Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

About some coercive inequalities for elementary elliptic and parabolic operators

A. Koshelev[1]

submitted: 9th October 1992

[1] This paper was written while the author was visiting at the Institute for Applied Analysis and Stochastic

Preprint No. 15
Berlin 1992

Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin

Fax: + 49 30 2004975
e-Mail (X.400): c=de;a=dbp;p=iaas-berlin;s=preprint
e-Mail (Internet): preprint@iaas-berlin.dbp.de

Consider at first in $\mathbb{R}^m$ $(m \geq 2)$ a ball $B_\delta = \{x : |x| < \delta\}$, where $x \in \mathbb{R}^m$ and δ is an arbitrary positive number. Suppose that a function $u(x)$, which is defined on B_δ belongs to the space $W^{(2)}_{2,\beta}(B_\delta)$ (all the second derivatives are square summable with the weight $|x|^\beta$) $(|\beta| < m)$ and satisfies the boundary condition

$$u|_{\partial B_\delta} = 0\,. \tag{1}$$

This condition gives us the possibility to define the norm of such function in $W^{(2)}_{2,\beta}$ by the equality

$$||u||^2_{\beta,2} = \int\limits_{B_\delta} |D^2 u|^2 |x|^\beta dx\,, \tag{2}$$

where

$$|D^2 u|^2 = \sum_{i,k=1}^{m} |u_{ik}|^2\,.$$

For such functions S. Chelkak (see for example [4], p. 28) under additional assumption

$$\nabla u|_{\partial B_\delta} = 0 \tag{3}$$

has proved the inequality

$$\int\limits_{B_\delta} |D^2 u|^2 |x|^\beta dx \leq \left[1 + \frac{4\beta(m-1)}{(m-\beta)^2}\right] \int\limits_{B_\delta} |\Delta u|^2 |x|^\beta dx\,, \tag{4}$$

which holds for all β with $0 < \beta < m$ and $m \geq 4$. This inequality holds also for $\beta = m - 2 + 2\gamma$ $(0 < \gamma < 1)$ for any $m \geq 2$.

The inequality (4) has many applications in the theory of quasilinear elliptic and parabolic systems. We shall show now that the relation of the type (4) holds also without assumption (3), but the constant before the right–hand side integral will be different in some cases.

It should be also mentioned that the inequalities of the type (4) can be obtained with the help of the methods of E. Stein [5] or V.A. Kondratjev [1]. But the above mentioned constant will have an implicit form.

Before formulating the main results we shall mention some necessary relations, which will be used later.

At first we would like to mention the so called Hardy's inequality

$$\int\limits_0^{+\infty} r^{-s} |F(r)|^p dr \leq \left(\frac{p}{|s-1|}\right)^p \int\limits_0^{+\infty} r^{-s} (r|f|)^p dr\,, \tag{5}$$

where

$$F(r) = \int\limits_0^r f(\varrho) d\varrho \quad (s > 1) \qquad \text{and} \qquad F(r) = \int\limits_r^{+\infty} f(\varrho) d\varrho \quad (s < 1)$$

and $f(\varrho)$ is suitable summable and defined on $(0,+\infty)$. Suppose that on $[0,\delta]$ is given a function $u(r)$, which satisfies the condition $u(\delta)=0$ and possesses the first derivative with a finite integral

$$\int_0^\delta |u'(r)|^2 r^{-s+2}\,dr$$

with some $s<1$. Expand this function on the whole axis $[0,+\infty)$ with the help of the equality $u(r)\equiv 0$ for $r>\delta$. Denoting $f(r)=u'(r)$ and using the identity

$$u(r)=-\int_r^{+\infty} u'(\varrho)d\varrho$$

we obtain from (5) the inequality

$$\int_0^\delta |u(r)|^2 r^{-s}dr<\frac{4}{(s-1)^2}\int_0^\delta |u'(r)|^2 r^{-s+2}dr\,. \tag{6}$$

