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Stability of multiple–pulse solutions

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Abstract

In this article, stability of multiple-pulse solutions in semilinear parabolic equations on the real line is studied. A system of equations is derived which determines stability of N-pulses bifurcating from a stable primary pulse. The system depends only on the particular bifurcation leading to the existence of the N-pulses.

As an example, existence and stability of multiple pulses are investigated if the primary pulse converges to a saddle-focus. It turns out that under suitable assumptions infinitely many N-pulses bifurcate for any fixed $N > 1$. Among them are infinitely many stable ones. In fact, any number of eigenvalues between 0 and $N - 1$ in the right half plane can be prescribed.

1 Introduction

This article is concerned with the stability of travelling waves of semilinear parabolic equations on the real line. Consider

$$(1.1) \qquad U_t = \mathcal{A} U + F(U, \epsilon),$$

where $U(x, t) \in \mathbb{R}^m$ and $x, t \in \mathbb{R}$ are space and time, respectively. The operator $\mathcal{A}$ is diagonal, the entries of which are homogeneous linear differential operators defining analytic semigroups on $BU(\mathbb{R}, \mathbb{R}^m)$, while the Nemitskii-operator F maps certain fractional power spaces associated to $\mathcal{A}$ smoothly into themselves, see [Hen81]. A particular example is the system

$$U_t = D\, U_{xx} + G(U),$$

where D is a diagonal $m \times m$ matrix and $G : \mathbb{R}^m \times \mathbb{R}^m$ denotes a smooth function. We are interested in pulse solutions of (1.1) which are travelling waves $U(\xi) = U(x - ct)$ of (1.1) decaying exponentially to the same constant vector $\bar{U}$, that is,

$$\lim_{\xi \to \pm\infty} U(\xi) = \bar{U}.$$

Transforming (1.1) into a moving coordinate frame $(x, t) \mapsto (x - ct, t) = (\xi, t)$ yields

$$(1.2) \qquad U_t = \mathcal{A} U + c U_\xi + F(U, \epsilon),$$

where $\mathcal{A}$ operates on $U(\xi)$ as before on $U(x)$. Therefore, a pulse solution U of (1.1) travelling with speed c corresponds to a steady state of (1.2)

$$(1.3) \qquad \mathcal{A} U + c U_\xi + F(U, \epsilon) = 0.$$

In order to investigate the stability properties of a given pulse U, there arise two problems. The first is wether linear stability implies nonlinear stability, the second one is concerned with the linear stability that is, the investigation of the spectrum of the linearization of (1.3) at the pulse U. We will only consider situations where the first problem is solved. Often, one can use center manifold theory to conclude this property, see e.g. [Hen81], [BJ89] or [AJ94] and the references therein. For the second problem, one has to investigate the spectrum of the operator

$$(1.4) \qquad \mathcal{A} V + c V_\xi + D_U F(U, \epsilon)\, V.$$

The spectrum decomposes into the essential and the normal part. The latter consists of all isolated eigenvalues of finite multiplicity, while the former is the complement of this set. We

assume that the essential spectrum is strictly contained in the left half plane. Again this situation arises frequently in applications. Indeed, the essential spectrum is determined by the spectrum of the steady state $\bar{U}$, see [Hen81, Theorem A.2]. If the steady state is stable, the essential spectrum of the pulse has negative realpart.

Under this two basic assumptions, the stability problem of a pulse reduces to determine the eigenvalues - isolated ones with finite multiplicity - of the linearization, that is, solutions of the equation

$$(1.5) \qquad \mathcal{A} V + c V_\xi + D_U F(U, \epsilon) V = \lambda V.$$

Note that (1.3) and (1.5) are ordinary differential equations in the variable ξ.

Suppose now that for $(\epsilon, c) = (\epsilon_1, c_1)$ a pulse U_1 of (1.3) exists. Under certain conditions, equation (1.3) admits N-pulse solutions U_N for values of (ϵ, c) close to (ϵ_1, c_1). These N-pulses look like N copies of the primary pulse U_1 widely spaced in ξ. Throughout, we will assume that the primary pulse U_1 is asymptotically stable. The question adressed in the present paper is the stability of the bifurcating N-pulses.

By the results of [AGJ90] and the basic assumptions stated above, the spectrum of (1.4) evaluated at the N-pulse is contained in the left half plane with the possible exception of eigenvalues close to zero. Indeed, this follows from the relation $c_1(\mathcal{E}(K)) = W(D(K))$ proved in [AGJ90] for general eigenvalue problems of the form (1.5). It turns out that there are precisely N eigenvalues close to zero, see [AGJ90], [GJ90]. One of these N eigenvalues is equal to zero with eigenfunction U_1'. The problem investigated here is the computation of the remaining $N-1$ eigenvalues close to zero.

To this end, we rewrite (1.3) and (1.5) as ordinary differential equations of first order

$$(1.6) \qquad \dot{u} = f(u, \mu) \qquad\qquad u \in \mathbb{R}^n, \ \mu = (\epsilon, c)$$

$$(1.7) \qquad \dot{v} = (D_u f(u, \mu) + \lambda B) v,$$

where B is a $n \times n$ matrix. We have to solve (1.7) evaluated at an N-pulse q_N of (1.6) for λ and v such that $\lambda \in \mathbb{C}$ is close to zero and v is a non-zero eigenfunction bounded on $\mathbb{R}$. Theorem 2 describes the eigenvalues λ of (1.7) close to zero by means of zeroes of $E(\lambda) = \det S(\lambda)$, where the $N \times N$ matrix $S(\lambda)$ is given by

$$S(\lambda) = A - M\lambda + h.o.t.$$

for some $M \in \mathbb{R}$. The entries of A are determined by the particular bifurcation scenario leading to the existence of the N-pulse q_N from a given primary pulse q_1. The method used to obtain this reduction was originally introduced by [Lin90]. In [San93], it was further developed such that bifurcations leading to N-pulses could be handled. The matrix A is

most easily computed if the existence proof of the N-pulses is done using Lin's method as presented in [San93]. In Lemma 4.1 we relate the order of a zero of $E(\lambda)$ to the algebraic multiplicity of the eigenvalue λ of the original PDE operator (1.4). Finally, Lemma 5.2 shows how to solve the reduced equation $S(\lambda) = 0$ by computing the eigenvalues of the matrix A.

As an application, we consider a stable primary homoclinic solution to a saddle-focus. It will be shown that infinitely many N-pulses bifurcate for any $N > 1$ under certain hypotheses. Moreover, we characterize the stability of these N-pulses. It turns out that any number of unstable eigenvalues between 0 and $N-1$ occurs for infinitely many of the N-pulses. In particular, infinitely many stable N-pulses bifurcate for any N.

Let us mention related results. In [EFF82] the existence of double pulses in nerve-axon equations was shown, while [YM89] solved the stability problem associated with these double pulses. Here, the primary pulse converges to a saddle-focus possessing a one-dimensional unstable manifold. In the same setting, [Fer86] proved the existence of infinitely many N-pulses for any N, see also [Gas83]. If the dimensions of stable and unstable manifolds are both equal to two and the eigenvalues of the equilibrium have non-vanishing imaginary part, existence of multiple pulses was shown by [Gle89]. Both existence and stability of double pulses are investigated in [AJ94] in this case. Under an additional assumption, [AJ93] proved the existence of stable triple pulses. A general theory of the Evans function was developed in [AGJ90].

The paper is organized as follows. In section 2 we state the basic assumptions and the main results concerning the ODE's (1.6) and (1.7), the proofs of which are contained in section 3. The multiplicity lemma is proved in section 4. We provide some basic strategies of solving the resulting reduced equations appearing in the main result and give relations to the Evans function in section 5. Finally, in section 6, we apply our results to a homoclinic orbit to a saddle-focus proving existence and stability of N-pulses. Section 7 is devoted to a discussion.

