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Regularization error estimates for semilinear elliptic optimal control problems with pointwise state and control constraints

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Abstract

In this paper a class of semilinear elliptic optimal control problem with pointwise state and control constraints is studied. A sufficient second order optimality condition and uniqueness of the dual variables are assumed for that problem. Sufficient second order optimality conditions are shown for regularized problems with small regularization parameter. Moreover, error estimates with respect to the regularization parameter are derived.

1 Introduction

In this paper we study the analysis of a class of optimal control problems governed by semilinear elliptic PDEs and pointwise state and control constraints:

$$\left.\begin{aligned} \min \quad J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \\ Ay + d(x,y) = u \quad \text{in } \Omega \\ \partial_{n_A} y = 0 \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ y(x) \ge y_c(x) \quad \text{a.e. in } \bar{\Omega}. \end{aligned}\right\} \tag{P}$$

In this setting $\Omega \subset \mathbb{R}^d$, $d = \{2,3\}$ is a bounded convex domain which has $C^{1,1}$-boundary Γ. The precise conditions on the given quantities in (P) are given in Assumption 1.1 below.

It is well known that problems with pointwise state constraints exhibit several difficulties caused by low regularity of the respective Lagrange multipliers, see [3]. Different regularization methods were proposed in the recent years to overcome this difficulty. We mention Lavrentiev-type regularization by Meyer, Rösch, and Tröltzsch, [17], or the Moreau-Yosida approximation by Ito and Kunisch, cf. [13]. We will apply the so called virtual control concept, first introduced in [16]. Instead of problem (P), we will investigate a family of regularized optimal control problems:

$$\left.\begin{aligned} \min \quad J_\varepsilon(y,u,v) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} \\ Ay + d(x,y) = u + \phi(\varepsilon)v \quad \text{in } \Omega \\ \partial_{n_A} y = 0 \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ y(x) \ge y_c(x) - \xi(\varepsilon)v \quad \text{a.e. in } \Omega, \end{aligned}\right\} \tag{P$_\varepsilon$}$$

with a regularization parameter $\varepsilon > 0$ and positive and real valued parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$. The remaining given quantities are defined as for problem (P), see Assumption 1.1.

Let us emphasize the differences to results that are known for the Lavrentiev regularization. In [8] error estimates were derived for linear-quadratic optimal control problems. The resulting general convex situation simplifies the analysis essentially. Plain convergence for semilinear problems is obtained in [11], and error estimates for parabolic optimal control problems were derived in [18]. In addition to convergence and error estimates, we show second order sufficient optimality conditions for locally optimal solutions of the regularized problems, where we require only assumptions on the unregularized problem (P). Consequently the results derived in this paper go essentially beyond the known theory.

Throughout the paper, we will use the following notation: By $\|\cdot\|$ we denote the usual norm in $L^2(\Omega)$, and $(\cdot,\cdot)$ is the associated inner product. The $L^\infty(\Omega)$-norm is specified by $\|\cdot\|_\infty$. Moreover, $\langle\cdot,\cdot\rangle$ represents the duality pairing in $C(\bar\Omega)$ and $C(\bar\Omega)^*$.

Assumption 1.1

- *The functions $y_d \in L^2(\Omega)$, and $y_c \in C^{0,1}(\bar\Omega)$ are given functions and $u_a \le u_b$, $\nu > 0$ are real numbers.*
- *A denotes a second order elliptic operator of the form*

$$Ay(x) = -\sum_{i,j=1}^{d} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y(x)),$$

where the coefficients a_{ij} belong to $C^{0,1}(\bar\Omega)$ with the ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2 \qquad \forall (x,\xi) \in \Omega \times \mathbb{R}^d, \quad \theta > 0.$$

Moreover, ∂_{n_A} denotes the conormal-derivative associated with A.

- *The function $d = d(x,y) : \Omega \times \mathbb{R}$ is measurable with respect to $x \in \Omega$ for all fixed $y \in \mathbb{R}$, and twice continuously differentiable with respect to y, for almost all $x \in \Omega$.*
- *Moreover, for $y = 0$ it is bounded of order 2 with respect to x, i.e. for d*

$$\|d(\cdot,0)\|_\infty + \|\frac{\partial d}{\partial y}(\cdot,0)\|_\infty + \|\frac{\partial^2 d}{\partial y^2}(\cdot,0)\|_\infty \le C \tag{1.1}$$

is satisfied.

- *Further, for a.a. $x \in \Omega$, it holds that $d_y(x,y) \ge 0$.*

- *Also, the derivatives of d w.r.t. y up to order two are uniformly Lipschitz on bounded sets, i.e. for all $M > 0$ there exists $L_M > 0$ such that d satisfies*

$$\|\frac{\partial^2 d}{\partial y^2}(\cdot, y_1) - \frac{\partial^2 d}{\partial y^2}(\cdot, y_2)\|_\infty \leq L_M |y_1 - y_2| \tag{1.2}$$

  for all $y_i \in \mathbb{R}$ with $|y_i| \leq M$, $i = 1, 2$.

- *There is a subset $E_\Omega \subset \Omega$ of positive measure with $d_y(x, y) > 0$ in $E_\Omega \times \mathbb{R}$.*

2 Analysis of problem (P)

2.1 The state equation

We will start by analyzing the state equation of problem (P). The proof of the following theorem can be found in [4].

Theorem 2.1 *Under Assumption 1.1 the semilinear elliptic boundary value problem*

$$\begin{aligned} Ay + d(x, y) &= u \quad \text{in } \Omega \\ \partial_{n_A} y &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.1}$$

admits for every right hand side $u \in L^2(\Omega)$ a unique solution $y \in H^1(\Omega) \cap C(\bar{\Omega})$.

Based on this theorem, we introduce the control-to-state operator

$$G : L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega}),\ u \mapsto y. \tag{2.2}$$

Let us reformulate the problem (P) with the help of the solution operator G to obtain the reduced formulation

$$\begin{aligned} \min \quad f(u) = J(Gu, u) &:= \frac{1}{2}\|Gu - y_d\|^2 + \frac{\nu}{2}\|u\|^2 \\ u_a \leq u(x) \leq u_b & \quad \text{a.e. in } \Omega \\ (Gu)(x) \geq y_c(x) & \quad \text{a.e. in } \bar{\Omega}. \end{aligned}$$

For future reference, let us define the set of admissible controls handling the box constraints on the control,

$$U_{ad} = \{u \in L^2(\Omega) \,:\, u_a \leq u \leq u_b \text{ a.e. in } \Omega\}.$$

We say that a control $u \in U_{ad}$ is feasible for problem (P) if the associated state $y = G(u)$ fulfills the state constraints $y(x) \geq y_c(x)$ a.e. in $\bar{\Omega}$. Due to the convexity of the cost functional with respect to the control u, the existence of at least one solution of problem (P) can be obtained by standard arguments if the set of feasible controls is nonempty and Assumption 1.1 is fulfilled. Let us first introduce the notation of a local solution:

Definition 2.2 *A control $\bar{u} \in U_{ad}$ satisfying $G(\bar{u}) \geq y_c$ in $\bar{\Omega}$ is called a local solution of problem* (P) *if there exists a $\rho > 0$ such that*

$$f(u) \geq f(\bar{u})$$

for all $u \in U_{ad}$ with $G(u) \geq y_c$ in $\bar{\Omega}$ and $\|u - \bar{u}\| \leq \rho$.

Theorem 2.3 *Let the Assumption 1.1 be satisfied. If the set of feasible controls is nonempty, then Problem* (P) *admits at least one local solution in the sense of Definition 2.2.*

The proof follows by standard arguments.

We proceed with recalling some results concerning differentiability of the nonlinear control-to-state mapping G.

Theorem 2.4 *Let Assumption 1.1 be fulfilled. Then the mapping $G : L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega})$, defined by $G(u) = y$ is of class C^2. Moreover, for all u, $h \in L^2(\Omega)$, $y_h = G'(u)h$ is defined as the solution of*

$$\begin{aligned} Ay_h + d_y(x,y)y_h &= h \quad \text{in } \Omega \\ \partial_{n_A} y_h &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.3}$$

Furthermore, for every $h_1, h_2 \in L^2(\Omega)$, $y_{h_1,h_2} = G''(u)[h_1, h_2]$ is the solution of

$$\begin{aligned} Ay_{h_1,h_2} + d_y(x,y)y_{h_1,h_2} &= -d_{yy}(x,y)y_{h_1}y_{h_2} \quad \text{in } \Omega \\ \partial_{n_A} y_{h_1,h_2} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.4}$$

where $y_{h_i} = G'(u)h_i$, $i = 1, 2$.

For later use, let us also state the following regularity result:

Theorem 2.5 *Let Ω be a bounded domain with $C^{1,1}$-boundary and $1 < q < \infty$. Then for every $u \in L^q(\Omega)$ the weak solution of* (2.1) *belongs to $W^{2,q}(\Omega)$.*

Proof. We note that $y(u) \in L^\infty(\Omega)$ due to Theorem 2.1, which implies $d(y) \in L^\infty(\Omega)$. Hence, the result is obtained by applying the regularity results from [9] to the linear equation

$$\begin{aligned} Ay &= u - d(x,y) \qquad \text{in } \Omega \\ \partial_{n_A} y &= 0 \qquad \text{on } \Gamma \end{aligned}$$

$\square$

2.2 First order necessary optimality conditions

In order to formulate first order optimality conditions, we have to state an additional assumption.

Assumption 2.6 *We assume that $\bar{u}$ satisfies the linearized Slater condition for* (P)*, i.e. there exists a control $\hat{u} \in L^2(\Omega)$ with $u_a \le \hat{u} \le u_b$, a.e. in Ω, such that*

$$(G\bar{u})(x) + G'(\bar{u})(\hat{u} - \bar{u})(x) \ge y_c(x) + \gamma \quad \forall x \in \bar{\Omega}$$

for some fixed $\gamma > 0$.

