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Constructing dynamical systems possessing homoclinic bifurcation points of codimension two

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Abstract

A procedure is derived which allows for a systematic construction of three-dimensional ordinary differential equations possessing homoclinic solutions. These are proved to admit homoclinic bifurcation points of codimension two. The examples include the non-orientable resonant bifurcation, the inclination-flip and the orbit-flip. In addition, an equation is constructed which admits a homoclinic orbit converging to a saddle-focus satisfying Shilnikov's condition. The vector fields are polynomial and non-stiff in that the eigenvalues are of moderate size.

1 Introduction

In recent years the analysis as well as the numerical computation of homoclinic bifurcations with codimension two or higher has attracted much attention, see e.g. [Yan87], [CDF90], [KKO93], [HKK94], [San93] and [San95] for analytical results and [Bey90], [DF91], [CK94] and [CKS95b] for the numerical work. Nevertheless, there are only a few equations known in the literature which one can prove to admit these bifurcations. For the orbit-flip bifurcation, in fact, no example is known where one finds such a bifurcation numerically. Another advantage of having sound examples realizing the above mentioned bifurcations is that they allow to test the recently developed algorithms for computing homoclinic orbits and detecting their bifurcation points, see [CK94], [CKS95b]. Before embarking on the computation of homoclinic orbits for equations arising in applications one should know, whether the algorithm gives reasonable results for those test equations. This program has been carried out for the code HomCont developed by [CKS95a] in the accompanying manual. Furthermore, these equations can be used to test algorithms to be developed in the future which enable switching onto branches of bifurcating N-homoclinic orbits. For these reasons we will provide a method of constructing vector fields for which the existence of generic unfoldings of the above mentioned bifurcations can be proved. In order to guarantee the existence of a homoclinic solution in $\mathbb{R}^2$, we choose an algebraic curve in the plane having the shape of a homoclinic orbit, see figure 2. By using the equation for this curve as a factor in the perturbations we can allow for many perturbations without loosing control over the homoclinic orbit. Based on the method presented here one can construct dynamical systems admitting homoclinic solutions to non-hyperbolic equilibria as well, see [CHS95].

Finally, let us mention related results. Deng [Den91] gave examples of vector fields possessing certain homoclinic bifurcations of codimension one. But these equations stem from singular perturbed ones and neither persistence nor genericity of the singular homoclinic solutions has been proved. Moreover, due to the singular nature, these equations are stiff. Terman [Ter92] studied a singular perturbed equation possessing a highly degenerate orbit-flip bifurcation. Dumortier, Kokubu and Oka [DKO91] have investigated a system, which admits a homoclinic bifurcation point of codimension three in a symmetric vector field.

Acknowledgement. I am much indebted to Floris Takens for a helpful hint.

2 The bifurcations of codimension two

We are interested in three different homoclinic bifurcations of codimension two. These are the resonant bifurcation, the inclination-flip and the orbit-flip. They were first investigated by Yanagida [Yan87]. We will give a short review of them for three-dimensional vector fields. Of course, the results obtained so far for these bifurcations are much more general. Consider

$$\dot{u} = F(u, \epsilon) \tag{2.1}$$

for $(u, \epsilon) \in \mathbb{R}^3 \times \mathbb{R}^2$ and a sufficiently smooth nonlinearity F satisfying $F(0,0) = 0$. Denote by $q(t)$ a homoclinic solution of (2.1) for $\epsilon = 0$, i.e. $q(t) \to 0$ for $t \to \pm\infty$. We denote the eigenvalues of $D_uF(0,0)$ by $-\lambda^{ss} < -\lambda^s < 0 < \lambda^u$. The bifurcations mentioned above will produce so-called N-homoclinic solutions. These are homoclinic orbits which are close to $q(t)$ in phase space, but will follow the orbit $q(t)$ N-times.

An important property of homoclinic solutions is their orientation, which is defined as follows. Consider the adjoint variational equation

$$\dot{w} = -D_uF(q(t), 0)^t w, \qquad w \in \mathbb{R}^3 \tag{2.2}$$

along $q(t)$. This equation possesses a unique bounded solution $\psi(t)$ (at least up to constant multiples), see [Pal84, Lemma 4.2]. It satisfies the relation

$$\psi(t) \perp T_{q(t)}W^s(0) + T_{q(t)}W^u(0),$$

i.e. it is perpendicular to the sum of the tangent spaces of stable and unstable manifolds along $q(t)$. The orientation of the homoclinic orbit is now defined by

$$\mathcal{O}(q) = \lim_{t\to\infty} \operatorname{sign}\langle \psi(t), q(-t)\rangle \cdot \langle \psi(-t), q(t)\rangle.$$

This limit is well-defined for generic homoclinic solutions to the equilibrium 0 satisfying the assumptions on the spectrum mentioned above. Now we define $q(t)$ to be orientable iff $\mathcal{O}(q) = 1$ otherwise we call it non-orientable. The orientation changes as a parameter is varied if $\psi(t)$ switches through the strong stable eigenspace associated with equation (2.2) or $q(t)$ switches through the strong stable manifold of (2.1). These are the bifurcations of codimension two we are interested in:

resonant bifurcation	$\lambda^s = \lambda^u$ and $\mathcal{O}(q) = -1$
inclination-flip	$\psi(t) \cdot e^{\lambda^s t} \to 0$ for $t \to \infty$
orbit-flip	$q(0) \in W^{ss}(0)$.