The analogous is true for $s>1$ and $u(0)=0$. We shall also use the inequality

$$ab<\eta a^2+\frac{1}{4\eta}b^2\,, \tag{7}$$

where $\eta>0$ is as always an arbitrary small number.
It is worthwhile to mention that we can from the beginning to consider u as a sufficiently smooth function satisfying the condition (1). Denote by S a sphere with the unit radius with the center at the origin in $\mathbb{R}^m$ and let $B_1=B$. Take a complete orthogonal set of spherical functions $\{Y_{j,l}(\theta)\}$ $(j=1,2,\cdots;l=1,\cdots,k_j;\ \theta\in S)$ and consider the expansion of $u(x)$

$$u(x)=\sum_{j=1}^{+\infty}\sum_{l=1}^{k_j} u_{j,l}(r)Y_{j,l}(\theta)\,, \tag{8}$$

where $r=|x|$. After elementary calculations we come to the following equalities

$$\begin{aligned}\int_B |\Delta u|^2 r^\beta dx &= (m-1)\sum_{j,l}|u'_{j,l}(1)|^2+\sum_{j,l}\int_0^1\Big\{|u''_{j,l}(r)|^2+\\ &+[(m-1)(1-\beta)+2j(j+m-2)]|u'_{j,l}(r)|^2r^2+\\ &+j(j+m-2)[j(j+m-2)-\beta+m-4)(\beta-2)]|u_{j,l}(r)|^2r^{-4}\Big\}r^{\beta+m-1}dr\end{aligned} \tag{9}$$

where $'$ denotes the derivative with respect to r and the summation for j,l is running in the same way as in (8);

$$\int_B (|D^2u|^2 - |\Delta u|^2) r^\beta dx = -(m-1) \sum_{j,l} |u'_{j,l}(1)|^2 +$$
$$+\beta \sum_{j,l} \int_0^1 [(m-1)|u'_{j,l}(r)|^2 + \tag{10}$$
$$+(\beta+m-3)j(j+m-2)|u_{j,l}(r)|^2 r^{-2}] r^{\beta+m-3} dr\,.$$

Let us show for example how to prove the equality (10). First of all integration by parts gives

$$\int_B u_{ik}u_{ik}r^\beta dx = \int_B (u_k u_{ik} r^\beta)_i dx - \int_B u_k u_{iik} r^\beta dx -$$
$$-\beta \int_B u_k u_{rk} r^{\beta-1} dx = \int_S u_k u_{rk} dS - \int_S u_r \Delta u dS +$$
$$+\int_B |\Delta u|^2 r^\beta dx + \beta \int_B u_r \Delta u r^{\beta-1} dx - \beta \int_B u_k u_{rk} r^{\beta-1} dx\,.$$

After simple calculations we come to the equality

$$\int_B u_{ik}u_{ik}r^\beta dx = \int_S \left(u_k u_{rk} - u_r\Delta u - \frac{\beta}{2}|\nabla u|^2\right) dS +$$
$$+\int_B |\Delta u|^2 r^\beta dx + \beta \int_B u_r \Delta u r^{\beta-1} dx + \frac{\beta(\beta+m-2)}{2} \int_B |\nabla u|^2 r^{\beta-2} dx\,.$$

Since $\beta > 0$ we have $\beta - 2 > -2 \geq -m$ and all the integrals are determined. Applying the boundary condition (1) we'll have

$$\left(u_k u_{rk} - u_r \Delta u - \frac{\beta}{2}|\nabla u|^2\right)|_{r=1} = -\left(m - 1 + \frac{\beta}{2}\right)(u')^2|_{r=1}\,.$$

Then

$$\int_B (|D^2u|^2 - |\Delta u|^2) r^\beta dx = -(m-1+\frac{\beta}{2}) \int_S (u')^2 dS +$$
$$+\beta \int_B u_r \Delta u r^{\beta-1} dx + \frac{\beta(\beta+m-2)}{2} \int_B |\nabla u|^2 r^{\beta-2} dx\,. \tag{11}$$

Substituting in the right–hand side for $m = 2$

$$|\nabla u|^2 = |u_r|^2 + r^{-2}|u_\theta|^2$$

and calculating the integral $\int_B u_r \Delta u r^{\beta-1} dx$ with the help of the expansion (8) we come to (10) in this case. For $m \geq 3$ the expansion (8) should be applied to the right–hand side of the identity

$$\int_B |\nabla u|^2 r^{\beta-2} dx = -\int_B u \Delta u r^{\beta-2} dx + \frac{(\beta-2)(\beta+m-4)}{2} \int_B |u|^2 r^{\beta-4} dx\,.$$

After substituting the expression of this integral and the analogous expression for the integral $\int\limits_B u_r \Delta u r^{\beta-1} dx$ in (11) we also come to (10) with $m \geq 3$. It should not be forgotten that for $j > 0$ we have the equality $u_{j,l}(0) = 0$.