Based on the linearized Slater condition, first order necessary optimality conditions for problem (P) can be established, which include the existence of a regular Borel measure as a Lagrange multiplier with respect to the state constraints. In order to formulate the first order necessary optimality conditions, we will use the classical Lagrange approach:

$$\min_{u \in U_{ad}} \mathcal{L}(u, \mu) = f(u) + \int_{\bar{\Omega}} (y_c - Gu)\, d\mu. \tag{2.5}$$

Adapting the theory of Casas in [4] and straightforward computation yields the following result.

Theorem 2.7 *Suppose that Assumption 2.6 is fullfilled. Moreover, let $\bar{u}$ be a solution of problem* (P) *and $\bar{y} = G\bar{u}$ the associated state. Then, a regular Borel measure $\bar{\mu} \in \mathcal{M}(\bar{\Omega})$ and an adjoint state $\bar{p} \in W^{1,s}(\Omega)$, $s < d/(d-1)$ exist, such that the following optimality system is satisfied:*

$$\begin{aligned} A\bar{y} + d(x, \bar{y}) &= \bar{u} & \qquad A^*\bar{p} + d_y(x, \bar{y})\bar{p} &= \bar{y} - y_d - \bar{\mu}_\Omega \\ \partial_{n_A}\bar{y} &= 0 & \partial_{n_{A^*}}\bar{p} &= -\bar{\mu}_\Gamma \end{aligned} \tag{2.6}$$

$$(\bar{p} + \nu\bar{u}\,,\, u - \bar{u}) \ge 0, \quad \forall u \in U_{ad} \tag{2.7}$$

$$\begin{gathered} \int_{\bar{\Omega}} (y_c - \bar{y})d\bar{\mu} = 0, \quad \bar{y}(x) \ge y_c(x) \quad \text{for all } x \in \bar{\Omega} \\ \int_{\bar{\Omega}} \varphi d\bar{\mu} \ge 0 \quad \forall \varphi \in C(\bar{\Omega})^+, \end{gathered} \tag{2.8}$$

where $C(\bar{\Omega})^+$ is defined by $C(\bar{\Omega})^+ := \{y \in C(\bar{\Omega}) \,|\, y(x) \ge 0 \,\forall\, x \in \bar{\Omega}\}$.

Here and in the following, A^* denotes the dual operator to the differential operator A. Note that every element $\mu \in \mathcal{M}(\bar{\Omega})$ can be decomposed as a sum of two measures $\mu = \mu_\Omega + \mu_\Gamma$, where the addends are regular Borel measures in $\bar{\Omega}$, concentrated in

Ω and Γ, respectively.

Before finishing this section, we provide the second derivative of the Lagrangian given in (2.5). By straightforward computation we obtain

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)[h_1,h_2] = f''(u)[h_1,h_2] - \int_{\bar{\Omega}} G''(u)[h_1,h_2]\, d\mu.$$

By the use of the adjoint state introduced in Theorem 2.7, the following formulation of the second derivative is well known.

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)[h_1,h_2] = \int_{\Omega} (y_{h_1} y_{h_2} + \nu h_1 h_2 - p d_{yy}(x,y(u)) y_{h_1} y_{h_2})\, dx, \tag{2.9}$$

with $y = Gu$, $y_{h_i} = G'(u)h_i$, $i = 1,2$ and p is the solution of

$$\begin{aligned} A^* p + d_y(x,y)p &= y - y_d - \mu && \text{in } \Omega \\ \partial_{n_{A^*}} p &= 0 && \text{on } \Gamma. \end{aligned}$$

We proceed with the formulation of the second order sufficient optimality conditions, that guarantees $\bar{u}$ to be a local minimum of problem (P).

Assumption 2.8 *Let $\bar{u} \in U_{ad}$ be a control satisfying the first order necessary optimality conditions* (2.6)-(2.8)*. We assume that there exists a constant $\alpha > 0$, such that*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\mu})h^2 \geq \alpha \|h\|^2$$

is valid for all $h \in L^2(\Omega)$.

We will note that the previous assumption is rather strong. For weaker assumptions we will refer to e.g. [5] and [19]. It is well known that the coercivity condition of Assumption 2.8 yields the quadratic growth condition for problem (P), cf. [21].

Proposition 2.9 *Let the Assumption 1.1 be fulfilled and let $\bar{u} \in U_{ad}$ be a control satisfying the first order necessary optimality conditions* (2.6)-(2.8)*. Additionally, $\bar{u}$ fulfills Assumption 2.8. Then there exist constants $\beta > 0$ and $\delta > 0$ such that*

$$f(u) \geq f(\bar{u}) + \beta \|u - \bar{u}\|^2 \tag{2.10}$$

for all feasible $u \in L^2(\Omega)$ with $\|u - \bar{u}\| \leq \delta$. Consequently, $\bar{u}$ is a locally optimal control of problem (P)*.*

Due to our Assumption 1.1 and the local Lipschitz-continuity of the second derivative of the Lagrangian (2.5) with respect to the control u, the coercivity condition of Assumption 2.8 is carried over to controls in the L^2-vicinity of $\bar{u}$.

Lemma 2.10 *Let Assumption 2.8 be fulfilled. There exist constants $\alpha' > 0$ and $\delta > 0$ such that*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\tilde{u}, \bar{\mu})h^2 \geq \alpha' \|h\|^2$$

is valid for all $h \in L^2(\Omega)$, provided that $\|\tilde{u} - \bar{u}\| \leq \delta$.

Proof. Under the general Assumption 1.1, one can easily verify that the second derivative of the Lagrangian 2.5 is locally Lipschitz continuous with respect to u. i.e. there exists a positive constant C_L such that the estimate

$$\left| \left(\frac{\partial^2 \mathcal{L}}{\partial u^2}(u_1, \bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(u_2, \bar{\mu}) \right) h^2 \right| \leq C_L \|u_1 - u_2\| \|h\|^2$$

is valid for $\|u_1 - u_2\| \leq \delta$ and $\delta > 0$ sufficiently small, see for instance [21, Lemma 4.24]. By means of the Lipschitz property and Assumption 2.8, we further conclude

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\tilde{u}, \bar{\mu})h^2 &= \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})h^2 + \left(\frac{\partial^2 \mathcal{L}}{\partial u^2}(\tilde{u}, \bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) \right) h^2 \\ &\geq \alpha \|h\|^2 - C_L \|\tilde{u} - \bar{u}\| \|h\|^2 \\ &\geq (\alpha - C_L \delta) \|h\|^2 := \alpha' \|h\|^2 \end{aligned}$$

provided that $\|\tilde{u} - \bar{u}\| \leq \delta$. For sufficiently small $\delta > 0$, we obtain $\alpha' > 0$, which completes the proof. ∎

3 Analysis of Problem (P_ε)

Throughout the following, we assume that the feasible set for the unregularized problem (P) is nonempty. Furthermore, we denote by $\bar{u}$ a local optimal solution of (P) in the sense of Definition 2.2 satsifying the first order optimality conditions of Theorem 2.7 as well as the second order sufficient optimality condition from Assumption 2.8. Moreover, let the linearized Slater condition of Assumption 2.6 be fulfilled by $\bar{u}$. In addition, we require:

Assumption 3.1 *We assume that the adjoint state $\bar{p}$ and the Lagrange multiplier $\bar{\mu}$ associated with $\bar{u}$ are unique.*

Remark 3.2 *Let us mention here that uniqueness of the Lagrange multipliers and adjoint states is a typical assumption in PDE control. It can for example be expected in cases where the active sets of the constraints are well separated. We mention [1], where uniqueness has been shown for an elliptic control problem subject to pointwise control and mixed control-state constraints under this assumption.*

3.1 The state equation

Let us now consider the state equation

$$\begin{aligned} Ay + d(x,y) &= u + \phi(\varepsilon)v \quad \text{in } \Omega \\ \partial_{n_A} y &= 0 \quad \text{on } \Gamma \end{aligned} \tag{3.1}$$

of problem (P_ε). Due to Theorem 2.1, the semilinear elliptic boundary value problem (3.1) admits for every pair $(u,v) \in L^2(\Omega)^2$ a unique solution $y \in H^1(\Omega) \cap C(\bar{\Omega})$. For convenience, we will use the same solution operator $G : L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega})$ as for the state equation of the original problem (P). Thus, we introduce the operator

$$T : L^2(\Omega)^2 \to L^2(\Omega) : \; (u,v) \mapsto w, \quad w = u + \phi(\varepsilon)v, \tag{3.2}$$

i.e. every pair $(u,v) \in L^2(\Omega)^2$ is assigned to the function $w := u + \phi(\varepsilon)v \in L^2(\Omega)$. One can easily see, that the operator T is linear and continuous. Using the solution operator defined in (2.2), the weak solution of (3.1) is given by

$$y = GT[u,v].$$

The differentiability of the control-to-state mapping directly results from Theorem 2.4 and the continuity of the linear operator T. We introduce the following denotation: for all (u,v), $(h_u,h_v) \in L^2(\Omega)^2$

$$y_h = G'(T[u,v])T[h_u,h_v] \tag{3.3}$$

is defined as the solution of

$$\begin{aligned} Ay_h + d_y(x,y)y_h &= h_u + \phi(\varepsilon)h_v \quad \text{in } \Omega \\ \partial_{n_A} y_h &= 0 \quad \text{on } \Gamma \end{aligned} \tag{3.4}$$

with $y = GT[u,v]$.
The reduced formulation of problem (P_ε) is given by

$$\begin{gathered} \min f_\varepsilon(u,v) := J(GT[u,v],u,v) = \frac{1}{2}\|GT[u,v] - y_d\|^2 + \frac{\nu}{2}\|u\|^2 + \frac{\psi(\varepsilon)}{2}\|v\|^2 \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ GT[u,v] \ge y_c - \xi(\varepsilon)v \quad \text{a.e. in } \Omega. \end{gathered}$$

Lemma 3.3 *The feasible set of* (P_ε) *is nonempty.*

Proof. We know that $\bar{u}$ is feasible for (P) and therefore conclude

$$\xi(\varepsilon)0 + \bar{y} = \bar{y} \ge y_c \quad \text{a.e. in } \Omega$$

for all $\varepsilon > 0$, which implies feasibility of $(\bar{u}, 0)$ for (P_ε). ☐
By means of the previous lemma, the existence of at least one optimal solution of Problem (P_ε) can be obtained by standard arguments, since the cost functional is convex with respect to the controls u and v, respectively.