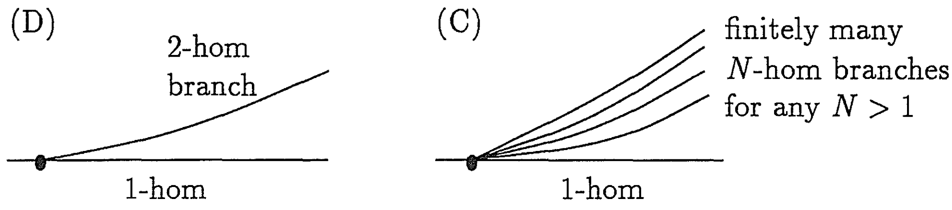

Figure 1: (D) homoclinic doubling, (C) homoclinic cascade

Under additional nondegeneracy conditions the following bifurcation diagrams occur, see figure 1.

resonant bifurcation	(D)		[CDF90]
inclination-flip	(D)	$\lambda^s < \lambda^u < \min(\lambda^{ss}, 2\lambda^s)$	[KKO93]
	(C)	$2\lambda^s < \min(\lambda^{ss}, \lambda^u)$	[HKK94]
	(C)	$\lambda^{ss} < \min(\lambda^u, 2\lambda^s)$	[San95]
orbit-flip	(D)	$\lambda^s < \lambda^u < \lambda^{ss}$	[San93]
	(C)	$\lambda^{ss} < \lambda^u$	[San93]

We refer to the cited articles for the precise statements and further information. The aim is now to construct equations which admit these homoclinic bifurcations. The difficulties stem from the fact that either the bifurcation points or the nondegeneracy conditions are in general given by global conditions. Therefore it is not clear how to fulfill these assumptions while guaranteeing the existence of a homoclinic orbit at the same time.

3 The construction

We would like to find a planar algebraic curve $\mathcal{C}(x) = 0$, which contains a homoclinic solution $q(\cdot)$ to some differential equation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2. \tag{3.1}$$

Of course, the shape of the algebraic curve should allow for a homoclinic orbit. For this reason we define $\mathcal{C}$ to be the cartesian leaf

$$\begin{aligned} \mathcal{C}(x) &:= x_1^2\,(1-x_1) - x_2^2 \\ \Gamma &:= \mathcal{C}^{-1}(0) = \left\{x \,|\, \mathcal{C}(x) = 0\right\}, \end{aligned} \tag{3.2}$$

see figure 2. Here and in the following we use the definition $x = (x_1, x_2) \in \mathbb{R}^2$. Moreover, we denote the inner unit normal of the zero level set Γ by $\nu(q)$ for $q \in \Gamma$, i.e.

$$\nu(q) = |\nabla\mathcal{C}(q)|^{-1}\,\nabla\mathcal{C}(q) = q_1^{-1}\sqrt{8 - 16\,q_1 + 9\,q_1^2}^{\,-1}\,(q_1\,(2 - 3\,q_1), -2\,q_2)^t. \tag{3.3}$$

Then we have the following lemma.

Lemma 3.1 *The vector field*

(3.4)
$$\dot{x} = f(x) = \begin{pmatrix} a\,x_1 + b\,x_2 - a\,x_1^2 \\ b\,x_1 + a\,x_2 - \frac{3}{2}\,b\,x_1^2 - \frac{3}{2}\,a\,x_1\,x_2 \end{pmatrix}$$

admits a homoclinic solution $q(t)$ to the equilibrium 0, which is contained in the zero level set Γ of the cartesian leaf. Here, a and b are arbitrary real numbers satisfying $a^2 \le b^2$ and $b \neq 0$.

Proof. The algebraic curve $\mathcal{C}$ must be invariant under the flow of (3.1), which yields the condition

(3.5)
$$\big\langle \nabla\mathcal{C}(q), f(q) \big\rangle = 0 \qquad \forall q \in \Gamma$$

for the vector field f. Indeed, the time-derivative $\frac{d}{dt}\mathcal{C}(x_1(t), x_2(t)) = 0$ must vanish along solutions $q(t)$ with $q(0) \in \Gamma$. Now we substitute the expression (3.4) for f into (3.5) evaluated at $x_2 = \pm x_1\sqrt{1-x_1}$ which is the solution of $\mathcal{C}(x) = 0$. A straightforward calculation shows that (3.5) is fulfilled for arbitrary a, $b \in \mathbb{R}$. Next we have to ensure that the algebraic curve does not contain any equilibria except for $x = 0$ in order to conclude the existence of a homoclinic orbit on the zero level set of $\mathcal{C}$. The first component of f evaluated at $(x_1, \pm x_1\sqrt{1-x_1})$ is given by

$$a\,x_1 \pm b\,x_1\sqrt{1-x_1} - a\,x_1^2 = x_1\sqrt{1-x_1}\Big(a\sqrt{1-x_1} \pm b\Big).$$

This expression is zero precisely for $x_1 = 0$, $x_1 = 1$ or $x_1 = 1 - (b/a)^2$. The assumption $a^2 \le b^2$ implies that the last zero is not contained in the right-hand half plane. Moreover, by the hypothesis $b \neq 0$ we obtain that the second component of f evaluated at $x_1 = 1$ equals $-\frac{1}{2}b \neq 0$. Thus the set

$$\Big\{(x_1, \pm x_1\sqrt{1-x_1}\,)\,\Big|\,x_1 > 0\Big\}$$

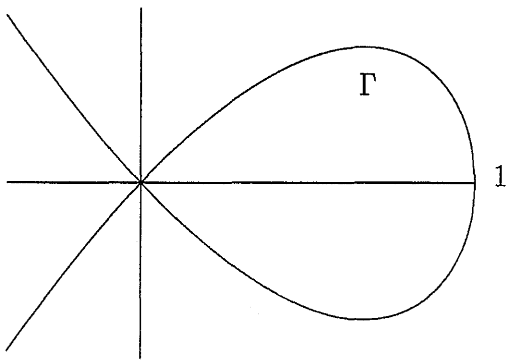

Figure 2: The cartesian leaf

contains no equilibria and therefore is the trace of a homoclinic solution $q(\cdot)$. □

Of course, the formula for f was found by substituting a general polynomial ansatz into equation (3.5).