2   The eigenvalue problem for the variational equation

Consider

(2.1) $$\dot{u} = f(u, \mu) \qquad\qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R}^p,$$

3

for a smooth function f. We assume that zero is a hyperbolic equilibrium for $\mu = \mu_1$. Hence, there exist real numbers α^s, $\alpha^u > 0$ such that

(H1) $\operatorname{Re} \sigma(D_u f(0, \mu_1)) \subset (-\infty, -\alpha^s) \cup (\alpha^u, \infty)$.

Let E^s and E^u be the generalized eigenspaces associated to the spectral sets appearing in (H1) and denote the corresponding projections by P_0^s and P_0^u, respectively. We denote by $p(\mu)$ the family of equilibria of (2.1) satisfying $p(\mu_1) = 0$.

Suppose that for $\mu = \mu_1$ a pulse solution $q_1(t)$ exists converging to zero for $t \to \pm\infty$. We assume that q_1 is non-degenerate:

(H2) $T_{q_1(0)} W^s(0, \mu_1) \cap T_{q_1(0)} W^u(0, \mu_1) = \mathbb{R}\dot{q}_1(0)$.

Here, W^s and W^u denote stable and unstable manifolds, respectively. (H2) is equivalent to the fact that $\dot{q}_1(t)$ is the unique bounded solution of the variational equation

$$(2.2) \qquad \dot{v} = D_u f(q_1(t), \mu_1)\, v.$$

Then the adjoint variational equation

$$(2.3) \qquad \dot{w} = -D_u f(q_1(t), \mu_1)^* w$$

possesses a unique bounded solution, too, which we denote by $\psi_1(t)$, see [San93]. The solution $\psi_1(t)$ satisfies

$$(2.4) \qquad \psi_1(t) \perp T_{q_1(t)} W^s(0, \mu_1) + T_{q_1(t)} W^u(0, \mu_1)$$

for all $t \in \mathbb{R}$. Next we decompose the tangent space at $q_1(0)$ according to

$$(2.5) \qquad \mathbb{R}^n = \mathbb{R}\psi_1(0) \oplus \mathbb{R}\dot{q}_1(0) \oplus Y^+ \oplus Y^-,$$

where Y^+ and Y^- are chosen such that

$$
\begin{aligned}
\mathbb{R}\dot{q}_1(0) \oplus Y^+ &= T_{q_1(0)} W^s(0, \mu_1) \\
\mathbb{R}\dot{q}_1(0) \oplus Y^- &= T_{q_1(0)} W^u(0, \mu_1).
\end{aligned}
$$

Then we have the following two lemmata.

Lemma 2.1 ([San93, Lemma 3.6]) *There exists a smooth change of coordinates such that $p(\mu) = 0$ holds. Moreover, assuming $p(\mu) = 0$, $W^s_{loc}(0, \mu) \subset E^s$, $W^u_{loc}(0, \mu) \subset E^u$ and for some $\delta > 0$*

$$
\begin{aligned}
W^s(0, \mu) \cap U_\delta(q_1(0)) &\subset q_1(0) + \mathbb{R}\dot{q}_1(0) \oplus Y^+ \\
W^u(0, \mu) \cap U_\delta(q_1(0)) &\subset q_1(0) + \mathbb{R}\dot{q}_1(0) \oplus Y^-
\end{aligned}
$$

are satisfied.

We assume that coordinates are changed according to Lemma 2.1.

Lemma 2.2 ([San93, Lemma 3.3]) *There exist smooth families of solutions $u_\infty^+(\mu)$ and $q^-(\mu)$ for μ close to μ_1 such that*

$$q^+(\mu)(t) \in W^s(0,\mu) \qquad t \geq 0$$
$$q^-(\mu)(t) \in W^u(0,\mu) \qquad t \leq 0$$

and

$$q^+(\mu)(0) - q^-(\mu)(0) \in \mathbb{R}\psi_1(0)$$
$$q^\pm(\mu)(0) - q_1(0) \in \mathbb{R}\psi_1(0) \oplus Y^+ \oplus Y^-$$
$$q^\pm(\mu_1)(0) = q_1(0).$$

The families $q^\pm(\mu)$ are unique with respect to these properties. Moreover, there is a unique bounded solution $\psi(\mu)$ of

$$\dot{w} = -D_u f(\tilde{u}(\mu), \mu)^* w$$

with $\psi(\mu)(0) = \psi_1(0)$. Here, $\tilde{u}$ is defined by $\tilde{u}(\mu) = q^-(\mu)(t)$ for $t < 0$ and $\tilde{u}(\mu) = q^+(\mu)(t)$ for $t > 0$. In addition, $\psi(\mu_1)(t) = \psi_1(t)$ holds.

Now we are in a position to characterize N-pulses which exist for parameters μ close μ_1 and are close to the trace of the primary pulse q_1 in phase space. Note that [San93] contains a sharper version of Theorem 1.

Let $U_\delta(q_1)$ denote the tubular neighborhood of the orbit $\{q_1(t) \mid t \in \mathbb{R}\}$ of size δ. The supremum norm is denoted by $\|\cdot\|$.

Theorem 1 ([San93]) *There exist δ_0, $\delta_1 > 0$ small with the following property.*

Suppose that $q_N(t)$ is a pulse solution of (2.1) satisfying $q_N(t) \in U_{\delta_0}(q_1)$ for all $t \in \mathbb{R}$ and existing for a parameter value μ_N with $|\mu_N - \mu_1| < \delta_0$. Moreover, assume that $q_N(t)$ intersects the transverse section

$$\{q_1(0) + x \mid x \in \mathbb{R}\psi_1(0) \oplus Y^+ \oplus Y^- \text{ and } |x| \leq \delta_0\}$$

N times. Throughout, let $i = 1, ..., N$ and $j = 1, ..., N-1$. Then there exist unique numbers $T_j > \frac{1}{\delta_1}$ for $j = 1, ..., N-1$ and unique functions

$$u_i^- : [-T_{i-1}, 0] \to \mathbb{R}^n$$
$$u_i^+ : [0, T_i] \to \mathbb{R}^n$$

for $i = 1, ..., N$ with $T_0 = T_N = \infty$ such that:

$$q^-(0) + u_i^-(0) = q^+(0) + u_i^+(0)$$
$$q^-(-T_j) + u_{j+1}^-(-T_j) = q^+(T_j) + u_j^+(T_j)$$

and $q_N(t)$ is defined piecewise via

$$q_N\left(t + 2\sum_{k=1}^{i-1} T_k\right) = \begin{cases} q^-(t) + u_i^-(t) & \text{for } t \in (-T_{i-1}, 0] \\ q^+(t) + u_i^+(t) & \text{for } t \in (0, T_i]. \end{cases}$$

Here, $q^\pm$ is evaluated at $\mu = \mu_N$. Moreover, the estimates

$$(i) \qquad \|u_i\| \leq C \sup_{t \geq T}(|q^+(\mu_N)(t)| + |q^-(\mu_N)(-t)|)$$

$$(2.6)(ii) \qquad |u_{j+1}^-(-T_j) - q^+(\mu_N)(T_j)| \leq C e^{-\alpha T} \sup_{t \geq T}(|q^+(\mu_N)(t)| + |q^-(\mu_N)(-t)|)$$

$$(iii) \qquad |u_j^+(T_j) - q^-(\mu_N)(-T_j)| \leq C e^{-\alpha T} \sup_{t \geq T}(|q^+(\mu_N)(t)| + |q^-(\mu_N)(-t)|)$$

hold with $T = \min_{j=1,\dots,N-1} T_j$ and $\alpha = \min(\alpha^s, \alpha^u)$. The constant C depends only on N, δ_0 and δ_1.