3.2 Convergence analysis

In this section we prove a convergence result for the solution of the regularized problem (P_ε) towards the solution of the unregularized Problem (P). Due to the nonlinearity of the state equation, for neither one of the problems uniqueness of the optimal solution can be expected. Therefore it is necessary to consider solutions that are associated with each other. We follow an idea from [6] and consider an auxiliary problem (P_ε^r). Let $\bar{u}$ be a local solution of (P) satisfying the first order optimality conditions from Theorem 2.7 and the second order optimality conditions from Assumption 2.8. We define (P_ε^r) as

$$\left.\begin{array}{c} \min \quad J_\varepsilon(y,u,v) := \dfrac{1}{2}\|y-y_d\|^2 + \dfrac{\nu}{2}\|u\|^2 + \dfrac{\psi(\varepsilon)}{2}\|v\|^2 \\ Ay + d(x,y) = u + \phi(\varepsilon)v \quad \text{in } \Omega \\ \partial_{n_A} y = 0 \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\ y(x) \ge y_c(x) - \xi(\varepsilon)v \quad \text{a.e. in } \Omega, \\ \|u-\bar{u}\| \le r, \quad \|v\| \le r \end{array}\right\} \qquad (\mathrm{P}_\varepsilon^r)$$

where r is small enough, such that the quadratic growth condition (2.10) is satisfied. We will consider a sequence of positive real numbers ε_n converging to zero as $n \to \infty$, and we will prove that the sequence of global solutions of (P_ε^r) associated with ε_n converges in some sense to the solution of the unregularized problem $\bar{u}$. We will also provide an estimate for the regularization error. Moreover, we will show that a global solution of (P_ε^r) is a local solution of (P_ε), which completes our analysis. This procedure is meanwhile standard technique also in the context of regularization of optimal control problems, applied for example in [23] to Lavrentiev-regularized elliptic problems. The main task of this section is to combine the techniques for nonlinear problems with the analysis for the virtual control regularization of linear-quadratic problems already at hand, cf. [8] and also [14].

First, however, let us note that (P_ε^r) admits at least one optimal control $(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r)$. For convenience, let us define the set of admissible controls for (P_ε^r)

$$U_{ad}^r := \{u \in U_{ad} \,:\, \|u-\bar{u}\| \le r\}$$

as well as an additional auxiliary set

$$V_{ad}^r := \{v \in L^2(\Omega) \,:\, \|v\| \le r\}.$$

We say that the pair $(u_\varepsilon, v_\varepsilon) \in U_{ad}^r \times V_{ad}^r$ is feasible for (P_ε^r) if the associated state $y_\varepsilon = G(T[u_\varepsilon, v_\varepsilon])$ satisfies the mixed control-state constraints $y_\varepsilon \ge y_c - \xi(\varepsilon)v_\varepsilon$ a.e. in Ω. Following the proof of Lemma 3.3, one can show that $(\bar{u}, 0)$ is feasible for (P_ε^r). Thus, we state:

Corollary 3.4 *Under Assumption 1.1, Problem* (P_ε^r) *admits at least one optimal solution* $(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r) \in U_{ad}^r \times V_{ad}^r$.

The proof follows by standard arguments.

To derive first order necessary optimality conditions we will again use the classical Lagrange approach. Hence, we consider

$$\min_{(u,v)\in U^r_{ad}\times V^r_{ad}} \mathcal{L}_\varepsilon(u,v,\mu) = f_\varepsilon(u,v) + \int_\Omega (y_c - GT[u,v] - \xi(\varepsilon)v)\mu\, dx. \tag{3.5}$$

Similar to problem (P), a regularity condition is necessary to ensure the existence of Lagrange multipliers. The following lemma shows that the linearized Slater condition of Assumption 2.6 can be carried over to feasible controls of problem (P^r_ε) provided that these controls are sufficiently close to the optimal control $\bar u$ of problem (P).

Lemma 3.5 *Let* $(u^r_\varepsilon, v^r_\varepsilon)$ *be a feasible control for* (P^r_ε)*. If* $r>0$ *is sufficiently small, then* $(u^r_\varepsilon, v^r_\varepsilon)$ *satisfies the linearized Slater condition for* (P^r_ε)*, i.e.*

$$GT[u^r_\varepsilon, v^r_\varepsilon] + G'(T[u^r_\varepsilon, v^r_\varepsilon])T[\hat u^r - u^r_\varepsilon, 0 - v^r_\varepsilon] \geq y_c + \frac{\gamma_r}{2},$$

where $\hat u^r$ *and* γ_r *are defined by*

$$\hat u^r := \bar u + \frac{r}{\max\{r, \|\hat u - \bar u\|\}}(\hat u - \bar u) \quad \text{and} \quad \gamma_r := \frac{r}{\max\{r, \|\hat u - \bar u\|\}}\gamma$$

with $\hat u$ *and* $\gamma > 0$ *from Assumption 2.6.*

Proof. We start with

$$\begin{aligned} GT[u^r_\varepsilon, v^r_\varepsilon] + G'(T[u^r_\varepsilon, v^r_\varepsilon])T[\hat u^r - u^r_\varepsilon, 0 - v^r_\varepsilon] = {} & G\bar u + G'(T[\bar u, 0])T[\hat u^r - \bar u, 0] \\ & + GT(u^r_\varepsilon, v^r_\varepsilon) - G\hat u^r + G'(T[u^r_\varepsilon, v^r_\varepsilon])T[\hat u^r - u^r_\varepsilon, -v^r_\varepsilon] \\ & + G\hat u^r - G\bar u + G'(T[\bar u, 0]) + G'(T[\bar u, 0])T[\bar u - \hat u^r, 0] \end{aligned}$$

Due to the definition of the operator T, we find

$$G'(T[\bar u, 0])T[\hat u^r - \bar u, 0] = G'(\bar u)(\hat u^r - \bar u).$$

We consider

$$\hat u_r := \bar u + \frac{r}{\max\{r, \|\hat u - \bar u\|\}}(\hat u - \bar u).$$

It can easily be verified, that $\hat u_r$ belongs to U^r_{ad} by construction. Straightforward computation and Assumption 2.6 imply

$$G(\bar u)(x) + G'(\bar u)(\hat u_r - \bar u)(x) \geq y_c(x) + \gamma_r$$

with $\gamma_r := \frac{r}{\max\{r, \|\hat u - \bar u\|\}}\gamma$. Hence, we derive

$$\begin{aligned} GT[u^r_\varepsilon, v^r_\varepsilon] + G'(T[u^r_\varepsilon, v^r_\varepsilon])T[\hat u^r - u^r_\varepsilon, 0 - v^r_\varepsilon] \geq {} & y_c + \gamma_r \\ & - \frac{1}{2}G''(\tilde u_1)T[\hat u^r - u^r_\varepsilon, -v^r_\varepsilon]^2 + \frac{1}{2}G''(\tilde u_2)(\hat u^r - \bar u)^2 \end{aligned}$$

with suitable controls $\tilde{u}_1$ and $\tilde{u}_2$. Using the feasibility of $(u^r_\varepsilon, v^r_\varepsilon)$ for $(\mathrm{P}^r_\varepsilon)$, $\|\hat{u}^r - \bar{u}\| \leq r$ as well as the boundedness of G'' in $C(\bar{\Omega})$, cf. Theorem 2.4, one obtains

$$GT[u^r_\varepsilon, v^r_\varepsilon] + G'(T[u^r_\varepsilon, v^r_\varepsilon])T[\hat{u}^r - u^r_\varepsilon, 0 - v^r_\varepsilon] \geq y_c + \gamma_r - Cr^2.$$

If $r > 0$ is chosen sufficiently small such that $\gamma_r - Cr^2 \geq \frac{\gamma_r}{2}$, we end up with the assertion

$$GT[u^r_\varepsilon, v^r_\varepsilon] + G'(T[u^r_\varepsilon, v^r_\varepsilon])T[\hat{u}^r - u^r_\varepsilon, 0 - v^r_\varepsilon] \geq y_c + \frac{\gamma_r}{2}.$$

□

We point out that the Lagrange multipliers are regular functions, see e.g. [2], [20], or [22]. By applying the analysis of [20], one obtains the following first order necessary optimality conditions for $(\mathrm{P}^r_\varepsilon)$:

Proposition 3.6 *Let $(\bar{u}^r_\varepsilon, \bar{v}^r_\varepsilon)$ be an optimal solution of $(\mathrm{P}^r_\varepsilon)$ and $\bar{y}^r_\varepsilon = GT[\bar{u}^r_\varepsilon, \bar{v}^r_\varepsilon]$ the associated state. Then, there exist a unique adjoint state $\bar{p}^r_\varepsilon \in H^1(\Omega) \cap C(\bar{\Omega})$ and a unique Lagrange multiplier $\bar{\mu}^r_\varepsilon \in L^2(\Omega)$ so that the following optimality system is satisfied*

$$\begin{aligned} A\bar{y}^r_\varepsilon + d(x, \bar{y}^r_\varepsilon) &= \bar{u}^r_\varepsilon + \phi(\varepsilon)\bar{v}^r_\varepsilon & \qquad A^*\bar{p}^r_\varepsilon + d_y(x, \bar{y}^r_\varepsilon)\bar{p}^r_\varepsilon &= \bar{y}^r_\varepsilon - y_d - \bar{\mu}^r_\varepsilon \\ \partial_{n_A}\bar{y}^r_\varepsilon &= 0 & \partial_{n_{A^*}}\bar{p}^r_\varepsilon &= 0 \end{aligned} \tag{3.6}$$

$$(\bar{p}^r_\varepsilon + \nu\bar{u}^r_\varepsilon\, ,\, u - \bar{u}^r_\varepsilon) \geq 0, \quad \forall u \in U^r_{ad} \tag{3.7}$$

$$(\phi(\varepsilon)\bar{p}^r_\varepsilon + \psi(\varepsilon)\bar{v}^r_\varepsilon - \xi(\varepsilon)\bar{\mu}^r_\varepsilon\, ,\, v - \bar{v}^r_\varepsilon) \geq 0, \quad \forall v \in V^r_{ad} \tag{3.8}$$

$$(\bar{\mu}^r_\varepsilon\, ,\, y_c - \bar{y}^r_\varepsilon - \xi(\varepsilon)\bar{v}^r_\varepsilon) = 0, \quad \bar{\mu}^r_\varepsilon \geq 0, \quad \bar{y}^r_\varepsilon \geq y_c - \xi(\varepsilon)\bar{v}^r_\varepsilon \quad \textit{a.e. in } \Omega. \tag{3.9}$$

