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Convergence of a finite volume scheme for the biharmonic problem

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Abstract

We propose a finite volume scheme for the approximation of a biharmonic problem, with Dirichlet boundary conditions. We prove that the piece-wise constant approximate solution converges in $L^2(\Omega)$ to the exact solution, as well as the discrete approximate of the gradient and the discrete approximate of the Laplacian of the exact solution. These results are confirmed by numerical results.

1 Introduction

We consider a polygonal open connected domain $\Omega \subset \mathbb{R}^d$, with d integer strictly positive. Let $f \in L^2(\Omega)$. The following biharmonic problem: find a function u, defined on Ω, such that

$$\Delta(\Delta u) = f \text{ on } \Omega, \tag{1}$$

$$u = \Delta u = 0 \text{ on } \partial\Omega. \tag{2}$$

can be solved by many methods, since it resumes to the consecutive resolution of two Laplace problems with homogeneous Dirichlet boundary condition, the first one for obtaining Δu, the second one for obtaining u. Unfortunately, this does no longer hold in the case of problem (1), with the full Dirichlet boundary conditions:

$$u = \nabla u \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega. \tag{3}$$

In order to provide a weak formulation of Problem (1) with boundary conditions (3), we introduce the function space $H_0^2(\Omega)$, which is defined as the closure of $C_c^\infty(\Omega)$ in $H^2(\Omega)$. Thanks to the Lipschitz regularity of the boundary, we get that

$$H_0^2(\Omega) = \{u \in H^2(\Omega) \cap H_0^1(\Omega),\ \nabla u \cdot \boldsymbol{n} = 0 \text{ a.e. on } \partial\Omega\}. \tag{4}$$

The weak formulation of Problem (1) with Dirichlet boundary conditions (3) is then given by

$$u \in H_0^2(\Omega),\ \forall v \in H_0^2(\Omega), \int_\Omega \Delta u(x)\Delta v(x)\mathrm{d}x = \int_\Omega f(x)v(x)\mathrm{d}x.$$

In this paper, we will consider the more general problem

$$\begin{gathered} u \in H_0^2(\Omega),\ \forall v \in H_0^2(\Omega), \int_\Omega \Delta u(x)\Delta v(x)\mathrm{d}x = \int_\Omega (f(x)v(x) + \boldsymbol{g}(x)\cdot\nabla v(x) + l(x)\Delta v(x))\mathrm{d}x, \\ f \in L^2(\Omega),\ \boldsymbol{g} \in L^2(\Omega)^d,\ l \in L^2(\Omega). \end{gathered} \tag{5}$$

Indeed, Problem (5) arises in the theory of the two-dimensional incompressible Navier-Stokes equations, since, for all divergence-free weakly differentiable vector fields $\boldsymbol{U} \in [H_0^1(\Omega)]^2$, there exists one and only one stream functions $u \in H_0^2(\Omega)$ (see [7]) such that $U_1 = \partial_2 u$ and $U_2 = -\partial_1 u$, which also is the solution of the problem

$$u \in H_0^2(\Omega),\ \forall v \in H_0^2(\Omega), \int_\Omega \Delta u(x)\Delta v(x)\mathrm{d}x = \int_\Omega (\partial_2 U_1(x) - \partial_1 U_2(x))\Delta v(x)\mathrm{d}x.$$

Then, from the Stokes problem $-\Delta U_1 + \partial_1 p = F_1$, $-\Delta U_2 + \partial_2 p = F_2$, one gets that u is the unique weak solution of the biharmonic problem

$$u \in H_0^2(\Omega),\ \forall v \in H_0^2(\Omega), \int_\Omega \Delta u(x)\Delta v(x)\mathrm{d}x = \int_\Omega (-F_2(x)\partial_1 v(x) + F_1(x)\partial_2 v(x))\mathrm{d}x.$$

We recall that Problem (5) has one and only one solution, due to Riesz theorem and to the fact that $\|\Delta u\|_{L^2(\Omega)}$ is an equivalent norm to $\|u\|_{H^2(\Omega)}$ in $H_0^2(\Omega)$. Indeed, the Poincaré inequality

$$\forall u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)\|\nabla u\|_{L^2(\Omega)^d}$$

and

$$\forall u \in H_0^2(\Omega),\ -\int_\Omega u\Delta u\,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla u\,\mathrm{d}x$$

imply

$$\forall u \in H_0^2(\Omega),\ \|\nabla u\|_{L^2(\Omega)^d} \le \mathrm{diam}(\Omega)\|\Delta u\|_{L^2(\Omega)}.$$

Besides, the following equality which is an immediate consequence of two integrations by parts

$$\forall \varphi \in C_c^\infty(\Omega),\ \int_\Omega (\Delta\varphi(x))^2\mathrm{d}x = \sum_{i=1}^d\sum_{j=1}^d \int_\Omega \partial_{ii}^2\varphi(x)\partial_{jj}^2\varphi(x)\mathrm{d}x = \sum_{i=1}^d\sum_{j=1}^d \int_\Omega (\partial_{ij}^2\varphi(x))^2\mathrm{d}x, \tag{6}$$

completes the proof of the equivalence of the norms.

The standard numerical method for the approximation of Problem (5) is the conformal finite element method. It consists in defining, on a partition of Ω with polyhedra, elementary basis functions such that the reconstructed basis functions on Ω belong to $C^1(\overline{\Omega})$. It is easy to define such a basis on cartesian meshes, since it suffices to consider the generalisation in 2D or 3D of the P^3 Hermite finite element in 1D. This task becomes much more difficult on more general meshes. For example, the Argyris finite element on triangles in 2D appears to have complex basis functions. Therefore, nonconformal methods have been studied on more general meshes. Indeed, only few works address the approximation of this problem, using different numerical methods (see [1, 10, 2]).

It is then worth to notice that the discretization of this problem, using a finite volume method on grids with some orthogonality property (see [5]), arises in the classical finite volume scheme [8, 9] for the incompressible Navier-Stokes equations [3]. In this situation, the finite volume method, used for the approximation of the discrete Laplace operator involved in the weak formulation (5), provides a discrete weak formulation, where no consistency is a priori proven on the discrete Laplace operator applied to the interpolation of a regular function.

Moreover, such a consistency property does not hold for this discrete Laplace operator in general (see [5]) on triangular meshes or rectangular meshes with nonconstant space steps. Hence, in the convergence proof, one has to prove that it is possible to strongly approximate $\Delta\varphi$, where φ is a regular function with compact support in Ω, in the discrete space, the approximation being strongly convergent to φ. This property is developped in Lemma 3.3. Error estimates are proposed in the case where the solution of the continuous problem has some regularity. These estimates are not sharp, as shown by the numerical results.

This paper is organized as follows. The scheme is presented in Section 2. The mathematical analysis is derived in Section 3, and finally, numerical results (in 1D, 2D and 3D, using various types of meshes) are provided in Section 4.

2 Approximation of the Dirichlet problem

The notations are summarized in Figure 1 for the particular case $d = 2$ (we recall that the case $d \geq 3$ is considered as well).

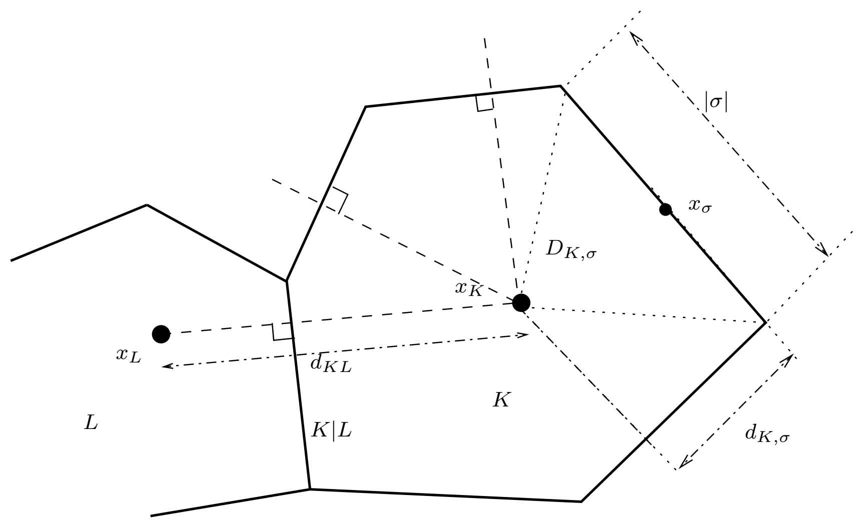

Figure 1: Notations for a control volume K in the case $d = 2$

We first define an admissible mesh in the sense of [5] and [6]. In the following definition, we say that a bounded subset of $\mathbb{R}^d$ is polygonal if its boundary is included in the union of a finite number of hyperplanes.

