1.4)-(1.5)*,* (1.6)-(1.7)*, and* (1.8)-(1.9)*, and the data* (Ω, f, ν) *is such that the formulations have unique solutions. Then the velocity solutions of these problems are identical.*

Remark 2.1. *Using the theorem, it is trivial to show the pressures are the same for the convective and skew-symmetric formulations. However, this is not true for the rotational form pressure since it approximates the Bernoulli pressure.*

Proof. The equivalence of the convective and skew-symmetric forms is trivial, since we have that $\nabla \cdot u_h = 0$ and thus $\frac{1}{2}((\nabla \cdot u_h)u_h, v_h) = 0$ in (1.6). Since this makes the two formulations identical, solution uniqueness of each formulation guarantees that computed velocity solutions of these two formulations must be identical as well.

For the equivalence of the rotational form's velocity, since (X_h, Q_h) is LBB stable, we pass to the equivalent V_h formulations, where the velocity is searched for in

$$V_h := \{v_h \in X_h \ : \ (\nabla \cdot v_h, q_h) = 0 \ \forall q_h \in Q_h\},$$

and the test functions are restricted to be in V_h. Thus consider

$$V_h \text{ convective form:} \qquad (u_h \cdot \nabla u_h, v_h) + \nu(\nabla u_h, \nabla v_h) = (f, v_h) \ \ \forall v_h \in V_h, \qquad (2.1)$$

$$V_h \text{ rotational form:} \qquad ((\nabla \times u_h) \times u_h, v_h) + \nu(\nabla u_h, \nabla v_h) = (f, v_h) \ \ \forall v_h \in V_h. \qquad (2.2)$$

Using the vector identity (1.10), these two formulations are identical since V_h contains only pointwise div-free functions and $v_h \in H_0^1$:

$$((\nabla \times u_h) \times u_h, v_h) = (u_h \cdot \nabla u_h, v_h) - \frac{1}{2}(\nabla |u_h|^2, v_h)$$
$$= (u_h \cdot \nabla u_h, v_h) + \frac{1}{2}(|u_h|^2, \nabla \cdot v_h) - \frac{1}{2}\int_{\partial\Omega} |u_h|^2 (v_h \cdot n) ds$$
$$= (u_h \cdot \nabla u_h, v_h). \tag{2.3}$$

Therefore we have that these formulations are equivalent, and since solutions exist uniquely, their respective velocity solutions must be identical. □

2.1 Numerical verification

We present here a numerical example to verify Theorem 2.1. The test problem we consider has solution

$$u = \begin{pmatrix} 2x^2(x-1)^2 y(2y-1)(y-1) \\ -2x(x-1)(2x-1)y^2(y-1)^2 \end{pmatrix}, \quad p = y,$$

and was taken from [8]. A plot of the true solution in given in figure 1.

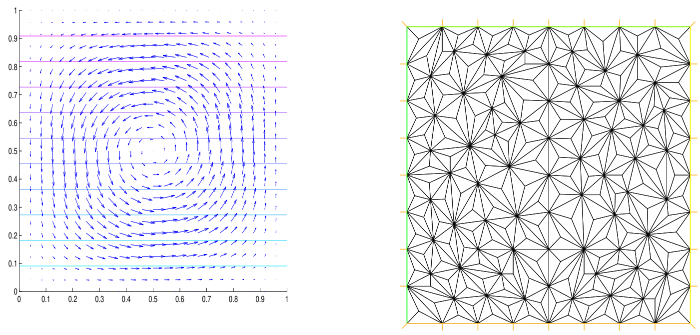

Figure 1: LEFT: Velocity vector field and pressure contours of true solution for the numerical experiments; RIGHT: The mesh used for first numerical experiment

Using $\nu = 0.01$, we calculate f from the NSE and the selected solution. Then using the domain $\Omega = (0,1)^2$, use it to solve each of the three formulations (1.4)-(1.5), (1.6)-(1.7), and (1.8)-(1.9), we use $((P_2)^2, P_1^{disc})$ elements and a Newton method. The mesh, shown in Figure 1, is a barycenter refinement of a quasi-uniform mesh, which provided 3,300 total degrees of freedom. Denoting the computed velocity solutions by u_h^{con}, u_h^{ss}, and u_h^{rot}, respectively, we found the SV velocity solutions are identical up to machine precision, as predicted by Theorem 2.1:

$$\|u_h^{ss} - u_h^{con}\| = 1.063E-16, \qquad \|u_h^{rot} - u_h^{con}\| = 8.403E-17$$

