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Optimal elliptic Sobolev regularity near three-dimensional, multi-material Neumann vertices

Robert Haller-Dintelmann,[1] Wolfgang Höppner,[2]

Hans-Christoph Kaiser,[2,3] Joachim Rehberg,[2,4] Günter M. Ziegler[5]

[1] Technische Universität Darmstadt, Institut für Mathematik, Schlossgartenstraße 7, 64289 Darmstadt, Germany. Email: haller@mathematik.tu-darmstadt.de

[2] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany.

[3] Email: Hans-Christoph.Kaiser@wias-berlin.de

[4] Email: Joachim.Rehberg@wias-berlin.de

[5] Freie Universität Berlin, Department of Mathematics and Computer Science, Arnimallee 2, 14195 Berlin, Germany. Email: ziegler@math.fu-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We investigate optimal elliptic regularity (within the scale of Sobolev spaces) of anisotropic div-grad operators in three dimensions at a multi-material vertex on the Neumann boundary part of a polyhedral spatial domain. The gradient of a solution to the corresponding elliptic PDE (in a neighbourhood of the vertex) is integrable to an index greater than three.

1 Introduction

In recent years several elliptic regularity results were established in the $W^{1,q}$ scale, see [12] and [30] for the pure Dirichlet/Neumann Laplacian on domains with Lipschitz boundary, [22] and [6] for Neumann and mixed problems on polyhedra, and [4, 19, 20, 24, 25, 3, 16, 21, 7, 9, 5] in case of discontinuous coefficients. Meanwhile, this covers quite a nice zoo of geometries and coefficient functions, even including mixed boundary conditions. What has not been treated, is optimal $W^{1,p} \leftrightarrow W^{-1,p}$ regularity in a neighbourhood of a Neumann vertex when additionally heterogeneous materials are involved.

The interest in such problems essentially comes from the natural sciences and engineering. Here, many phenomena are described by elliptic or parabolic equations, and the influence of heterogeneous materials often is an important issue. For a detailed list of such problems we refer to the introduction of [9], see also [4]. In particular, in the investigations of quite a few nonlinear problems (see [10]) it is relevant that $W^{1,p}$ admits nice multiplier properties if p is larger than the space dimension. Alternatively, the reader may think of a parabolic equation with a quadratic gradient term on the right hand side. Then it is important to have, on one hand, $L^{p/2} \hookrightarrow W^{-1,p}$ (what implies, by Sobolev embedding, that p must be larger than the space dimension) and, on the other hand, an embedding for the domain of the elliptic operator into $W^{1,p}$, see [10], [11]. Recall that spaces of type $W^{-1,q}$ include distributional objects as e.g. surface densities – necessary in many applications, see [29, Chapter 1].

In our main result we show, roughly spoken, that there is a $p > 3$, such that, for any $f \in (W^{1,p'}(\Pi))'$, every solution v of $-\nabla \cdot \mu \nabla v = f$ is in $W^{1,p}$ locally around any vertex $\mathrm{a}_\blacktriangle$ of Π, provided that $\overline{\Pi}$ is a polyhedral manifold with boundary and the coefficient function μ satisfies some properties which allow to limitate the corresponding elliptic edge singularities, see further details in Theorem 5.1.

Let us emphasise that the matrices which constitute the coefficient function μ may be not diagonal and, in particular, not multiples of the identity, see [15, Ch. IV/V]. This is motivated by the applications. Moreover, anisotropic coefficients are unavoidable in view of (local) deformation of the domain, see Proposition 3.1. It should be noted that in case of an anisotropic coefficient matrix μ the generic properties of the elliptic operator $\nabla \cdot \mu \nabla$ possibly differ dramatically from the case of a scalar coefficient function, see [7, Remark 5.1], [8, §4], and [27, Ch. 5].

We solve the problem by localization, deformation and a reflection argument. In particular, one has to flatten a part of the boundary in a way that this piece then becomes part of a plane. In order to preserve the cellular structure of the constancy domains for the transformed coefficient function, one must, additionally, take a piecewise linear homeomorphism for the transformation. Since our aim was the treatment of general vertices, we make heavily use of nontrivial – but classical – instruments from geometric topology in dimensions 2 and 3. As a by-product, we get the (affirmative) answer (see Theorem 3.10) to the question: "Does every TOP manifold have a LIP structure?" (see [17, Ch. 9]) in case of three-dimensional polyhedra which form, additionally, a 3-manifold with boundary.

2 Notation

Throughout the text we will employ the following notation. By $\mathcal{C} :=]-1,1[^3$ we denote the open cube in $\mathbb{R}^3$, centered at 0, while $\mathcal{C}_\pm := \mathcal{C} \cap \{\mathrm{x} = (x,y,z) \in \mathbb{R}^3 : y \gtrless 0\}$ and $\Sigma := \mathcal{C} \cap \{\mathrm{x} = (x,0,z) : x, z \in \mathbb{R}\}$.

Concerning the notion *Lipschitz domain*, we follow the terminology in [18, Ch. 1.1.9]. In all what follows, $\Gamma \subseteq \partial\Omega$ is always a relatively open part of the boundary $\partial\Omega$. The symbol $W^{1,p}(\Omega)$ denotes the usual (complex) *Sobolev space* on Ω and we use $W^{1,p}_\Gamma(\Omega)$ for the closure of

$$\big\{v|_\Omega : v \in C^\infty(\mathbb{R}^3),\ \mathrm{supp}(v) \cap (\partial\Omega\backslash\Gamma) = \emptyset\big\}$$

in $W^{1,p}(\Omega)$. If $\Gamma = \emptyset$ we write as usual $W^{1,p}_0(\Omega)$ instead of $W^{1,p}_\emptyset(\Omega)$. Finally, $W^{-1,p'}_\Gamma(\Omega)$ denotes the space of continuous antilinear forms on $W^{1,p}_\Gamma(\Omega)$.

The expression $\langle\cdot,\cdot\rangle_X$ always indicates the pairing between a Banach space X and its (anti-)dual; in case of $X = \mathbb{C}^d$ we mostly write $\langle\cdot,\cdot\rangle$. If ω is a Lebesgue measurable, essentially bounded function on Ω taking its values in the set of $d \times d$ matrices, then we define $-\nabla\cdot\omega\nabla : W^{1,2}_\Gamma(\Omega) \to W^{-1,2}_\Gamma(\Omega)$ by

$$\langle -\nabla\cdot\omega\nabla v, w\rangle_{W^{1,2}_\Gamma} := \int_\Omega \omega\nabla v \cdot \nabla\overline{w}\,\mathrm{dx}\ , \qquad v, w \in W^{1,2}_\Gamma(\Omega).$$

3 Transformation of the problem

All our transformation techniques heavily rely on the fact that the general structure of the problem is not altered by bi-Lipschitz transformations. Thus we first quote from [9] the essential lemma that allows to transform elliptic divergence operators under bi-Lipschitz mappings maintaining optimal regularity.

Proposition 3.1 ([9, Prop. 16]). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and Γ be an open subset of its boundary. Assume that ϕ is a mapping from a neighbourhood of $\overline{\Omega}$ into $\mathbb{R}^d$, which is bi-Lipschitz and denote $\phi(\Omega) = \Omega_\star$ and $\phi(\Gamma) = \Gamma_\star$. Then the following assertions are true.*

1. *For any $p \in]1,\infty[$ the mapping ϕ induces a linear, topological isomorphism*

$$\Psi_p : W^{1,p}_{\Gamma_\star}(\Omega_\star) \to W^{1,p}_\Gamma(\Omega)$$

that is given by $(\Psi_p f)(x) = f(\phi(x)) = (f \circ \phi)(x)$.

2. *The operator $\Psi^*_{p'}$ is a linear, topological isomorphism between $W^{-1,p}_\Gamma(\Omega)$ and $W^{-1,p}_{\Gamma_\star}(\Omega_\star)$.*

3. *If ω is a bounded measurable function on Ω, taking its values in the set of $d \times d$ matrices, then*

$$\Psi^*_{p'}\nabla\cdot\omega\nabla\Psi_p = \nabla\cdot\underline{\omega}\nabla$$

with

$$\underline{\omega}(\mathrm{y}) = (D\phi)(\phi^{-1}(\mathrm{y}))\omega(\phi^{-1}(\mathrm{y}))\big(D\phi\big)^T(\phi^{-1}(\mathrm{y}))\frac{1}{\big|\det(D\phi)(\phi^{-1}\mathrm{y})\big|}, \tag{3.1}$$

where $D\phi$ denotes the Jacobian of ϕ and $\det(D\phi)$ *the corresponding determinant. Furthermore, if* $-\nabla\cdot\omega\nabla : W^{1,p}_{\Gamma}(\Omega) \to W^{-1,p}_{\Gamma}(\Omega)$ *is a topological isomorphism, then* $-\nabla\cdot\underline{\omega}\nabla : W^{1,p}_{\Gamma_\star}(\Omega_\star) \to W^{-1,p}_{\Gamma_\star}(\Omega_\star)$ *also is (and vice versa).*

3.1 Local flattening of the boundary by piecewise linear maps

In this section we will prove – under very general conditions – that the boundary of a polyhedron in $\mathbb{R}^3$ may be locally flattened around any boundary point by means of a piecewise linear homeomorphism. This means in particular that if the polyhedron is subdivided into cells, on each of which the coefficient function of an elliptic operator is constant, then one can find a mapping which locally flattens the boundary and, additionally, does not destroy this configuration.