Definition 2.1 **[Admissible discretization]** *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:*

- *$\mathcal{M}$ is a finite family of non empty open polygonal convex disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K and $|K| > 0$ denote the measure of K.*
- *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, there exists a hyperplane E of $\mathbb{R}^d$ and $K \in \mathcal{M}$ with $\overline{\sigma} = \partial K \cap E$ and σ is a non empty open subset of E. We then denote by $|\sigma| > 0$ the $(d-1)$-dimensional measure of σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\rm int}$ (resp. $\mathcal{E}_{\rm ext}$), that is $\mathcal{E}_{\rm int} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\rm ext} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. We then denote by $\mathcal{E}_{K,\rm ext} = \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$ and $\mathcal{E}_{K,\rm int} = \mathcal{E}_K \cap \mathcal{E}_{\rm int}$. It then results from the previous hypotheses that, for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \overline{\sigma}$; we denote in the latter case $\sigma = K|L$.*
- *$\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$. We assume that $x_K \in K$ for all $K \in \mathcal{M}$. Furthermore, for all $\sigma \in \mathcal{E}$ such that there exists $(K, L) \in \mathcal{M}^2$ with $\sigma = K|L$, it is assumed that $x_K \neq x_L$, that the straight line (x_K, x_L) going through x_K and x_L is perpendicular to $K|L$ and that the vector from x_K to x_L points outward of K. For all $K \in \mathcal{M}$ and all $\sigma \in \mathcal{E}_K$, let z_σ be the orthogonal projection of x_K on σ. We suppose that $z_\sigma \in \sigma$ if $\sigma \subset \partial\Omega$.*

Remark 2.1 *In the above definition, we could relax the hypothesis $x_K \in K$, since this does not necessarily occur in the case of Delaunay triangulations. This leads to a few technical difficulties in the proofs below: in particular, the property*

$$\sum_{\sigma\in\mathcal{E}_K} |\sigma| d_{K,\sigma} = d\ |K| \tag{7}$$

does no longer hold, which makes necessary some additional geometric hypotheses.

The following notations are used. The size of the discretization is defined by:

$$h_{\mathcal{D}} = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}.$$

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\boldsymbol{n}_{K,\sigma}$ the unit vector normal to σ outward to K. We denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ.

For all $\sigma \in \mathcal{E}_{\rm int}$, an orientation is chosen by defining one of the two unit normal vectors $\boldsymbol{n}_\sigma$, for each $\sigma \in \mathcal{E}_{\rm int}$, and we denote by K_σ^- and K_σ^+ the two adjacent control volumes such that $\boldsymbol{n}_\sigma$ is oriented from K_σ^- to K_σ^+. We then set

$$d_\sigma = d(x_{K_\sigma^-}, x_{K_\sigma^+}) = \mathrm{d}(x_{K_\sigma^-}, \sigma) + \mathrm{d}(x_{K_\sigma^+}, \sigma). \tag{8}$$

For all $\sigma \in \mathcal{E}_{\rm ext}$, we denote the control volume $K \in \mathcal{M}$ such that $\sigma \in \mathcal{E}_K$ by K_σ; we define

$$d_\sigma = \mathrm{d}(x_{K_\sigma}, \sigma), \tag{9}$$

and we define $\boldsymbol{n}_\sigma$ by $\boldsymbol{n}_\sigma = \boldsymbol{n}_{K_\sigma,\sigma}$ For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we define

$$D_{K,\sigma} = \{t x_K + (1-t) y, t \in (0,1),\ y \in \sigma\},$$

For all $\sigma \in \mathcal{E}_{\rm int}$, let $K, L \in \mathcal{M}$ be such that $\sigma = K|L$; we set $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$. For all $\sigma \in \mathcal{E}_{\rm ext}$, let $K \in \mathcal{M}$ be such that $\sigma \in \mathcal{E}_K$; we define $D_\sigma = D_{K,\sigma}$.

For all $\sigma \in \mathcal{E}$, we define

$$x_\sigma = \frac{1}{|\sigma|} \int_\sigma x \ \mathrm{d}\gamma(x). \tag{10}$$

We shall measure the regularity of the mesh through the function $\theta_\mathcal{D}$ defined by

$$\theta_\mathcal{D} = \inf \left\{ \frac{d_{K,\sigma}}{\mathrm{diam}(K)}, \frac{d_{K,\sigma}}{d_\sigma},\ K \in \mathcal{M},\ \sigma \in \mathcal{E}_K \right\}. \tag{11}$$

Definition 2.2 *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, and $\mathcal{D}$ an admissible discretization of Ω in the sense of Definition (2.1). We define $H_\mathcal{D}$ as the set of functions $u \in L^2(\Omega)$ which are constant in each control volume. For $u \in H_\mathcal{D}$, we denote by u_K the constant value of u in K. We define the interpolation operator $P_\mathcal{D} : C(\overline{\Omega}) \to H_\mathcal{D}$, by $u \mapsto P_\mathcal{D} u$ such that*

$$P_\mathcal{D} u(x) = u(x_K) \text{ for a.e. } x \in K,\ \forall K \in \mathcal{M}. \tag{12}$$

For any $u \in H_\mathcal{D}$, we denote

$$\delta_\sigma u = u_{K_\sigma^+} - u_{K_\sigma^-}, \forall \sigma \in \mathcal{E}_{\rm int} \text{ and } \delta_\sigma u = 0 - u_{K_\sigma}, \forall \sigma \in \mathcal{E}_{\rm ext}, \tag{13}$$

and

$$\delta_{K,\sigma} v = -v_K,\ \forall \sigma \in \mathcal{E}_{K,\rm ext} \text{ and } \delta_{K,\sigma} v = v_L - v_K,\ \forall \sigma = K|L \in \mathcal{E}_{K,\rm int},\ \forall K \in \mathcal{M}. \tag{14}$$

We introduce the following symmetric bilinear form:

$$[u,v]_\mathcal{D} = \sum_{\sigma \in \mathcal{E}} \frac{|\sigma|}{d_\sigma} \delta_\sigma u \delta_\sigma v,\ \forall u, v \in (H_\mathcal{D})^2, \tag{15}$$

which defines a scalar product in $H_{\mathcal{D}}$. We then denote $\|u\|_{\mathcal{D}} = ([u,u]_{\mathcal{D}})^{1/2}$ for all $u \in H_{\mathcal{D}}$. *We define $\nabla_{\mathcal{D}} : H_{\mathcal{D}} \to (H_{\mathcal{D}})^d$ and $\widetilde{\nabla}_{\mathcal{D}} : H_{\mathcal{D}} \to (L^2(\Omega))^d$ respectively by*

$$|K|\nabla_K u = \sum_{\sigma\in\mathcal{E}_K} \frac{|\sigma|}{d_\sigma}\ \delta_{K,\sigma} u\ \ (x_\sigma - x_K),\ \forall K \in \mathcal{M},\ \forall u \in H_{\mathcal{D}}, \tag{16}$$

and

$$\widetilde{\nabla}_{\mathcal{D}} u(x) = d\frac{\delta_\sigma u_m}{d_\sigma}\boldsymbol{n}_\sigma,\ \text{for a.e. } x \in D_\sigma,\ \forall \sigma \in \mathcal{E},\ \forall u \in H_{\mathcal{D}}. \tag{17}$$