Proof. The statements of the theorem are consequences of [San93, section 3]. The estimate (2.6)(i) follows from [San93, (3.38),(3.39) and (3.42)] and the definition [San93, (3.17)]. The other two estimates follow from the proof of [San93, Lemma 3.20]. Indeed, in the notation of [San93], we have

$$\begin{aligned} w_i(\omega_{i+1}) &= w_i^s(\omega_{i+1}) + d_{i+1}^u + R_{i+1}^u \\ \hat{w}_i(-\omega_i) &= \hat{w}_i^u(-\omega_i) + d_i^s + R_i^s \end{aligned}$$

and the estimates for R_i^s and R_i^u can be written as

$$\|R\| \leq C e^{-\alpha T} \|d\|,$$

see [San93, Lemma 3.20]. $\qquad\qquad\square$

The stability of the pulses described in Theorem 1 is considered in the next theorem, which is the main result of the present article. Assume that

$$B : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n \times n}$$

is a smooth function. We are interested in bounded solutions v of

$$(2.7) \qquad \dot{v} = D_u f(q_N(t), \mu_N) v + \lambda B(q_N(t), \mu_N) v$$

for $\lambda \in \mathbb{C}$, where q_N is a N-pulse existing for the parameter value μ_N. However, we applied a change of coordinates to (2.1). The consequences of this transformation for (2.7) are given in the next lemma, the proof of which is straightforward.

Lemma 2.3 *Consider the equation $\dot{u} = f(u, \mu)$ and change coordinates according to the transformation $x = h(u, \mu)$ yielding $\dot{x} = g(x, \mu)$. Then the equations*

$$\begin{aligned} \dot{v} &= (D_u f(u, \mu) + \lambda B(u, \mu)) v \\ \dot{w} &= -D_u f(u, \mu)^* w \end{aligned}$$
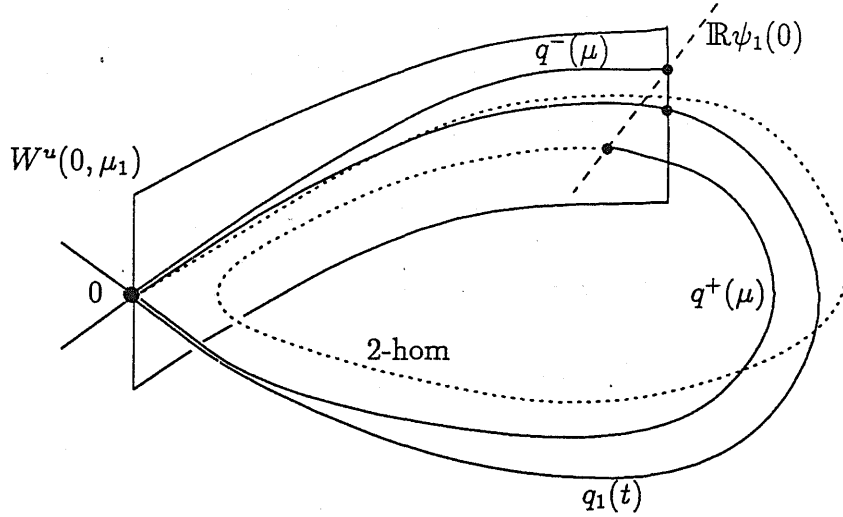
6

Figure 1: A 2-homoclinic solution

transform by the change $y = D_u h(u, \mu) v$ and $z = (D_u h(u, \mu)^*)^{-1} w$ into the system

$$
\begin{aligned}
\dot{y} &= (D_u g(x, \mu) + \lambda \tilde{B}(x, \mu)) y \\
\dot{z} &= -D_u g(x, \mu)^* z
\end{aligned}
$$

for some function $\tilde{B}$. In addition, $M = \int_{-\infty}^{\infty} \langle \psi_1(t), B(q_1(t), \mu_1) \dot{q}_1(t) \rangle \, dt$ does not depend on the coordinate system chosen.

Therefore, the solution set of (2.7) remains unchanged if we transform (2.7) according to the lemma.

Theorem 2 *There exists a $\delta_2 > 0$ small with the following property.*

Let (q_N, μ_N) denote a N-pulse as described in Theorem 1 with associated return times T_j for $j = 1, ..., N-1$ and $T_0 = T_N = \infty$. Then there exists a bounded non-zero solution v of (2.7) for $\lambda \in U_{\delta_2}(0) \subset \mathbb{C}$ if and only if

$$
E(\lambda) = \det S(\lambda) = \det(A - M\lambda \, id + R(\lambda)) = 0.
$$

Here, the tridiagonal matrix $A = (a_{ij})_{i,j=1,...,N}$ is defined by

(2.8)
$$
A = \begin{pmatrix}
-a_1 & a_1 & & & \\
-\tilde{a}_1 & \tilde{a}_1 - a_2 & a_2 & & \\
& -\tilde{a}_2 & \tilde{a}_2 - a_3 & a_3 & \\
& & \ddots & \ddots & \\
& & & -\tilde{a}_{N-1} & \tilde{a}_{N-1}
\end{pmatrix}
$$

with

$$a_j = \langle \psi(T_j), \dot{q}^-(-T_j) \rangle$$
$$\tilde{a}_j = \langle \psi(-T_j), \dot{q}^+(T_j) \rangle,$$

where ψ and $q^\pm$ are evaluated at the parameter value μ_N. Moreover,

$$M = \int_{-\infty}^{\infty} \langle \psi_1(t), B(q_1(t), \mu_1) \, \dot{q}_1(t) \rangle \, dt \in \mathbb{R}.$$

The remainder term $R(\lambda)$ is analytic in λ and can be estimated by

$$\|R(\lambda)\| \leq C \left(e^{-2\alpha T} \sup_{t \geq T}(|q^+(\mu_N)(t)| + |q^-(\mu_N)(-t)|) + |\lambda| \left(e^{-\alpha T} + |\mu_N - \mu_1| + |\lambda|\right) \right),$$

with the notation introduced in Theorem 1.

In the appendix A, an improved estimate of the remainder term is given.

An interpretation of M in terms of the Evans function associated to the primary pulse q_1 is given in section 5. Note that M does not depend on the actual coordinate system due to Lemma 2.3.

3   The Proof

The main idea for proving Theorem 2 consists in interpreting eigenfunctions of the linearized system (2.7) associated to an N-pulse q_N as N-homoclinic solutions again. Indeed, it will turn out that eigenfunctions corresponding to eigenvalues close to zero look like N copies of $\dot{q}_1(t)$. Therefore, they can be considered as N-homoclinic solutions with respect to $\dot{q}_1(t)$ in the same way as $q_N(t)$ is a multiple pulse with respect to $q_1(t)$. This leads to the idea of applying the theory developed to prove Theorem 1 to the non-autonomous equation (2.7).

To this end, we are going to reformulate the eigenvalue problem (2.7) in section 3.1. Then we are essentially in the setting considered in [Lin90, Lemma 2.2 and Theorem 2.4]. However, since we need to exploit the special structure of (2.7), we will give full proofs in section 3.2 instead of refering to [Lin90]. Another reason for proceeding in this way is that sharper estimates will be needed later on which will not follow from [Lin90] but require techniques developed in [San93]. The equations considered in 3.2 are of general type and we give the relation to the original system in section 3.3 finally.

3.1 The reformulation

We have to investigate bounded solutions (v, λ) of

$$(3.1) \qquad \dot{v} = \left(D_u f(q_N(t), \mu_N) + \lambda B(q_N(t), \mu_N)\right) v \qquad t \in \mathbb{R},$$

where (q_N, μ_N) is a N-pulse of

$$(3.2) \qquad \dot{u} = f(u, \mu)$$

close to (q_1, μ_1). We are interested in $\lambda \in \mathbb{C}$ close to zero. By Theorem 1, it is possible to describe q_N as follows

$$q_N\left(t + 2 \sum_{k=1}^{i-1} T_k\right) = \begin{cases} q^-(\mu_N)(t) + u_i^-(t) & \text{for } t \in (-T_{i-1}, 0] \\ q^+(\mu_N)(t) + u_i^+(t) & \text{for } t \in (0, T_i] \end{cases}$$

and $i = 1, ..., N$. Remember that $T_0 = T_N = \infty$. Therefore, we can rewrite (3.1) in the equivalent form

$$(3.3) \qquad \begin{aligned} \dot{v}_i^- &= \left(D_u f(q^-(\mu_N)(t) + u_i^-(t), \mu_N) + \lambda B(q^-(\mu_N)(t) + u_i^-(t), \mu_N)\right) v_i^- \\ &\qquad \text{for } t \in (-T_{i-1}, 0) \\ \dot{v}_i^+ &= \left(D_u f(q^+(\mu_N)(t) + u_i^+(t), \mu_N) + \lambda B(q^+(\mu_N)(t) + u_i^+(t), \mu_N)\right) v_i^+ \\ &\qquad \text{for } t \in (0, T_i) \\ v_i^-(0) &= v_i^+(0) \\ v_j^+(T_j) &= v_{j+1}^-(-T_j) \end{aligned}$$

for $i = 1, ..., N$ and $j = 1, ..., N-1$ considered as equations over the complex field. Then

$$(3.4) \qquad v\left(t + 2 \sum_{k=1}^{i-1} T_k\right) = \begin{cases} v_i^-(t) & \text{for } t \in (-T_{i-1}, 0] \\ v_i^+(t) & \text{for } t \in (0, T_i] \end{cases}$$

is a solution of (3.1) and vice versa.