In order to do so, we will need some notions and results from geometric topology, which we introduce briefly. All this material is in the spirit of the books by Moise [23] and Alexandroff & Hopf [1], see also Bing [2].

3.1.1 Some notions and results from geometric topology

If $\mathrm{v}_0, \ldots, \mathrm{v}_m \in \mathbb{R}^3$, and the convex hull of these points contains a d-dimensional ball and no $(d+1)$-dimensional ball, then this convex hull is called the *d-cell* generated by $\mathrm{v}_0, \ldots, \mathrm{v}_m$. The sides, edges and vertices from the boundary of the d-cell are called *faces.* If $m \le 3$ and the points $\mathrm{v}_0, \ldots, \mathrm{v}_m \in \mathbb{R}^3$ lie in general position, then we call the cell a simplex. Simplices in $\mathbb{R}^3$ are either tetrahedra, triangles, edges, or vertices.

A *euclidean complex* K is a locally finite collection of cells in $\mathbb{R}^3$ such that K contains all faces of all elements of K and if σ and τ are two cells in K with $\sigma\cap\tau \neq \emptyset$, then $\sigma\cap\tau$ must be a face both of σ and τ (see [1, Ch. III]). We call a complex a *simplicial complex*, if all cells involved are simplices. For a complex K in $\mathbb{R}^3$ we denote by $|K| := \bigcup_{\sigma\in K}\sigma \subseteq \mathbb{R}^3$ the *polyhedron* given by the complex K. If v is a vertex of the euclidean complex K, then the set of all cells from K which contain v, together with all their faces, form the *star* $K^\bigstar_{\mathrm{v}}$ of v in K. All those faces in the star that do not have v as a vertex form the *link* of v. The *open star* is the set-theoretic difference of star minus link. (The open star is an open set in $|K|$ and not carried by a subcomplex.)

If K is a finite simplicial complex such that $|K|$ is additionally an m-dimensional topological manifold (with boundary), then the complex K is a *triangulated m-manifold (with boundary)* and the topological space $|K|$ a *polyhedral m-manifold (with boundary).* We denote by K_∂ its *boundary complex*, which is generated by all the $(m-1)$-cells of the complex that lie in exactly one m-cell. All the manifolds M considered in the following are compact.

If K and K' are euclidean complexes in $\mathbb{R}^3$ with $|K| = |K'|$ and every $\sigma \in K'$ is contained in some element from K, then K' is called a *subdivision* of K. Below we will repeatedly need the following important relation between euclidean complexes and simplicial complexes:

Proposition 3.2 ([1, Ch. III.2]). *Every euclidean complex K admits a simplicial subdivision K' without new vertices. Moreover, K' can be constructed such that all edges of K' that come to lie in the open star of one specified vertex* $\mathrm{v} \in K$ *have* v *as an endpoint.*

Let K be a complex in $\mathbb{R}^d$. A continuous mapping f from $|K|$ onto a subset of $\mathbb{R}^m$ is *piecewise linear*, if there is a subdivision K' of K such that the restricted function $f|_\sigma$ is linear for every $\sigma \in K'$.

Remark 3.3. By Proposition 3.2 we may always assume that the cells on which a piecewise linear mapping is linear are simplices.

If K is a finite complex and f is injective, then on $f(|K|)$ one has the structure of a complex, induced from K by f. In the case of simplicial complexes this definition coincides with that in [23]; compare also [2, Ch. II]. See [23, Ch. 5] also for basics about *piecewise linear homeomorphisms*.

Finally, we need different notions of "boundary" for manifolds and complexes. Let M be an m-dimensional topological manifold in $\mathbb{R}^3$, with or without boundary. Then all points in M having an open neighbourhood in M that is homeomorphic to $\mathbb{R}^m$ form the *interior* $\mathrm{Int}(M)$ of M and the rest $\mathrm{Bd}(M) := M \backslash \mathrm{Int}(M)$ is the *manifold-theoretic boundary* of M.

The expression ∂A stands for the *topological frontier* of a set $A \subseteq \mathbb{R}^d$, i.e. $\partial A = \overline{A} \cap \overline{\mathbb{R}^d \backslash A}$, where the closure has to be taken in $\mathbb{R}^d$.

Remark 3.4. Let K be a triangulated 3-manifold with boundary. Assume that μ is a coefficient function on $|K|$ that is constant on the interiors of all 3-cells from K, and that $\phi : |K| \to \mathbb{R}^3$ is a piecewise linear mapping which establishes a homeomorphism from $|K|$ onto its image. Then the resulting coefficient function on $\phi(|K|)$ (see (3.1)) is constant on the interiors of 3-cells whose pre-images are contained in the 3-cells of K.

In the sequel, we will exploit the following results from geometrical topology:

Proposition 3.5 ([2, Thm. I.2.A]). *If K is a finite simplicial complex in $\mathbb{R}^3$, then there is a triangulation $K_{\mathbb{R}^3}$ of $\mathbb{R}^3$ which contains K as a subcomplex.*

Proposition 3.6 ([17, Thm. 2.18], [28, p. 504]). *Let K be a finite euclidean complex in $\mathbb{R}^d$ and let $\phi : |K| \to \mathbb{R}^m$ be piecewise linear and continuous. Then ϕ is Lipschitz continuous.*

Proposition 3.7 (The 3D PL Schoenflies Theorem [23, Thm 17.12] [2, Thm. XIV.I]). *Let S be a polyhedron in $\mathbb{R}^3$, whose boundary is topologically a 2-sphere. Then there is a piecewise linear homeomorphism $\phi_S : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the interior of S onto the interior of a tetrahedron σ^3 and ∂S onto $\partial\sigma^3$.*

Remark 3.8. In the following, we will apply the 3D PL Schoenflies Theorem to get a homeomorphism not to the boundary of a tetrahedron, but instead to the boundary of the brick $\overline{\mathcal{C}}_+ = [-1, 1] \times [0, 1] \times [-1, 1] = \{(x, y, z) \in [-1, 1]^3 : y \geq 0\}$. Moreover, we need that a specified point $\mathrm{a} \in S$ is mapped to the point $0 \in \partial\mathcal{C}_+$. Both these properties may be achieved by constructing a piecewise linear homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ that maps σ^3 onto $\overline{\mathcal{C}}_+$ and carries a given point $\mathrm{a} \in \partial\sigma^3$ to 0. This is easily achieved by explicit geometric construction, where we distinguish the three cases when a lies in the relative interior of a triangle face, lies on an edge, or is a vertex of σ^3.

Next, we prove the key lemma needed for the flattening theorem in the next subsection.

Lemma 3.9. *Let K be a triangulated 3-manifold with boundary in $\mathbb{R}^3$ and v be a vertex in the boundary of $|K|$. Then the polyhedron $|K^\bigstar_\mathrm{v}|$ of the star of v in K is homeomorphic to the closed unit ball in $\mathbb{R}^3$. In particular, the boundary of $K^\bigstar_\mathrm{v}$ is a polyhedral 2-sphere.*

Proof. By [23, Thm. 17.1] and [23, Thm. 23.3] the boundary complex K_∂ of K is a 2-manifold, and hence the star of the vertex $\mathrm{v} \in K_\partial$, denoted $K^\bigstar_{\partial,\mathrm{v}}$, is a disk (see [23, Thm. 4.8]). Consider a sufficiently small ball B centered at v, small enough so that it intersects only those faces of K that lie in $K^\bigstar_\mathrm{v}$, i.e., that contain v. One first observes that $K^\bigstar_{\partial,\mathrm{v}} \cap B$ is homeomorphic to $K^\bigstar_{\partial,\mathrm{v}}$. Hence, the intersection of the disk $K^\bigstar_{\partial,\mathrm{v}}$ with the boundary of B yields a Jordan curve in the 2-sphere ∂B, which by the 2D Schoenflies theorem ([23, Thm 10.2]) divides the sphere into two disks, one of them being $\partial B \cap K^\bigstar_\mathrm{v}$. Thus radial projection from v shows that the star $K^\bigstar_\mathrm{v}$ is a cone (with apex v) over a disk, hence it is a ball; its boundary, which is the union of two discs, namely the star of v in $K^\bigstar_{\partial,\mathrm{v}}$ and the link of v in $K^\bigstar_\mathrm{v}$, is a 2-sphere. □

3.1.2 The PL flattening theorem

The aim of this section is to establish the following result.

Theorem 3.10. *Let $\Pi \subseteq \mathbb{R}^3$ be a domain such that $\overline{\Pi}$ is a polyhedral 3-manifold with boundary satisfying $\Pi = \mathrm{Int}(\overline{\Pi})$. Then Π is a Lipschitz domain and the local bi-Lipschitz charts around the boundary points may be taken as piecewise linear homeomorphisms, as follows.*

Let K_Π be a euclidean complex with $|K_\Pi| = \overline{\Pi}$ and extend this to a triangulation $K_{\mathbb{R}^3} \supset K_\Pi$ of $\mathbb{R}^3$. Then for every $\mathrm{a} \in \partial\Pi$ there is a continuous, piecewise linear map φ_a of the cube $\overline{\mathcal{C}} = [-1,1]^3 \subseteq \mathbb{R}^3$ onto a closed neighborhood $\mathcal{W}_\mathrm{a}$ of a in $\mathbb{R}^3$ with the following properties:

(1) 0 *maps to* a,

(2) $\Sigma =]-1,1[\times \{0\} \times]-1,1[$ *maps to the boundary* $\partial\Pi$ *of the domain* Π,

(3) $\mathcal{C}_+ =]-1,1[\times]0,1[\times]-1,1[$ *maps to the domain* Π,

(4) $\mathcal{C}_- =]-1,1[\times]-1,0[\times]-1,1[$ *maps to the exterior* $\mathbb{R}^3 \backslash \overline{\Pi}$ *of the domain* Π.