3 A connection between formulations using Taylor-Hood elements

Recent work in [10, 14] has shown that when using TH elements, accuracy and physical fidelity can be increased through the use of grad-div stabilization, that is, adding the term $\gamma(\nabla \cdot u_h, \nabla \cdot v_h)$ to (1.4), (1.6), (1.8). This term, which can be derived for use in the discrete scheme by adding $0 = -\gamma\nabla(\nabla \cdot u)$ to the continuous NSE, and also arises in the SUPG formulations, penalizes for lack of mass conservation and relaxes the effect of the pressure error on the velocity error. Since rotational form uses Bernoulli pressure, which is typically much more complex than usual pressure, this stabilization seems natural and has proven quite effective when the rotational formulation is used [10]. Although its effect is less dramatic when used with the other formulations, grad-div stabilization can still have a significant positive effect [13, 7].

Related work in [4] has shown that on a barycentric mesh and for $k \geq d$, the grad-div stabilized skew-symmetric form solutions corresponding to a sequence of grad-div stabilization parameters $\gamma_i \to \infty$, converge to the convective form SV solution, when the SV solution is unique; otherwise convergence is for a subsequence to a SV solution. This leads us to the following result.

Theorem 3.1. *Suppose the mesh is a barycenter refinement of a quasi-uniform mesh, and grad-div stabilized TH elements with $k \geq dim$ are used to solve* (1.6)-(1.7)*, and* (1.8)-(1.9)*. If the solution to* (1.4)-(1.5) *using SV elements (with the same k) is unique, then for any sequence of stabilization parameters $\gamma_i \to \infty$, the corresponding sequences of grad-div stabilized TH velocity solutions of the skew-symmetric and rotational formulations converge to each other and to the SV solution.*

Remark 3.1. *For TH elements, the skew-symmetric formulation should be used in place of the convective formulation for unconditionally stable, as the convective formulation is (only) conditionally stable, e.g. for h small enough [11]. Thus for h small enough, Theorem 3.1 can be extended to include the convective formulation. Moreover, following [4], in this case one can show the sequence of modified pressures $p_h^{\gamma_i} := p_h + \gamma_i(\nabla \cdot u_h)$ of the the convective and skew symmetric formulations converge to each other and to the SV pressure of the convective formulation.*

Proof. For either the rotational or skew symmetric formulations with grad-div stabilization and TH elements, taking $v_h = u_h$ gives the a priori bound

$$\nu\|\nabla u_h\|^2 + \gamma_i\|\nabla \cdot u_h\|^2 \leq C(data), \tag{3.1}$$

for any γ_i chosen from the sequence. Thus both sequences $\{u_h^{ss,i}\}_i$ and $\{u_h^{rot,i}\}_i$ are bounded and therefore have convergent subsequences, and moreover their limits must be divergence free.

Denote by U_h^{rot} the limit of the convergent subsequence $\{\{u_h^{rot}\}_i\}_j$, and consider the V_h formulation satisfied by an element of the subsequence $u_h^{rot,j}$,

$$\nu(\nabla u_h^{rot,j}, \nabla v_h) + \gamma_j(\nabla \cdot u_h^{rot,j}, \nabla \cdot v_h) + ((\nabla \times u_h^{rot,j}) \times u_h^{rot,j}, v_h) = (f, v_h) \ \forall v_h \in V_h^{TH}. \tag{3.2}$$

Since $V_h^{SV} \subset V_h^{TH}$, we can restrict $v_h \in V_h^{SV}$ in (3.2). This vanishes the grad-div term, and then using the uniform boundedness and finite dimensionality of $u_h^{rot,j}$, we can take the limit as $\gamma_j \to \infty$ to get

$$\nu(\nabla U_h^{rot}, \nabla v_h) + (U_h^{rot} \cdot \nabla U_h^{rot}, v_h) = (f, v_h) \; \forall v_h \in V_h^{SV}. \tag{3.3}$$