Theorem 1. *Let $\beta = m-2+2\gamma$ and $0 < \gamma < 1/2$. For any $u \in W^{(2)}_{2,\beta}(B_\delta)$ satisfying the boundary condition (1) the inequality*

$$\int\limits_{B_\delta} |D^2 u|^2 r^\beta dx \leq \left\{ 1 + \frac{4\beta(m-1)}{(m-\beta)^2} + \right.$$

$$\left. \frac{4\beta(\beta+m-2)^4(m-1)}{(m-\beta)^2(m+\beta-3)(m-\beta-1)^2\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]} + O(\gamma) \right\} \times \tag{12}$$

$$\times \int\limits_{B_\delta} |\Delta u|^2 r^\beta dx$$

holds.

Proof. It is enough to prove (12) for $\delta = 1$. Evidently

$$u'_{j,l}(r) - u'_{j,l}(1) = -\int\limits_r^1 u''_{j,l}(\varrho) d\varrho \,. \tag{13}$$

For all β ($0 < \beta < m$) the inequality $s = -\beta - m + 3 < 1$ holds. Then, according to (6) we have

$$\int\limits_0^1 |u''_{j,l}|^2 r^{\beta+m-1} dr \geq \frac{(\beta+m-2)^2}{4} \int\limits_0^1 |u'_{j,l}(r) - u'_{j,l}(1)|^2 r^{\beta+m-3} dr \,. \tag{14}$$

Estimating on the right–hand side of (9) the term with the second derivative with the help of (14) we come to the inequality

$$\begin{aligned} \int\limits_B |\Delta u|^2 r^\beta dx \geq \left(m-1+\frac{\beta+m-2}{4}\right) \sum_{j,l} |u'_{j,l}(1)|^2 - \\ -\frac{(\beta+m-2)^2}{2} \sum_{j,l} \int\limits_0^1 u'_{j,l} r^{\beta+m-3} dr u'_{j,l}(1) + \\ +\sum_{j,l} \int\limits_0^1 \left\{ \left[\frac{(m-\beta)^2}{4} + 2j(j+m-2) \right] |u'_{j,l}|^2 + \right. \\ \left. +j(j+m-2)\left[j(j+m-2) + (\beta+m-4)(2-\beta) \right] |u^2_{j,l}| r^{-2} \right\} r^{\beta+m-3} dr \,. \end{aligned} \tag{15}$$

Since for $m \geq 4$ (all $\beta > 0$) we have that $s = -\beta - m + 5 < 1$ also from (6) follows

$$\int\limits_0^1 |u'_{j,l}|^2 r^{\beta+m-3} dr \geq \frac{(\beta+m-4)^2}{4} \int\limits_0^1 |u_{j,l}|^2 r^{\beta+m-5} dr \,. \tag{16}$$

The same will happen for $\beta > 1$ and $m = 3$. For $m = 2$ and $0 < \beta < 2$ and $m = 3$ and $0 < \beta < 1$ using the fact that $u_{j,l}(0) = 0$ for $j > 0$ instead of the representation analogous to (13) we can write

$$u_{j,l}(r) = \int_0^r u'_{j,l}(\varrho)d\varrho\,.$$

These cases give us $s = -\beta - m + 5 > 1$ and we can apply (6). With the help of the last equality we also come to (16) (with the exception for $m = 3$ and $\beta = 1$). Estimating the right–hand side of (15) from below with the help of (16) we come to