Moreover, the complex $L_{\overline{\mathcal{C}}}$ which supports the mapping φ_a has the following properties:

(5) *For every $A \in L_{\overline{\mathcal{C}}}$, $\varphi_\mathrm{a}(A)$ is a subset of an element from the star of* a *in $K_{\mathbb{R}^3}$. In particlular, if $A \in L_{\overline{\mathcal{C}}}$ is a subset of $\overline{\mathcal{C}_+}$, then $\varphi_\mathrm{a}(A)$ is a subset of an element from the star of* a *in K_Π.*

(6) *Every* 3*-cell of* $L_{\overline{\mathcal{C}}}$ *has one of its vertices in* 0*.*

(7) *Every edge from* $L_{\overline{\mathcal{C}}}$ *which intersects* $\mathcal{C}$ *connects* 0 *with a vertex on* $\partial\mathcal{C}$*.*

Remark 3.11. The crucial conditions are $\Pi = \mathrm{Int}(\overline{\Pi})$ and that $\overline{\Pi}$ is a manifold with boundary. The first excludes cracks while the second forbids e.g. a prism with basis as indicated in Figure 1, cf. Remark 4.3.

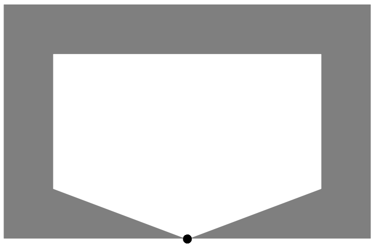

Figure 1: A pair of pincers is not a manifold with boundary, cf. Remark 3.11.

Proof. All the claims of this theorem are easy to establish if a is a relative interior point of a boundary triangle of K_Π, and quite easy to prove in the case where a lies on an edge of K_Π, since then a projection parallel to the edge reduces the situation to a 2-dimensional problem. The hard case is if a is a vertex of K_Π:

Lemma 3.9 in conjunction with the 3D PL Schoenflies Theorem – adapted according to Remark 3.8 – provides a piecewise linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ that carries the star $K^\bigstar_{\mathrm{a}}$ onto the brick $\overline{\mathcal{C}_+} = [-1, 1] \times [0, 1] \times [-1, 1]$, while mapping a to the boundary point $0 \in \partial\mathcal{C}_+$. Let L_{ref} be a simultaneous refinement of both the complexes $K_{\mathbb{R}^3}$ and a triangulation of $\mathbb{R}^3$ that supports the piecewise linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ (with a subcomplex triangulating $K^\bigstar_{\mathrm{a}}$). Modulo another refinement (which does not reduce the polyhedron of the star of a) we may assume that the 3-cells of L_{ref} are tetrahedra; we perform this refinement, but maintain the notation L_{ref}.

Let $L^\bigstar_{\mathrm{a}}$ denote the star of a in L_{ref}. Now we consider the complex $h(L^\bigstar_{\mathrm{a}})$. The polyhedron $|h(L^\bigstar_{\mathrm{a}})|$ is a neighbourhood of 0. Obviously, all cells of $h(L^\bigstar_{\mathrm{a}})$ are again tetrahedra – this time with one vertex in 0. Moreover, all edges which intersect the interior of the polyhedron $|h(L^\bigstar_{\mathrm{a}})|$ originate in 0. Let $\varepsilon\overline{\mathcal{C}}$ be a cube that is situated in the interior of $|h(L^\bigstar_{\mathrm{a}})|$. Thus, the intersection with $|h(L^\bigstar_{\mathrm{a}})|$ induces on $\varepsilon\overline{\mathcal{C}}$ the structure of a complex the cells of which have one vertex in 0. Furthermore, the mapping $\varphi_{\mathrm{a}} := h^{-1}|_{\varepsilon\overline{\mathcal{C}}}$ is linear on each of these cells. Thus, finally, rescaling $\mathrm{x} \mapsto \varepsilon\mathrm{x}$ and defining $\mathcal{W}_{\mathrm{a}} := \varphi_{\mathrm{a}}(\overline{\mathcal{C}})$, one obtains the asserted neighbourhood. The stated bi-Lipschitz property of φ_{a} follows from Proposition 3.6. □

4 Edge singularities

In this section, after some preparations, we recall the optimal regularity result from [21] for heterogeneous Dirichlet problems on polyhedral domains and explain how to identify the occurring edge singularities. We first introduce some notions and notation corresponding to our geometric situation of a polyhedral domain Π and the piecewise constant coefficient function μ.

Definition 4.1. For $\iota, \vartheta \in \,]-\pi,\pi]$ with $\iota < \vartheta$ we define the *sector*

$$S_\iota^\vartheta := \{(r\cos\theta, r\sin\theta) : \; r > 0, \theta \in \,]\iota,\vartheta[\}.$$

Definition 4.2. Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain, such that $\overline{\Omega}$ is a polyhedral 3-manifold with boundary, associated to the finite complex K. If $\{\sigma_k\}_k$ is the collection of all 3-cells from K, then let μ be a matrix function on Ω which is constant on the interior of each cell σ_k and takes real, symmetric, positive definite 3×3 matrices as values. Take an edge E of any of the σ_k's and consider an arbitrary point P of this edge that is not an endpoint of it.

Choose a new orthogonal coordinate system (x, y, z) with origin at the point P such that the direction of E coincides with the z-axis. We denote by $\mathcal{O}_E$ the corresponding orthogonal transformation matrix and by $\mu_{E,P}$ the piecewise constant matrix function, which coincides for x from a neighbourhood of $0 \in \mathbb{R}^3$ with $A_{E,P}\mathrm{x} := \mathcal{O}_E\mu(\mathcal{O}_E^{-1}\mathrm{x}+P)\mathcal{O}_E^{-1}$ and which satisfies

$$\mu_{E,P}(tx, ty, z) = \mu_{E,P}(x, y, 0), \quad \text{for all } (x, y, z) \in \mathbb{R}^3,\; t > 0. \tag{4.1}$$

By $\mu_E(\cdot,\cdot)$ we denote the upper left 2×2 block of $\mu_{E,P}(\cdot,\cdot,0)$.

Remark 4.3. It is essential that – as the above notation suggests – the coefficient function μ_E is the same for every point P from the (relative) interior of the edge E.

There exist angles $\theta_0 < \theta_1 < \ldots < \theta_n \le \theta_0 + 2\pi$, such that μ_E is constant on each of the sectors $S_{\theta_j}^{\theta_{j+1}}$ and takes real, symmetric, positive definite matrices as values. In the sequel we call such coefficient functions on a sector $S \subseteq \mathbb{R}^2$ *sectorwise constant*.

If μ_E corresponds to an interior edge E, then we have $\theta_n = \theta_0 + 2\pi$, otherwise μ_E is given on an infinite sector $S_{\theta_0}^{\theta_n}$ that coincides near P with the intersection of the transformed Ω with the x-y-plane.

Note that the appearance of a whole sector $S_{\theta_0}^{\theta_n}$ in case of boundary edges is indeed due to the fact that $\overline{\Omega}$ is a polyhedral 3-manifold with boundary.

In order to cite the optimal regularity result from [21], we now introduce the Sturm-Liouville operator associated to an edge and to the coefficient function μ.

Definition 4.4. Let numbers $\theta_0 < \theta_1 < \ldots < \theta_n \le \theta_0 + 2\pi$ be given and, additionally, real, symmetric, positive definite 2×2 matrices $\rho^1, \ldots, \rho^n$, which are asscociated to the sectors $S_{\theta_0}^{\theta_1}, \ldots, S_{\theta_{n-1}}^{\theta_n}$. We introduce on $]\theta_0, \theta_n[\, \setminus \{\theta_1, \ldots, \theta_{n-1}\}$ coefficient functions b_0, b_1, b_2, whose restrictions to the interval $]\theta_j, \theta_{j+1}[$, $j = 0, \ldots, n-1$, are given by

$$\begin{aligned}
b_0(\theta) &= \rho_{11}^j \cos^2\theta + 2\rho_{12}^j \sin\theta\cos\theta + \rho_{22}^j \sin^2\theta\,,\\
b_1(\theta) &= (\rho_{22}^j - \rho_{11}^j)\sin\theta\cos\theta + \rho_{12}^j(\cos^2\theta - \sin^2\theta)\,,\\
b_2(\theta) &= \rho_{11}^j \sin^2\theta - 2\rho_{12}^j \sin\theta\cos\theta + \rho_{22}^j \cos^2\theta\,.
\end{aligned}$$

If $\theta_n \neq \theta_0 + 2\pi$, then we define the space $\mathcal{H}$ as $W^{1,2}(]\theta_0, \theta_n[)$ in case of a Neumann condition and as $W_0^{1,2}(]\theta_0, \theta_n[)$ in case of a Dirichlet condition. If $\theta_n = \theta_0 + 2\pi$ – the case of an interior edge – we define $\mathcal{H}$ as the periodic Sobolev space $W^{1,2}(]\theta_0, \theta_n[) \cap \{\upsilon : \upsilon(\theta_0) = \upsilon(\theta_n)\}$, which clearly may be identified with the Sobolev space $W^{1,2}(S^1)$ on the unit circle S^1. For every $\lambda \in \mathbb{C}$ we define the quadratic form $\mathfrak{t}_\lambda$ on $\mathcal{H}$ by

$$\mathfrak{t}_\lambda[\upsilon] := \int_{\theta_0}^{\theta_n} \left(b_2\, \upsilon' \, \overline{\upsilon'} + \lambda b_1 \upsilon \, \overline{\upsilon'} - \lambda b_1 \upsilon' \, \overline{\upsilon} - \lambda^2 b_0 \upsilon \overline{\upsilon} \right) \mathrm{d}\theta \tag{4.2}$$

and $\mathcal{A}_\lambda$ as the operator which is induced by $\mathfrak{t}_\lambda$ on $L^2(]\theta_0, \theta_n[)$, or on $L^2(S^1)$, respectively.