We define $\Delta_{\mathcal{D}} : H_{\mathcal{D}} \to H_{\mathcal{D}}$ by

$$|K|\Delta_K u = \sum_{\sigma\in\mathcal{E}_K} \frac{|\sigma|}{d_\sigma}\delta_{K,\sigma} u,\ \forall K \in \mathcal{M},\ \forall u \in H_{\mathcal{D}}. \tag{18}$$

We define

$$H_{\mathcal{D},0} = \{u \in H_{\mathcal{D}},\ u_K = 0 \text{ for all } K \in \mathcal{M} \text{ such that } \mathcal{E}_{K,\mathrm{ext}} \neq \emptyset\} \tag{19}$$

Thanks to the above definitions, we have

$$-\int_\Omega u(x)\Delta_{\mathcal{D}} v(x)\mathrm{d}x = [u,v]_{\mathcal{D}},\ \forall u,v \in (H_{\mathcal{D}})^2. \tag{20}$$

We now approximate Problem (5) by

$$u \in H_{\mathcal{D},0},\ \forall v \in H_{\mathcal{D},0},\ \int_\Omega \Delta_{\mathcal{D}} u(x)\Delta_{\mathcal{D}} v(x)\mathrm{d}x = \int_\Omega (f(x)v(x) + \boldsymbol{g}(x)\cdot\nabla_{\mathcal{D}} v(x) + l(x)\Delta_{\mathcal{D}} v(x))\mathrm{d}x. \tag{21}$$

Remark 2.2 *It is possible to replace $\nabla_{\mathcal{D}} v$ (defined by (16)) in (21) by $\widetilde{\nabla}_{\mathcal{D}} v$ (defined by (17)). We can see that the difference between $\int_\Omega \boldsymbol{g}(x)\cdot\nabla_{\mathcal{D}} v(x)\mathrm{d}x$ and $\int_\Omega \boldsymbol{g}(x)\cdot\widetilde{\nabla}_{\mathcal{D}} v(x)\mathrm{d}x$ resumes to different averaging formula of $\boldsymbol{g}$ on D_σ. One choice or the other one could be preferred, depending on the regularity of $\boldsymbol{g}$.*

Remark 2.3 *Considering the particular case $\boldsymbol{g} = 0$ and $l = 0$, we notice that (21) can also be written as*

$$u \in H_{\mathcal{D},0},\ \forall v \in H_{\mathcal{D},0},\ \sum_{K\in\mathcal{M}} |K|\Delta_K u \Delta_K v = \sum_{K\in\mathcal{M}} v_K \int_K f(x)\mathrm{d}x.$$

We can then regard the above relation as a finite volume scheme. Indeed, we take in (21), $v = 1_K$ for $K \in \mathcal{M}$ with $\mathcal{E}_{K,\mathrm{ext}} = \emptyset$. We have

$$\sum_{L\in\mathcal{M}} |L|\Delta_L u \Delta_L v = \sum_{\sigma\in\mathcal{E}_K} \frac{|\sigma|}{d_\sigma}\delta_{K,\sigma}\Delta_{\mathcal{D}} u,$$

which is a discrete equivalent of

$$\int_K \Delta(\Delta u)(x)\mathrm{d}x = \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \nabla(\Delta u)(x)\cdot \boldsymbol{n}_{K,\sigma}\mathrm{d}\gamma(x).$$

The scheme can then also be written as

$$|K|\Delta_K(\Delta_{\mathcal{D}} u) = \int_K f(x)\mathrm{d}x,\ \forall K\in\mathcal{M} \text{ such that } \mathcal{E}_{K,\mathrm{ext}} = \emptyset,$$

and

$$u_K = 0,\ \forall K\in\mathcal{M} \text{ such that } \mathcal{E}_{K,\mathrm{ext}} \neq \emptyset.$$

We can now derive the mathematical properties of the scheme, thanks to that of the discrete operator $\Delta_{\mathcal{D}}$, already studied in [5].

3 Study of the convergence of the scheme

We have the following estimate.

Lemma 3.1 (Existence, uniqueness and estimate on the solution of (21)) *Let Ω be an open bounded connected polygonal subset of $\mathbb{R}^d$, $d\in\mathbb{N}^\star$, let $f\in L^2(\Omega)$, $\boldsymbol{g}\in L^2(\Omega)^d$, $l\in L^2(\Omega)$ and let $\mathcal{D}$ be an admissible finite volume discretization of Ω in the sense of Definition 2.1 and let $\theta<\theta_{\mathcal{D}}$. Then there exists $C>0$, only depending on Ω, such that, for any $u\in H_{\mathcal{D},0}$ such that (21) holds, then*

$$\|u\|_{L^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|l\|_{L^2(\Omega)}), \tag{22}$$

$$\|u\|_{\mathcal{D}} \le C(\|f\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|l\|_{L^2(\Omega)}), \tag{23}$$

and

$$\|\Delta_{\mathcal{D}} u\|_{L^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|l\|_{L^2(\Omega)}). \tag{24}$$

and there exists $C'>0$, only depending on Ω and θ, such that

$$\|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d} \le C'(\|f\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|l\|_{L^2(\Omega)}). \tag{25}$$

As a consequence, there exists one and only one $u\in H_{\mathcal{D},0}$ such that (21) holds.

Proof. We first recall the discrete Poincaré inequality [5]:

$$\|v\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)\|v\|_{\mathcal{D}},\ \forall v\in H_{\mathcal{D}}. \tag{26}$$

therefore, thanks to (20) and (26), we get:

$$\|u\|_{\mathcal{D}} \le \mathrm{diam}(\Omega)\|\Delta_{\mathcal{D}} u\|_{L^2(\Omega)}, \tag{27}$$

We also recall that the following inequality, given in [6],

$$\|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d} \le \theta\sqrt{d}\|u\|_{\mathcal{D}}, \tag{28}$$

is a consequence of the Cauchy-Schwarz inequality and of Definitions (11) and (16). Hence, setting $v = u$ in (21), and using the Cauchy-Schwarz inequality, we get

$$\|\Delta_{\mathcal{D}} u\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)^2\|f\|_{L^2(\Omega)} + \theta\sqrt{d}\,\mathrm{diam}(\Omega)\|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|l\|_{L^2(\Omega)},$$

which proves

$$\|u\|_{\mathcal{D}} \le \mathrm{diam}(\Omega)(\mathrm{diam}(\Omega)^2\|f\|_{L^2(\Omega)} + \theta\sqrt{d}\,\mathrm{diam}(\Omega)\|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|l\|_{L^2(\Omega)})$$

and

$$\|u\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)^2(\mathrm{diam}(\Omega)^2\|f\|_{L^2(\Omega)} + \theta\sqrt{d}\,\mathrm{diam}(\Omega)\|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|l\|_{L^2(\Omega)}).$$

The three above inequalities provide (24), (23) and (22) (note that the example provided in Section 4.1 indicates that the above inequalities lead to the optimal orders with respect to $\mathrm{diam}(\Omega)$). We then get (25) using (28). Finally, we conclude the existence and uniqueness of the solution to (21), which leads to a square linear system, from the estimate (22), setting $f = 0$. $\square$

Lemma 3.2 (Compactness of a sequence of approximate solutions)

Let Ω be an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$, let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of admissible finite volume discretizations of Ω in the sense of Definition 2.1 such that $h_{\mathcal{D}_m}$ tends to 0 as $m \to \infty$ and there exists $\theta > 0$ with $\theta < \theta_{\mathcal{D}_m}$ for all $m \in \mathbb{N}$. We assume that there exists $C > 0$ and $u_m \in H_{\mathcal{D}_m,0}$, for all $m \in \mathbb{N}$, such that $\|\Delta_{\mathcal{D}_m} u_m\|_{L^2(\Omega)} \le C$ for all $m \in \mathbb{N}$. Then there exists a subsequence of $(\mathcal{D}_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m)_{m\in\mathbb{N}}$, and $u \in H^2_0(\Omega)$, such that the corresponding subsequence $(u_m)_{m\in\mathbb{N}}$ satisfies

1. *the sequence $(u_m)_{m\in\mathbb{N}}$ converges in $L^2(\Omega)$ to u,*
2. *the sequence $(\nabla_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges in $L^2(\Omega)^d$ to ∇u,*
3. *the sequence $(\Delta_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ weakly converges in $L^2(\Omega)$ to Δu.*