3.2.1 Construction of feasible controls

In this section we construct feasible controls for (P) and $(\mathrm{P}^r_\varepsilon)$ to be used for the convergence analysis. We begin by some preliminary results. We define first a violation function, which measures the possible violation of the pure state constraints by a regularized solution. The maximal violation of the constraints by a given function u can be expressed as

$$d[u, (\mathrm{P})] = \|(y_c - G(u))_+\|_\infty.$$

In order to estimate this violation later on, let us mention a helpful regularity result:

Lemma 3.7 *Let $f \in C^{0,\kappa}(\Omega)$ for some $0 < \kappa \leq 1$ be given. Then the estimate*

$$\|f\|_\infty \leq c\|f\|^{\frac{\kappa}{\frac{d}{2}+\kappa}}$$

is satisfied.

For a proof, we refer to [16].

We now construct an auxiliary sequence of controls feasible for (P) and close to the optimal control of (P_ε^r). For that purpose, let us first show an estimate for the maximal constraint violation

$$d[\bar{u}_\varepsilon^r, (\mathrm{P})] = \|(y_c - G\bar{u}_\varepsilon^r)_+\|_\infty$$

of the control $\bar{u}_\varepsilon^r$ w.r.t. problem (P).

Lemma 3.8 *The maximal violation $d[\bar{u}_\varepsilon^r, (\mathrm{P})]$ of the control $\bar{u}_\varepsilon^r$ w.r.t. problem* (P) *can be estimated by*

$$d[\bar{u}_\varepsilon^r, (\mathrm{P})] \le c(\phi(\varepsilon) + \xi(\varepsilon))^{2/(2+d)} \|\bar{v}_\varepsilon^r\|^{2/(2+d)}.$$

Proof. The proof is similar to the one in [7] adapted to the nonlinear case. By $\bar{u}_\varepsilon^r \in L^\infty(\Omega)$ and Theorem 2.5 we obtain

$$d[\bar{u}_\varepsilon^r, (\mathrm{P})] \le c\|(y_c - G\bar{u}_\varepsilon^r)_+\|^{2/(2+d)},$$

where we used the embedding

$$W^{2,q}(\Omega) \hookrightarrow C^{0,1}(\Omega) \quad \text{for } q > d$$

and Lemma 3.7. Making use of the Fréchet differentiability of the control-to-state operator from Theorem 2.4, we obtain with $u = \bar{u}_\varepsilon^r - t\phi(\varepsilon)\bar{v}_\varepsilon^r$ for some $0 < t < 1$

$$\begin{aligned} d[\bar{u}_\varepsilon^r, (\mathrm{P})] &\le c(\|(y_c - G(\bar{u}_\varepsilon^r + \phi(\varepsilon)\bar{v}_\varepsilon^r) + G'(u)\phi(\varepsilon)v_\varepsilon^r)_+\|^{2/(2+d)} \\ &\le c(\|\xi(\varepsilon)\bar{v}_\varepsilon^r\| + \|G'(u)\phi(\varepsilon)v_\varepsilon^r\|)^{2/(2+d)} \\ &\le c(\phi(\varepsilon) + \xi(\varepsilon))^{2/(2+d)}\|\bar{v}_\varepsilon^r\|^{2/(2+d)}. \end{aligned}$$

□

Lemma 3.9 *Let the Assumptions of Lemma 3.5 be satisfied. Then, for every $\varepsilon > 0$ there exists a $\delta_\varepsilon \in (0,1)$, such that $u_\delta^r := (1-\delta)\bar{u}_\varepsilon^r + \delta\hat{u}^r$ is feasible for* (P) *for all $\delta \in [\delta_\varepsilon, 1]$.*

Proof. Consider

$$\begin{aligned} y(u_\delta^r) - y_c &= G(\bar{u}_\varepsilon^r + \delta(\hat{u}^r - \bar{u}_\varepsilon^r)) - y_c \\ &= (1-\delta)(G(\bar{u}_\varepsilon^r) - y_c) + \delta(G(\bar{u}_\varepsilon^r) - y_c + G'(\bar{u}_\varepsilon^r)(\hat{u}^r - \bar{u}_\varepsilon^r)) + \frac{\delta^2}{2}G''(\tilde{u})(\hat{u}^r - \bar{u}_\varepsilon^r)^2 \\ &\ge -(1-\delta)d[\bar{u}_\varepsilon^r, (P)] + \delta(G(\bar{u}_\varepsilon^r) - y_c + G'(\bar{u}_\varepsilon^r)(\hat{u}^r - \bar{u}_\varepsilon^r)) + \frac{\delta^2}{2}G''(\tilde{u})(\hat{u}^r - \bar{u}_\varepsilon^r)^2 \end{aligned}$$

with $\tilde{u} = \bar{u}_\varepsilon^r + t(\hat{u}^r - \bar{u}_\varepsilon^r)$ for a $t \in (0,1)$ and the maximal violation of $\bar{u}_\varepsilon^r$ w.r.t. problem (P). Due to Lemma 3.5, $\|\hat{u}^r - \bar{u}_\varepsilon^r\| \le 2r$ and the boundedness of G'' in $C(\bar{\Omega})$, cf. Theorem 2.4, one derives

$$y(u_\delta^r) - y_c \ge -(1-\delta)d[\bar{u}_\varepsilon^r, (P)] + \delta(\frac{\gamma_r}{2} - Cr^2).$$

Take now r small enough, such that $\frac{\gamma_r}{2} - Cr^2 \geq \frac{\gamma_r}{4}$. Setting $-(1-\delta_\varepsilon)d[\bar{u}^r_\varepsilon, (P)] + \delta_\varepsilon \frac{\gamma_r}{4} = 0$ leads to

$$\delta_\varepsilon = \frac{d[\bar{u}^r_\varepsilon, (P)]}{d[\bar{u}^r_\varepsilon, (P)] + \frac{\gamma_r}{4}} \tag{3.10}$$

and hence to

$$y(u^r_\delta) - y_c \geq 0$$

for all $\delta \geq \delta_\varepsilon$. ∎

We mention that the choice of δ_ε in the previous Lemma depends on the radius r of the auxiliary problem $(\mathrm{P}^r_\varepsilon)$, see γ_r in (3.10). In addition, we already know from the proof of Lemma 3.3 that $(\bar{u}, 0)$ is feasible for $(\mathrm{P}^r_\varepsilon)$.

3.2.2 Convergence result and error estimate

In this section, we develop the error estimates of the optimal regularized controls with the help of the feasibility results of the last section. Let us begin with an auxiliary convergence result for the controls of the auxiliary problem $(\mathrm{P}^r_\varepsilon)$.

We apply the following assumption:

Assumption 3.10 *For sufficiently small $\varepsilon > 0$, we assume that*

$$\frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} < 1$$

Lemma 3.11 *Let $\bar{u}$ be a locally optimal control of* (P) *satisfying the quadratic growth condition* (2.10) *and Assumption 2.6. Moreover, consider a fixed $r > 0$ sufficiently small. If $(\bar{u}^r_\varepsilon, \bar{v}^r_\varepsilon)$ is a (globally) optimal control for* $(\mathrm{P}^r_\varepsilon)$*, then it fulfills*

$$\beta\|\bar{u} - \bar{u}^r_\varepsilon\|^2 + \frac{\psi(\varepsilon)}{4}\|\bar{v}^r_\varepsilon\|^2 \leq c(\phi(\varepsilon) + \xi(\varepsilon))^{\frac{2}{(2+d)}}\|\bar{v}^r_\varepsilon\|^{\frac{2}{(2+d)}} + c\frac{(\phi(\varepsilon))^2}{\psi(\varepsilon)}.$$

Proof. We apply the quadratic growth condition (2.10) and obtain

$$f(u) \geq f(\bar{u}) + \alpha\|u - \bar{u}\|^2 \quad \forall\, u \in B_r(\bar{u}),$$

for r sufficiently small and u feasible for (P). Choosing $u = u^r_\delta$ with $\delta = \delta_\varepsilon$ as defined in Lemma 3.9 and (3.10), we obtain

$$\begin{aligned} f(u^r_\delta) &\geq f(\bar{u}) + \beta\|u^r_{\delta_\varepsilon} - \bar{u}\|^2 \\ &\geq f(\bar{u}) + \beta\|u^r_{\delta_\varepsilon} - \bar{u}^r_\varepsilon + \bar{u}^r_\varepsilon - \bar{u}\|^2 \\ &\geq f(\bar{u}) + \beta(\|\bar{u}^r_\varepsilon - \bar{u}\|^2 - 2|(\bar{u}^r_\varepsilon - \bar{u}, u^r_{\delta_\varepsilon} - \bar{u}^r_\varepsilon)| + \|u^r_{\delta_\varepsilon} - \bar{u}^r_\varepsilon\|^2) \\ &\geq f(\bar{u}) + \beta\|\bar{u}^r_\varepsilon - \bar{u}\|^2 - c\|\bar{u}^r_\varepsilon - u^r_{\delta_\varepsilon}\| \end{aligned}$$

by the boundedness of $\bar{u}_\varepsilon^r$ and $\bar{u}$ due to the control constraints. Noting that

$$\delta_\varepsilon = \frac{d[\bar{u}_\varepsilon^r, (P)]}{d[\bar{u}_\varepsilon^r, (P)] + \frac{\gamma_r}{4}} \leq 4\frac{d[\bar{u}_\varepsilon^r, (P)]}{\gamma_r}.$$