As usual [19], we refer to the entity $\{\mathcal{A}_\lambda\}_\lambda$ as the corresponding *operator pencil* in the sequel.

Definition 4.5. Let E be any edge of the triangulation of $\overline{\Omega}$ and $\mathcal{A}_\lambda$ as defined in Definition 4.4 with $\rho = \mu_E$. Then we call the number

$$\inf\{\Re\lambda > 0 : \ker(\mathcal{A}_\lambda) \neq \{0\}\}$$

the *singularity exponent* associated to E. Generally, we call a number λ with $\Re\lambda > 0$ for which $\ker(\mathcal{A}_\lambda) \neq \{0\}$ a *singular value* of the operator pencil $\mathcal{A}_\lambda$.

We proceed by quoting the central linear regularity result [21, Thm. 2.3] – which serves later on as the the main tool for the proof of our regularity result.

Proposition 4.6. *Let Ω, $\{\sigma_k\}_k$ and μ be as in Definition 4.2. If for every edge, belonging to the trinangulation $\{\sigma_k\}_k$, the associated singularity exponent is larger than $\frac{1}{3}$, then there is a $p > 3$, such that*

$$-\nabla \cdot \mu \nabla : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$$

is a topological isomorphism.

Remark 4.7. It is known that in every strip $\{\lambda = \lambda_1 + i\lambda_2 : |\lambda_1| \le c\}$ there is only a finite number of values λ, for which the kernel of $\mathcal{A}_\lambda$ is not trivial, see Corollary 3.11 from [21]. Therefore, it suffices to show that no $\mathcal{A}_\lambda$ with $0 < \Re\lambda \le \frac{1}{3}$ does admit a nontrivial kernel.

In the next sections all edges resulting from our problem are inspected, concerning the triviality of $\ker(\mathcal{A}_\lambda)$. Here, two essential types are monomaterial edges and bimaterial outer edges:

Definition 4.8. Let E be an edge in $\overline{\Omega}$ that lies in $\partial\Omega$. Then we define in the terminology of Definition 4.2:

1. E is a *monomaterial edge*, if all relative interior points of E possess a neighbourhood in $\overline{\Omega}$ on which μ is constant a.e. with respect to 3-dimensional Lebesgue measure.

2. E is a *bimaterial outer edge*, if the function $[\theta_0, \theta_n] \ni \theta \mapsto \mu_E(\cos(\theta), \sin(\theta))$, has exactly 1 jump, and the lengths of both the constancy intervals of this function do not exceed π. Here θ_0 and θ_n are again the angles associated to E in Remark 4.3.

For the treatment of the corresponding singularities, we have the following two results, see [7, Lemma 2.3] or [9, Thm. 24/25].

Proposition 4.9. *For any monomaterial edge E the kernels of the associated operators $\mathcal{A}_\lambda$ are trivial, when either a pure Dirichlet condition or a pure Neumann condition is imposed, and if $\Re\lambda \in]0, 1/2]$.*

Proposition 4.10. *Let $S_{\theta_0}^{\theta_1}, S_{\theta_1}^{\theta_2}$ be two neighbouring sectors in $\mathbb{R}^2$ with $\theta_1 - \theta_0$, $\theta_2 - \theta_1 \leq \pi$ and $\theta_2 - \theta_0 < 2\pi$. Let ρ^1, ρ^2 be two real, symmetric, positive definite 2×2 matrices corresponding to the sectors $S_{\theta_0}^{\theta_1}$ and $S_{\theta_1}^{\theta_2}$, respectively. Let $\mathfrak{t}_\lambda$ be the form defined in (4.2) either on $W_0^{1,2}(]\theta_0, \theta_2[)$ or on $W^{1,2}(]\theta_0, \theta_2[)$. Then there is an $\varepsilon > 0$, such that the kernel of the corresponding operator $\mathcal{A}_\lambda$ (see Definition 4.4) is trivial for $\Re\lambda \in]0, 1/2 + \varepsilon]$.*

In the rest of this section we collect some results on edge singularities which will be needed in the sequel.

First we establish an invariance principle for the set of singular values of an operator pencil $\mathcal{A}_\lambda$ when a sectorwise linear mapping is applied to the geometrical and coefficient constellation.

Definition 4.11. Let $S \subseteq \mathbb{R}^2$ be a sector which splits up in the subsectors $\hat{S}_1, \ldots, \hat{S}_m$, such that $\hat{S}_l$, is neighbour of $\hat{S}_{l-1}$ and $\hat{S}_{l+1}$, $l = 2, \ldots, m-1$. Let A be a homeomorphism from S onto another sector $S_\bullet$. We call A *sectorwise linear* if the restriction of A to each $\hat{S}_l$ is a linear mapping, i.e. if for $\mathrm{x}, \mathrm{y} \in \hat{S}_l$ also $\mathrm{x}+\mathrm{y}$ belongs to $\hat{S}_l$, then $A\mathrm{x}+A\mathrm{y} = A(\mathrm{x}+\mathrm{y})$, and $A(t\mathrm{x}) = tA\mathrm{x}$ for $\mathrm{x} \in \hat{S}_l$ and $t > 0$.

Remark 4.12. Note that the inverse of a sectorwise linear mapping is also a sectorwise linear mapping.

Having this at hand, one can show the following

Theorem 4.13. *Let, for the sector $S \subseteq \mathbb{R}^2$ a sector partition $S_1, \ldots, S_n$ be given. Let ρ be a coefficient function on S which takes on each sector S_l constantly the symmetric, positive definite 2×2-matrix ρ_l as value. Let $A : S \to S_\bullet$ be a homeomorphism which is sectorwise linear.*

1. *The transformed coefficient function $\underline{\rho}$ (see Proposition 3.1) is also sectorwise constant on $S_\bullet$.*

2. *If $\mathcal{A}_\lambda$ is the operator pencil associated to the coefficient function ρ and $\underline{\mathcal{A}_\lambda}$ is the corresponding operator pencil for the coefficient function $\underline{\rho}$ (cf. Definition 4.4) then the singular values (cf. Definition 4.5) for $\mathcal{A}_\lambda$ and $\underline{\mathcal{A}_\lambda}$ in fact coincide. In particular, their singularity exponents are the same.*

Proof. The first assertion follows from the transformation formula (3.1). Furthermore, the second assertion is proved in [13, Thm. 4.3] for the case $\theta_n \neq \theta_0 + 2\pi$ and that the sector S splits up into only two subsectors $\hat{S}_1, \hat{S}_2$. The general case needs only obvious modifications of this proof. The crucial point is the fact that a sectorwise linear mapping is homogeneous of degree 1. □

Remark 4.14. Note that the sector partitions $S_1, \dots, S_n$ and $\hat{S}_1, \dots, \hat{S}_m$ are completely independent from each other. Nevertheless, one can introduce a common refinement.

Theorem 4.15. *Let K_E be a euclidean complex in $\mathbb{R}^3$ and let the following hypotheses be satisfied.*

1. *Every 3-cell $\sigma \in K_E$ contains the edge E.*
2. *If $P \in E$ is an abitrary (relative) interior point and $\mathcal{H}(P, E)$ is the plane which contains P and is perpendicular to E, then $|K_E| \cap \mathcal{H}(P, E)$ coincides locally around P with a complete sector.*
3. *ω is a coefficient function on $|K_E|$ which takes its values in the set of real, symmetric, positive definite 3×3-matrices and is constant on every $\sigma \in K_E$.*
   Assume that $\phi : |K_E| \to \mathbb{R}^3$ maps $|K_E|$ homeomorphically onto its image and is linear on every $\sigma \in K_E$.

Then the transformed coefficient function (see Proposition 3.1)

$$\rho(\mathrm{y}) = (D\phi)(\phi^{-1}(\mathrm{y}))\omega(\phi^{-1}(\mathrm{y}))\big(D\phi\big)^T(\phi^{-1}(\mathrm{y}))\frac{1}{\big|\det(D\phi)(\phi^{-1}\mathrm{y})\big|}$$

on the polyhedron $\phi(|K_E|)$ satisfies (mutatis mutandis) the above hypotheses 3. and, additionally, the singular values for the corresponding operator pencils $\mathcal{A}_\lambda$ and $\underline{\mathcal{A}_\lambda}$ (cf. Definition 4.4) are the same. In particular, their singular exponents coincide, (cf. Definition 4.5).