Remark 3.2 The eigenvalues $\lambda_{1,2}$ of the linearisation of f at the origin are given by

$$\lambda_{1,2} = a \pm b. \tag{3.6}$$

Thus any two given numbers $\lambda_{1,2}$ satisfying $\lambda_1 \leq 0 \leq \lambda_2$ can occur as eigenvalues for suitably chosen a, b fulfilling $a^2 \leq b^2$. In particular, semi-hyperbolic equilibria can be constructed.

Lemma 3.3 *We observe, that the three-dimensional equation*

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = F(x,z) := \begin{pmatrix} f(x) \\ 0 \end{pmatrix} + C(x) \cdot G(x,z) + z \cdot H(x,z) \tag{3.7}$$

still possesses the original homoclinic solution $q(t)$ for arbitrary functions $G, H : \mathbb{R}^3 \to \mathbb{R}^3$.

Proof. Indeed, the vector field on the set $\Gamma = \{(x,z) \,|\, C(x) = 0, z = 0\}$ has not changed. Here and in the sequel we will not distinguish between $q(t) \in \mathbb{R}^2$, $\Gamma \subset \mathbb{R}^2$ and $q(t) \in \mathbb{R}^2 \times \{0\}$, $\Gamma \subset \mathbb{R}^2 \times \{0\}$. □

Thus we have obtained our basic equations (3.7). In the next sections we will find suitable functions G and H as well as parameter-dependent perturbations in order to guarantee existence of homoclinic bifurcations of codimension two. To that end we consider the adjoint equation in the following lemma.

Lemma 3.4 *We decompose the unique bounded solution $\psi(t)$ of (2.2) as follows*

$$\psi(t) = v_1(t) \cdot \nu(q(t)) + v_2(t) \cdot (0,0,1)^t. \tag{3.8}$$

Then (v_1, v_2) solve the two-dimensional system

$$\dot{v} = -\begin{pmatrix} \langle Df(q)\,\nu(q), \nu(q)\rangle + \langle G(q), \nabla C(q)\rangle & |\nabla C(q)|\, G_3(q) \\ \langle \nu(q), H(q)\rangle & H_3(q) \end{pmatrix} v. \tag{3.9}$$

Proof. Because the solution $\psi(t)$ is perpendicular to $\dot{q}(t)$ for all times t, the decomposition

$$\psi(t) = v_1(t) \cdot \nu(q(t)) + v_2(t) \cdot (0,0,1)^t$$

exist for all t. Now we differentiate this relation with respect to time omitting the time-dependence

$$
\begin{aligned}
\dot{\psi} &= \dot{v}_1\, \nu(q) + v_1\, \dot{\nu}(q) + \dot{v}_2\, (0,0,1)^t \\
&= -\left(\begin{pmatrix} Df(q)^t & 0 \\ 0 & 0 \end{pmatrix} + \nabla C(q) \cdot G(q)^t + (0,0,1)^t \cdot H(q)^t \right) \psi \\
&= -\left(\begin{pmatrix} Df(q)^t & 0 \\ 0 & 0 \end{pmatrix} + \nabla C(q) \cdot G(q)^t + (0,0,1)^t \cdot H(q)^t \right) \cdot \\
&\quad \left(v_1\, \nu(q) + v_2\, (0,0,1)^t \right).
\end{aligned} \tag{3.10}
$$

Due to $|\nu(q)| = 1$ we have

$$\langle \nu(q), \dot{\nu}(q) \rangle = 0.$$

Thus by taking the scalar product of (3.10) with $\nu(q)$ and $(0,0,1)^t$, we obtain

$$
\begin{aligned}
\dot{v}_1 &= -\langle \nu(q), Df(q)^t\, \nu(q) \rangle\, v_1 - \langle \nu(q), \nabla C(q) \rangle\, \langle G(q), \nu(q) \rangle\, v_1 - \langle \nu(q), \nabla C(q) \rangle\, G_3(q)\, v_2 \\
\dot{v}_2 &= -H_3(q)\, v_2 - \langle H(q), \nu(q) \rangle\, v_1,
\end{aligned}
$$

respectively. This proves the lemma. □

Remark 3.5 By using the lemniscate

$$\tilde{C}(x_1, x_2) = x_1^2(1 - x_1^2) - x_2^2$$

instead of the cartesian leaf it is possible to construct $\mathbb{Z}_2$-equivariant equations possessing pairs of symmetric homoclinic orbits. Indeed, the nonlinear vector field

$$\dot{x} = \tilde{f}(x) = \begin{pmatrix} a\, x_1 + b\, x_2 - a\, x_1^3 \\ b\, x_1 + a\, x_2 - 2\, b\, x_1^3 - 2\, a\, x_1^2\, x_2 \end{pmatrix} \tag{3.11}$$

is equivariant under the action $x \mapsto -x$ and leaves $\tilde{C}^{-1}(0) =: \tilde{\Gamma}$ invariant. Moreover, $\tilde{f}$ does not vanish on $\tilde{\Gamma} \setminus \{0\}$ for $a^2 < b^2$. This allows one to realize, for example, orientable or non-orientable figure of eight bifurcations.