Notation. The superscripts $^+$ and $^-$ always correspond to $t > 0$ and $t < 0$, respectively. Moreover, families (x_i) will be abbreviated by x. Throughout, the indices i and j run through $i = 1, ..., N$ and $j = 1, ..., N-1$. All vector spaces will be considered over the complex field. Many different constants, all of which are independent of the family (T_i) and the parameter μ, will be denoted by C in this section. We will often suppress the dependence of $q^\pm(\mu)$ on the parameter μ by setting $q^\pm = q^\pm(\mu)$.

In order to exploit the fact that q_N is a solution of (3.2) we employ the decomposition

$$(3.5) \qquad v_i^\pm(t) = \left(\dot{q}^\pm(t) + \dot{u}_i^\pm(t)\right) d_i + w_i^\pm(t)$$

9

for $i = 1, ..., N$ and arbitrary vectors $d = (d_i) \in \mathbb{C}^N$. Then $(d_i, w_i^\pm)$ solves

$$
\begin{aligned}
\dot{w}_i^\pm &= D_u f(q^\pm, \mu_N) w_i^\pm + \lambda B(q^\pm + u_i^\pm, \mu_N)(\dot{q}^\pm + \dot{u}_i^\pm) d_i + \\
&\quad \left(D_u f(q^\pm + u_i^\pm, \mu_N) - D_u f(q^\pm, \mu_N) + \lambda B(q^\pm + u_i^\pm, \mu_N) \right) w_i^\pm \\
&\quad \text{on } t \in (-T_{i-1}, 0) \text{ and } t \in (0, T_i), \text{ respectively} \\
w_i^-(0) &= w_i^+(0) \\
w_i^\pm(0) &\in \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^- \\
w_j^+(T_j) &= w_{j+1}^-(-T_j) + (\dot{q}^-(-T_j) + \dot{u}_{j+1}^-(-T_j)) d_{j+1} - (\dot{q}^+(T_j) + \dot{u}_j^+(T_j)) d_j
\end{aligned}
$$
(3.6)

if and only if $(v_i^\pm)$ solves (3.3). Indeed, if $(d_i, w_i^\pm)$ solve (3.6), then $(v_i^\pm)$ defined by (3.5) solve (3.3). On the other hand, suppose that $(v_i^\pm)$ is a solution of (3.3). Let Q denote the projection onto $\mathbb{C}\dot{q}_1(0)$ with kernel $\mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^-$. Define d_i by

$$
d_i := \frac{\langle \dot{q}_1(0), Q\, v_i^+(0) \rangle}{\langle \dot{q}_1(0), Q\, (\dot{q}^+(\mu_N)(0) + \dot{u}_i^+(0)) \rangle}
$$

and note that the denominator is non-zero. Then define $(w_i^\pm)$ according to (3.5) and observe that $(d_i, w_i^\pm)$ solve (3.6).

Note that $w_i^\pm = 0$ and $d_i = 1$ for $i = 1, ..., N$ is a bounded solution of (3.6) for $\lambda = 0$ due to the fact that q_N solves (3.2). In the following, we investigate the system

$$
\begin{aligned}
&(i) && \dot{w}_i^\pm = D_u f(q^\pm(\mu_N)(t), \mu_N) w_i^\pm + (G_i^\pm(t) + \lambda B_i^\pm(t)) w_i^\pm + \lambda h_i^\pm(t) d_i \\
&(ii) && w_i^\pm(0) \in \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^- \\
&(iii) && w_i^+(0) - w_i^-(0) \in \mathbb{C}\psi_1(0) \\
&(iv) && w_j^+(T_j) - w_{j+1}^-(-T_j) = D_j d,
\end{aligned}
$$
(3.7)

where we used the notations

$$
\begin{aligned}
G_i^\pm(t) &= D_u f(q^\pm(\mu_N)(t) + u_i^\pm(t), \mu_N) - D_u f(q^\pm(\mu_N)(t), \mu_N) \\
B_i^\pm(t) &= B(q^\pm(\mu_N)(t) + u_i^\pm(t), \mu_N) \\
h_i^\pm(t) &= B(q^\pm(\mu_N)(t) + u_i^\pm(t), \mu_N)(\dot{q}^\pm(\mu_N)(t) + \dot{u}_i^\pm(t)) \\
D_j d &= (\dot{q}^-(\mu_N)(-T_j) + \dot{u}_{j+1}^-(-T_j)) d_{j+1} - (\dot{q}^+(\mu_N)(T_j) + \dot{u}_j^+(T_j)) d_j.
\end{aligned}
$$
(3.8)

Note that a solution of (3.7) solves (3.6) if and only if

$$
\xi_i := \langle \psi_1(0), w_i^+(0) - w_i^-(0) \rangle = 0
$$

vanishes for $i = 1, ..., N$, see also Lemma 3.6. Indeed, $w_i^+(0) - w_i^-(0) \in \mathbb{C}\psi_1(0)$ if w solves (3.7).

Thus, the question of determining the eigenvalues reduces to solving $\xi = 0$ once (3.7) has been solved. For the functions defined in (3.8) we obtain the estimates collected in the next lemma.

Lemma 3.1 *We have the estimates*

$$|G_i^{\pm}(t)| \leq C|u_i^{\pm}(t)| \leq C \sup_{t \geq T}(|q^+(\mu_N)(t)| + |q^-(\mu_N)(-t)|)$$

$$|B_i^{\pm}(t)| \leq C$$

$$|h_i^{\pm}(t) - H(t)| \leq C(|\mu_N - \mu_1| + \sup_{t \geq T}(|q^+(\mu_N)(t)| + |q^-(\mu_N)(-t)|))$$

$$D_i d = (\dot{q}^+(\mu_N)(T_i) + \dot{q}^-(\mu_N)(-T_i))(d_{i+1} - d_i) +$$

$$\mathcal{O}\left(e^{-\alpha T}|d| \sup_{t \geq T}(|q^+(\mu_N)(t)| + |q^-(\mu_N)(-t)|)\right)$$

using the definition $H(t) = B(q_1(t), \mu_1)\,\dot{q}_1(t)$.

Proof. The first estimate follows from Theorem 1, (2.6)(i), the last one from (2.6)(ii) and (iii). Moreover, the third inequality is a consequence of Lemma 2.2 and again (2.6)(i). The second inequality is trivial. $\qquad\square$

3.2 The reduction

Consider the general system

(3.9)
$$\begin{aligned}
(i) \quad & \dot{w}_i^{\pm} = D_u f(q^{\pm}(\mu)(t), \mu)\,w_i^{\pm} + (G_i^{\pm}(t) + \lambda\,B_i^{\pm}(t))\,w_i^{\pm} + \lambda\,h_i^{\pm}(t)\,d_i \\
(ii) \quad & w_i^{\pm}(0) \in \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^- \\
(iii) \quad & w_i^+(0) - w_i^-(0) \in \mathbb{C}\psi_1(0) \\
(iv) \quad & w_j^+(T_j) - w_{j+1}^-(-T_j) = D_j\,d
\end{aligned}$$

for arbitrary functions $G = (G_i^{\pm})$, $h = (h_i^{\pm})$, matrices $D = (D_j)$ and $\lambda \in \mathbb{C}$ satisfying

(3.10)
$$\begin{aligned}
|G_i^{\pm}(t)| &\leq \delta & |B_i^{\pm}(t)| &\leq C \\
|h_i^{\pm}(t) - H(t)| &\leq \delta & |D_j| &\leq \delta \\
|\lambda| &\leq \delta &
\end{aligned}$$

for some $\delta > 0$ sufficiently small and a bounded function H defined on $\mathbb{R}$. In section 3.3, we substitute the expressions (3.8) into (3.9).