Proof. We shall show that the coefficient functions ω_E and $\rho_{\phi(E)}$ (cf. Definition 4.2) are related via a sectorwise linear mapping – what makes Theorem 4.13 applicable.

Let us recall how the coefficient function ω_E, defining the operator pencil $\mathcal{A}_\lambda$, is obtained: one considers the affine transformation $A_{E,P}$ that shifts an arbitrary point $P \in E$ to the origin and rotates the (shifted) edge E into the z-axis, see Definition 4.2. The coefficient function $\omega_{E,P}$ coincides on a sufficiently small neighbourhood $\mathcal{V}$ of$0 \in \mathbb{R}^3$ with the coefficient function, that results from ω under the transformation $A_{E,P}$, cf. Proposition 3.1, and is extended to the whole of $\mathbb{R}^3$ by (4.1). Finally, ω_E is the upper left block of $\omega_{E,P}(\cdot, \cdot, 0)$ — thus, not depending on P anymore, cf. Definition 4.2.

The same construction is applied to the complex $\phi(K_E)$ and the corresponding coefficient function ρ — in this way defining the mapping $A_{\phi(E),\phi(P)}$ and the operator pencil $\underline{\mathcal{A}_\lambda}$. The mapping $B := A_{\phi(E),\phi(P)}\phi A_{E,P}^{-1} : A_{E,P}|K_E| \to A_{\phi(E),\phi(P)}\phi(|K_E|)$ is a homeomorphism which acts linearly on any wedge $S_l \times\,]-\infty, \infty[$, where S_l is any from the sectors defined in Remark 4.3 — here corresponding to the coefficient function ω. Note that B maps $0 \in \mathbb{R}^3$ onto itself and transforms the vector $(0,0,1)^T$ into a vector $(0,0,\tau)^T$ with $\tau \neq 0$. Hence, the restriction of B to any of these wedges must be a linear mapping of the form

$$B_l = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & \tau \end{pmatrix}.$$

The value τ does not depend on the specific wedge, but $|\tau|$ is the stretching factor of E with respect to ϕ. According to the transformation formula (3.1), one has, for points x from the corresponding image wedge which are, additionally, close to $A_{\phi(E),\phi(P)}\phi(E)$,

$$\rho_{\phi(E),\phi(P)}(\mathrm{x}) = \frac{1}{|\det(B_l)|} B_l \omega_{E,P}(B_l^{-1}(\mathrm{x})) B_l^T. \tag{4.3}$$

But, due to (4.1), equation (4.3) extends to the whole image wedge. For a 3×3 matrix M let $\widehat{M}$ denote the upper left 2×2 block. $\widehat{B}$ induces a sectorwise linear mapping on the corresponding sector – which is part of the x-y-plane. The special form of B, namely the property $b_{13} = b_{23} = 0$, then yields the equation

$$\widehat{\rho}_{\phi(E),\phi(P)}(\mathrm{x}) = \frac{1}{|\det(B)|} \widehat{B} \widehat{\omega}_{E,P}(B^{-1}(\mathrm{x})) \widehat{B}^T = \frac{1}{|\tau|} \frac{1}{|\det(\widehat{B})|} \widehat{B} \widehat{\omega}_{E,P}(B^{-1}(\mathrm{x})) \widehat{B}^T,$$

for $\mathrm{x} \in A_{\phi(E),\phi(P)}\phi(|K_E|)$. This gives

$$\rho_{\phi(E)}(x,y) = \widehat{\rho}_{\phi(E),\phi(P)}(x,y,0) = \frac{1}{|\tau|} \frac{1}{|\det(\widehat{B})|} \widehat{B} \widehat{\omega}_{E,P}(B^{-1}(x,y,0)) \widehat{B}^T. \tag{4.4}$$

Further, again the special form of B provides

$$B^{-1} = \begin{pmatrix} \widehat{B}^{-1} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \begin{smallmatrix} c_{1,3} & c_{2,3} \end{smallmatrix} & 1/\tau \end{pmatrix}.$$

Inserting this into (4.4) one obtains

$$\rho_{\phi(E)}(x,y) = \frac{1}{|\tau|} \frac{1}{|\det(\widehat{B})|} \widehat{B} \widehat{\omega}_{E,P}(\widehat{B}^{-1}(x,y),\ z) \widehat{B}^T \tag{4.5}$$

for $z = c_{1,3}x + c_{2,3}y \in \mathbb{R}$. However, the coefficient functions $\omega_{E,P}$ and $\widehat{\omega}_{E,P}$ do not depend on z such that we finally get

$$\rho_{\phi(E)}(x,y) = \frac{1}{|\tau|} \frac{1}{|\det(\widehat{B})|} \widehat{B} \widehat{\omega}_{E,P}(\widehat{B}^{-1}(x,y)\ ,\ 0) \widehat{B}^T = \frac{1}{|\tau|} \frac{1}{|\det(\widehat{B})|} \widehat{B} \omega_E(\widehat{B}^{-1}(x,y)) \widehat{B}^T$$

Thus, the coefficient function $\rho_{\phi(E)}$ results from the coefficient function ω_E via a sectorwise linear mapping (modulo the constant, scalar prefactor $\frac{1}{|\tau|}$) – and vice versa. Hence, $\ker \underline{\mathcal{A}_\lambda}$ is trivial if $\ker \mathcal{A}_\lambda$ is trivial, due to Theorem 4.13. $\square$

The final result in this section reverses in some sense the reflection argument (cf. Lemma 6.3/Lemma 6.4 below): it allows us to delimitate the singularities at edges in Σ which result from the even reflection across the former Neumann boundary part.

Lemma 4.16. *Assume that the half space $\{(x,y) : y > 0\}$ splits up into the sectors $S_0^{\theta_1}, \dots, S_{\theta_{n-1}}^{\pi}$ with the associated matrices*

$$\begin{pmatrix} \rho_{11}^1 & \rho_{12}^1 \\ \rho_{12}^1 & \rho_{22}^1 \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \rho_{11}^n & \rho_{12}^n \\ \rho_{12}^n & \rho_{22}^n \end{pmatrix}. \tag{4.6}$$

Assume that to the sectors $S_0^{-\theta_1}, \dots, S_{-\theta_{n-1}}^{-\pi}$ the matrices

$$\begin{pmatrix} \rho_{11}^1 & -\rho_{12}^1 \\ -\rho_{12}^1 & \rho_{22}^1 \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \rho_{11}^n & -\rho_{12}^n \\ -\rho_{12}^n & \rho_{22}^n \end{pmatrix} \tag{4.7}$$

are assigned. Let $\mathcal{A}_\lambda$ denote the operator that corresponds to the matrices (4.6), (4.7) *within the given sector partition of $\mathbb{R}^2$. Let further $\mathcal{A}_\lambda^D$ and $\mathcal{A}_\lambda^N$ denote the operators, which correspond to the matrices* (4.6) *within the sectors in the half space $\{(x,y) : y > 0\}$, once combined with Dirichlet and once with Neumann boundary conditions. Then for any number λ with $\Re\lambda > 0$ the kernel of the operator $\mathcal{A}_\lambda$ is trivial, if for this same λ the kernels of $\mathcal{A}_\lambda^D$ and $\mathcal{A}_\lambda^N$ are trivial.*

Proof. A proof is given in [9, Lemma 22] in the case of two sectors in the half space. Mutatis mutandis, the proof can be carried over to the case of many sectors. □

5 The optimal regularity result

We are now in the position to state our main result on optimal regularity of the elliptic problem near a heterogeneous Neumann vertex.

Theorem 5.1. *Let $\Pi \subseteq \mathbb{R}^3$ be a bounded domain, such that Π and $\overline{\Pi}$ have the same boundary and its closure $\overline{\Pi}$ is a polyhedral 3-manifold with boundary. Let $\mathrm{a}_\blacktriangle$ be a vertex of $\overline{\Pi}$ and suppose:*

1. *The coefficient function μ on Π is elliptic and takes its values in the set of real, symmetric, positive definite 3×3 matrices.*

2. *There is a triangulation of $\overline{\Pi}$ by a (finite) euclidean complex K, such that μ is constant on the interior of each 3-cell belonging to K.*

3. *Any edge in K, belonging to the boundary of Π and having one endpoint in $\mathrm{a}_\blacktriangle$, is a monomaterial edge or a bimaterial outer edge.*

4. *For every interior edge with endpoint $\mathrm{a}_\blacktriangle$, resulting from the triangulation K, the singularity exponent, associated to this edge, is larger than $\frac{1}{3}$.*

Then there is an open neighbourhood $\mathcal{U}$ of $\mathrm{a}_\blacktriangle \in \mathbb{R}^3$ such that, setting $\Pi_\bullet := \Pi \cap \mathcal{U}$ and $\Gamma := \partial\Pi \cap \mathcal{U}$, the operator

$$-\nabla \cdot \mu|_{\Pi_\bullet} \nabla : W^{1,p}_\Gamma(\Pi_\bullet) \to W^{-1,p}_\Gamma(\Pi_\bullet) \tag{5.1}$$

is a topological isomorphism for some $p > 3$.

Remark 5.2. 1. Condition 2. in Theorem 5.1 says by no means that the coefficient function μ has to take different values on different cells.

2. In case of the Laplacian (i.e. $\mu \equiv 1_{\mathbb{R}^3}$) Theorem 5.1 says that only the geometric suppositions on Π suffice to guarantee the ismorphism property (5.1) – irrespective how 'wild' the local geometry of Π really is (compare [6]). Thus, one is here in the same situation as in the Dirichlet case, cf. Proposition 4.6 and Proposition 4.9, compare also [7, Thm. 2.1].