4 The non-orientable resonant bifurcation

Remember, that this bifurcation is characterized by the relation $\lambda_1 = -\lambda_2$. Hence by (3.6) we have $a = 0$ in (3.4) and obtain

$$f(x) = \begin{pmatrix} b\, x_2 \\ b\, x_1 - \frac{3}{2}\, b\, x_1^2 \end{pmatrix}. \tag{4.1}$$

Of course, (4.1) is a Hamiltonian system with Hamilton function $E(u) = \frac{1}{2}C(x)$. Now in order to obtain a non-orientable orbit in three dimensions and to unfold it generically we consider the following vector field

$$(4.2) \qquad \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(x) \\ \mu\, x_1 \end{pmatrix} + C(x) \cdot \begin{pmatrix} \gamma\, \nabla C(x) \\ \alpha \end{pmatrix} + z \cdot \begin{pmatrix} -\alpha\, \nabla C(x) \\ -\lambda^{ss} + \gamma\, |\nabla C(x)|^2 \end{pmatrix},$$

which is of the form described in (3.7) for $\mu = 0$, hence Lemma 3.3 applies. Let us consider (4.2) in more detail. We consider $(a, \mu) =: (\epsilon_1, \epsilon_2)$ as the unfolding parameters. The first term is just the vectorfield f. The parameter a will break the resonance of the eigenvalues, while μ breaks the homoclinic orbit. For suitable choices of α, the terms depending on α will force the homoclinic orbit to be non-orientable. Lastly the term depending on γ allows one to vary the coefficient a (in the notation of [CDF90]) in the bifurcation equations. From the results in [CDF90] it follows that there exists a curve in parameter space consisting of 2-homoclinic solutions provided the homoclinic solution is non-orientable. The coefficient a gives the leading term in the transcendent expansion of this flat curve.

Lemma 4.1 *Assume* $|\lambda^{ss}| > b$. *Then there exist an open range of coefficients* α, *such that equation (4.2) possesses a non-orientable homoclinic orbit, which undergoes a resonant homoclinic-doubling bifurcation for any* γ. *Moreover, the value of* a *in [CDF90] can be varied by changing* γ.

Proof. First we consider the adjoint variational equation along $q(t)$. By substituting (4.2) into (3.9) we obtain the two-dimensional system

$$(4.3) \qquad \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} g_3(t) - \gamma\, g_1(t) & -\alpha\, g_2(t) \\ \alpha\, g_2(t) & \lambda^{ss} - \gamma\, g_1(t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

for $\psi(t) = v_1(t)\, \nu(q(t)) + v_2(t)\, (0, 0, 1)^t$. The functions $g_i(t)$ for $i = 1, 2, 3$ are given by

$$(4.4) \qquad \begin{aligned} g_1(t) &= |\nabla C(q(t))|^2 & > 0 \\ g_2(t) &= |\nabla C(q(t))| & > 0 \\ g_3(t) &= -\langle Df(q)\, \nu(q), \nu(q) \rangle & \end{aligned}$$

evaluated at $(x_1, x_2)(t) = q(t)$. Thus, increasing the coefficient α causes solutions to rotate, while increasing γ imposes an additional contraction on both components. Indeed, introduce new coordinates by

$$(4.5) \qquad \begin{aligned} u_1(t) &= e^{-\int_0^t (g_3(\tau) - \gamma\, g_1(\tau))\, d\tau}\, v_1(t) \\ u_2(t) &= e^{-\int_0^t (\lambda^{ss} - \gamma\, g_1(\tau))\, d\tau}\, v_2(t). \end{aligned}$$

Equation (4.3) in the new coordinates is given by

$$\dot{u} \;=\; \alpha\,|\nabla C(q(t))| \cdot \begin{pmatrix} 0 & -e^{\int_0^t (\lambda^{ss} - g_3(\tau))\,d\tau} \\ e^{-\int_0^t (\lambda^{ss} - g_3(\tau))\,d\tau} & 0 \end{pmatrix} u.$$

First of all we observe that this equation is independent of γ. Therefore, on changing γ the direction of $v(t)$ does not change. Increasing γ causes both coordinates $v_1(t)$ and $v_2(t)$ to decrease by the same factor

$$e^{-\gamma \int_0^t g_1(\tau)\,d\tau} < 1, \qquad t > 0$$

for positive time or to increase $v(t)$ by multiplication with

$$e^{\gamma \int_t^0 g_1(\tau)\,d\tau} > 1, \qquad t < 0$$

for negative time. Thus the bounded solution $\psi(t)$ of the adjoint equation will be multiplied asymptotically by

$$e^{\pm\gamma \int_0^\infty g_1(\pm\tau)\,d\tau} \qquad \text{for } t \to \pm\infty.$$

This changes a by the factor

$$e^{\gamma \int_{-\infty}^\infty g_1(\tau)\,d\tau},$$

see [San93]. Next we consider equation (4.6) in order to determine the direction of $\psi(t)$. Of course, the direction will not change for $\alpha = 0$. On the other hand, if we choose α sufficiently large, any nonzero solution of (4.6) will wind several times around the origin. Indeed, equation (4.6) is close to

$$\dot{u} = \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot u$$

near $x_1 = 1$, i.e. $t = 0$. Thus our claim follows easily.