We concentrate on $|\lambda| < \delta$, whence $|G(t) + \lambda\,B(t)| < \delta$. Hence, (3.9)(i) is a small perturbation of

(3.11)
$$\dot{w}^{\pm} = D_u f(q^{\pm}(\mu), \mu)\,w^{\pm},$$

which is independent of i and N. In particular, it does not depend on the N-pulse q_N under consideration. This fact is exploited next.

Lemma 3.2 *The evolution $\Phi_\pm(\mu; t, s)$ of (3.11) can be decomposed into $\Phi_\pm(\mu; t, s) = \Phi_\pm^s(\mu; t, s) + \Phi_\pm^u(\mu; t, s)$ defined for $s, t \geq 0$ or $s, t \leq 0$, respectively. The functions*

$$\Phi_\pm^s(\mu; t, t) =: P_\pm^s(\mu)(t) \qquad \Phi_\pm^u(\mu; t, t) =: P_\pm^u(\mu)(t)$$

are projections and the estimates

$$\left.\begin{array}{rcl} |\Phi_+^s(\mu; t, s)| & \leq & Ce^{-\alpha^s(t-s)} \\ |\Phi_+^u(\mu; s, t)| & \leq & Ce^{-\alpha^u(t-s)} \end{array}\right\} \quad 0 \leq s \leq t$$

$$\left.\begin{array}{rcl} |\Phi_-^s(\mu; s, t)| & \leq & Ce^{-\alpha^s(t-s)} \\ |\Phi_-^u(\mu; t, s)| & \leq & Ce^{-\alpha^u(t-s)} \end{array}\right\} \quad s \leq t \leq 0$$

hold. The ranges of the projections at $t = 0$ are given by

$$RP_+^s(\mu)(0) = \mathbb{C}\dot{q}_1(0) \oplus Y^+ \qquad RP_+^u(\mu)(0) = \mathbb{C}\psi_1(0) \oplus Y^-$$
$$RP_-^u(\mu)(0) = \mathbb{C}\dot{q}_1(0) \oplus Y^- \qquad RP_-^s(\mu)(0) = \mathbb{C}\psi_1(0) \oplus Y^+.$$

In addition,

$$\begin{array}{rcll} |P_+^k(\mu)(t) - P_0^k| & \leq & Ce^{-\alpha^s t} & t \geq 0 \\ |P_-^k(\mu)(t) - P_0^k| & \leq & Ce^{\alpha^u t} & t \leq 0 \end{array}$$

holds for $k = s, u$.

Proof. See [San93, Lemma 1.1] and Lemma 2.1. Note that equation (3.11) is defined on $\mathbb{R}^n$. The estimates immediately carry over to the complex field. $\square$

We define the spaces

$$(3.12) \quad \begin{array}{rcl} V_w & := & \bigoplus_{i=1}^N C^0([-T_{i-1}, 0], \mathbb{C}^n) \oplus C^0([0, T_i], \mathbb{C}^n) \\ V_a & := & \bigoplus_{i=1}^{N-1} E^s \oplus E^u \\ V_b & := & \bigoplus_{i=1}^N RP_-^u(\mu)(0) \oplus RP_+^s(\mu)(0) = \bigoplus_{i=1}^N (\mathbb{C}\dot{q}_1(0) \oplus Y^-) \oplus (\mathbb{C}\dot{q}_1(0) \oplus Y^+) \\ V_\lambda & := & U_\delta(0) \subset \mathbb{C} \end{array}$$

considered over the complex field. The product spaces are endowed with the maximum norm. We consider elements

$$(3.13) \quad \begin{array}{rcl} w = (w_i^-, w_i^+) \in V_w & \qquad & a = (a_j^-, a_j^+) \in V_a \\ b = (b_i^-, b_i^+) \in V_b & \qquad & \lambda \in V_\lambda, \end{array}$$

where $i = 1, ..., N$ and $j = 1, ..., N-1$. We set $a_0^- = a_N^+ = 0$ in the following.

In order to solve (3.9), we first consider the ordinary differential equation (3.9)(i)

$$\dot{w}_i^\pm = D_u f(q^\pm(\mu)(t), \mu) w_i^\pm + (G_i^\pm(t) + \lambda B_i^\pm(t)) w_i^\pm + \lambda h_i^\pm(t) d_i$$

for $i = 1, ..., N$ on the intervals $(-T_{i-1}, 0)$ and $(0, T_i)$, respectively. The general solution bounded uniformly in (T_j) is given by the fixed point equation

$$
\begin{aligned}
(3.14) \quad w_i^-(t) &= \Phi_-^s(t, -T_{i-1})\, a_{i-1}^- + \Phi_-^u(t, 0)\, b_i^- + \\
&\quad \int_0^t \Phi_-^u(t, s)\Big(\big(G_i^-(s) + \lambda B_i^-(s)\big) w_i^-(s) + \lambda h_i^-(s)\, d_i\Big)\, ds + \\
&\quad \int_{-T_{i-1}}^t \Phi_-^s(t, s)\Big(\big(G_i^-(s) + \lambda B_i^-(s)\big) w_i^-(s) + \lambda h_i^-(s)\, d_i\Big)\, ds \\
w_i^+(t) &= \Phi_+^u(t, T_i)\, a_i^+ + \Phi_+^s(t, 0)\, b_i^+ + \\
&\quad \int_0^t \Phi_+^s(t, s)\Big(\big(G_i^+(s) + \lambda B_i^+(s)\big) w_i^+(s) + \lambda h_i^+(s)\, d_i\Big)\, ds + \\
&\quad \int_{T_i}^t \Phi_+^u(t, s)\Big(\big(G_i^+(s) + \lambda B_i^+(s)\big) w_i^+(s) + \lambda h_i^+(s)\, d_i\Big)\, ds
\end{aligned}
$$