The definition of γ_r, see Lemma 3.5, and $\|\hat{u}^r - \bar{u}_\varepsilon^r\| \leq 2r$ imply

$$\|\bar{u}_\varepsilon^r - \bar{u}_\delta^r\| = \|\delta_\varepsilon(\hat{u}^r - \bar{u}_\varepsilon^r)\| \leq c\frac{rd[\bar{u}_\varepsilon^r, (\mathrm{P})]}{\gamma_r} \leq cd[\bar{u}_\varepsilon^r, (\mathrm{P})]$$

due to the control constraints with a positive constant c independent of ε and r. We proceed with

$$\begin{aligned} f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r) &= f(u_\delta^r) - (f(u_\delta^r) - f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r)) \\ &\geq f(\bar{u}) + \beta\|\bar{u}_\varepsilon^r - \bar{u}\|^2 - cd[\bar{u}_\varepsilon^r, (\mathrm{P})] - (f(u_\delta^r) - f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r)) \\ &= f_\varepsilon(\bar{u}, 0) + \beta\|\bar{u}_\varepsilon^r - \bar{u}\|^2 - cd[\bar{u}_\varepsilon^r, (\mathrm{P})] - (f_\varepsilon(u_\delta^r, 0) - f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r)), \end{aligned}$$

which yields

$$\beta\|\bar{u}_\varepsilon^r - \bar{u}\|^2 \leq f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r) - f_\varepsilon(\bar{u}, 0) + f_\varepsilon(u_\delta^r, 0) - f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r) + cd[\bar{u}_\varepsilon^r, (\mathrm{P})].$$

Noting that $(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r)$ is optimal for $(\mathrm{P}_\varepsilon^r)$, and $(\bar{u}, 0)$ is feasible, we obtain

$$\begin{aligned} \beta\|\bar{u}_\varepsilon^r - \bar{u}\|^2 &\leq f_\varepsilon(u_\delta^r, 0) - f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r) + cd[\bar{u}_\varepsilon^r, (\mathrm{P})] \\ &= f_\varepsilon(u_\delta^r, 0) - f_\varepsilon(\bar{u}_\varepsilon^r, 0) + f_\varepsilon(\bar{u}_\varepsilon^r, 0) - f_\varepsilon(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r) + cd[\bar{u}_\varepsilon^r, (\mathrm{P})] \\ &\leq cd[\bar{u}_\varepsilon^r, (\mathrm{P})] + c\|\phi(\varepsilon)v_\varepsilon^r\| - \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon^r\|^2 \end{aligned}$$

by the definition of f_ε and the Lipschitz continuity of the solution operator G and the norm. This yields by Young's inequality and Lemma 3.8

$$\beta\|\bar{u}_\varepsilon^r - \bar{u}\|^2 + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon^r\|^2 \leq c(\phi(\varepsilon) + \xi(\varepsilon))^{2/(2+d)}\|\bar{v}_\varepsilon^r\|^{2/(2+d)} + c\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} + \frac{\psi(\varepsilon)}{4}\|v_\varepsilon^r\|^2,$$

and finally the assertion

$$\beta\|\bar{u}_\varepsilon^r - \bar{u}\|^2 + \frac{\psi(\varepsilon)}{4}\|\bar{v}_\varepsilon^r\|^2 \leq c(\phi(\varepsilon) + \xi(\varepsilon))^{2/(2+d)}\|\bar{v}_\varepsilon^r\|^{2/(2+d)} + c\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}.$$

$\square$

Corollary 3.12 *Let the assumptions of Lemma 3.11 as well as Assumption 3.10 be fulfilled. Then for sufficiently small $\varepsilon > 0$ the estimate*

$$\|\bar{v}_\varepsilon^r\| \leq c\frac{1}{\sqrt{\psi(\varepsilon)}}\left(\frac{\xi(\varepsilon) + \phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}\right)^{\frac{1}{1+d}}$$

holds for some positive constant c.

For a proof, we refer to [16]. We immediately obtain an error estimate for $\bar{u}_\varepsilon^r$. Furthermore, the continuity of the solution operator G implies an analogous estimate for the respective state $\bar{y}_\varepsilon^r$.

Corollary 3.13 *Let the assumptions of Lemma 3.11 as well as Assumption 3.10 be fulfilled. Then for sufficiently small $\varepsilon > 0$ the estimate*

$$\|\bar{u} - \bar{u}_\varepsilon^r\| + \|\bar{y} - \bar{y}_\varepsilon^r\| \le c \left(\frac{\xi(\varepsilon) + \phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{1}{d+1}}$$

holds for some positive constant c.

The estimate in the previous corollary delivers conditions for the parameter functions ensuring our convergence result.

Assumption 3.14 *Let ε be a sequence of positive real numbers. We assume that*

$$\lim_{\varepsilon \to 0} \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} = 0, \quad \lim_{\varepsilon \to 0} \frac{\xi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} = 0.$$

Lemma 3.15 *Let the assumptions of Lemma 3.11 as well as Assumption 3.14 be satisfied. There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ the auxiliary solution $(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r)$ is a local solution to* (P_ε).

Proof. From Corollaries 3.12 and 3.13 we conclude the existence of an $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$ we have $\|\bar{u} - \bar{u}_\varepsilon^r\| < r$ and $\|\bar{v}_\varepsilon^r\| < r$, i.e. $\bar{u}_\varepsilon^r$ and $\bar{v}_\varepsilon^r$ are in the interior of the closed ball with radius r around $\bar{u}$ and 0, respectively. This directly implies that $(\bar{u}_\varepsilon^r, \bar{v}_\varepsilon^r)$ is a local solution of (P_ε). ∎

Now, we can formulate our main result.

Theorem 3.16 *Let the assumptions of Lemma 3.11 as well as Assumption 3.14 be satisfied. Moreover, let $\bar{u}$ be a local solution of Problem P, and $\{\varepsilon_n\}$ be an arbitrary sequence of positive real numbers converging to zero. There exists a sequence $\{(\bar{u}_{\varepsilon_n}, \bar{v}_{\varepsilon_n})\}$ of local solutions to Problem (P_{ε_n}), such that $\bar{u}_{\varepsilon_n}$ converges strongly in $L^2(\Omega)$ to $\bar{u}$. Moreover, the following error estimate holds:*

$$\|\bar{u} - \bar{u}_{\varepsilon_n}\| + \|\bar{y} - \bar{y}_{\varepsilon_n}\| \le c \left(\frac{\xi(\varepsilon_n) + \phi(\varepsilon_n)}{\sqrt{\psi(\varepsilon_n)}} \right)^{\frac{1}{d+1}}.$$

Proof. Under Assumption 3.14, this is a direct consequence of Corollaries 3.12 and 3.13 as well as Lemma 3.15. ∎

For completeness, we state the first order necessary optimality conditions for a local solution $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ of (P_ε) for a fixed parameter $\varepsilon > 0$. This is follows directly from Theorem 3.6 and the fact that the restriction to the closed ball with radius r is inactive.

Proposition 3.17 *Let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be an optimal solution of* (P_ε) *and let $\bar{y}_\varepsilon = GT[\bar{u}_\varepsilon, \bar{v}_\varepsilon]$ be the associated state. Then, there exist a unique adjoint state $\bar{p}_\varepsilon \in H^1(\Omega) \cap C(\bar{\Omega})$ and a unique Lagrange multiplier $\bar{\mu}_\varepsilon \in L^2(\Omega)$ so that the following optimality system is satisfied*

$$\begin{aligned} A\bar{y}_\varepsilon + d(x, \bar{y}_\varepsilon) &= \bar{u}_\varepsilon + \phi(\varepsilon)\bar{v}_\varepsilon \qquad & A^*\bar{p}_\varepsilon + d_y(x, \bar{y}_\varepsilon)\bar{p}_\varepsilon &= \bar{y}_\varepsilon - y_d - \bar{\mu}_\varepsilon \\ \partial_{n_A}\bar{y}_\varepsilon &= 0 & \partial_{n_{A^*}}\bar{p}_\varepsilon &= 0 \end{aligned} \tag{3.11}$$

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon\, ,\, u - \bar{u}_\varepsilon) \geq 0, \quad \forall u \in U_{ad} \tag{3.12}$$

$$\phi(\varepsilon)\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\bar{\mu}_\varepsilon = 0, \quad a.e.\ in\ \Omega \tag{3.13}$$

$$(\bar{\mu}_\varepsilon\, ,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon) = 0, \quad \bar{\mu}_\varepsilon \geq 0, \quad \bar{y}_\varepsilon \geq y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad a.e.\ in\ \Omega. \tag{3.14}$$

Remark 3.18 *By means of* (3.13)*, one derives $\bar{\mu}_\varepsilon = \frac{\phi(\varepsilon)}{\xi(\varepsilon)}\bar{p}_\varepsilon + \frac{\psi(\varepsilon)}{\xi(\varepsilon)}\bar{v}_\varepsilon$ for a fixed regularization parameter $\varepsilon > 0$. Consequently, the adjoint state $\bar{p}_\varepsilon$ fulfills*

$$A^*\bar{p}_\varepsilon + d_y(\bar{y}_\varepsilon)\bar{p}_\varepsilon + \frac{\phi(\varepsilon)}{\xi(\varepsilon)}\bar{p}_\varepsilon = \bar{y}_\varepsilon - y_d - \frac{\psi(\varepsilon)}{\xi(\varepsilon)}\bar{v}_\varepsilon, \quad in\ \Omega, \quad \partial_{n_{A^*}}\bar{p}_\varepsilon = 0 \quad on\ \Gamma.$$

Note that from the above representation of the Lagrange multiplier and the adjoint state it is easily seen that the dual variables are uniquely determined. Moreover, by well known regularity results for elliptic equations, we obtain

$$\|\bar{p}_\varepsilon\| \leq c \left\| \bar{y}_\varepsilon - y_d - \frac{\psi(\varepsilon)}{\xi(\varepsilon)}\bar{v}_\varepsilon \right\|.$$

Using again (3.13)*, one derives*

$$\|\bar{\mu}_\varepsilon\| \leq c_\varepsilon(\|\bar{y}_\varepsilon - y_d\| + \|\bar{v}_\varepsilon\|),$$

where the constant depends on the regularization parameter ε.