Unfortunately, for interior edges it seems to be extremely difficult to decide, whether the kernels of $\mathcal{A}_\lambda$ are trivial or not in generality, see the detailed discussion in [7]. However, there are several important constellations, where this assumption is known to be true:

Definition 5.3. Let angles $-\pi = \theta_0 < \theta_1 < \cdots < \theta_n = \pi$ be given and let $\hat{\mu}$ be a constant matrix on every sector $S_j := S^{\theta_j}_{\theta_{j-1}}$, $j = 1, \ldots, n$, in $\mathbb{R}^2$. Then $\hat{\mu}$ is distributed *quasi-monotonely*, if there exist indices $j_{\min}, j_{\max} \in \{1, \ldots, n\}$, such that

$$\hat{\mu}|_{S_{j_{\max}}} \geq \hat{\mu}|_{S_{j_{\max}+1}} \geq \cdots \geq \hat{\mu}|_{S_{j_{\min}-1}} \geq \hat{\mu}|_{S_{j_{\min}}} \leq \hat{\mu}|_{S_{j_{\min}+1}} \leq \cdots \leq \hat{\mu}|_{S_{j_{\max}-1}} \leq \hat{\mu}|_{S_{j_{\max}}}$$

and there exists a point $\mathrm{x} \in \mathbb{R}^2$, such that $\mathrm{x} \in S_{j_{\max}}$ and $-\mathrm{x} \in S_{j_{\min}}$. Here the indices are to be understood modulo n, i.e. S_{n+1} is again S_1.

Example 5.4. Condition 4. of Theorem 5.1 is satisfied in each of the following cases.

1. There is exactly one plane containing $\mathrm{a}_\blacktriangle$, which splits up Π in a neighbourhood of $\mathrm{a}_\blacktriangle$ into two pieces, such that the coefficient function is locally constant on both intersections of Π with the half spaces, induced by the plane. Of course, then no interior edges appear around $\mathrm{a}_\blacktriangle$.

2. There are exactly two planes containing $\mathrm{a}_\blacktriangle$ and intersecting Π. The coefficient function is constant in the induced four quarter spaces and scalar valued. This sequence of scalars is monotonously increasing if starting with the smallest scalar and then running from sector to sector, see [26, Thm. 6.4].

3. The matrices are distributed quasi-monotonely (compare [7, Lemma 2.1], see also [14]). This is in particular the case, if μ is scalar valued, and $\mathbb{R}^2$ splits up into three sectors, each of which has an opening angle less than π.

Remark 5.5. 1. Condition 3. of Theorem 5.1 in particular forbids a material constellation M_+, M_-, M_+ in three neighbouring sectors, despite the fact that only two different materials are involved.

6 Proof of Theorem 5.1

6.1 Localization and reflection

Having in mind our aim to show optimal elliptic regularity in a neighbourhood of a vertex a of the polyhedral domain Π, the next task will be to define a suitable neighbourhood $\mathcal{U}$ of the point a under consideration, where we investigate the regularity of the solution.

Definition 6.1. From now on we always suppose that $\mathrm{a}_\blacktriangle$ is a vertex of Π and that $\varphi_{\mathrm{a}_\blacktriangle}$ is chosen as in Theorem 3.10. Moreover, we define the euclidean complex $K^+_{\overline{\mathcal{C}}}$ as the complex $L_{\overline{\mathcal{C}}} \cap \overline{\mathcal{C}_+}$.

Now, let a cut-off function $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\eta) \subseteq \mathcal{U}$ and a right hand side $f \in (W^{1,p'}(\Pi))'$ for some $p \in \,]3,6]$ be given. The following lemma establishes an equation for the truncated function ηv.

Lemma 6.2 ([10, Ch. 5.1])**.** *Let $p \in \,]3,6]$ and assume that $\mathcal{U} \subseteq \mathbb{R}^3$ is open such that $\Pi_\bullet := \Pi \cap \mathcal{U}$ is again a Lipschitz domain. Define $\Gamma := \partial\Pi \cap \mathcal{U}$ and let $v \in W^{1,2}(\Pi)$ be the solution to*

$$-\nabla \cdot \mu \nabla v + v = f \in \big(W^{1,p'}(\Pi)\big)'. \tag{6.1}$$

Then $\eta v \in W^{1,2}_\Gamma(\Pi_\bullet)$, and it holds

$$-\nabla \cdot \mu|_{\Pi_\bullet} \nabla(\eta v) = g \in W^{-1,p}_\Gamma(\Pi_\bullet), \tag{6.2}$$

where g continuously depends on f.

This Lemma states that, given $f \in (W^{1,p'}(\Pi))'$ and the variational solution v of (6.1), the function ηv is the variational solution of problem (6.2) with mixed boundary conditions (in the sense that $g \in L^2(\Pi_\bullet)$ implies Neumann condition on Γ and Dirichlet condition on $\partial\Pi_\bullet \backslash \Gamma$). So, we may rephrase our original regularity problem about the behaviour in the neighbourhood of a Neumann edge into a question about optimal regularity for a mixed boundary value problem.

In a next step — from now on taking $\mathcal{U} := \varphi_{\mathrm{a}_\blacktriangle}(\mathcal{C})$ — we transform our problem by the piecewise linear homeomorphism $\phi = \varphi_{\mathrm{a}_\blacktriangle}^{-1}$ constructed in Theorem 3.10. As pointed out at the beginning of Section 3, equation (6.2) then transforms into

$$-\nabla \cdot \underline{\mu} \nabla u = h \in W^{-1,p}_\Sigma(\mathcal{C}_+) \tag{6.3}$$

with $u := (\eta v) \circ \phi^{-1}$. Additionally we know that the regularity behaviour for constellations which are related by only a bi-Lipschitz transform is just the same, cf. Proposition 3.1. So, our decisive instrument for the proof of our main result in Theorem 5.1 will be to prove

Lemma 6.3. *Let $\Pi \subseteq \mathbb{R}^3$, $\mathrm{a}_\blacktriangle \in \partial\Pi$ and μ as in Theorem 5.1 and $\varphi_{\mathrm{a}_\blacktriangle}$ as in Theorem 3.10. Define $\Pi_\bullet := \mathcal{U} \cap \Pi = \varphi_{\mathrm{a}_\blacktriangle}(\mathcal{C}) \cap \Pi$ and $\underline{\mu}$ as the coefficient function which is obtained on $\mathcal{C}_+$ from $\mu|_{\Pi_\bullet}$ under the bi-Lipschitzian transformation $\varphi_{\mathrm{a}_\blacktriangle}^{-1}$, cf. Proposition 3.1.*

Then $-\nabla \cdot \underline{\mu} \nabla : W^{1,p}_\Sigma(\mathcal{C}_+) \to W^{-1,p}_\Sigma(\mathcal{C}_+)$ is a topological isomorphism for some $p > 3$.

This is again a problem with mixed boundary conditions, but now the Neumann boundary part Σ is planar. Thus, we may apply the standard reflection argument, see [9, Proposition 17]. After this we end up with a Dirichlet problem on the cube $\mathcal{C}$, which will be treated afterwards. Thus, in order to prove Lemma 6.3, it suffices to show

Lemma 6.4.

$$-\nabla \cdot \hat{\mu} \nabla : W_0^{1,p}(\mathcal{C}) \to W^{-1,p}(\mathcal{C}) \tag{6.4}$$

is a topological isomorphism for some $p > 3$*, where* $\hat{\mu}$ *is the coefficient function which is obtained from* $\underline{\mu}$ *by even reflection.*

Obviously, (6.4) is a Dirichlet problem on a polyhedral domain. In order to identify a triangulation of $\overline{\mathcal{C}}$, such that the coefficient function $\hat{\mu}$ is constant on its 3-cells, we introduce the following

Definition 6.5. We define the complex $K_{\overline{\mathcal{C}}}$ as the union of all cells from the complex $K^+_{\overline{\mathcal{C}}}$ together with all cells which are images of cells from $K^+_{\overline{\mathcal{C}}}$ under the mapping $(x, y, z) \mapsto (x, -y, z)$.

Remark 6.6. The triangulation $K_{\overline{\mathcal{C}}}$ of $\overline{\mathcal{C}}$ is different from $L_{\overline{\mathcal{C}}}$ in general, but it coincides with $L_{\overline{\mathcal{C}}}$ on $\overline{\mathcal{C}_+}$. Furthermore, the triangulation $K_{\overline{\mathcal{C}}}$ is of such kind that the coefficient function $\hat{\mu}$ is constant on the interior of all its 3-cells. Thus, this complex determines the edges which are then the relevant ones for the Dirichlet problem in Lemma 6.4. For this it is essential that the whole coefficient configuration on $\mathcal{C}$ relates to the original coefficient configuration on Π in a way that lateron will allow us to restrict the elliptic edge singularities via the invested assumptions on the coefficient function μ on Π.

According to Proposition 4.6, for the proof of Lemma 6.4 it suffices to delimitate the edge singularities. We present the details in the next two subsections.

6.2 Identification of edges

In order to identify all these edges, we consider for a given vertex $\mathrm{a}_\blacktriangle$ the coefficient function $\hat{\mu}$ from Lemma 6.4. Recall in this context Definition 6.5 and Remark 6.6, as well as the piecewise linear map $\varphi_{\mathrm{a}_\blacktriangle}$ and the complex $K_{\overline{\mathcal{C}}}$ from the Definitions 6.1 and 6.5.