for $i = 1, ..., N$ and arbitrary elements $a \in V_a$, $b \in V_b$ and $d \in \mathbb{C}^N$. Here and in the sequel, we will suppress the dependence of the evolution on μ. Note that the integrals in (3.14) exist even for $i = 1, N$ for which $T_0 = T_N = \infty$ due to the exponential decay of the evolution and the definition $a_0^- = a_N^+ = 0$. Indeed, the corresponding equations are given
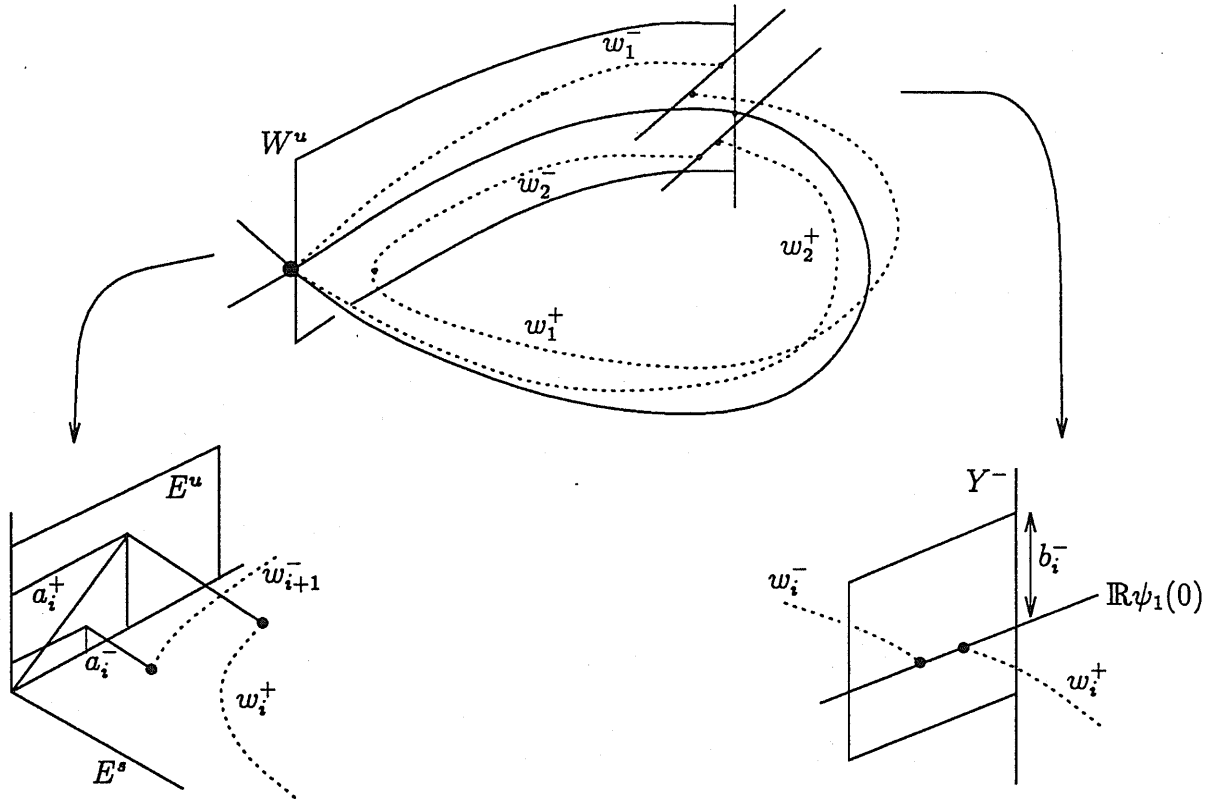

Figure 2: The interpretation of eigenfunctions as N-homoclinic solutions

by

$$w_1^-(t) = \Phi_-^u(t,0)\, b_1^- + \int_0^t \Phi_-^u(t,s)\Big((G_1^-(s) + \lambda B_1^-(s))\, w_1^-(s) + \lambda h_1^-(s)\, d_1\Big)\, ds +$$
$$\int_{-\infty}^t \Phi_-^s(t,s)\Big((G_1^-(s) + \lambda B_1^-(s))\, w_1^-(s) + \lambda h_1^-(s)\, d_1\Big)\, ds$$
$$w_N^+(t) = \Phi_+^s(t,0)\, b_N^+ + \int_0^t \Phi_+^s(t,s)\Big((G_N^+(s) + \lambda B_N^+(s))\, w_N^+(s) + \lambda h_N^+(s)\, d_N\Big)\, ds +$$
$$\int_\infty^t \Phi_+^u(t,s)\Big((G_N^+(s) + \lambda B_N^+(s))\, w_N^+(s) + \lambda h_N^+(s)\, d_N\Big)\, ds.$$

See figure 2 for a geometric interpretation. Instead of inverting the system (3.9) at once, we are going to invert it step by step in a sequence of Lemmata for clarity.

Lemma 3.3 *There exists an operator* $W_1 : V_\lambda \times V_a \times V_b \times \mathbb{C}^N \to V_w$ *such that*

$$(3.15) \qquad\qquad w := W_1(\lambda)(a,b,d)$$

solves (3.14) for any (a,b,d) *and* λ. *Moreover, any bounded solution of (3.14) is given by (3.15). The operator* W_1 *is analytic in* λ, *linear in* (a,b,d) *and admits the estimate*

$$(3.16) \qquad\qquad \|W_1(\lambda)(a,b,d)\| \le C(|a| + |b| + |\lambda|\,|d|)$$

uniformly in (T_j).

Proof. We rewrite (3.14) as

$$(3.17) \qquad\qquad (id - L_1(\lambda))\, w = L_2(\lambda)(a,b,d),$$

where the linear operators $L_1(\lambda)$ and $L_2(\lambda)$ are given by

$$(3.18) \quad \begin{aligned} (L_1(\lambda)\, w)_i^-(t) &= \int_0^t \Phi_-^u(t,s)\Big((G_i^-(s) + \lambda B_i^-(s))\, w_i^-(s)\Big)\, ds + \\ &\quad \int_{-T_{i-1}}^t \Phi_-^s(t,s)\Big((G_i^-(s) + \lambda B_i^-(s))\, w_i^-(s)\Big)\, ds \\ (L_1(\lambda)\, w)_i^+(t) &= \int_0^t \Phi_+^s(t,s)\Big((G_i^+(s) + \lambda B_i^+(s))\, w_i^+(s)\Big)\, ds + \\ &\quad \int_{T_i}^t \Phi_+^u(t,s)\Big((G_i^+(s) + \lambda B_i^+(s))\, w_i^+(s)\Big)\, ds \end{aligned}$$

and

$$(3.19) \quad \begin{aligned} (L_2(\lambda)(a,b,d))_i^-(t) &= \Phi_-^s(t,-T_{i-1})\, a_{i-1}^- + \Phi_-^u(t,0)\, b_i^- + \\ &\quad \Big(\int_0^t \Phi_-^u(t,s)\, \lambda\, h_i^-(s)\, ds + \int_{-T_{i-1}}^t \Phi_-^s(t,s)\, \lambda\, h_i^-(s)\, ds\Big)\, d_i \\ (L_2(\lambda)(a,b,d))_i^+(t) &= \Phi_+^u(t,T_i)\, a_i^+ + \Phi_+^s(t,0)\, b_i^+ + \\ &\quad \Big(\int_0^t \Phi_+^s(t,s)\lambda\, h_i^+(s)\, ds + \int_{T_i}^t \Phi_+^u(t,s)\, \lambda\, h_i^+(s)\, ds\Big)\, d_i. \end{aligned}$$

Both operators are analytic in λ and bounded uniformly in (T_j) considered as operators with range in V_w. In fact, the estimates

$$(3.20) \quad \begin{aligned} \|L_1(\lambda)\, w\| &\le C(|G| + |\lambda|)\,\|w\| \le C\,\delta\,\|w\| \\ \|L_2(\lambda)(a,b,d)\| &\le C(|a| + |b| + |\lambda|\,|d|) \end{aligned}$$

14

hold owing to (3.10) and Lemma 3.2. Thus, the operator $id - L_1(\lambda)$ is invertible on V_w. The inverse $(id - L_1(\lambda))^{-1}$ is analytic in λ and we obtain the solution of (3.9)(i)

$$w = (id - L_1(\lambda))^{-1} L_2(\lambda)(a, b, d) =: W_1(\lambda)(a, b, d)$$

depending linearly on $(a, b, d) \in V_a \times V_b \times V_d$ and analytically on $\lambda \in V_\lambda$. The estimate

$$(3.21) \qquad \|W_1(\lambda)(a, b, d)\| \le C(|a| + |b| + |\lambda| \, |d|)$$

follows from (3.20). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the next step, we solve equation (3.9)(iv)

$$(3.22) \qquad\qquad w_j^+(T_j) - w_{j+1}^-(-T_j) = D_j d$$

for $j = 1, ..., N-1$.