Proof. We first extract a subsequence of $(\mathcal{D}_m)_{m\in\mathbb{N}}$, such that $(\Delta_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ weakly converges to some $w \in L^2(\Omega)$. Let $u \in H^1_0(\Omega)$ such that

$$\forall v \in H^1_0(\Omega),\ \int_\Omega \nabla u(x)\cdot\nabla v(x)\mathrm{d}x = -\int_\Omega w(x)v(x)\mathrm{d}x.$$

Then, from an immediate adaptation of the results of [6], we get that $(u_m)_{m\in\mathbb{N}}$ converges in $L^2(\Omega)$ to u and $(\nabla_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges in $L^2(\Omega)^d$ to ∇u. We then get that $\Delta u(x) = w(x)$

for a.e. $x \in \Omega$, which proves that $\Delta u \in L^2(\Omega)$. Let us prove that $u \in H_0^2(\Omega)$. Using definition (17), we prolongate $\widetilde{\nabla}_{\mathcal{D}_m} u_m$ by 0 in $\mathbb{R}^d \setminus \Omega$. For $\varphi \in C_c^\infty(\mathbb{R}^d)$ (hence φ does not necessarily vanish at the boundary of Ω), we define

$$\widehat{\nabla}_{\mathcal{D}_m} \varphi(x) = \frac{\delta_\sigma P_{\mathcal{D}_m} \varphi}{d_\sigma} \boldsymbol{n}_\sigma + \nabla \varphi(x_\sigma) - (\nabla \varphi(x_\sigma) \cdot \boldsymbol{n}_\sigma) \boldsymbol{n}_\sigma,$$

for a.e. $x \in D_\sigma$, for all $\sigma \in \mathcal{E}$. We set $\widehat{\nabla}_{\mathcal{D}_m} \varphi(x) = \nabla \varphi(x)$ for a.e. $x \in \mathbb{R}^d \setminus \Omega$.

Using the results of [4], we get that the sequence $(\widetilde{\nabla}_{\mathcal{D}_m} u_m)_{m \in \mathbb{N}}$ weakly converges to ∇u in $L^2(\mathbb{R}^d)^d$, where ∇u is prolonged by 0 outwards from Ω. We consider the expression

$$T_m = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \frac{|\sigma|}{d_\sigma} \delta_\sigma u_m \delta_\sigma P_{\mathcal{D}_m} \varphi.$$

On the one hand, we have that

$$T_m = \int_{\mathbb{R}^d} \widetilde{\nabla}_{\mathcal{D}_m} u_m(x) \cdot \widehat{\nabla}_{\mathcal{D}_m} \varphi(x) \mathrm{d}x,$$

which implies

$$\lim_{m \to \infty} T_m = \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla \varphi(x) \mathrm{d}x.$$

On the other hand, thanks to $u_m \in H_{\mathcal{D}_m,0}$, we have

$$T_m = -\sum_{K \in \mathcal{M}} \varphi(x_K) \Delta_K u_m = -\int_\Omega P_{\mathcal{D}_m} \varphi(x) \Delta_{\mathcal{D}_m} u_m(x) \mathrm{d}x.$$

Hence, passing to the limit, we get

$$-\int_\Omega \varphi(x) \Delta u(x) \mathrm{d}x = \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla \varphi(x) \mathrm{d}x.$$

This proves that $\nabla u \in H_{\mathrm{div}}(\mathbb{R}^d)$ and that, prolonging Δu by 0 outwards of Ω, we have $\Delta u \in L^2(\mathbb{R}^d)$. Since $u \in H^1(\mathbb{R}^d)$, this implies that $u \in H^2(\mathbb{R}^d)$ (this also is a consequence of (6), which holds with $\Omega = \mathbb{R}^d$). Since $\nabla u = 0$ in $\mathbb{R}^d \setminus \Omega$, we get that $\nabla u \cdot \boldsymbol{n}_{\partial\Omega} = 0$ in the appropriate sense. Hence $u \in H_0^2(\Omega)$. □

Lemma 3.3 (Interpolation of regular functions with compact support)
Let Ω be an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$, let $\mathcal{D}$ be an admissible finite volume discretization of Ω in the sense of Definition 2.1 and let $\theta > 0$ with $\theta < \theta_{\mathcal{D}}$. Let $\varphi \in C_c^2(\Omega)$ and let $a = \mathrm{d}(\mathrm{support}(\varphi), \partial\Omega)$. Then there exists $C > 0$, only depending on θ, and $v \in H_{\mathcal{D},0}$ such that

1.
$$\|v - \varphi\|_{L^2(\Omega)} \leq C h_{\mathcal{D}} \frac{|\varphi|_2}{a^2}, \tag{29}$$

2.

$$\|v - P_{\mathcal{D}}\varphi\|_{\mathcal{D}} \leq C h_{\mathcal{D}} \frac{|\varphi|_2}{a^2}, \tag{30}$$

3.

$$\|\Delta_{\mathcal{D}} v - \Delta\varphi\|_{L^2(\Omega)} \leq C h_{\mathcal{D}} \frac{|\varphi|_2}{a^2}, \tag{31}$$

where $|\varphi|_2 = \max_{i,j=1,d} \|\partial^2_{ij}\varphi\|_{L^\infty(\Omega)}$.

PROOF. Let $\rho \in C_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$ be the function defined by

$$\rho(x) = \frac{\exp(-1/(1-|x|^2))}{\int_{B(0,1)} \exp(-1/(1-|y|^2))\mathrm{d}y}, \ \forall x \in B(0,1),$$

and $\rho(x) = 0$ for $x \notin B(0,1)$. Let ψ (see Figure 2) be the function defined by

$$\psi(y) = \int_{x\in\Omega, \mathrm{d}(x,\partial\Omega) > \frac{a}{2}} \left(\frac{4}{a}\right)^d \rho\left(\frac{4}{a}(y-x)\right) \mathrm{d}x, \ \forall y \in \Omega. \tag{32}$$

Then the function ψ satisfies that $\psi \in C_c^\infty(\Omega)$, $\psi(x) \in [0,1]$ for all $x \in \Omega$, $\psi(x) = 0$ for

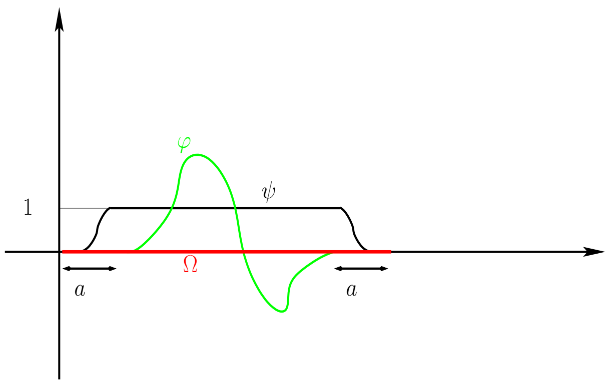

Figure 2: Functions φ and ψ

all $x \in \Omega$ such that $\mathrm{d}(x,\partial\Omega) < \frac{a}{4}$ and $\psi(x) = 1$ for all $x \in \Omega$ such that $\mathrm{d}(x,\partial\Omega) > \frac{3a}{4}$. The idea of the proof is to consider the discrete solution of the Laplace problem with the right hand side $-\Delta\varphi$, and to multiply it by ψ. Then the proof mimicks the identity $\Delta(\psi v) = v\Delta\psi + 2\nabla\psi\cdot\nabla v + \psi\Delta v$.