Definition 6.7. For all $t_0 > 0$ and any vectors $\mathrm{a} \in \mathbb{R}^3$ and $\mathrm{b} \in \mathbb{R}^3 \setminus \{0\}$ we call the set $\{\mathrm{a} + t\mathrm{b}, t \in]0, t_0[\}$ *segment* with *starting point* a. A segment in Π is *monomaterial* if it is contained in the interior of one 3-cell from K_Π. A segment in Π is a *bi-cellular segment* if it lies in the interior of a 2-face of K_Π.

Remark 6.8. Recall that the coefficient matrix μ need not have different values on the two 3-cells which are adjacent to the 2-face which includes a bi-cellular segment.

The following facts about the edges from $K_{\overline{\mathcal{C}}}$ are essential in what follows.

Lemma 6.9. *Let for the edge* E *the symbol* $\mathring{E}$ *denote its relative interior.*

1. *Any edge from the complex $K_{\overline{\mathcal{C}}}$, lying in $\partial\mathcal{C}$, is either a monomaterial edge or a bimaterial outer edge.*

2. *Any edge E from $K_{\overline{\mathcal{C}}}$, which intersects $\mathcal{C}_+$ has one endpoint in 0. Consequently, the segment $\varphi_{\mathrm{a}_\blacktriangle}\big(\mathring{E}\big)$ has starting point $\mathrm{a}_\blacktriangle$. Thus, $\varphi_{\mathrm{a}_\blacktriangle}\big(\mathring{E}\big)$ is either a monomaterial or a bi-cellular segment in Π or it is contained in an edge of K_Π, i.e. in the starting triangulation of $\overline{\Pi}$.*

3. *For every edge $E \subseteq \Sigma$, $\varphi_{\mathrm{a}_\blacktriangle}\big(\mathring{E}\big) \subseteq \partial\Pi$ is a segment with starting point $\mathrm{a}_\blacktriangle$ and is, hence, a monomaterial segment or part of a monomaterial edge or part of a bimaterial outer edge which was already present in the starting triangulation K_Π.*

Proof. All three assertions follow from the definition of the triangulation of the polyhedron $\varepsilon\overline{\mathcal{C}}$ which is induced by the complex $h(L^\bigstar_{\mathrm{a}})$, see the proof of Theorem 3.10.

1. Since the edges from $\partial\mathcal{C}_- \setminus \overline{\Sigma}$ are the reflected edges from $\partial\mathcal{C}_+ \setminus \overline{\Sigma}$, it suffices to show the assertion only for the edges within the latter set and for edges from $\partial\Sigma$. Concerning the edges from $\partial\mathcal{C}_+ \setminus \overline{\Sigma}$, it is important to remember that the value of ε was chosen to assure that the whole link of $h(L^\bigstar_{\mathrm{a}})$ is disjoint to $\varepsilon\overline{\mathcal{C}}$. Thus, any edge, occurring on $\partial\mathcal{C}_+\setminus\overline{\Sigma}$, is either a "natural" edge of $\partial\mathcal{C}$ or results from the intersection of a 2-cell from $h(L^\bigstar_{\mathrm{a}})$, having one of its vertices in 0, with $\partial(\varepsilon\mathcal{C})$. Since such 2-cells are adjacent to exactly 2 tetrahedra, the resulting edge can be only monomaterial or bimaterial. Finally, each edge E from $\partial\Sigma$ is, by construction, an edge of exactly one element from $K^+_{\overline{\mathcal{C}}}$ (see Definition 6.1). Recalling the Definition 6.5 of the complex $K_{\overline{\mathcal{C}}}$ it is then clear that E is a monomaterial or a bimaterial edge within $K_{\overline{\mathcal{C}}}$ – depending whether the coefficient function on the reflected cell is different from that on the coresponding cell in $K^+_{\overline{\mathcal{C}}}$ or not.

2. By construction, all edges which intersect the interior of the polyhedron $|h(L^\bigstar_{\mathrm{a}})|$ originate in 0. Clearly, this property does not get lost when intersecting with $\varepsilon\overline{\mathcal{C}}$. Hence, if E intersects $\mathcal{C}_+$, then $\varphi_{\mathrm{a}_\blacktriangle}(\mathring{E})$ must be a segment with $\mathrm{a}_\blacktriangle$ as one of its endpoints. But all such segments can only be monomaterial or bi-cellular or have to be part of an edge of the original triangulation K_Π of $\overline{\Pi}$.

3. Every edge in Σ is already present in $h(L^\bigstar_{\mathrm{a}})$ and must, hence, have its starting point in $0 \in \mathbb{R}^3$. Thus, the assertion follows from (5) in Theorem 3.10. □

6.3 Estimates for the edge singularities

In this section we want to finish the proof of Theorem 5.1. Basing on our considerations in Section 4, we are done, if for all edges from the triangulation $K_{\overline{\mathcal{C}}}$ of $\overline{\mathcal{C}}$ the induced operators $\mathcal{A}_\lambda$ have a trivial kernel for all λ with $\Re\lambda \in]0, 1/3]$, see also Remark 4.7. The occurring edges are the following:

I	edges from $\partial\mathcal{C}$,	II	edges from $\mathcal{C}_+$,
III	edges from Σ,	IV	edges from $\mathcal{C}_-$.

It is not hard to see that the edges in IV may be treated analogously to the edges in II. Thus, we will discuss the cases I – III in the following.

I Edges from $\partial\mathcal{C}$ According to 1. of Lemma 6.9, any edge contained in $\partial\mathcal{C}$ is a monomaterial edge or a bimaterial outer edge. Thus the corresponding operators $\mathcal{A}_\lambda$ have a trivial kernel if $\Re\lambda \in]0, 1/2]$ by Proposition 4.9 and Proposition 4.10.

II Edges from $\mathcal{C}_+$ Let E be an edge from $\mathcal{C}_+$. Then E is an interior edge with one endpoint in $0 \in \mathbb{R}^3$, cf. 2. of Lemma 6.9. We consider the euclidean complex $K_E \subseteq K_{\mathcal{C}}^+$ which contains exactly those tetrahedra $\sigma \in K_{\mathcal{C}}^+$ for which $E \subseteq \sigma$. The corresponding two-dimensional sector coincides in this case with the whole space $\mathbb{R}^2$. Now we transform back to the original constellation in Π: i.e. we apply the piecewise linear homeomorphism $\phi := \varphi_{\mathrm{a}_\blacktriangle}$ associated to $\mathrm{a}_\blacktriangle$ by Theorem 3.10. Accordingly, E passes then either to a monomaterial or a bi-cellular segment in Π (both viewed as edges with a very peculiar coefficient constellation around in the sequel) or to an open part of an edge of K_Π, i.e. in the starting triangulation of $\overline{\Pi}$, cf. Lemma 6.9. In the first two cases there are no edge singularities, i.e. $\ker\mathcal{A}_\lambda = \{0\}$ if $\Re\lambda \in]0, 1[$ (cf. [7, Ch. 5.1]), while in the third case we have $\ker\mathcal{A}_\lambda = \{0\}$ if $\Re\lambda \leq \frac{1}{3} + \epsilon$ by supposition. One now simply applies Theorem 4.15.

III Edges from Σ Obviously, every edge from Σ is an interior edge relative to $\mathcal{C}$ and by 2. of Lemma 6.9 one of its endpoints is $0 \in \mathbb{R}^3$. If, for a given edge E from Σ, the coefficient matrices, belonging to the corresponding sectors in the half space $\{(x, y, z) \in \mathbb{R}^3 : y > 0\}$, are

$$M^1 = \begin{pmatrix} m_{11}^1 & m_{12}^1 & m_{13}^1 \\ m_{12}^1 & m_{22}^1 & m_{23}^1 \\ m_{13}^1 & m_{23}^1 & m_{33}^1 \end{pmatrix}, \quad \dots, \quad M^n = \begin{pmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{12}^n & m_{22}^n & m_{23}^n \\ m_{13}^n & m_{23}^n & m_{33}^n \end{pmatrix},$$

then the corresponding matrices in the reflected sectors are

$$M_-^1 = \begin{pmatrix} m_{11}^1 & -m_{12}^1 & m_{13}^1 \\ -m_{12}^1 & m_{22}^1 & -m_{23}^1 \\ m_{13}^1 & -m_{23}^1 & m_{33}^1 \end{pmatrix}, \quad \dots, \quad M_-^n = \begin{pmatrix} m_{11}^n & -m_{12}^n & m_{13}^n \\ -m_{12}^n & m_{22}^n & -m_{23}^n \\ m_{13}^n & -m_{23}^n & m_{33}^n \end{pmatrix}$$

see [9, Proposition 17]. According to Proposition 4.6, one has to perform a rotation in the x-z-plane, which moves the edge under consideration to the z-axis. This means, one has to consider the matrices

$$\begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} M \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix},$$

M taken as $M^1, \dots, M^n, M_-^1, \dots, M_-^n$, respectively, and α being the angle between the corresponding edge and the z-axis. A straightforward calculation shows that the resulting upper 2×2 blocks look alike

$$\begin{pmatrix} \rho_{11}^1 & \rho_{12}^1 \\ \rho_{12}^1 & \rho_{22}^1 \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \rho_{11}^n & \rho_{12}^n \\ \rho_{12}^n & \rho_{22}^n \end{pmatrix}$$

in the sectors within the half space $\{(x,y,z) \in \mathbb{R}^3 : y > 0\}$ and

$$\begin{pmatrix} \rho_{11}^1 & -\rho_{12}^1 \\ -\rho_{12}^1 & \rho_{22}^1 \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \rho_{11}^n & -\rho_{12}^n \\ -\rho_{12}^n & \rho_{22}^n \end{pmatrix}$$

on the reflected sectors within the half space $\{(x,y,z) \in \mathbb{R}^3 : y < 0\}$. Concerning this constellation, we may apply Lemma 4.16, in order to reduce the problem, given on a sector partition of the whole of $\mathbb{R}^2$, to a Dirichlet and a Neumann problem on a sector partition of the half space.