$$\begin{gathered}\int_B |\Delta u|^2 r^\beta dx \geq \left(m - 1 + \frac{\beta + m - 2}{4}\right) \sum_{j,l} |u'_{j,l}(1)|^2 + \\ + \sum_{j,l} \int_0^1 \left\{ \frac{(m-\beta)^2}{4} |u'_{j,l}|^2 + j(j+m-2)\left[j(j+m-2) + \right.\right. \\ \left.\left. + \frac{(m-\beta)(m+\beta-4)}{2}\right] |u_{j,l}|^2 r^{-2} \right\} r^{\beta+m-3} dr - \\ - \frac{(\beta+m-2)^2}{2} \sum_{j,l} \int_0^1 u'_{j,l} r^{\beta+m-3} dr u'_{j,l}(1)\,.\end{gathered} \tag{17}$$

The middle term on the right—hand side can be estimated in the same way as it was done by Chelkak ([4], p. 29)

$$\begin{gathered}\sum_{j,l} \int_0^1 \left\{ \frac{(m-\beta)^2}{4} |u'_{j,l}|^2 + j(j+m-2)\left[j(j+m-2) + \right.\right. \\ \left.\left. + \frac{(m-\beta)(\beta+m-4)}{2}\right] |u_{j,l}|^2 r^{-2} \right\} r^{\beta+m-3} dr \geq \\ \geq \frac{(m-\beta)^2}{4\beta(m-1)} \cdot \beta \sum_{j,l} \int_0^1 \left[(m-1)|u'_{j,l}|^2 + \right. \\ \left. + (\beta+m-3) j(j+m-2) |u_{j,l}|^2 r^{-2} \right] r^{\beta+m-3} dr\,.\end{gathered}$$

Applying now the equality (10) from (17) we come to the estimate

$$\int_B |\Delta u|^2 r^\beta dx \geq \left(m-1+\frac{\beta+m-2}{4}\right)\sum_{j,l}|u'_{j,l}(1)|^2+$$

$$+\frac{(m-\beta)^2}{4\beta(m-1)}\left[\int_B (|D^2u|^2-|\Delta u|^2)r^\beta dx+(m-1)\sum_{j,l}|u'_{j,l}(1)|^2\right]-$$

$$-\frac{(\beta+m-2)^2}{2}\sum_{j,l}\int_0^1 u'_{j,l}r^{\beta+m-3}dru'_{j,l}(1)\,.$$

From this follows that for all $m \geq 2$ (except $m=3$ and $\beta=1$) the inequality

$$\left[1+\frac{4\beta(m-1)}{(m-\beta)^2}\right]\int_B |\Delta u|^2 r^\beta dx \geq \int_B |D^2u|^2 r^\beta dx+$$

$$+\frac{4\beta(m-1)}{(m-\beta)^2}\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]\sum_{j,l}|u'_{j,l}(1)|^2- \tag{18}$$

$$-\frac{2\beta(\beta+m-2)^2(m-1)}{(m-\beta)^2}\sum_{j,l}\int_0^1 u'_{j,l}(r)r^{\beta+m-3}dru'_{j,l}(1)$$

takes place. Consider now the integral

$$I=\int_0^1 u'_{j,l}(r)r^{\beta+m-3}dr \tag{19}$$

for $m \geq 3$. After integrating by parts we get

$$\int_0^1 u'_{j,l}(r)r^{\beta+m-3}dr=-(\beta+m-3)\int_0^1 u_{j,l}(r)r^{\beta+m-4}dr\,.$$

Applying the Hölder's inequality we come to the following relation

$$|\int_0^1 u'_{j,l}(r)r^{\beta+m-3}dr| \leq \left(\int_0^1 |u_{j,l}(r)|^2 r^{\beta+m-4}dr\right)^{1/2}(\beta+m-3)^{1/2}\,.$$

Then from (18) we'll have

$$\left[1+\frac{4\beta(m-1)}{(m-\beta)^2}\right]\int_B |\Delta u|^2 r^\beta dx \geq \int_B |D^2u|^2 r^\beta dx+\frac{4\beta(m-1)}{(m-\beta^2)}+$$

$$\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]\sum_{j,l}|u'_{j,l}(1)|^2-\frac{2\beta(\beta+m-2)^2(m-1)}{(m-\beta)^2(\beta+m-3)^{1/2}}\times$$

$$\times\sum_{j,l}\left(\int_0 |u_{j,l}(r)|^2 r^{\beta+m-4}dr\right)^{1/2}|u'_{j,l}(1)|$$