Lemma 3.4 *There exist operators*

$$A_1 \ : \ V_\lambda \times V_b \times \mathbb{C}^N \to V_a$$
$$W_2 \ : \ V_\lambda \times V_b \times \mathbb{C}^N \to V_w,$$

such that
$$(3.23) \qquad\qquad (a, w) := (A_1(\lambda)(b, d), W_2(\lambda)(b, d))$$

solves (3.14) and (3.22) for any (b, d) and λ. In addition, any bounded solution of these equations is given by (3.23). The operators A_1 and W_2 are analytic in λ, linear in (b, d) and fulfill

$$(3.24) \qquad
\begin{aligned}
|A_1(\lambda)(b, d)| &\le C\big((e^{-\alpha T} + |G| + |\lambda|) \, |b| + (|D| + |\lambda|) \, |d|\big) \\
\|W_2(\lambda)(b, d)\| &\le C(|b| + (|D| + |\lambda|) \, |d|)
\end{aligned}$$

uniformly in (T_j). Moreover, we have

$$(3.25) \qquad
\begin{aligned}
(A_1(\lambda)(b, d))_j^- &= -P_0^s D_j d + (A_2(\lambda)(b, d))_j^- \\
(A_1(\lambda)(b, d))_j^+ &= P_0^u D_j d + (A_2(\lambda)(b, d))_j^+,
\end{aligned}$$

for a bounded operator A_2 admitting the estimate

$$(3.26) \qquad |A_2(\lambda)(b, d)| \le C\big((e^{-\alpha T} + |G| + |\lambda|) \, |b| + ((p(T) + |G|) \, |D| + |\lambda|) \, |d|\big)$$

with $p(T) := \sup_{t \ge T}(|P_+^u(t) - P_0^u| + |P_-^s(-t) - P_0^s|)$.

15

Proof. Using (3.14) and (3.13) we obtain

$$(3.27) \quad D_j\, d \;=\; a_j^+ - a_j^- + \left(P_+^u(T_j) - P_0^u\right) a_j^+ + \left(P_0^s - P_-^s(-T_j)\right) a_j^- +$$

$$\Phi_+^s(T_j, 0)\, b_j^+ - \Phi_-^u(-T_j, 0)\, b_{j+1}^- +$$

$$\int_0^{T_j} \Phi_+^s(T_j, s)\Big(\big(G_j^+(s) + \lambda B_j^+(s)\big) w_j^+(s) + \lambda\, h_j^+(s)\, d_j\Big)\, ds \;+$$

$$\int_{-T_j}^0 \Phi_-^u(-T_j, s)\Big(\big(G_{j+1}^-(s) + \lambda B_{j+1}^-(s)\big) w_{j+1}^-(s) + \lambda\, h_{j+1}^-(s)\, d_{j+1}\Big)\, ds$$

for $j = 1, ..., N-1$. Here, $P_+^s(t)$ and $P_-^u(t)$ are defined in Lemma 3.2. We substitute $w = W_1(\lambda)(a, b, d)$ from Lemma 3.3 into (3.27). Then (3.27) defines an equation

$$(3.28) \qquad\qquad D_j\, d = a_j^+ - a_j^- + \left(L_3(\lambda)(a, b, d)\right)_j$$

linear in (a, b, d) and analytic in λ. Owing to the estimates in Lemma 3.2 and (3.18), (3.16) we obtain

$$
\begin{aligned}
(3.29) \quad |L_3(\lambda)(a, b, d)| &\leq C\Big(p(T)\,|a| + e^{-\alpha T}\,|b| + (|G| + |\lambda|)\,\|W_1(\lambda)(a, b, d)\| + |\lambda|\,|d|\Big) \\
&\leq C\Big((p(T) + |G| + |\lambda|)\,|a| + (e^{-\alpha T} + |G| + |\lambda|)\,|b| + |\lambda|\,|d|\Big) \\
&\leq C\,\delta\,(|a| + |b| + |d|)
\end{aligned}
$$

for T large, where $p(T)$ was defined in the statement of the Lemma. The map J_1 defined by

$$
\begin{aligned}
V_a &\;\to\; \oplus_{j=1}^{N-1} \mathbb{C}^n \\
(a_j^+, a_j^-) &\;\mapsto\; (a_j^+ - a_j^-)
\end{aligned}
$$

is a linear isomorphism due to $E^u \times E^s = \mathbb{C}^n$. For $\delta > 0$ sufficiently small, we can therefore invert the operator

$$a \mapsto J_1\, a + L_3(\lambda)(a, 0, 0),$$

yielding the solution operator

$$(3.30) \qquad a = \left(J_1 + L_3(\lambda)I_1\right)^{-1} \left(D\, d - L_3(\lambda)(0, b, d)\right) =: A_1(\lambda)(b, d)$$

of (3.28) using the definition $I_1\, a = (a, 0, 0)$. Moreover, we define

$$W_2(\lambda)(b, d) := W_1(\lambda)(A_1(\lambda)(b, d), b, d).$$

Then

$$(\lambda, a, b, d, w) = (\lambda, A_1(\lambda)(b, d), b, d, W_2(\lambda)(b, d))$$

solve (3.14) and (3.22) by Lemma 3.3 and the statements above. From (3.29) we conclude

$$|A_1(\lambda)(b, d)| \leq C\Big((e^{-\alpha T} + |G| + |\lambda|)\,|b| + (|D| + |\lambda|)\,|d|\Big),$$

16

which proves the first part of (3.24). Using this estimate together with (3.16) and the definition of W_2 yields the second part of (3.24), namely

$$\|W_2(\lambda)(a,b,d)\| \leq C(|b| + (|D| + |\lambda|)\,|d|).$$

It remains to prove the decomposition (3.25). Projecting (3.28) into E^s and E^u by P_0^s and P_0^u yields

(3.31)
$$
\begin{aligned}
a_j^- &= -P_0^s D_j\, d + P_0^s\, (L_3(\lambda)(a,b,d))_j \\
a_j^+ &= P_0^u D_j\, d - P_0^u\, (L_3(\lambda)(a,b,d))_j.
\end{aligned}
$$

Substituting $a = A_1(\lambda)(b,d)$ into (3.31) gives

$$
\begin{aligned}
(A_1(\lambda)(b,d))_j^- &= -P_0^s D_j\, d + P_0^s\, (L_3(\lambda)(A_1(\lambda)(b,d),b,d))_j \\
&=: -P_0^s D_j\, d + (A_2(\lambda)(b,d))_j^- \\
(A_1(\lambda)(b,d))_j^+ &= P_0^u D_j\, d - P_0^u\, (L_3(\lambda)(A_1(\lambda)(b,d),b,d))_j \\
&=: P_0^u D_j\, d + (A_2(\lambda)(b,d))_j^+.
\end{aligned}
$$

Using the estimates (3.29) for L_3 and (3.24) for A_1 finally gives (3.26). This proves the lemma. $\qquad\square$

We have to investigate the remaining equations (3.9)(ii) and (iii)

(3.32)
$$
\begin{aligned}
w_i^-(0) &\in \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^- \\
w_i^+(0) &\in \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^- \\
w_i^+(0) - w_i^-(0) &\in \mathbb{C}\psi_1(0),
\end{aligned}
$$

for $i = 1, ..., N$, where we substitute $w = W_2(\lambda)(b,d)$ defined in Lemma 3.4. Owing to (3.14), we have

(3.33)
$$
\begin{aligned}
w_i^-(0) &= \Phi_-^s(0, -T_{i-1})\, a_{i-1}^- + b_i^- + \\
&\quad \int_{-T_{i-1}}^0 \Phi_-^s(0,s)\big((G_i^-(s) + \lambda B_i^-(s))\, w_i^-(s) + \lambda h_i^-(s)\, d_i\big)\, ds \\
w_i^+(0) &= \Phi_+^u(0, T_i)\, a_i^+ + b_i^+ + \\
&\quad \int_{T_i}^0 \Phi_+^u(0,s)\big((G_i^+(s) + \lambda B_i^+(s))\, w_i^+(s) + \lambda h_i^+(s)\, d_i\big)\, ds,
\end{aligned}
$$

with w and a given by (3.23). By definition (3.12) and (3.13), we can decompose b in a unique way according to

(3.34)
$$
\begin{aligned}
b_i^- &= x_i^- + y_i^- \\
b_i^+ &= x_i^+ + y_i^+,
\end{aligned}
$$

such that $x_i^\pm \in \mathbb{C}\dot{q}_1(0)$ and $y_i^\pm \in Y^\pm$, respectively. In other words,

$$
\begin{aligned}
b &= x + y \\
x &\in \bigoplus_{i=1}^N \mathbb{C}\dot{q}_1(0) \times \mathbb{C}\dot{q}_1(0) \\
y &\in \bigoplus_{i=1}^N Y^- \times Y^+.
\end{aligned}
$$

Next we use the fact (2.5), namely that

$$\mathbb{C}^n = \mathbb{C}\psi_1(0) \oplus \mathbb{C}\dot{q}_1(0) \oplus Y^+ \oplus Y^-.$$

Then (3.32) is equivalent to the system of equations

$$
\begin{aligned}
P(\mathbb{C}\dot{q}_1(0), \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^-) \, w_i^-(0) &= 0 \\
P(\mathbb{C}\dot{q}_1(0), \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^-) \, w_i^+(0) &= 0 \\
P(Y^+ \oplus Y^-, \mathbb{C}\dot{q}_1(0) \oplus \mathbb{C}\psi_1(0)) \, (w_i^+(0) - w_i^-(0)) &= 0,
\end{aligned}
$$

(3.35)

where $P(X, Y)$ denotes the projection onto X with kernel Y. Indeed, note that the component of $w_i^-(0)$ and $w_i^+(0)$ in $\mathbb{C}\dot{q}_1(0)$ vanish due to the first two equations, whence it is sufficient to project onto $Y^+ \oplus Y^-$ in the third equation.