We first suppose that $\mathcal{D}$ is such that $h_{\mathcal{D}} < \frac{a}{4}$. We denote in the following $\psi_K = \psi(x_K)$, $\varphi_K = \varphi(x_K)$ for all $K \in \mathcal{M}$ and $\psi_{\mathcal{D}} = P_{\mathcal{D}}\psi$, $\varphi_{\mathcal{D}} = P_{\mathcal{D}}\varphi$. Let us define $\widetilde{v} \in H_{\mathcal{D}}$ such that

$$-|K|\Delta_K \widetilde{v} = -\int_K \Delta\varphi(x)\mathrm{d}x, \ \forall K \in \mathcal{M}, \tag{33}$$

which is equivalent to

$$\forall w \in H_{\mathcal{D}}, [\widetilde{v}, w]_{\mathcal{D}} = -\int_{\Omega} \Delta\varphi(x) w(x) \mathrm{d}x. \tag{34}$$

Let us remark that $\widetilde{v}$ satisfies thanks to (33)

$$|K|\psi_K \Delta_K \widetilde{v} = \int_K \Delta\varphi(x)\mathrm{d}x, \ \forall K \in \mathcal{M}. \tag{35}$$

Indeed, if $\int_K \Delta\varphi(x)\mathrm{d}x \neq 0$, then $K \cap \mathrm{support}(\varphi) \neq \emptyset$, which implies $\mathrm{d}(x_K, \partial\Omega) > \frac{3a}{4}$, and therefore $\psi_K = 1$. Otherwise, $\Delta_K \widetilde{v} = \int_K \Delta\varphi(x)\mathrm{d}x = 0$.

Using the results of [5], since the solution of the continuous Laplace problem is $\varphi \in C^2(\overline{\Omega})$, we can write the following error estimates:

$$\sum_{K\in\mathcal{M}} |K|(\widetilde{v}_K - \varphi_K)^2 \leq C_\Omega h_{\mathcal{D}}^2 \ |\varphi|_2^2, \tag{36}$$

and

$$\sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}(\delta_\sigma(\widetilde{v} - \varphi_{\mathcal{D}}))^2 \leq C_\Omega h_{\mathcal{D}}^2 \ |\varphi|_2^2, \tag{37}$$

where C_Ω only depends on Ω. We define $v \in H_{\mathcal{D},0}$ by its values in all $K \in \mathcal{M}$, given by $v_K = \psi_K \widetilde{v}_K$ (recall that, for all $K \in \mathcal{M}$ such that $\mathcal{E}_{K,\mathrm{ext}} \neq \emptyset$, then $\mathrm{d}(x_K, \partial\Omega) < \frac{a}{4}$, hence $\psi_K = 0$). We first remark that

$$|v_K - \varphi_K| = |\psi_K \widetilde{v}_K - \psi_K \varphi_K| \leq |\widetilde{v}_K - \varphi_K|,$$

which proves (29) thanks to (36) since $a \leq \mathrm{diam}(\Omega)$. Let us notice that the identity $ab - cd = c(b-d) + d(a-c) + (a-c)(b-d)$ yields

$$\delta_{K,\sigma} v = \psi_K \delta_{K,\sigma} \widetilde{v} + \widetilde{v}_K \delta_{K,\sigma}\psi_{\mathcal{D}} + \delta_{K,\sigma}\widetilde{v} \ \delta_{K,\sigma}\psi_{\mathcal{D}}.$$

Hence we get

$$|K|\Delta_K v = |K|\psi_K \Delta_K \widetilde{v} + |K|\widetilde{v}_K \Delta_K \psi_{\mathcal{D}} + \sum_{\sigma\in\mathcal{E}_K} \frac{|\sigma|}{d_\sigma} \delta_{K,\sigma}\psi_{\mathcal{D}} \delta_{K,\sigma}\widetilde{v}.$$

We remark that, for all $K \in \mathcal{M}$ such that $\Delta_K \psi_{\mathcal{D}} \neq 0$, then $\varphi_K = 0$, and for all $\sigma \in \mathcal{E}$ such that $\delta_\sigma \psi_{\mathcal{D}} \neq 0$, then $\varphi_{K_\sigma^+} = \varphi_{K_\sigma^-} = 0$. This leads, using (35), to

$$|K|\Delta_K v = \int_K \Delta\varphi(x)\mathrm{d}x + |K|(\widetilde{v}_K - \varphi_K)\Delta_K \psi_{\mathcal{D}} + \sum_{\sigma\in\mathcal{E}_K} \frac{|\sigma|}{d_\sigma} \delta_{K,\sigma}\psi_{\mathcal{D}} \delta_{K,\sigma}(\widetilde{v} - \varphi_{\mathcal{D}}).$$

Moreover, a Taylor expansion provides

$$\delta_{K,\sigma}\psi_{\mathcal{D}} = d_\sigma \nabla\psi_K \cdot \boldsymbol{n}_{K,\sigma} + d_\sigma^2 \frac{C_{K,\sigma}}{a^2},$$

with $C_{K,\sigma}$ bounded by a constant. Since $\sum_{\sigma\in\mathcal{E}_K}|\sigma|\boldsymbol{n}_{K,\sigma}=0$, $\sum_{\sigma\in\mathcal{E}_K}|\sigma|d_{K,\sigma}=d\,|K|$ and $d_\sigma\le d_{K,\sigma}/\theta$, we get

$$|K||\Delta_K\psi_{\mathcal{D}}|=\left|\sum_{\sigma\in\mathcal{E}_K}|\sigma|d_\sigma\frac{C_{K,\sigma}}{a^2}\right|\le\frac{C_2}{a^2}|K|,$$

where C_2 only depends on θ. Hence we get

$$\begin{aligned}\sum_{K\in\mathcal{M}}|K|\left(\Delta_K v-\frac{1}{|K|}\int_K\Delta\varphi(x)\mathrm{d}x\right)^2\le\;& 2\frac{C_2^2}{a^4}\sum_{K\in\mathcal{M}}|K|(\widetilde{v}_K-\varphi_K)^2\\ &+2\sum_{K\in\mathcal{M}}\frac{1}{|K|}\left(\sum_{\sigma\in\mathcal{E}_K}\frac{|\sigma|}{d_\sigma}\delta_{K,\sigma}\psi_{\mathcal{D}}\delta_{K,\sigma}(\widetilde{v}-\varphi_{\mathcal{D}})\right)^2.\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(\sum_{\sigma\in\mathcal{E}_K}\frac{|\sigma|}{d_\sigma}\delta_{K,\sigma}\psi_{\mathcal{D}}\delta_{K,\sigma}(\widetilde{v}-\varphi_{\mathcal{D}})\right)^2&\le\sum_{\sigma\in\mathcal{E}_K}\frac{|\sigma|}{d_\sigma}(\delta_{K,\sigma}\psi_{\mathcal{D}})^2\sum_{\sigma\in\mathcal{E}_K}\frac{|\sigma|}{d_\sigma}(\delta_{K,\sigma}(\widetilde{v}-\varphi_{\mathcal{D}}))^2\\&\le\frac{C_1^2}{a^2}|K|\sum_{\sigma\in\mathcal{E}_K}\frac{|\sigma|}{d_\sigma}(\delta_{K,\sigma}(\widetilde{v}-\varphi_{\mathcal{D}}))^2,\end{aligned}$$

where C_1 only depends on θ. Hence

$$\begin{aligned}\sum_{K\in\mathcal{M}}|K|\left(\Delta_K v-\frac{1}{|K|}\int_K\Delta\varphi(x)\mathrm{d}x\right)^2\le\;& 2\frac{C_2^2}{a^4}\sum_{K\in\mathcal{M}}|K|(\widetilde{v}_K-\varphi_K)^2\\ &+2\frac{C_1^2}{a^2}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\frac{|\sigma|}{d_\sigma}(\delta_{K,\sigma}(\widetilde{v}-\varphi_{\mathcal{D}}))^2.\end{aligned}$$

This leads, thanks to (36) and (37), to

$$\sum_{K\in\mathcal{M}}|K|\left(\Delta_K v-\frac{1}{|K|}\int_K\Delta\varphi(x)\mathrm{d}x\right)^2\le\frac{2C_2^2+4C_1^2a^2}{a^4}C_\Omega h_{\mathcal{D}}^2\;|\varphi|_2^2,\tag{38}$$

hence proving (31) thanks to the regularity of $\Delta\varphi$ and thanks to $a\le\mathrm{diam}(\Omega)$. Finally, since (38) can also be written

$$\|\Delta_{\mathcal{D}}v-\Delta_{\mathcal{D}}\widetilde{v}\|_{L^2(\Omega)}^2\le\frac{C}{a^4}h_{\mathcal{D}}^2\;|\varphi|_2^2,$$

we get, thanks to (20) and (26),

$$\|v-\widetilde{v}\|_{\mathcal{D}}^2\le\mathrm{diam}(\Omega)^2\frac{C}{a^4}h_{\mathcal{D}}^2\;|\varphi|_2^2.$$

Hence we deduce (30) from (37) using the triangle inequality and $a\le\mathrm{diam}(\Omega)$.