Applying the inequality (7) we come to the relation

$$\left[1+\frac{4\beta(m-1)}{(m-\beta)^2}\right]\int\limits_B |\Delta u|^2 r^\beta dx \geq \int\limits_B |D^2u|^2 r^\beta dx +$$
$$+\frac{4\beta(m-1)}{(m-\beta)^2}\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]\sum_{j,l}|u'_{j,l}(1)|^2 -$$
$$-\frac{2\beta(\beta+m-2)^2(\beta+m-3)^{1/2}(m-1)}{(m-\beta)^2}\cdot\eta\sum_{j,l}|u'_{j,l}(1)|^2 -$$
$$-\frac{\beta(\beta+m-2)^2(m-1)(\beta+m-3)^{1/2}}{2(m-\beta)^2\eta}\times\sum_{j,l}\int\limits_0^1 |u_{j,l}|^2 r^{\beta+m-y} dr\,.$$

Take

$$\eta=\frac{2\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]}{(\beta+m-2)^2(\beta+m-3)^{1/2}}\,.$$

Then the terms with $\sum|u'_{j,l}(1)|^2$ will be abolished and we'll have

$$\left[1+\frac{4\beta(m-1)}{(m-\beta)^2}\right]\int\limits_B |\Delta u|^2 r^\beta dx \geq \int\limits_B |D^2u|^2 r^\beta dx -$$
$$-\frac{\beta(\beta+m-2)^4(m-1)(\beta+m-3)}{4(m-\beta)^2\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]}\sum_{j,l}\int\limits_0^1 |u_{j,l}|^2 r^{\beta+m-4} dr\,.$$

Using the equality

$$\sum_{j,l}\int\limits_0^1 |u_{j,l}|^2 r^{\beta+m-4} dr = \int\limits_B |u|^2 r^{\beta-3} dx$$

we come to

$$\left[1+\frac{4\beta(m-1)}{(m-\beta)^2}\right]\int\limits_B |\Delta u|^2 r^\beta dx \geq$$
$$\int\limits_B |D^2u|^2 r^\beta dx - \frac{\beta(\beta+m-2)^4(m-1)(\beta+m-3)}{4(m-\beta)^2\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]}\int\limits_B |u|^2 r^{\beta-3} dx\,. \tag{20}$$

So, now we have to estimate the integral

$$\int\limits_B |u|^2 r^{\beta-3} dx\,.$$

Integrating by parts we come to

$$-\int\limits_B \Delta u\cdot u r^{\beta-1} dx = \int\limits_B |\nabla u|^2 r^{\beta-1} dx + (\beta-1)\int\limits_B u' u r^{\beta-2} dx\,.$$

Using the condition (1) we can integrate by parts once more in the second term on the right–hand side. Then we get

$$-\int_B \Delta u \cdot u r^{\beta-1} dx = \int_B |\nabla u|^2 r^{\beta-1} dx - \frac{(\beta-1)(\beta+m-3)}{2} \int_B |u|^2 r^{\beta-3} dx .$$

Since

$$u(r) = -\int_r^1 u'(\varrho) d\varrho$$

and $-\beta - m + 4 < 1$ then from the inequality (6) we have

$$\int_0^1 |u|^2 r^{\beta+m-4} dr \le \frac{4}{(\beta+m-3)^2} \int_0^1 |u'|^2 r^{\beta+m-2} dr .$$

Therefore

$$-\int_B \Delta u u r^{\beta-1} dx \ge \frac{(m+\beta-3)(m-\beta-1)}{4} \int_B |u|^2 r^{\beta-3} dx .$$

Since $\beta = m - 2 + 2\gamma$ $(0 < \gamma < 1/2)$ the coefficient on the right–hand side will be positive and we get

$$\int_B |u|^2 r^{\beta-3} dx \le -\frac{4}{(m+\beta-3)(m-\beta-1)} \int_B \Delta u \cdot u r^{\beta-1} dx .$$