Therefore, for suitably chosen α, equation (4.2) possesses a non-orientable homoclinic orbit $q(t)$ such that the unique bounded solution $\psi(t)$ of the adjoint equation just winds once around the origin. Moreover, α can be chosen in such a way that the third component of $\psi(t)$ is nonzero, e.g. $\psi_3(t) \geq 0$ for all $t \in \mathbb{R}$. Thus the Melnikov integral

$$\int_{-\infty}^\infty \left\langle \psi(t), D_{\mu_2} F(q(t), 0) \right\rangle dt = \int_{-\infty}^\infty \psi_3(t)\, x_1(t)\, dt > 0 \tag{4.6}$$

is positive. Next we compute the linearization at the origin and obtain

$$DF(0) = \begin{pmatrix} \mu_1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ \mu_2 & 0 & -\lambda^{ss} \end{pmatrix}.$$

The eigenvalues are given by

$$\begin{aligned} \lambda_{1,2} &= \mu_1 \pm 1 \\ \lambda_3 &= -\lambda^{ss} \end{aligned}$$

and thus do not depend on μ_2. Moreover, the resonance of $\lambda_{1,2}$ is broken by μ_1 with nonzero speed.
These facts together with property (4.6) imply that the μ-dependent terms are the unfolding of the resonant bifurcation. This concludes the proof of the lemma. □

Remark 4.2 The complicated term depending on γ in equation (4.2) could be replaced by the expression

$$\gamma\,(0,0,x_1\,z)^t$$

which is supposed to change a, too. Moreover, one can add this term in order to suppress the strong expansion or contraction of the variational equation near $x_1 = 1$. Then the non-orientability could be verified more easily using a computer.

Remark 4.3 The assumption on the existence of a non-orientable homoclinic orbit in Lemma 4.1 can be verified numerically in a stable way. In principle, this computation can be done in a strict manner using the fact that the assumptions are known to be fulfilled in an open range of coefficients.

5 The orbit-flip bifurcation

At the orbit-flip bifurcation the homoclinic orbit is contained in the strong stable (or strong unstable) manifold by definition. This can be realized as follows.

Lemma 5.1 *Assume* $-\lambda^{ss} < -\lambda^s < 0 < \lambda^u$. *Then the vector field*

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(x) \\ -\lambda^s\,z \end{pmatrix}, \tag{5.1}$$

where f is given by (4.1), possesses a homoclinic solution for $a = \frac{1}{2}(\lambda^u - \lambda^{ss})$ *and* $b = \frac{1}{2}(\lambda^u + \lambda^{ss})$, *which is contained in the strong stable manifold of the origin. Moreover, the eigenvalues of the linearization at the origin are given by* $-\lambda^{ss}$, $-\lambda^s$ *and* λ^u.

Proof. Observe that when $z = 0$ (5.1) coincides with (3.4). Thus the claim about the eigenvalues follows from remark 3.2. The homoclinic solution $q(t)$ lies in the invariant x-coordinate plane and is therefore contained in the strong stable manifold by Lemma 3.1. □

From now on we assume, that a and b are chosen as in the previous lemma. We have to consider two different cases of the orbit-flip, called the inward and outward twist, respectively, see [San95]. There are defined by the following condition. Choose $\psi(t)$ such that $\langle\psi(-t), q(t)\rangle > 0$ for $t \to \infty$. Then the inward twist is characterized by

$$\lim_{t\to\infty} e^{\lambda^{ss}t}\langle\psi(t), q(-t)\rangle > 0$$

while the outward is defined by

$$\lim_{t\to\infty} e^{\lambda^{ss}t}\langle\psi(t), q(-t)\rangle < 0.$$

We investigate here the inward twist.

Lemma 5.2 (Inward twist) *Suppose a and b are chosen as in Lemma 5.1. Then for all sufficiently small α the homoclinic orbit $q(t)$ of*

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(x) \\ 0 \end{pmatrix} - z \cdot \begin{pmatrix} \alpha\,\nabla C(x) \\ \lambda^s \end{pmatrix} + \begin{pmatrix} \tilde{\mu}\,\nabla C(x) \\ \mu\, x_1 \end{pmatrix} \tag{5.2}$$

undergoes a generic orbit-flip bifurcation with inward twist with respect to the parameter $\epsilon = (\mu, \tilde{\mu})$.

Proof. By the previous lemma it is sufficient to show the genericity assumption as well as the generic dependence on the parameters. Consider the variational equation

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} D_x f(q) & -\alpha\,\nabla C(q) \\ 0 & -\lambda^s \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \tag{5.3}$$

evaluated at $q(t)$. First, we prove that the unique bounded solution $\psi(t)$ of the adjoint variational equation

$$\dot{w} = \begin{pmatrix} -D_x f(q)^t & 0 \\ \alpha\,\nabla C(q)^t & \lambda^s \end{pmatrix} w \tag{5.4}$$

along $q(t)$ converges generically to zero for $t \to -\infty$, i.e.

$$\lim_{t\to-\infty} e^{-\lambda^s t} \cdot \psi(t) \neq 0. \tag{5.5}$$

Using the decomposition $\psi(t) = v_1(t)\,\nu(q(t)) + v_2(t)\,(0,0,1)^t$ together with Lemma 3.4, we obtain the system

$$\begin{aligned} \dot{v} &= \begin{pmatrix} \langle -D_x f(q)^t\,\nu(q), \nu(q)\rangle & 0 \\ \langle\alpha\,\nabla C(q)^t, \nu(q)\rangle & \lambda^s \end{pmatrix} v \\ &= \begin{pmatrix} \langle -D_x f(q)^t\,\nu(q), \nu(q)\rangle & 0 \\ \alpha\,|\nabla C(q)| & \lambda^s \end{pmatrix} v. \end{aligned} \tag{5.6}$$