In the case where $h_{\mathcal{D}}\ge\frac{a}{4}$, we set $v=0$. Since $\|\varphi\|_{L^2(\Omega)}$ and $\|\varphi_{\mathcal{D}}\|_{\mathcal{D}}$ are bounded, up to some constants only depending on Ω, by $\|\Delta\varphi\|_{L^2(\Omega)}$, and using $\frac14\le\frac{h_{\mathcal{D}}}{a}$, we conclude that the lemma holds for all $h_{\mathcal{D}}>0$. $\square$

Remark 3.1 *Lemma 3.3 is the main tool for a similar interpolation result which is needed in the convergence proof [3] for a finite volume discretization of the incompressible Navier-Stokes equations. In that case, we have to construct discrete test functions with homogeneous Dirichlet boundary values, which are discretely divergence-free and converge to regular divergence-free test functions with compact support.*

We can now state the convergence of the scheme.

Theorem 3.1 (Convergence of the scheme)

Let Ω be an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$, let $f \in L^2(\Omega)$, $\boldsymbol{g} \in L^2(\Omega)^d$ and $l \in L^2(\Omega)$. Let $u \in H_0^2(\Omega)$ be the solution of Problem (5).

Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of admissible finite volume discretizations of Ω in the sense of Definition 2.1 such that $h_{\mathcal{D}_m}$ tends to 0 as $m \to \infty$ and there exists $\theta > 0$ with $\theta < \theta_{\mathcal{D}_m}$ for all $m \in \mathbb{N}$. Let $u_m \in H_{\mathcal{D}_m,0}$, for all $m \in \mathbb{N}$, be the solution of (21). Then the following holds:

1. *the sequence $(u_m)_{m\in\mathbb{N}}$ converges in $L^2(\Omega)$ to u,*
2. *the sequence $(\nabla_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges in $L^2(\Omega)^d$ to ∇u,*
3. *the sequence $(\Delta_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges in $L^2(\Omega)$ to Δu.*

Proof. Thanks to Lemmas 3.1 and 3.2, we get the existence of a subsequence of $(\mathcal{D}_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m)_{m\in\mathbb{N}}$, and of $u \in H_0^2(\Omega)$ such that the conclusion of Lemma 3.2 hold. Let $\varphi \in C_c^\infty(\Omega)$ be given. We take, in (21), $v = v_m$ where v_m is given by Lemma 3.3 for $\mathcal{D} = \mathcal{D}_m$. Passing to the limit (thanks to weak/strong convergence) and by density of $C_c^\infty(\Omega)$ in $H_0^2(\Omega)$, we get that u is the solution of Problem (5). By a classical argument of uniqueness, we get that all the sequence converges. Setting $v = u_m$ in (21), we get the convergence of $\|\Delta_{\mathcal{D}_m} u_m\|_{L^2(\Omega)}^2$ to $\int_\Omega (f(x)u(x) + \boldsymbol{g}(x)\cdot\nabla u(x) + l(x)\Delta u(x))\mathrm{d}x = \int_\Omega (\Delta u(x))^2 \mathrm{d}x$. In addition to the weak convergence of $\Delta_{\mathcal{D}_m} u_m$ to Δu, this provides the convergence in $L^2(\Omega)$ of $\Delta_{\mathcal{D}_m} u_m$ to Δu. $\square$

Let us now state error estimate results, that, for the sake of simplicity, we only provide in the case $\boldsymbol{g} = 0$ and $l = 0$.

Theorem 3.2 (Error estimate in the case where $u \in C_c^4(\Omega)$)

Let Ω be an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$. Let us assume that $u \in C_c^4(\Omega)$ is given and that $f = \Delta(\Delta u)$. Let $\mathcal{D}$ be an admissible finite volume discretization of Ω in the sense of Definition 2.1 and let $\theta > 0$ with $\theta < \theta_{\mathcal{D}}$. Let $u_{\mathcal{D}} \in H_{\mathcal{D},0}$ be the solution of (21). Then there exists $C > 0$, only depending on Ω, θ and u such that

1.
$$\|u_{\mathcal{D}} - u\|_{L^2(\Omega)} \le C h_{\mathcal{D}}, \tag{39}$$

2.

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla u\|_{L^2(\Omega)^d} \le C h_{\mathcal{D}}, \tag{40}$$

3.

$$\|\Delta_{\mathcal{D}} u_{\mathcal{D}} - \Delta u\|_{L^2(\Omega)} \le C h_{\mathcal{D}}. \tag{41}$$

PROOF. In this proof, we denote by C_i various positive quantities only depending on Ω, u and θ. Let us first take any $w \in H_{\mathcal{D},0}$. We have

$$\int_\Omega w(x)\Delta(\Delta u)(x)\mathrm{d}x = \int_\Omega w(x) f(x)\mathrm{d}x,$$

which leads, thanks to $w_K = 0$ if K has a common boundary with $\partial\Omega$, to

$$-\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \delta_\sigma w \int_\sigma \nabla(\Delta u)(x)\cdot \boldsymbol{n}_\sigma \mathrm{d}\gamma(x) = \sum_{K\in\mathcal{M}} w_K \int_K f(x)\mathrm{d}x.$$

We set, for $\sigma \in \mathcal{E}_{\mathrm{int}}$,

$$R_\sigma = \frac{1}{|\sigma|}\int_\sigma \nabla(\Delta u)(x)\cdot \boldsymbol{n}_\sigma \mathrm{d}\gamma(x) - \frac{\delta_\sigma P_{\mathcal{D}}\Delta u}{d_\sigma}.$$

We have the existence of C_4, only depending on u (as in [5]), such that

$$|R_\sigma| \le C_4 d_\sigma. \tag{42}$$

Using $w \in H_{\mathcal{D},0}$, we have

$$\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \frac{|\sigma|}{d_\sigma}\delta_\sigma w \delta_\sigma P_{\mathcal{D}}\Delta u = [w, P_{\mathcal{D}}\Delta u]_{\mathcal{D}}.$$

Therefore, using (20), we have

$$\sum_{K\in\mathcal{M}} |K|\Delta u(x_K)\Delta_K w = \sum_{K\in\mathcal{M}} w_K \int_K f(x)\mathrm{d}x + \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |\sigma| R_\sigma \delta_\sigma w.$$

Let us now introduce some $v \in H_{\mathcal{D},0}$, which will be chosen later as some discrete interpolation of u. We have

$$\sum_{K\in\mathcal{M}} |K|\Delta_K v \Delta_K w = \sum_{K\in\mathcal{M}} w_K \int_K f(x)\mathrm{d}x + \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |\sigma| R_\sigma \delta_\sigma w + \sum_{K\in\mathcal{M}} |K|(\Delta_K v - \Delta u(x_K))\Delta_K w.$$

We now subtract the above equation with (21), in which we replace v by w and we get

$$\sum_{K\in\mathcal{M}} |K|\Delta_{\mathcal{D}}(v - u_{\mathcal{D}})\Delta_K w = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |\sigma| R_\sigma \delta_\sigma w + \sum_{K\in\mathcal{M}} |K|(\Delta_K v - \Delta u(x_K))\Delta_K w.$$

Thanks to the Cauchy-Schwarz inequality, we have the existence of C_5 such that

$$\left| \sum_{\sigma \in \mathcal{E}_{\rm int}} |\sigma| R_\sigma \delta_\sigma w \right| \le C_5 h_{\mathcal{D}} \|w\|_{\mathcal{D}},$$

which provides, thanks to (20), (26) and (27)

$$\left| \sum_{\sigma \in \mathcal{E}_{\rm int}} |\sigma| R_\sigma \delta_\sigma w \right| \le C_6 h_{\mathcal{D}} \|\Delta_{\mathcal{D}} w\|_{L^2(\Omega)}.$$