Hence, it remains to prove the following

Lemma 6.10. *Let $\underline{\mu}$ be the transformed coefficient function on $\mathcal{C}_+$ and $\underline{\mathcal{A}_\lambda^D}$, $\underline{\mathcal{A}_\lambda^N}$ be the Sturm-Liouville operator pencils which correspond to an edge from Σ – once combined with Dirichlet and once combined with Neumann condition. Then the kernels of $\underline{\mathcal{A}_\lambda^D}$ and $\underline{\mathcal{A}_\lambda^N}$ are trivial, if $\Re\lambda \in \left]0, \frac{1}{2}\right]$.*

Proof. Let E be an edge from Σ. This time, we consider the euclidean complex K_E which contains exactly all tetrahedra $\sigma \in K_{\underline{\mathcal{C}}}^+$, for which $E \subseteq \sigma$, cf. Definition 6.1. The projection of the polyhedron $|K_E|$ onto any plane that is perpendicular to E, is locally around E, a sector, namely the corresponding half space in $\mathbb{R}^2$. This assures supposition 2. of Theorem 4.15.

Now, we transform back the corresponding edge to the original setting in Π, i.e., we apply the map $\phi := \varphi_{\mathrm{a}_\blacktriangle}$ in the sense of Section 3. The image of the edge E under this map is necessarily either contained in a planar face of one cell from K_Π or part of a monomaterial edge or part of a bimaterial outer edge which was already present in the starting triangulation K_Π, see 3. of Lemma 6.9. Thus, a cut trough $\Pi_\bullet$ perpendicular to $\phi(E)$ in a neighbourhood of $\phi(E)$ looks like indicated in Figure 2. Note that, due to the

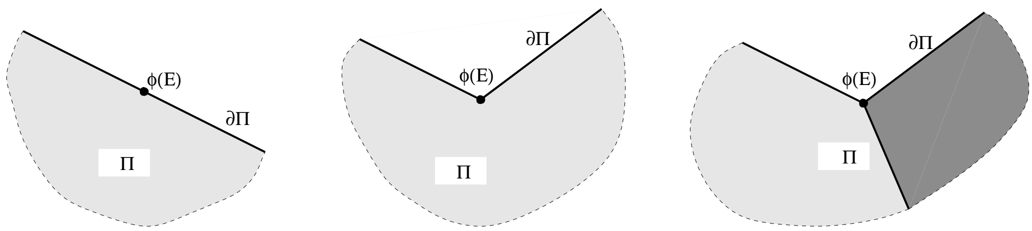

Figure 2: Cut through $\Pi_\bullet$ perpendicular to $\phi(E)$ for a monomaterial segment, a monomaterial edge and a bimaterial edge, respectively

definition of a bimaterial outer edge, the opening angles of the sectors in the third case both do not exceed π.

For these constellations, the strip $\{\lambda : 0 < \Re\lambda \le \frac{1}{2}\}$ does not contain singular values of the corresponding operator pencils $\mathcal{A}_\lambda$, thanks to Proposition 4.9 and Proposition 4.10. Thus, again an application of Theorem 4.15 gives the assertion. $\square$

7 Concluding remarks

Remark 7.1. We suggest that our suppositions are in general also necessary. Namely, if one of the appearing edges admits a singularity index $< \frac{1}{3}$, then [9, Lemma 14] permits to construct a function on $\mathbb{R}^3$ whose gradient is not from L^3. Applying an appropriate cutoff-function one can arrange the support property in a suitable cone – with the singularity also appearing at the vertex.

Remark 7.2. The regularity result of this paper easily carries over to problems with Robin boundary conditions because the involved boundary operator is relativlely compact with respect to the main part.

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References

[1] P. Alexandroff, H. Hopf, Topologie, vol. 45 of Grundlehren der math. Wissenschaften, Springer, Berlin, 1935; berichtigter Reprint, Springer, 1974.

[2] R. H. Bing, The Geometric Topology of 3-Manifolds, vol. 40 of AMS Colloquium Publications, Amer. Math. Soc., Providence/RI, 1983.

[3] M. Chipot, D. Kinderlehrer, G. V. Caffarelli, Smoothness of linear laminates, Arch. Rational Mech. Anal. 96 (1986) 81–96.

[4] M. Costabel, M. Dauge, S. Nicaise, Singularities of Maxwell interface problems. M2AN Math. Model. Numer. Anal. 33 no.3 (1999) 627-649

[5] A. Cianchi, V. Maz'ya, Neumann problems and isocapacitary inequalities, J. Math. Pures Appl. 89 (2008) 71–105.

[6] M. Dauge, Neumann and mixed problems on curvilinear polyhedra, Integral Equations Operator Theory 15 (1992) 227–261.

[7] J. Elschner, H.-C. Kaiser, J. Rehberg, G. Schmidt, $W^{1,q}$ regularity results for elliptic transmission problems on heterogeneous polyhedra, Math. Models Methods Appl. Sci. 17 (2007) 593–615.

[8] J. Elschner, J. Rehberg, G. Schmidt, Optimal regularity for elliptic transmission problems including C^1 interfaces, Interfaces Free Bound. 9, no. 2, (2007) 233–252.

[9] R. Haller-Dintelmann, H.-C. Kaiser, J. Rehberg, Elliptic model problems including mixed boundary conditions and material heterogeneities, J. Math. Pures Appl. 89 (2008) 25–48.

[10] R. Haller-Dintelmann, J. Rehberg, Maximal parabolic regularity for divergence operators including mixed boundary conditions, J. Differential Equations 247 (2009) 1354–1396.

[11] D. Hömberg, Ch. Meyer, J. Rehberg, W. Ring, Optimal control for the thermistor problem, SIAM J. Control Optim. 48 (2010) 3449–3481.

[12] D. Jerison, C. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.

[13] H.-C. Kaiser, J. Rehberg, Optimal elliptic regularity at the crossing of a material interface and a Neumann boundary edge, J. Math. Sci. (N.Y.) 169 (2010) 145–166.

[14] D. Knees, On the regularity of weak solutions of quasi-linear elliptic transmission problems on polyhedral domains, Z. Anal. Anwendungen 23 (2004) 509–546.

[15] D. Leguillon, E. Sanchez-Palenzia, Computation of Singular Solutions in Elliptic Problems and Elasticity, Wiley, Chichester, 1987.

[16] Y. Y. Li, M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Ration. Mech. Anal. 153 (2000) 91–151.

[17] J. Lukkainen, J. Väisälä, Elements of Lipschitz topology, Ann. Acad. Sci. Fenn., Ser. A I 3 (1977) 85–122.

[18] V. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[19] V. Maz'ya, J. Rossmann, Elliptic equations in polyhedral domains, vol. 162 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2010.

[20] V. Maz'ya, J. Rossmann, Weighted estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains. ZAMM Z. Angew. Math. Mech. 83, no. 7, (2003), 435–467.

[21] V. Maz'ya, J. Elschner, J. Rehberg, G. Schmidt, Solutions for quasilinear evolution systems in L^p, Arch. Ration. Mech. Anal. 171 (2004) 219–262.

[22] V. G. Maz'ya and S. A. Nazarov, *Singularities of solutions of the Neumann problem at a conic point*, Sibirsk. Mat. Zh. 30 (1989) 52–63, 218.

[23] E. E. Moise, Geometric Topology in Dimensions 2 and 3, vol. 47 of Graduate Texts in Math., Springer, New York, 1977.

[24] S. Nicaise, Polygonal interface problems, vol. 39 of Methoden und Verfahren der Mathematischen Physik, Verlag Peter D. Lang, Frankfurt/M, 1993.

[25] S Nicaise, A.-M. Sändig, General interface problems. I, II. Math. Methods Appl. Sci. 17 (1994), no. 6, 395-429, 431-450

[26] M. Petzoldt, Regularity results for Laplace interface problems in two dimensions, Z. Anal. Anwendungen 20 (2001) 431–455.

[27] G. Savaré, Regularity results for elliptic equations in Lipschitz domains, J. Funct. Anal. 152 (1998) 176–201.

[28] L. Siebenmann, D. Sullivan: On complexes that are Lipschitz manifolds, in: "Geometric Topology", Proc. 1977 Georgia Topology Conf., Athens/GA 1977 (J. C. Cantrell, ed.), Academic Press, New York, 1979, 503–525.

[29] I. E. Tamm, Fundamentals of the theory of electricity. Mir Publishers, Moscow, 1979.

[30] D. Zanger, The inhomogeneous Neumann problem in Lipschitz domains, Comm. Partial Differential Equations 25 (2000) 1771–1808.