The second component of the bounded solution of (5.6) is given by

$$v_2(t) = \alpha \int_t^\infty e^{\lambda^s(t-\tau)} \, |\nabla C(\tau)| \, v_1(\tau) \, d\tau =: \alpha \, \hat{v}_2(t). \tag{5.7}$$

Moreover, $v_1(t)$ solves the linear decoupled equation

$$\dot{v_1} = \langle -D_x f(q)^t \, \nu(q), \nu(q) \rangle \, v_1$$

and hence is of a definite sign, say $v_1(t) > 0$ without loss of generality. Therefore $v_2(t) > 0$ is positive for $\alpha > 0$. By multiplication of (5.7) with $e^{-\lambda^s t}$ we obtain

$$\begin{aligned} \psi_3(t) \, e^{-\lambda^s t} &= v_2(t) \, e^{-\lambda^s t} \\ &= \alpha \int_t^\infty e^{-\lambda^s \tau} \, |\nabla C(\tau)| \, v_1(\tau) \, d\tau \\ &\to \alpha \int_{-\infty}^\infty e^{-\lambda^s \tau} \, |\nabla C(\tau)| \, v_1(\tau) \, d\tau > 0 \end{aligned} \tag{5.8}$$

for $t \to -\infty$. Therefore, (5.5) is fulfilled and $q(t)$ satisfies the strong inclination property, see [San93] or [San95].

Next we investigate the dependence on the parameter ϵ. For this we consider the variational equation (5.3) along $q(t)$. This system possesses the solution $\dot{q}(t)$ which lies in the tangent space of the strong stable manifold. Moreover, we choose two further solutions $\varphi^u(t)$ and $\varphi^s(t)$ in the following way

$$\begin{aligned} &\varphi^s(t) \in T_{q(t)} W^s(0) \cap T_{q(t)} W^{ss}(0)^\perp \\ &\varphi^u(0) = (-1, 0, 0)^t \in T_{q(0)} W^{ss}(0)^\perp. \end{aligned}$$

We denote the projection onto X with kernel Y by $P(X, Y)$ and define

$$\begin{aligned} P^s(t) &:= P\Big(\mathbb{R}\varphi^s(t), \mathbb{R}\varphi^u(t) \oplus \mathbb{R}\dot{q}(t)\Big) \\ P^{ss}(t) &:= P\Big(\mathbb{R}\dot{q}(t), \mathbb{R}\varphi^u(t) \oplus \mathbb{R}\varphi^s(t)\Big) \\ P^u(t) &:= 1 - P^s(t) - P^{ss}(t) \\ Q_0 &:= P\Big(\mathbb{R}\varphi^s(0), \mathbb{R}\psi(0) \oplus \mathbb{R}\dot{q}(0)\Big). \end{aligned}$$

Then we have to prove that the vectors

$$\begin{aligned} M &:= \textstyle\int_{-\infty}^\infty \langle \psi(t), D_\epsilon F(q(t), 0) \rangle \, dt \\ N &:= \textstyle\int_{-\infty}^\infty Q(0) \, \Phi(0, t) \, (1 - P^{ss}(t)) \, D_\epsilon F(q(t), 0) \, dt \end{aligned} \tag{5.9}$$

are linearly independent in parameter space. Here $\Phi(t, s)$ denotes the evolution of the variational equation. By [San93] or [San95] this implies that the parameter ϵ unfolds the orbit-flip in a generic way.

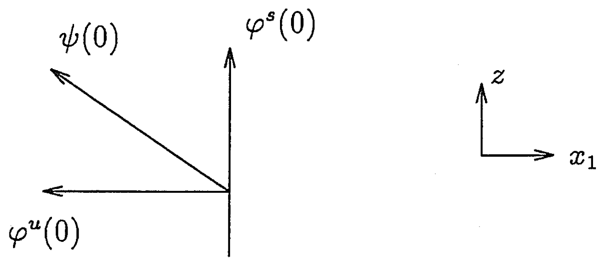

Figure 3: Sign of $\left\langle Q_0\,\varphi^u(0),(0,0,1)^t\right\rangle$ at $t=0$

First, we compute the signs of the Melnikov integrals M with respect to $\epsilon=(\mu,\tilde{\mu})$. Using (3.8) we obtain

$$
\begin{aligned}
M_1 &= \textstyle\int_{-\infty}^{\infty}\langle\psi(t),D_\mu F(q(t),0)\rangle\,dt = \int_{-\infty}^{\infty} v_1(t)\,x_1(t)\,dt && >0\\
M_2 &= \textstyle\int_{-\infty}^{\infty}\langle\psi(t),D_{\tilde{\mu}} F(q(t),0)\rangle\,dt = \int_{-\infty}^{\infty} v_2(t)\,|\nabla C(q(t))|\,dt && >0.
\end{aligned}
\tag{5.10}
$$

Second, we have to examine the unfolding of the flip. We choose $\varphi^u(0)\in\mathbb{R}^2\times\{0\}$ and obtain

$$
\begin{aligned}
P^s(t) &= \varphi^s(t)\,\langle(0,0,1)^t,\cdot\rangle\\
P^u(t) &= \varphi^u(t)\,\langle\psi(t),\cdot\rangle.
\end{aligned}
$$

Indeed, due to the decoupling in equation (5.3) we have $\varphi^u(t)\in\mathbb{R}^2\times\{0\}$ for all t. Moreover, note that

$$
\langle Q_0\,\varphi^u(0),(0,0,1)^t\rangle<0,
\tag{5.11}
$$

due to $\psi_3(0)>0$, see figure 3.