Now by (3.1), $\nabla \cdot U_h^{rot} = 0$. Since $U_h^{rot} \in X_h^{TH} = X_h^{SV}$, and is divergence free, $U_h^{rot} \in V_h^{SV}$, and by (3.3) also satisfies the V_h convective formulation of the NSE with SV elements. Thus U_h^{rot} is a rotational form SV solution, and by assumption and Theorem 2.1, U_h^{rot} is *the* SV solution. Therefore all convergent subsequences of $\{u_h^{rot,i}\}_i$ must converge to the SV solution, and hence so must the entire sequence.

From [4] and Theorem 2.1, we have that the entire sequence of skew-symmetric solutions also converges to the SV solution. Thus both the skew symmetric and rotational forms' sequences of velocity solutions converge to the same limit, and thus the theorem is proven. $\square$

3.1 Numerical verification

To experimentally verify Theorem 3.1, we use the same test problem as in Section 2. For $\gamma_i = \{0,\ 1,\ 10,\ 100,\ 1{,}000,\ 10{,}000\}$, we find solutions to all three grad-div stabilized TH formulations, then calculate the differences between their velocity solutions. Results are given in Table 1, and confirm the theory.

γ	$\|u_{h,\gamma}^{ss} - u_{h,\gamma}^{con}\|$	$\|u_{h,\gamma}^{ss} - u_{h,\gamma}^{rot}\|$	$\|u_{SV} - u_{h,\gamma}^{ss}\|$	$\|\nabla \cdot u_{h,\gamma}^{ss}\|$
0	1.3569E-06	2.7352E-04	2.8780E-04	1.5530E-02
1	1.5149E-08	6.9127E-06	6.9911E-06	2.1946E-04
10	1.5472E-09	7.2828E-07	7.3620E-07	2.2585E-05
100	1.5506E-10	7.3227E-08	7.4022E-08	2.2654E-06
1,000	1.5477E-11	7.3267E-09	7.4062E-09	2.2661E-07
10,000	6.3895E-12	7.3306E-10	7.4106E-10	2.2661E-08

Table 1: Convergence of the grad-div stabilized TH solutions for the different formulations toward each other, toward the SV solution, and toward a divergence free velocity field as $\gamma \to \infty$.

4 Further extensions

It is important to note that the results herein can easily be extended to the time-dependent case. With the restriction that the time-step be small enough to ensure solution uniqueness, analogous theorems can be proven in a similar manner to those herein for the steady case. Moreover, these results are also extendable to time-dependent equations that include NSE-type systems where velocity-pressure is approximated by LBB stable element pairs, e.g. the α models of turbulence, and MHD.

One important extension is for general TH elements, i.e. not on a barycenter refined mesh or $k = 2$ in 3D. In this case, similar proofs hold, with the limit solution living in the divergence free subspace of the velocity spaces. However, one cannot expect this limit solution to be accurate (see [4, 13]), as except in special cases, large grad-div stabilization parameters can over-stabilize.

The results herein can also be extended to other element choices. For example, the two-dimensional $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ element is a common element choice that has found success in two-dimensional fluid flow computations. This element pair is LBB stable, but does not exactly conserve mass; it only provides local mass conservation, although for many problems this is much better than only global mass conservation [15]. If grad-div stabilization is added, then the limiting results for the velocity solution would be the solution from using (P_2, P_1^{disc}) Scott-Vogelius elements. For, the cubic volume bubble function can be represented on the reference element as $(xy - x^2y - xy^2)(u_0, v_0)^T$ and its divergence is in the linear pressure space only, if $u_0 = 0$ and $v_0 = 0$. Therefore, a large grad-div stabilization eliminates all the degrees of freedom corresponding to the element bubble functions. Thus the different formulations of the nonlinear convection term would have to converge to each other. Similar arguments can also be made for other element choices such as (P_2, P_0), (P_3, P_1), and so forth, since in the limit they converge, e.g., to the Scott-Vogelius elements (P_2, P_1^{disc}) and (P_3, P_2^{disc}).

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