Replacing w by $(v - u_{\mathcal{D}})$ we obtain

$$\|\Delta_{\mathcal{D}}(v - u_{\mathcal{D}})\|_{L^2(\Omega)} \le C_6 h_{\mathcal{D}} + \left(\sum_{K \in \mathcal{M}} |K| (\Delta_K v - \Delta u(x_K))^2 \right)^{\frac{1}{2}}$$

Finally, we use the triangle inequality and obtain

$$\|\Delta u - \Delta_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)} \le \left(\sum_{K \in \mathcal{M}} \int_K (\Delta u - \Delta u(x_K))^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + C_6 h_{\mathcal{D}} + 2 \left(\sum_{K \in \mathcal{M}} |K| (\Delta_K v - \Delta u(x_K))^2 \right)^{\frac{1}{2}}$$

Now we choose $v \in H_{\mathcal{D},0}$ according to Lemma 3.3 using $\varphi = u$. Thanks to (31) and $\Delta u \in C^2(\Omega)$, we get the existence of C_7 such that

$$\left(\sum_{K \in \mathcal{M}} |K| (\Delta_K v - \Delta u(x_K))^2 \right)^{\frac{1}{2}} \le C_7 h_{\mathcal{D}}.$$

Gathering the above results, we get

$$\|\Delta u - \Delta_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)} \le C_8 h_{\mathcal{D}},$$

and, thanks to (20) and (26),

$$\|P_{\mathcal{D}} u - u_{\mathcal{D}}\|_{\mathcal{D}} \le C_9 h_{\mathcal{D}},$$

and

$$\|P_{\mathcal{D}} u - u_{\mathcal{D}}\|_{L^2(\Omega)} \le C_{10} h_{\mathcal{D}}.$$

Using (28), we conclude the proof of the theorem. □

Theorem 3.3 (Error estimate in the case where $u \in C^4(\overline{\Omega}) \cap H_0^2(\Omega)$)
Let Ω be an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$. Let us assume that $u \in C^4(\overline{\Omega}) \cap H_0^2(\Omega)$ is given and that $f = \Delta(\Delta u)$. Let $\mathcal{D}$ be an admissible finite volume discretization of Ω in the sense of Definition 2.1 and let $\theta > 0$ with $\theta < \theta_{\mathcal{D}}$. Let $u_{\mathcal{D}} \in H_{\mathcal{D},0}$ be the solution of (21). Then there exists $C > 0$, only depending on Ω, θ and u such that

1.
$$\|u_{\mathcal{D}} - u\|_{L^2(\Omega)} \le C h_{\mathcal{D}}^{1/5}, \tag{43}$$

2.
$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla u\|_{L^2(\Omega)^d} \le C h_{\mathcal{D}}^{1/5}, \tag{44}$$

3.
$$\|\Delta_{\mathcal{D}} u_{\mathcal{D}} - \Delta u\|_{L^2(\Omega)} \le C h_{\mathcal{D}}^{1/5}. \tag{45}$$

PROOF. For a given $a > 0$ (which will be chosen later), we define the function ψ_a by (32). We remark that the function u_a defined by $u_a(x) = u(x)\psi_a(x)$ for all $x \in \Omega$ is such that
$$\|\Delta u - \Delta u_a\|_{L^2(\Omega)} \le C\sqrt{a}, \tag{46}$$
where C only depends on u. Indeed, we have
$$\Delta u_a(x) = \psi_a(x)\Delta u(x) + 2\nabla\psi_a(x)\cdot\nabla u(x) + u(x)\Delta\psi_a(x),$$
which gives
$$\Delta u_a(x) - \Delta u(x) = (\psi_a(x) - 1)\Delta u(x) + 2\nabla\psi_a(x)\cdot\nabla u(x) + u(x)\Delta\psi_a(x).$$
Since there exists $C_u > 0$, only depending on u, such that for all $x \in \Omega$, $|\nabla u(x)| \le C_u \mathrm{d}(x,\partial\Omega)$ and $|u(x)| \le C_u \mathrm{d}(x,\partial\Omega)^2$, we get the existence of C'_u, only depending on u, such that
$$|\Delta u_a(x) - \Delta u(x)| \le C'_u, \ \forall x \in \Omega \text{ such that } \mathrm{d}(x,\partial\Omega) \le a,$$
remarking that $|\nabla\psi_a(x)| \le C_0/a$ and $|\Delta\psi_a(x)| \le C_0/a^2$, with C_0 being a constant. Using $\Delta u_a(x) = \Delta u(x)$ if $\mathrm{d}(x,\partial\Omega) > a$, we get
$$\|\Delta u - \Delta u_a\|^2_{L^2(\Omega)} \le \mathrm{meas}(\partial\Omega)\ a\ (C'_u)^2.$$
We now reproduce the proof of Theorem 3.2 until the choice of $v \in H_{\mathcal{D},0}$, which is now given by Lemma 3.3 for $\varphi = u_a$. We then get that
$$\sum_{K\in\mathcal{M}} |K|(\Delta_K v - \frac{1}{|K|}\int_K \Delta u_a(x)\mathrm{d}x)^2 \le C\frac{h_{\mathcal{D}}^2}{(a/4)^4}.$$
Using the triangle inequality we thus get the existence of C_{11}, only depending on u, such that
$$\sum_{K\in\mathcal{M}} |K|(\Delta_K v - \Delta u(x_K))^2 \le C_{11}\left(h_{\mathcal{D}}^2 + a + \frac{h_{\mathcal{D}}^2}{a^4}\right).$$
It now suffices to choose $a = h_{\mathcal{D}}^{2/5}$ (note that, for small values of $h_{\mathcal{D}}$ the case $h_{\mathcal{D}} \le a/4$ holds, which allows the function v given by Lemma 3.3 to be different from 0), which leads to the conclusion of the proof. □

4 Numerical results

We consider in this section 1D, 2D and 3D examples with various types of meshes in the 2D case. Note that, in particular, the discrete Laplace operator is not consistent in the case of triangular meshes or rectangular meshes with nonconstant space steps. In the tables below we use for the difference of the approximate solution $u_{\mathcal{D}} \in H_{\mathcal{D},0}$ and the exact solution $u \in H_0^2(\Omega)$ the following discrete norms defined by

$$E_0 = \left(\sum_{K\in\mathcal{M}} |K|(u_K - u(x_K))^2 / \sum_{K\in\mathcal{M}} |K| u(x_K)^2 \right)^{1/2},$$

$$E_1 = \left(\sum_{K\in\mathcal{M}} |K|\, |\nabla_K u_{\mathcal{D}} - \nabla u(x_K)|^2 \,/ \sum_{K\in\mathcal{M}} |K| |\nabla u(x_K)|^2 \right)^{1/2},$$

and

$$E_2 = \left(\sum_{K\in\mathcal{M}} |K|(\Delta_K u_{\mathcal{D}} - \Delta u(x_K))^2 / \sum_{K\in\mathcal{M}} |K| (\Delta u(x_K))^2 \right)^{1/2}.$$

4.1 1D example

We solve the problem

$$u^{(4)}(x) = -1,\ \ x \in [0, L],$$
$$u(0) = u(1) = u'(0) = u'(L) = 0,$$

which is the classical problem of the completely fixed beam, under uniform load. The analytical solution is given by

$$u(x) = -\frac{(x(L-x))^2}{24}.$$

The exact minimum value of u is $-L^4/(2^4\ 24) \simeq -0.002604167\ L^4$.

n	$N_{\rm mat}$	E_0	order	E_1	order	E_2	order	$u_{\rm min}$	$u_{\rm max}$
100	484	4.29E-4	-	6.27E-4	-	1.12E-4	-	-0.0026031	0
200	984	1.07E-4	$\simeq 2$	1.57E-4	$\simeq 2$	2.80E-5	$\simeq 2$	-0.0026039	0
400	1984	2.68E-5	$\simeq 2$	3.92E-5	$\simeq 2$	6.99E-6	$\simeq 2$	-0.0026041	0

Table 1: Convergence orders, in the case $L = 1$

In this standard example, we get convergence with order 2 for u, ∇u and Δu. This convergence order is lower than that obtained using conformal H^2 finite element methods, but may be sufficient in practice. Note that $\|f\|_{L^2(\Omega)} = \sqrt{L}$, and that $\|\Delta u\|_{L^2(\Omega)}$, $\|\nabla u\|_{L^2(\Omega)}$ and $\|u\|_{L^2(\Omega)}$ behave with L as $L^i \|f\|_{L^2(\Omega)}$ with respectively $i = 2, 3, 4$, which shows that the constants found in the proof of Lemma 3.1 have the optimal order.