Therefore by substituting the expressions for the projections into the second integral in (5.9) we obtain

$$
\begin{aligned}
N_1 &= \int_{-\infty}^{\infty} Q(0)\,\Phi(0,t)\,(1-P^{ss}(t))\,D_\mu F(q(t),0)\,dt\\
&= \int_{-\infty}^{\infty} Q(0)\,\Phi(0,t)\,(P^s(t)+P^u(t))\,(0,0,x_1(t))^t\,dt\\
&= \int_{-\infty}^{\infty} Q(0)\,\Phi(0,t)\,\varphi^s(t)\,x_1(t)\,dt+\int_{-\infty}^{\infty} Q(0)\,\Phi(0,t)\,\varphi^u(t)\,\langle\psi(t),(0,0,x_1(t))^t\rangle\,dt\\
&= \int_{-\infty}^{\infty} e^{\lambda^s t}\,x_1(t)\,dt\,(0,0,1)^t+\alpha\int_{-\infty}^{\infty} Q(0)\,\Phi(0,t)\,\varphi^u(t)\,x_1(t)\,\hat{v}_2(t)\,dt,
\end{aligned}
$$

where we have used (5.7) together with the decomposition (3.8) of $\psi(t)$. Therefore

$$
\langle\varphi^s(0),N_1\rangle>0
$$

for α sufficiently small. On the other hand

$$
N_2 = \int_{-\infty}^{\infty} Q(0)\,\Phi(0,t)\,(1-P^{ss}(t))\,D_{\tilde{\mu}}F(q(t),0)\,dt
$$

$$= \int_{-\infty}^{\infty} Q(0)\, \Phi(0,t)\, (P^s(t) + P^u(t))\, \nabla C(q(t))\, dt$$
$$= \int_{-\infty}^{\infty} Q(0)\, \Phi(0,t)\, \varphi^u(t)\, \langle \psi(t), \nabla C(q(t)) \rangle\, dt$$
$$= \int_{-\infty}^{\infty} Q(0)\, \Phi(0,t)\, \varphi^u(t)\, v_1(t)\, |\nabla C(q(t))|\, dt.$$

Here we have used the identity $D_{\mu_2} F(q,0) = \nabla C(q)$ together with the decomposition of $\psi(t)$. Hence

$$\Big\langle \varphi^s(0), N_2 \Big\rangle < 0,$$

due to $v_1(t) > 0$ and (5.11). Thus the two vectors M and N are linearly independent for small α, because M_1, M_2 and $N_1 > 0$ are strictly positive while $N_2 < 0$ is negative. This finishes the proof of the lemma. □

For the outward twist one has to add an additional rotation as for the non-orientable resonant bifurcation to ensure that

$$\lim_{t \to \infty} e^{\lambda^{ss} t} \Big\langle \psi(t), q(-t) \Big\rangle < 0.$$

However, we are not able to prove that the parameter ϵ leads to a generic unfolding in this case, although it seems likely to be true.

Lemma 5.3 (Outward twist) *Suppose that a and b are chosen as in Lemma 5.1. Then for α in an open domain the homoclinic orbit $q(t)$ of*

$$(5.12) \qquad \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(x) \\ 0 \end{pmatrix} - z \cdot \begin{pmatrix} \alpha \nabla C(x) \\ \lambda^s \end{pmatrix} + C(x) \cdot \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} \tilde{\mu} \nabla C(x) \\ \mu x_1 \end{pmatrix}$$

fulfills the conditions for an orbit-flip bifurcation with outward twist for $\epsilon = (\mu, \tilde{\mu}) = 0$. Furthermore, ϵ breaks the homoclinic orbit.

Proof. The proof follows the same lines as the two previous proofs, so we will not go into the details. As in Lemma 4.1 the parameter α causes a rotation of the solutions of the variational equation in the plane perpendicular to $\dot{q}(t)$. Therefore the outward-twist condition is fulfilled for an open set of coefficients α. The Melnikov integrals with respect to both parameters are non-zero, if one further restricts to the set of allowed coefficients α. □

We remark that it is quite likely that the parameter ϵ unfolds the outward orbit-flip in a generic way, i.e. that the two integrals (5.9) are linearly independent.

6 The inclination-flip bifurcation

In this section we apply the method to construct vector fields possessing an inclination-flip. At an inclination-flip point the strong stable foliation of the homoclinic orbit does not exist by definition. In terms of the bounded solution of the adjoint equation we have

$$\lim_{t\to\infty} e^{\lambda^s t}\,\psi(-t) = 0. \tag{6.1}$$

Lemma 6.1 *Assume $-\lambda^{ss} < -\lambda^s < 0 < \lambda^u$ and consider the vector field*

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(x) \\ \mu\, x_1 \end{pmatrix} + C(x)\cdot \begin{pmatrix} 0 \\ \alpha \end{pmatrix} - z\cdot \begin{pmatrix} \alpha\,\nabla C(x) \\ -\lambda^{ss} \end{pmatrix}, \tag{6.2}$$

for $a = \frac{1}{2}(\lambda^u - \lambda^s)$ and $b = \frac{1}{2}(\lambda^u + \lambda^s)$. Then there exists an α_0 such that the homoclinic solution $q(t)$ of (6.2) for $\alpha = \alpha_0$ fulfills (6.1). Moreover, $\epsilon = (\alpha - \alpha_0, \mu)$ unfolds the inclination-flip in a generic way near $\epsilon = 0$.