From the Hölder inequality follows

$$\int_B |u|^2 r^{\beta-3} dx \le \frac{16}{(m+\beta-3)^2 (m-\beta-1)^2} \int_B |\Delta u|^2 r^{\beta+1} dx .$$

Since $r \le 1$ we have

$$\int_B |u|^2 r^{\beta-3} dx \le \frac{16}{(m+\beta-3)^2 (m-\beta-1)^2} \int_B |\Delta u|^2 r^{\beta} dx .$$

Using the estimate (20) we come to the inequality (12) for $m \ge 3$.
Let us consider now the case $m = 2$. In the inequality (17) we shall estimate the integral I (19) in a different way. Evidently

$$I \le (\beta+m-2)^{-1/2} \sum_{j,l} \left(\int_0^1 |u|_{j,l}|^2 r^{\beta+m-3} dr \right)^{1/2} |u'_{j,l}(1)| .$$

Applying (7) we get

$$\frac{(\beta+m-2)^2}{2} I \le \eta \sum_{j,l} \int_0^1 |u'_{j,l}|^2 r^{\beta+m-3} dr + \frac{(\beta+m-2)^3}{16\eta} \sum_{j,l} |u'_{j,l}(1)|^2 .$$

Then from (17) we get

$$\int_B |\Delta u|^2 r^\beta dx \geq \left[m-1+\frac{\beta+m-2}{4}-\frac{(\beta+m-2)^3}{16\eta}\right]\sum_{j,l}|u'_{j,l}(1)|^2+$$

$$+\sum_{j,l}\int_0^1\left\{\left[\frac{(m-\beta)^2}{4}-\eta\right]|u'_{j,l}|^2+j(j+m-2)\right.$$

$$\left.\left[j(j+m-2)+\frac{(m-\beta)(m+\beta-4)}{2}\right]|u_{j,l}|^2r^{-2}\right\}r^{\beta+m-3}dr\,.$$

Take $\eta = O(\gamma^2)$. Since

$$\min\left\{\frac{(m-\beta)^2}{4\beta(m-1)},\ \min_j\frac{j(j+m-2)+\frac{(m-\beta)(m+\beta-4)}{2}}{\beta(\beta+m-3)}\right\}=\frac{(m-\beta)^2}{4\beta(m-1)}$$

then for small $\gamma > 0$

$$\int_B |\Delta u|^2 r^\beta dx \geq \left[m-1+\frac{\beta+m-2}{4}-\frac{(\beta+m-2)^3}{4\eta}\right]\sum_{j,l}|u'_{j,l}(1)|^2+$$

$$+\left[\frac{(m-\beta)^2}{4\beta}-O(\gamma)\right]\beta\sum_{j,l}\int_0^1$$

$$\left[(m-1)|u'_{j,l}(r)|^2+(\beta+m-3)j(j+m-2)\times|u_{j,l}(r)|^2r^{-2}\right]r^{\beta+m-3}dr\,.$$

According to (10) we come to the inequality

$$\int_B |\Delta u|^2 r^\beta dx \geq \left[m-1+\frac{\beta+m-2}{4}-\frac{(\beta+m-2)^3}{4\eta}\right]\sum_{j,l}|u'_{j,l}(1)|^2+$$

$$+\left[\frac{(m-\beta)^2}{4\beta}-O(\gamma)\right]\left[\int_B(|D^2u|^2-|\Delta u|^2)r^\beta dx+(m-1)\sum_{j,l}|u'_{j,l}(1)|^2\right].$$

As far as $\eta = 0(\gamma^2)$ the first bracket on the right–hand side will be positive. Then

$$\int_B |\Delta u|^2 r^\beta dx \geq \left[\frac{(m-\beta)^2}{4\beta}-O(\gamma)\right]\int_B(|D^2u|^2-|\Delta u|^2)r^\beta dx$$

and the proof of the theorem is completed. □

Consider now in the space (t, x) where $t \geq 0$ and $x \in \mathbb{R}^m$ the cylinder $(0 < t < \delta^2) \times B_\delta$. Suppose that the boundary condition

$$u|_{t=0} = 0 \tag{21}$$

holds.