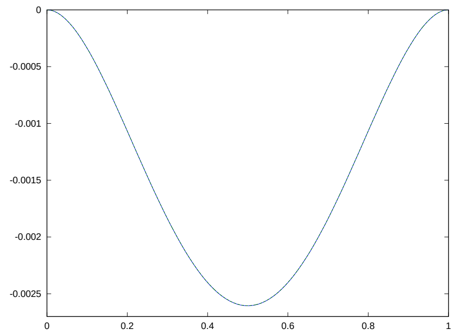

Figure 3: Exact and approximate solutions with $n = 200$.

4.2 2D example

Let us consider the 2D problem, where the solution is given by (5) with $f = \Delta(\Delta u)$, $\boldsymbol{g} = 0$, $l = 0$, $\Omega =]0, 1[^2$ and

$$u(x_1, x_2) = (1 - \cos(2\pi x_1))(1 - \cos(2\pi x_2)),\ \forall (x_1, x_2) \in [0, 1]^2.$$

We then have

$$\Delta(\Delta u)(x_1, x_2) = (2\pi)^4 (4 \cos(2\pi x_1) \cos(2\pi x_2) - (\cos(2\pi x_1) + \cos(2\pi x_2)))$$

We then have the following numerical results, for different meshes (squares or triangles).

Mesh	N_{mat}	E_0	order	E_1	order	E_2	order	$u_{\min}$	$u_{\max}$
20x20	3856	1.04E-2	-	6.03E-3	-	1.03E-2	-	0	3.991
40x40	18016	2.58E-3	$\simeq 2$	1.49E-3	$\simeq 2$	2.56E-3	$\simeq 2$	0	3.998
1400 tr.	12736	3.99E-3	-	5.27E-2	-	5.97E-3	-	0	3.998
5600 tr.	53456	9.89E-4	$\simeq 2$	2.63E-2	$\simeq 1$	2.53E-3	≥ 1	0	3.9995
22400 tr.	218896	2.47E-4	$\simeq 2$	1.31E-2	$\simeq 1$	1.20E-3	≥ 1	0	3.9999

Table 2: Convergence orders

In this example, we again get convergence with order 2 for u, ∇u and Δu using square meshes, but only order 1 using triangular meshes. Again, this convergence order is lower than that obtained using conformal H^2 finite element methods, but we remark that the complexity of conformal finite element methods increases with that of the chosen element. For example, the Argyris triangular finite element should be used for getting conformal approximation in $H^2(\Omega)$. This element has complex degrees of freedom, and the computation of the elementary stiffness matrix is much more complex than the implementation of this finite volume method.

It is worth noticing that in this case, for any function $\varphi \in C_c^\infty(\Omega)$, the function $\Delta_{\mathcal{D}} P_{\mathcal{D}} \varphi$ has no chance to converge to $\Delta\varphi$ in $L^2(\Omega)$ using the triangular meshes.

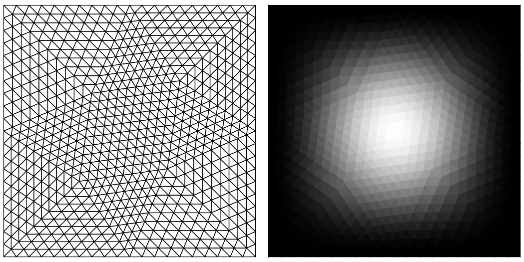

Figure 4: The solution for mesh with 1400 tr., grid (left), solution (right)

4.3 3D example

Let us consider the 3D problem, where the solution is given by (5) with $f = \Delta(\Delta u)$, $\boldsymbol{g} = 0$, $l = 0$, $\Omega =]0,1[^3$ and

$$u(x_1, x_2, x_3) = (1 - \cos(2\pi x_1))(1 - \cos(2\pi x_2))(1 - \cos(2\pi x_3)),\ \forall (x_1, x_2, x_3) \in [0,1]^3.$$

We then have

$$\begin{aligned}\Delta(\Delta u)(x_1, x_2, x_3) = (2\pi)^4(\quad & 4(\cos(2\pi x_1)\cos(2\pi x_2) + \cos(2\pi x_2)\cos(2\pi x_3) + \cos(2\pi x_3)\cos(2\pi x_1)) \\ & -(\cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi x_3)) \\ & -9\cos(2\pi x_1)\cos(2\pi x_2)\cos(2\pi x_3))\end{aligned}$$

We then have the following numerical results, for cubic meshes with n^3 control volumes.

Mesh	N_{mat}	E_0	order	E_1	order	E_2	order	$u_{\min}$	$u_{\max}$
8x8x8	3960	0.721E-01	-	0.564E-01	-	7.49E-2	-	0	7.57
16x16x16	60536	0.175E-01	$\simeq 2$	0.134E-01	$\simeq 2$	1.82E-2	$\simeq 2$	0	7.90
32x32x32	637560	0.435E-02	$\simeq 2$	0.329E-02	$\simeq 2$	4.52E-3	$\simeq 2$	0	7.98
2000 Vor.	78597	0.958E-01	-	0.238	-	0.281	-	-0.015	7.83
16000 Vor.	955719	0.475E-01	$\simeq 1$	0.114	$\simeq 1$	0.172	≤ 1	-0.002	7.85

Table 3: Convergence orders

In this 3D example, we again get convergence with order 2 for u, ∇u and Δu with cubic meshes. We recall that it is not possible to consider tetrahedral admissible meshes in 3D in the sense of Definition 2.1. The more general meshes that we can consider here are the Voronoï meshes (recall that the control volumes are defined, for any point x_K, as the set of the points of Ω closer to x_K than to any point x_L for $L \neq K$). Note that, for such meshes, no standard finite element techniques are available. In Table 3, we present the results obtained using two Voronoï meshes, with respectively 2000 and 16000 control volumes. The centers of the control volumes are randomly generated. The convergence orders remain significant, although in this case again, for any function $\varphi \in C_c^\infty(\Omega)$, the function $\Delta_{\mathcal{D}} P_{\mathcal{D}} \varphi$ has no chance to converge to $\Delta\varphi$ in $L^2(\Omega)$. We observe that the maximum and

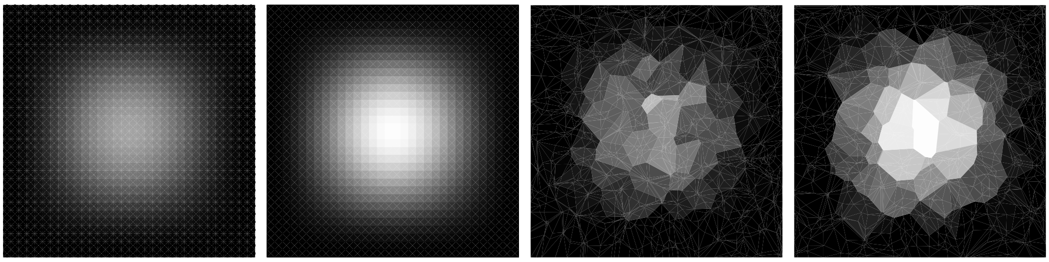

Figure 5: From left to right: solution obtained with mesh 32x32x32, at $x_1 = .2$, at $x_1 = .5$, solution obtained with Voronoï mesh with 16000 control volumes, at $x_1 = .2$ and at $x_1 = .5$

minimum values are not as precise as those obtained using cubic meshes. It is interesting to notice that the nonzero terms in the matrix are much more numerous, in comparison with cubic meshes, for comparable mesh sizes.

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