Proof. The proof follows easily from the proofs of the Lemmata 4.1 and 5.2. □

Remark 6.2 In the cases $\lambda^s < \lambda^{ss} < \min(\lambda^u, 2\lambda^s)$ or $2\lambda^s < \min(\lambda^u, \lambda^{ss})$ additional non-degeneracy conditions are needed in [San95] and [HKK94], respectively, in order to show the existence of a horseshoe in the unfolding. In the first case, $\lambda^s < \lambda^{ss} < \min(\lambda^u, 2\lambda^s)$, the condition

$$\lim_{t\to\infty} e^{\lambda^{ss} t}\,\langle \psi(-t), q(t)\rangle \neq 0$$

stated in [San95] cannot be satisfied. However, see [SS95] for a system satisfying this condition. In the second case studied by [HKK94], a quadratic tangency of the stable manifold and an invariant manifold tangent to the eigenspace associated to the eigenvalues $-\lambda^s$ and λ^u is required. It is likely that this holds, see [CHS95, appendix] for a possible approach to this. If $\lambda^s < \lambda^u < \min(\lambda^{ss}, 2\lambda^s)$ no additional conditions are needed.

7 The saddle-focus bifurcation

Consider the vector field

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(x) \\ (a-b)\,z \end{pmatrix} + \begin{pmatrix} \tilde{\mu}\,\nabla C(x) \\ \mu\, x_1 \end{pmatrix} + z \begin{pmatrix} e \\ 0 \end{pmatrix}, \tag{7.1}$$

for $e = (1,-1)^t$ and with $0 < a < b$. The eigenvalues of the linearization of (7.1) at zero are then given by

$$\lambda_{1,2} = a - b \pm \sqrt{\mu} \qquad \lambda_3 \approx a + b. \tag{7.2}$$

Indeed, the linearisation restricted to the space spanned by $e_1 = (1,-1,0)^t$, $e_2 = (0,0,1)^t$ is given by

$$\begin{pmatrix} a-b & 1 \\ \mu & a-b \end{pmatrix}.$$

In particular, the eigenvalues become complex for $\mu < 0$. Owing to the term depending on $\tilde{\mu}$, the homoclinic orbit is transversally constructed. Therefore, it persists along a branch $(\mu, \tilde{\mu}_*(\mu))$ parametrized by μ. For $\mu < 0$ the homoclinic solutions on the branch converge to a saddle-focus equilibrium satisfying $\text{Re}\,\lambda_{1,2} = a - b < \lambda_3 \approx a + b$. Hence the system fulfills the conditions stated in [Shi70] for $\mu < 0$ and possesses horseshoes nearby. Note that the system does not undergo a generic Belyakov transition [Bel80] as the homoclinic orbit converges without the additional algebraic order to zero.

8 Summary

By substituting formula (3.2) for the algebraic curve into the several equations obtained in the previous lemmata we finally obtain the following system

$$\begin{aligned} \dot{x} &= a\,x + b\,y - a\,x^2 + (\tilde{\mu} - \alpha\,z)\,x\,(2 - 3x) + \delta\,z \\ \dot{y} &= b\,x + a\,y - \tfrac{3}{2}\,b\,x^2 - \tfrac{3}{2}\,a\,x\,y - (\tilde{\mu} - \alpha\,z)\,2\,y - \delta\,z \\ \dot{z} &= c\,z + \mu\,x + \gamma\,x\,z + \alpha\beta\,(x^2\,(1-x) - y^2), \end{aligned} \tag{8.1}$$

which contains all the equations constructed in the sections above. The eigenvalues of the linearisation at 0 for $(\mu, \tilde{\mu}) = 0$ are given by

$$a \pm b,\ c$$

for $\delta = 0$, see (7.2) for $\delta = 1$. Under the conditions

$$a^2 \leq b^2,\ b > 0,\ c < 0$$

there exists a homoclinic solution of (8.1) on Γ for $(\mu, \tilde{\mu}) = 0$. The different bifurcations are realized as follows

bifurcation	conditions				free parameters
resonant	$a = 0$	$c < -b$	$\beta = 1$	$\delta = 0$	$\mu \quad a$
orbit-flip	$a - b < c$	$\gamma = 0$	$\beta = 0$	$\delta = 0$	$\mu \quad \tilde{\mu}$
inclination-flip	$c < a - b$	$\gamma = 0$	$\beta = 1$	$\delta = 0$	$\mu \quad \alpha$
saddle-focus	$c = a - b$	$\gamma = 0$	$\alpha = 0$	$\delta = 1$	$\mu \quad \tilde{\mu}$

The parameter $\alpha \neq 0$ breaks the invariance of the xy-plane, while γ allows for adjusting certain constants for the resonant bifurcation. The orientation can be changed by performing a pathfollowing in the free parameters $(\tilde{\mu}, \alpha)$ for fixed $\beta = 1$. By this procedure we obtain suitable values for α at which the homoclinic solution is non-orientable or undergoes an inclination-flip. Indeed, $\mu = 0$ holds along the branch by construction. Moreover, it is possible to reduce the differential equation on Γ to a piecewise one-dimensional one in order in order to compute starting data for the homoclinic orbit. Another possibility is to use the explicit solution

$$(x(t), y(t), z(t)) = \left(1 - \left(\frac{1-e^t}{1+e^t}\right)^2, 4\,e^t \frac{1-e^t}{(1+e^t)^3}, 0\right),$$

which exists for $a = 0$, $b = 1$, and perform a continuation afterwards. For numerical results, we refer to [CKS95a].

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