4 Second order sufficient conditions of local solutions for (P_ε)

In this section, we prove that the second order sufficient conditions from Assumption 2.8 for the unregularized problem (P) are robust, i.e. they can be carried over to the regularized problem (P_ε).

4.1 Convergence properties of the dual variables

In the following, let ε_n be an arbitrary sequence of positive real numbers tending to zero for $n \to \infty$. The associated regularized problems are denoted by (P_n) and their local solutions will be referred to as $(\bar{u}_n, \bar{v}_n, \bar{y}_n)$ with respective adjoint state

$\bar{p}_n$ and Lagrange multiplier $\bar{\mu}_n$ with respect to the mixed control-state constraints. According to Theorem 3.16, there is a sequence of local solutions $(\bar{u}_n, \bar{v}_n, \bar{y}_n)$, such that

$$\bar{u}_n \to \bar{u} \quad \text{in } L^2(\Omega) \quad \text{and} \quad \bar{y}_n \to \bar{y} \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty.$$

We will establish corresponding results for the respective dual variables $\bar{\mu}_n$ and $\bar{p}_n$. First, we state the uniform boundedness of the Lagrange multipliers $\bar{\mu}_n$ in some function space.

Lemma 4.1 *The sequence of Lagrange multipliers $\{\bar{\mu}_n\}$ associated to the mixed constraints of (P_n) is uniformly bounded in $L^1(\Omega)$.*

Except the nonlinearity in the state equation, we follow the ideas of [15, Lemma 2.12]. Hence, the proof is done in the appendix of this work.

Lemma 4.2 *There is a subsequence $\{\bar{\mu}_{n_k}\}$ of $\{\bar{\mu}_n\}$ converging weakly-$*$ in $C(\bar{\Omega})^*$ to a weak-$*$ limit $\tilde{\mu} \in C(\bar{\Omega})^*$ as $k \to \infty$, i.e.*

$$\int_\Omega \bar{\mu}_{n_k} \phi dx \to \langle \tilde{\mu} \,, \, \phi \rangle \qquad \forall \phi \in C(\bar{\Omega}) \text{ as } k \to \infty.$$

For a proof, we refer to [12]. Based on this lemma, the convergence of the associated adjoint states is discussed in the following lemma.

Lemma 4.3 *The sequence of adjoint states $\bar{p}_{n_k}$ associated to problem (P_{n_k}) converges strongly in $L^2(\Omega)$ to the limit $\tilde{p} \in L^2(\Omega)$ which is the solution of*

$$\begin{aligned} A^* \tilde{p} + d_y(x, \bar{y}) \tilde{p} &= \bar{y} - y_d - \tilde{\mu}_\Omega \\ \partial_{n_{A^*}} \tilde{p} &= -\tilde{\mu}_\Gamma. \end{aligned}$$

Proof. We will discuss here only the convergence of the part of $\bar{p}_{n_k}$ with respect to the Lagrange multiplier $\bar{\mu}_{n_k}$, since the strong convergence in $L^2(\Omega)$ of the remaining part associated to $\bar{y}_{n_k} - y_d$ is clear. First we state: Due to the embedding $W^{1,s'}(\Omega) \hookrightarrow C(\bar{\Omega})$, $s' > d$ the Lagrange multipliers $\bar{\mu}_{n_k}$ converge even more weakly-$*$ in $W^{1,s'}(\Omega)^*$. However, this is equivalent to the weak convergence of the multipliers in $W^{1,s'}(\Omega)^*$. According to Gröger [10] for $d = 2$ and Zanger [24] for $d = 3$, the solution operators associated to the adjoint equations in (2.6) and (3.11), respectively, are continuous from $W^{1,s'}(\Omega)^*$ to $W^{1,s}(\Omega)$, $s < d/(d-1)$. This fact implies the weak convergence of the adjoint states associated to the $\bar{\mu}_{n_k}$ in $W^{1,s}(\Omega)$. Concluding, the compact embedding of $W^{1,s}(\Omega)$ in $L^2(\Omega)$ completes the proof. $\square$

Next, it is to be shown that the weak-$*$ limit $\tilde{\mu}$ represents a Lagrange multiplier with respect to the pure state constraints of problem (P), and that furthermore, $\tilde{p}$ defined in the previous lemma is an adjoint state associated to problem (P).

Theorem 4.4 *Let Assumption 3.1 be satisfied. Then, the sequence of Lagrange multipliers $\{\bar\mu_n\}$ associated to the regularized pointwise state constraints in (P_n) converge weakly-$*$ in $C(\bar\Omega)^*$ to the Lagrange multiplier $\bar\mu$ with respect to the pure state constraints of problem* (P). *Moreover, the respective adjoint states of problem (P_n) converge strongly in $L^2(\Omega)$ to the adjoint state $\bar p$ of the unregularized problem* (P).

The proof can be done along the lines of [12, Theorem 2.7]. Note, that we had to require the uniqueness of the dual variables in Assumption 3.1, contrary to the problem in [12], where the uniqueness is given by construction.

4.2 Second order sufficient conditions

Before discussing second order sufficient optimality conditions for problem (P_ε), the second derivative of the particular Lagrangian, given in (3.5), is needed. By straightforward computation, we obtain for $h_1 = (h_{u,1}, h_{v,1})$, $h_2 = (h_{u,2}, h_{v,2}) \in L^2(\Omega)^2$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}_\varepsilon}{\partial(u,v)^2}(u,v,\mu)[h_1,h_2] &= f_\varepsilon''(u,v)[h_1,h_2] \\ &\quad - \int_\Omega G''(T[u,v])[T[h_{u,1},h_{v,1}],T[h_{u,2},h_{v,2}]]\mu\,dx\end{aligned}$$

A formulation of the second derivative similarly to (2.9) is given by

$$\begin{aligned}\frac{\partial^2 \mathcal{L}_\varepsilon}{\partial(u,v)^2}(u,v,\mu)[h_1,h_2] &= \int_\Omega y_{h_1}y_{h_2} + \nu h_{u,1}h_{u,2} + \psi(\varepsilon)h_{v,1}h_{v,2}\,dx \\ &\quad - \int_\Omega p d_{yy}(x,y)y_{h_1}y_{h_2}\,dx,\end{aligned}\tag{4.1}$$

with $y = GT[u,v]$, $y_{h_i} = G'(T[u,v])T[h_{u,i},h_{v,i}]$, $i = 1,2$ and p is the solution of

$$\begin{aligned}A^*p + d_y(x,y)p &= y - y_d - \mu \qquad &&\text{in } \Omega\\ \partial_{n_{A^*}}p &= 0 &&\text{on } \Gamma.\end{aligned}$$

Theorem 4.5 *Let Assumption 2.8 be fulfilled and let $\bar u$ be an optimal control of problem* (P). *Furthermore, let $(\bar u_\varepsilon, \bar v_\varepsilon)$ be an optimal control satisfying the first order necessary optimality conditions* (3.11)-(3.14). *Then, there exists a constant $\alpha'' > 0$, such that*

$$\frac{\partial^2 \mathcal{L}_\varepsilon}{\partial(u,v)^2}(\bar u_\varepsilon,\bar v_\varepsilon,\bar\mu_\varepsilon)h^2 \ge \alpha''(\|h_u\|^2 + \|h_v\|^2)\tag{4.2}$$

is valid for all $h = (h_u, h_v) \in L^2(\Omega)^2$, provided that $\varepsilon > 0$ is chosen sufficiently small.

Proof. We start with the introduction of an auxiliary function $\hat{p}$ which is the unique solution of

$$\begin{aligned} A^*\hat{p} + d_y(x, \bar{y}_\varepsilon)\hat{p} &= \bar{y}_\varepsilon - y_d - \bar{\mu} && \text{in } \Omega \\ \partial_{n_{A^*}} p &= 0 && \text{on } \Gamma. \end{aligned}$$

Hence, we find

$$\begin{aligned} \frac{\partial^2 \mathcal{L}_\varepsilon}{\partial(u,v)^2}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)h^2 &= \int_\Omega \nu h_u^2 + \psi(\varepsilon)h_v^2 + (1 - \bar{p}_\varepsilon d_{yy}(x, \bar{y}_\varepsilon))y_h^2\,dx \\ &= \int_\Omega \nu h_u^2 + \psi(\varepsilon)h_v^2 + (1 - \hat{p} d_{yy}(x, \bar{y}_\varepsilon))y_h^2\,dx \\ &\quad + \int_\Omega (\hat{p} - \bar{p}_\varepsilon)d_{yy}(x, \bar{y}_\varepsilon)y_h^2\,dx \\ &= \int_\Omega \nu(T[h_u, h_v])^2 + (1 - \hat{p} d_{yy}(x, \bar{y}_\varepsilon))y_h^2\,dx \\ &\quad + \int_\Omega \nu h_u^2 + \psi(\varepsilon)h_v^2 - \nu(T[h_u, h_v])^2 + (\hat{p} - \bar{p}_\varepsilon)d_{yy}(x, \bar{y}_\varepsilon)y_h^2\,dx \\ &= \frac{\partial^2 \mathcal{L}}{\partial u^2}(T[\bar{u}_\varepsilon, \bar{v}_\varepsilon], \bar{\mu})(T[h_u, h_v])^2 \\ &\quad + \int_\Omega \nu h_u^2 + \psi(\varepsilon)h_v^2 - \nu(T[h_u, h_v])^2 + (\hat{p} - \bar{p}_\varepsilon)d_{yy}(x, \bar{y}_\varepsilon)y_h^2\,dx, \end{aligned}$$

since $\bar{y}_\varepsilon = GT[\bar{u}_\varepsilon, \bar{v}_\varepsilon]$ and $y_h = G'(T[\bar{u}_\varepsilon, \bar{v}_\varepsilon])T[h_u, h_v]$. According to Lemma 2.10, there exist constants $\alpha' > 0$ and $\delta' > 0$ such that