Let the function $u(x, t)$ possesses in Sobolev's sense all the second derivatives with respect to x and the first derivative with respect to t.

Assume that the conditions (1) and (21) are satisfied and the integral

$$\int_{Q_\delta} (|u_t|^2 + |D^2u|^2)|x|^\beta dxdt \quad (|\beta| < m)$$

is finite. We shall tell that the functions of this class belongs to the $W^{(1,2)}_{2,\beta,0}(Q_\delta)$.

Theorem 2. *If* $u \in W^{(1,2)}_{2,\beta,0}(Q_\delta)$ $(\beta = m - 2 + 2\gamma, 0 < \gamma < 1/2)$ *then the following inequalities take place: for* $m \geq 3$

$$\int_{Q_\delta} |D^2u|^2|x|^\beta dxdt \leq \frac{m}{m-\beta}\left\{1 + \frac{4\beta(m-1)}{(m-\beta)^2} + \right.$$

$$\left. + \frac{4\beta(\beta+m-2)^4(m-1)}{(m-\beta)^2(m+\beta-3)(m-\beta-1)^2\left[m-1+\frac{\beta+m-2}{4}+\frac{(m-\beta)^2}{4\beta}\right]} + O(\gamma)\right\} \times \tag{22}$$

$$\int_{Q_\delta} |u_t - \Delta u|^2|x|^\beta dxdt;$$

for $m = 2$ *and small* $\gamma > 0$

$$\int_{Q_\delta} |D^2u|^2|x|^\beta dxdt \leq 2[1 + O(\gamma)] \int_{Q_\delta} |u_t - \Delta u|^2|x|^\beta dxdt. \tag{23}$$

The inequalities (22) and (23) follow from (12) and lemma 2 and theorem 2 in [3] and [2].

References

1. V.A. Kondratjev, *Kraewie zadachi dlja ellipticheskichwravneni s konicheskimi ili ugovymi tochkami*, Trudy Moskov. Mat. Obshch. **16** (1967), 209–292.
2. A. Koshelev, *Regularity of solutions for some quasilinear parabolic systems*, to appear.
3. ______, *Regularity of solutions for some quasilinear parabolic systems*, Universität Heidelberg, Preprint 633, 1991.
4. A. Koshelev and S. Chelkak, *Regularity of solutions of quasilinear elliptic systems*, Teubner–Texte zur Mathematik, Bd. 77, 1985.
5. E. Stein, *Singular integrals and differentiability properties on functions*, Princeton–University Press, NJ, 1970.

Veröffentlichungen des Instituts für Angewandte Analysis und Stochastik

Preprints 1992

1. D.A. Dawson and J. Gärtner: Multilevel large deviations.
2. H. Gajewski: On uniqueness of solutions to the drift–diffusion–model of semiconductor devices.
3. J. Fuhrmann: On the convergence of algebraically defined multigrid methods.
4. A. Bovier and J.-M. Ghez: Spectral properties of one–dimensional Schrödinger operators with potentials generated by substitutions.
5. D.A. Dawson and K. Fleischmann: A super–Brownian motion with a single point catalyst.
6. A. Bovier, V. Gayrard: The thermodynamics of the Curie–Weiss model with random couplings.
7. W. Dahmen, S. Prößdorf, R. Schneider: Wavelet approximation methods for pseudodifferential equations I: stability and convergence.
8. A. Rathsfeld: Piecewise polynomial collocation for the double layer potential equation over polyhedral boundaries. Part I: The wedge, Part II: The cube.
9. G. Schmidt: Boundary element discretization of Poincaré–Steklov operators.
10. K. Fleischmann, F. I. Kaj: Large deviation probability for some rescaled superprocesses.
11. P. Mathé: Random approximation of finite sums.
12. C.J. van Duijn, P. Knabner: Flow and reactive transport in porous media induced by well injection: similarity solution.
13. G.B. Di Masi, E. Platen, W.J. Runggaldier: Hedging of options under discrete observation on assets with stochastic volatility.
14. J. Schmeling, R. Siegmund–Schultze: The singularity spectrum of self-affine fractals with a Bernoulli measure.