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(T[\bar{u}_\varepsilon, \bar{v}_\varepsilon], \bar{\mu})(T[h_u, h_v])^2 \geq \alpha' \|T[h_u, h_v]\|^2 \tag{4.3}$$

is satisfied, if $\|T[\bar{u}_\varepsilon, \bar{v}_\varepsilon] - \bar{u}\| \leq \delta'$. Due to the definition (3.2) of the operator T, the previous coercivity condition remains valid if $\|\bar{u} - \bar{u}_\varepsilon\| \leq \delta'$ and $\phi(\varepsilon)\|\bar{v}_\varepsilon\| \leq \delta'$. Note that the existence of such a constant $\delta' > 0$ is guaranteed if $\varepsilon > 0$ is sufficiently small, see Theorem 3.16. By $T[h_u, h_v] = h_u + \phi(\varepsilon)h_v$, we find

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial u^2}(T[\bar{u}_\varepsilon, \bar{v}_\varepsilon], \bar{\mu})(T[h_u, h_v])^2 &+ \int\limits_\Omega \nu h_u^2 + \psi(\varepsilon)h_v^2 - \nu(T[h_u, h_v])^2 \, dx \\
&\geq \alpha' \|T[h_u, h_v]\|^2 + \int\limits_\Omega \nu h_u^2 + \psi(\varepsilon)h_v^2 - \nu(T[h_u, h_v])^2 \, dx \\
&= \int\limits_\Omega \alpha' h_u^2 + ((\alpha' - \nu)\phi(\varepsilon)^2 + \psi(\varepsilon))h_v^2 + 2(\alpha' - \nu)h_u\phi(\varepsilon)h_v \, dx \\
&\geq \int\limits_\Omega \alpha' h_u^2 + ((\alpha' - \nu)\phi(\varepsilon)^2 + \psi(\varepsilon))h_v^2 \, dx \\
&\quad - 2(\alpha' + \nu)\left|\int\limits_\Omega h_u\phi(\varepsilon)h_v \, dx\right|
\end{aligned}$$

The last term is estimated by Young's inequality and an appropriate constant $\kappa > 0$

$$2\left|\int\limits_\Omega h_u\phi(\varepsilon)h_v \, dx\right| \leq \frac{1}{\kappa}\int\limits_\Omega h_u^2 \, dx + \kappa\int\limits_\Omega \phi(\varepsilon)^2 h_v^2 \, dx.$$

Sorting in terms of h_u and h_v respectively, we arrive at:

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial u^2}(T[\bar{u}_\varepsilon, \bar{v}_\varepsilon], \bar{\mu})(T[h_u, h_v])^2 &+ \int\limits_\Omega \nu h_u^2 + \psi(\varepsilon)h_v^2 - \nu(T[h_u, h_v])^2 \, dx \\
&\geq \left(\alpha' - \frac{\alpha' + \nu}{\kappa}\right)\|h_u\|^2 \\
&\quad + \big((\alpha' - \nu)\phi(\varepsilon)^2 + \psi(\varepsilon) - (\alpha' + \nu)\phi(\varepsilon)^2\kappa\big)\|h_v\|^2
\end{aligned}$$

The positivity of the constants in front of the norms is guaranteed, if $\kappa > 0$ is chosen in way such that

$$\frac{\alpha' + \nu}{\alpha'} < \kappa < \frac{(\alpha' - \nu)\phi(\varepsilon)^2 + \psi(\varepsilon)}{(\alpha' + \nu)\phi(\varepsilon)^2}.$$

The existence of such a constant κ is given, if

$$\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} < \frac{\alpha'}{3\alpha'\nu + \nu^2}.$$

Choosing ε sufficiently small, the previous condition is satisfied by Assumption 3.14. Summarizing, there exists a constant $\tilde{\alpha} > 0$ such that

$$\frac{\partial^2 \mathcal{L}_\varepsilon}{\partial (u, v)^2}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)h^2 \geq \tilde{\alpha}(\|h_u\|^2 + \|h_v\|^2) + \int\limits_\Omega (\hat{p} - \bar{p}_\varepsilon)d_{yy}(x, \bar{y}_\varepsilon)y_h^2 \, dx$$

is valid, provided that ε is chosen sufficiently small. Due to the Lipschitz continuity of the solution operator G and its derivative G' from $L^2(\Omega)$ to $H^1(\Omega) \cap C(\bar{\Omega})$, the last term can be estimated by

$$\begin{aligned}\left| \int_\Omega (\hat{p} - \bar{p}_\varepsilon)\, d_{yy}(x, \bar{y}_\varepsilon) y_h^2 \, dx \right| &\leq c\|\hat{p} - \bar{p}_\varepsilon\| \|d_{yy}(x, \bar{y}_\varepsilon)\|_\infty \|y_h\|_\infty^2 \\ &\leq c\|\hat{p} - \bar{p}_\varepsilon\|(\|h_u\|^2 + \phi(\varepsilon)^2\|h_v\|^2) \\ &\leq c(\|\hat{p} - \bar{p}\| + \|\bar{p} - \bar{p}_\varepsilon\|)(\|h_u\|^2 + \phi(\varepsilon)^2\|h_v\|^2).\end{aligned} \tag{4.4}$$

Note that $\bar{p}$ denotes the optimal adjoint state of the unregularized problem (P). According to Theorem 4.4, the adjoint state $\bar{p}_\varepsilon$ converges strongly in $L^2(\Omega)$ to $\bar{p}$ as $\varepsilon \to 0$. Furthermore, the strong convergence of the auxiliary function $\hat{p}$ in $L^2(\Omega)$ to $\bar{p}$ for $\varepsilon \to 0$ is obtained by similar arguments as in the proofs of Lemma 4.3 and Theorem 4.4. Consequently, choosing ε sufficiently small there exists a constant $\alpha'' > 0$ such that

$$\frac{\partial^2 \mathcal{L}_\varepsilon}{\partial(u, v)^2}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)h^2 \geq \alpha''(\|h_u\|^2 + \|h_v\|^2)$$

is valid for all $h = (h_u,\, h_v) \in L^2(\Omega)^2$, which is the assertion. ☐

Let us now conclude with a result on local uniqueness of stationary points for the regularized problem, which is an important issue for numerical methods. Thanks to the specific structure of our optimality system we can show the local uniqueness of stationary points by a direct argumentation.

Theorem 4.6 *Let the assumptions of Theorem 4.5 be satisfied, and let $\varepsilon > 0$ be a fixed regulization parameter. Then, there exists a radius r_ε such that the optimal control $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ is the unique stationary point of* (P_ε) *in the open ball around $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ with radius r_ε.*

Proof. Let $\bar{\mu}_\varepsilon$ be the Lagrange multipliers associated with $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$, and assume that for every $r > 0$ there exists another stationary point $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{\mu}_\varepsilon)$ of Problem (P_ε) with $\|\bar{u}_\varepsilon - \tilde{u}_\varepsilon\| \leq r$ and $\|\bar{v}_\varepsilon - \tilde{v}_\varepsilon\| \leq r$. Therefore, with $h = [\tilde{u}_\varepsilon - \bar{u}_\varepsilon, \tilde{v}_\varepsilon - \bar{v}_\varepsilon]$, we have that

$$\begin{aligned} 0 &\leq f_\varepsilon'(\bar{u}_\varepsilon, \bar{v}_\varepsilon)h + (-\xi(\varepsilon)(\tilde{v}_\varepsilon - \bar{v}_\varepsilon) - G'(T[\bar{u}_\varepsilon, \bar{v}_\varepsilon]Th, \bar{\mu}_\varepsilon) \\ 0 &\leq -f_\varepsilon'(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)h + (\xi(\varepsilon)(\tilde{v}_\varepsilon - \bar{v}_\varepsilon) + G'(T[\tilde{u}_\varepsilon, \tilde{v}_\varepsilon]Th, \tilde{\mu}_\varepsilon), \end{aligned}$$

which follows in a standard way by testing the variational inequality for $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)$ with $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ and vice versa. Adding both inequalities, we obtain by Taylor expansion with $u_\varepsilon^{t_{i,u}} = \bar{u}_\varepsilon + t_{i,u}(\bar{u}_\varepsilon - \tilde{u}_\varepsilon)$ and $v_\varepsilon^{t_{i,v}} = \bar{v}_\varepsilon + t_{i,v}(\bar{v}_\varepsilon - \tilde{v}_\varepsilon)$ with $0 < t_{i,u}, t_{i,v} < 1$,

$i = 1, 2$

$$
\begin{aligned}
0 \leq & (f_\varepsilon'(\bar{u}_\varepsilon, \bar{v}_\varepsilon) - f_\varepsilon'(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon))Th + ((G'(T[\tilde{u}_\varepsilon, \tilde{v}_\varepsilon])) - G'(T[\bar{u}_\varepsilon, \bar{v}_\varepsilon])))Th, \bar{\mu}_\varepsilon) \\
& + (\xi(\varepsilon)(\tilde{v}_\varepsilon - \bar{v}_\varepsilon) + G'(T[\tilde{u}_\varepsilon, \tilde{v}_\varepsilon]Th), \tilde{\mu}_\varepsilon - \bar{\mu}_\varepsilon) \\
= & - \mathcal{L}_\varepsilon''(u_\varepsilon^{t_{1,u}}, v_\varepsilon^{t_{1,v}}, \bar{\mu}_\varepsilon)[Th]^2 + (GT[\bar{u}_\varepsilon, \bar{v}_\varepsilon]) - GT[\tilde{u}_\varepsilon, \tilde{v}_\varepsilon] - G'(T[\tilde{u}_\varepsilon, \tilde{v}_\varepsilon])Th, \tilde{\mu}_\varepsilon - \bar{\mu}_\varepsilon) \\
& + (\xi(\varepsilon)\tilde{v}_\varepsilon + GT[\tilde{u}_\varepsilon, \tilde{v}_\varepsilon] - y_c, \tilde{\mu}_\varepsilon - \bar{\mu}_\varepsilon) + (y_c - \xi(\varepsilon)\bar{v}_\varepsilon - GT[\bar{u}_\varepsilon, \bar{v}_\varepsilon], \tilde{\mu}_\varepsilon - \bar{\mu}_\varepsilon) \\
\leq & - \mathcal{L}_\varepsilon''(u_\varepsilon^{t_{1,u}}, v_\varepsilon^{t_{1,v}}, \bar{\mu}_\varepsilon)[Th]^2 + \frac{1}{2}(G''(T[u_\varepsilon^{t_{2,u}}, v_\varepsilon^{t_{2,v}}])[Th]^2, \tilde{\mu}_\varepsilon - \bar{\mu}_\varepsilon),
\end{aligned}
$$

where the last inequality follows by the positivity of $\tilde{\mu}_\varepsilon$ and $\bar{\mu}_\varepsilon$, as well as by the complementary slackness condition fulfilled for both stationary points. It can be proven analogously to Lemma 2.10 that for r_ε small enough we have $\mathcal{L}_\varepsilon''(u_\varepsilon^{t_{1,u}}, v_\varepsilon^{t_{1,v}}, \bar{\mu}_\varepsilon)[Th]^2 \geq \alpha'''\|h\|^2$ for some fixed $\alpha''' > 0$. Hence, we obtain

$$
0 \leq \alpha''' \leq \frac{1}{2} \frac{\|(G''(T[u_\varepsilon^{t_{2,u}}, v_\varepsilon^{t_{2,v}}])[Th]^2\|}{\|h\|} \frac{\|\tilde{\mu}_\varepsilon - \bar{\mu}_\varepsilon\|}{\|h\|}. \tag{4.5}
$$

Applying similar arguments as in Remark 3.18 to the difference of Lagrange multipliers, one derives the estimate

$$
\|\tilde{\mu}_\varepsilon - \bar{\mu}_\varepsilon\| \leq c_\varepsilon \|h\|
$$

with a positive constant dependent on the regularization parameter ε. Concluding, (4.5) yields

$$
0 \leq \alpha''' \leq c_\varepsilon \frac{\|(G''(T[u_\varepsilon^{t_{2,u}}, v_\varepsilon^{t_{2,v}}])[Th]^2\|}{\|h\|}.
$$

From the properties of the second derivative of G and $\|h\| \leq 2r$, we arrive at

$$
r_\varepsilon \geq \frac{\alpha'''}{2c_\varepsilon},
$$

which contradicts the consideration of arbitrarily small neighbourhoods around the local optimal control $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$. ∎

Appendix

We present the proof of Lemma 4.1.

Proof. It is well known that the adjoint state $\bar{p}_n$ can be represented by the adjoint of the derivative of the control-to-state mapping and we obtain for $\bar{p}_n$:

$$
\bar{p}_n = G'(T[\bar{u}_n, \bar{v}_n])^*(\bar{y}_n - y_d - \bar{\mu}_n). \tag{4.6}
$$

Next, we rewrite the equation (3.13) in a variational form

$$
(\phi(\varepsilon_n)\bar{p}_n + \psi(\varepsilon_n)\bar{v}_n - \xi(\varepsilon_n)\bar{\mu}_n \, , \, v - \bar{v}_n) = 0 \quad \forall v \in L^2(\Omega).
$$

Adding the previous variational equation and (3.12) and using the representation (4.6) of the adjoint state $\bar{p}_n$, we arrive at

$$\begin{aligned}&(\phi(\varepsilon_n)G'(T[\bar{u}_n,\bar{v}_n])^*(\bar{y}_n - y_d - \bar{\mu}_n) + \psi(\varepsilon_n)\bar{v}_n - \xi(\varepsilon_n)\bar{\mu}_n\,,\, v - \bar{v}_n) +\\ &\qquad (G'(T[\bar{u}_n,\bar{v}_n])^*(\bar{y}_n - y_d - \bar{\mu}_n) + \nu\bar{u}_n\,,\, u - \bar{u}_n) \geq 0 \quad \forall (u,v) \in U_{ad} \times L^2(\Omega).\end{aligned}$$

Sorting all terms where the multiplier arises and applying the adjoint operator, we deduce

$$\begin{aligned}(\bar{\mu}_n\,,\, \xi(\varepsilon_n)(v - \bar{v}_n) + G'(T[\bar{u}_n,\bar{v}_n])(\phi(\varepsilon_n)(v - \bar{v}_n)) + G'(T[\bar{u}_n,\bar{v}_n])(u - \bar{u}_n))&\\ \leq (\psi(\varepsilon_n)\bar{v}_n + \phi(\varepsilon_n)G'(T[\bar{u}_n,\bar{v}_n])^*(\bar{y}_n - y_d)\,,\, v - \bar{v}_n)&\\ + (\nu\bar{u}_n + G'(T[\bar{u}_n,\bar{v}_n])^*(\bar{y}_n - y_d)\,,\, u - \bar{u}_n),&\end{aligned} \tag{4.7}$$

for all $(u,v) \in U_{ad} \times L^2(\Omega)$. Choosing the special test function $(\hat{u},\, 0) \in U_{ad} \times L^2(\Omega)$, where $\hat{u}$ is the Slater-point with respect to the linearized pure state constraints defined in Assumption 2.6. By means (3.2), we find for the left hand side of the previous inequality (4.7)

$$\begin{aligned}(\bar{\mu}_n\,,\, \xi(\varepsilon_n)(-\bar{v}_n) &+ G'(T[\bar{u}_n,\bar{v}_n])(-\phi(\varepsilon_n)\bar{v}_n) + G'(T[\bar{u}_n,\bar{v}_n])(\hat{u} - \bar{u}_n))\\ &= (\bar{\mu}_n\,,\, \xi(\varepsilon_n)(-\bar{v}_n) + G'(T[\bar{u}_n,\bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n])\\ &= (\bar{\mu}_n\,,\, y_c - GT[\bar{u}_n,\bar{v}_n] - \xi(\varepsilon_n)\bar{v}_n)\\ &\quad + (\bar{\mu}_n\,,\, GT[\bar{u}_n,\bar{v}_n] + G'(T[\bar{u}_n,\bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n] - y_c)\\ &= (\bar{\mu}_n\,,\, GT[\bar{u}_n,\bar{v}_n] + G'(T[\bar{u}_n,\bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n] - y_c),\end{aligned} \tag{4.8}$$

since the first term in the third line vanishes by (3.14) and $\bar{y}_n = GT[\bar{u}_n, \bar{v}_n]$. With the help of Lemma 3.5 and the positivity of the Lagrange multiplier, one derives the estimate

$$\frac{\gamma}{2}\|\bar{\mu}_n\|_{L^1(\Omega)} \leq (\bar{\mu}_n\,,\, GT[\bar{u}_n,\bar{v}_n] + G'(T[\bar{u}_n,\bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n] - y_c). \tag{4.9}$$

Note that the multiplier is zero in $\Omega\backslash\Omega$. Summarizing (4.7) for $(\hat{u},\, 0) \in U_{ad} \times L^2(\Omega)$, (4.8) and (4.9), we conclude

$$\begin{aligned}\gamma\|\bar{\mu}_n\|_{L^1(\Omega)} &\leq (\bar{\mu}_n\,,\, GT[\bar{u}_n,\bar{v}_n] + G'(T[\bar{u}_n,\bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n] - y_c)\\ &\leq (\psi(\varepsilon_n)\bar{v}_n + \phi(\varepsilon_n)G'(T[\bar{u}_n,\bar{v}_n])^*(\bar{y}_n - y_d)\,,\, -\bar{v}_n)\\ &\quad + (\nu\bar{u}_n + G'(T[\bar{u}_n,\bar{v}_n])^*(\bar{y}_n - y_d)\,,\, \hat{u} - \bar{u}_n).\end{aligned}$$

This implies

$$\begin{aligned}\gamma\|\bar{\mu}_n\|_{L^1(\Omega)} \leq &- \psi(\varepsilon_n)\|\bar{v}_n\|^2 + (\bar{y}_n - y_d\,,\, G'(T[\bar{u}_n,\bar{v}_n])(-\phi(\varepsilon_n)\bar{v}_n))\\ &+ (\bar{y}_n - y_d\,,\, G'(T[\bar{u}_n,\bar{v}_n])(\hat{u} - \bar{u}_n)) + \nu(\bar{u}_n\,,\, \hat{u} - \bar{u}_n)\\ \leq\ & (\bar{y}_n - y_d\,,\, G'(T[\bar{u}_n,\bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n]) + \nu(\bar{u}_n\,,\, \hat{u} - \bar{u}_n)\\ \leq\ & \|\bar{y}_n - y_d\|\|G'(T[\bar{u}_n,\bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n]\| + \nu\|\bar{u}_n\|\|\hat{u}\|.\end{aligned}$$

Due to $\bar{u}_n,\, \hat{u} \in U_{ad}$, the second term is bounded independent of ε_n. Furthermore, $\|\bar{y}_n - y_d\|$ is bounded by the cost functional of problem (P_ε), since $(\bar{u}_n,\, \bar{v}_n)$ is an optimal control. Finally, $y_h := G'(T[\bar{u}_n, \bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n]$ is the weak solution of the PDE (3.4) with respect to the right hand side $\hat{u} - \bar{u}_n + \phi(\varepsilon_n)\bar{v}_n$ such that we obtain

$$\|G'(T[\bar{u}_n, \bar{v}_n])T[\hat{u} - \bar{u}_n, -\bar{v}_n]\| \leq C(\|\hat{u} - \bar{u}_n\| + \phi(\varepsilon_n)\|\bar{v}_n\|).$$

The first norm was already discussed. The L^2-norm of the virtual control can be estimated easily by the cost functional and we deduce

$$\phi(\varepsilon_n)\|\bar{v}_n\| \leq C\frac{\phi(\varepsilon_n)}{\sqrt{\psi(\varepsilon_n)}} \leq C,$$

where the boundedness of $\frac{\phi(\varepsilon_n)}{\sqrt{\psi(\varepsilon_n)}}$ follows from Assumption 3.14. □

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