Lemma 3.5 *There exist operators*

$$
\begin{aligned}
B_1 &: \; V_\lambda \times \mathbb{C}^N \to V_b \\
A_3 &: \; V_\lambda \times \mathbb{C}^N \to V_a \\
W_3 &: \; V_\lambda \times \mathbb{C}^N \to V_w,
\end{aligned}
$$

such that

(3.36) $$(a, b, w) := (A_3(\lambda)\, d, B_1(\lambda)\, d, W_3(\lambda)\, d)$$

solves (3.14), (3.22) and (3.32) for any d and λ. Any bounded solution of these equations is given by (3.36). The operators A_3, B_1 and W_3 are analytic in λ and linear in d. They satisfy

$$
\begin{aligned}
|A_3(\lambda)\, d| &\leq C(|D| + |\lambda|)\, |d| \\
|B_1(\lambda)\, d| &\leq C((e^{-\alpha T} + |G|)\, |D| + |\lambda|)\, |d| \\
\|W_3(\lambda)\, d\| &\leq C(|D| + |\lambda|)\, |d|
\end{aligned}
$$

(3.37)

uniformly in (T_j). Again $A_3(\lambda)$ can be written according to

(3.38) $$
\begin{aligned}
(A_3(\lambda)\, d)_j^- &= -P_0^s D_j\, d + (A_4(\lambda)\, d)_j^- \\
(A_3(\lambda)\, d)_j^+ &= P_0^u D_j\, d + (A_4(\lambda)\, d)_j^+,
\end{aligned}
$$

for some operator A_4 linear in d and analytic in λ such that

(3.39) $$|A_4(\lambda)\, d| \leq C((e^{-\alpha T} + |G|)\, |D| + |\lambda|)\, |d|$$

holds.

18

Proof. First we substitute (3.33) into (3.35). Using (3.34) we obtain therefore the equations

$$
\begin{aligned}
0 &= x_i^- + P(\mathbb{C}\dot{q}_1(0), \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^-)\Big(\Phi_-^s(0, -T_{i-1})\, a_{i-1}^- + \\
&\qquad \int_{-T_{i-1}}^0 \Phi_-^s(0,s)\Big(\big(G_i^-(s) + \lambda B_i^-(s)\big) w_i^-(s) + \lambda h_i^-(s)\, d_i\Big)\, ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
(3.40) \quad 0 &= x_i^+ + P(\mathbb{C}\dot{q}_1(0), \mathbb{C}\psi_1(0) \oplus Y^+ \oplus Y^-)\Big(\Phi_+^u(0, T_i)\, a_i^+ + \\
&\qquad \int_{T_i}^0 \Phi_+^u(0,s)\Big(\big(G_i^+(s) + \lambda B_i^+(s)\big) w_i^+(s) + \lambda h_i^+(s)\, d_i\Big)\, ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
0 &= y_i^+ - y_i^- + \\
&\quad P(Y^+ \oplus Y^-, \mathbb{C}\dot{q}_1(0) \times \mathbb{C}\psi_1(0))\Big(\Phi_+^u(0, T_i)\, a_i^+ - \Phi_-^s(0, -T_{i-1})\, a_{i-1}^- + \\
&\qquad \int_{T_i}^0 \Phi_+^u(0,s)\Big(\big(G_i^+(s) + \lambda B_i^+(s)\big) w_i^+(s) + \lambda h_i^+(s)\, d_i\Big)\, ds - \\
&\qquad \int_{-T_{i-1}}^0 \Phi_-^s(0,s)\Big(\big(G_i^-(s) + \lambda B_i^-(s)\big) w_i^-(s) + \lambda h_i^-(s)\, d_i\Big)\, ds\Big).
\end{aligned}
$$

They can be written in the form

$$
(3.41) \qquad \begin{pmatrix} x_i^- \\ x_i^+ \\ y_i^+ - y_i^- \end{pmatrix} + (L_4(\lambda)(b,d))_i = 0
$$

by substituting (3.23). Owing to Lemma 3.2 and (3.24), we obtain the estimate

$$
\begin{aligned}
(3.42) \quad |L_4(\lambda)(b,d)| &\leq C\Big(e^{-\alpha T}|a| + (|G| + |\lambda|)\,|w| + |\lambda|\,|d|\Big) \\
&\leq C\Big(e^{-\alpha T}\big((e^{-\alpha T} + |G| + |\lambda|)\,|b| + (|D| + |\lambda|)\,|d|\big) + \\
&\qquad (|G| + |\lambda|)\big(|b| + (|D| + |\lambda|)\,|d|\big) + |\lambda|\,|d|\Big) \\
&\leq C\Big((e^{-2\alpha T} + |G| + |\lambda|)\,|b| + \big((e^{-\alpha T} + |G|)\,|D| + |\lambda|\big)\,|d|\Big) \\
&\leq C\Big((e^{-2\alpha T} + |G| + |\lambda|)\,(|x| + |y|) + \big((e^{-\alpha T} + |G|)\,|D| + |\lambda|\big)\,|d|\Big) \\
&\leq C\,\delta\,(|x| + |y|) + C\big((e^{-\alpha T} + |G|)\,|D| + |\lambda|\big)\,|d|.
\end{aligned}
$$

Note that $b = x + y$ by (3.34). Moreover, the map J_2 defined by

$$
\begin{aligned}
\Big(\oplus_{i=1}^N \mathbb{C}\dot{q}_1(0) \oplus \mathbb{C}\dot{q}_1(0)\Big) \oplus \Big(\oplus_{i=1}^N Y^- \oplus Y^+\Big) &\to \oplus_{i=1}^N \mathbb{C}\dot{q}_1(0) \oplus \mathbb{C}\dot{q}_1(0) \oplus (Y^- \oplus Y^+) \\
((x_i^-, x_i^+), (y_i^-, y_i^+)) &\mapsto (x_i^-, x_i^+, y_i^+ - y_i^-)
\end{aligned}
$$

is an isomorphism due to (2.5). Thus, the operator

$$
(x, y) \mapsto J_2(x + y) + L_4(\lambda)(x + y, 0)
$$

is invertible due to (3.42), whence

$$
(3.43) \qquad b = -(J_2 + L_4(\lambda)I_1)^{-1}\, L_4(\lambda)(0, d) =: B_1(\lambda)\, d
$$

19

solves (3.41). Here, we used the definition $I_1 b = (b,0)$. From (3.42) we conclude the estimate

(3.44)
$$|B_1(\lambda)\,d| \leq C((e^{-\alpha T} + |G|)\,|D| + |\lambda|)\,|d|.$$

Substituting $b = B_1(\lambda)\,d$ into the operators A_1 and W_2 given in (3.23) defines A_3 and W_3. The estimate (3.37) follows now easily from (3.44) and (3.24). In the same way, one obtains

$$A_4(\lambda)\,d = A_2(\lambda)(B_1(\lambda)\,d, d)$$

using (3.25) and the estimate (3.39) by using (3.26). The lemma is proved. □

Let us summarize the results obtained so far. By the Lemmata 3.3, 3.4 and 3.5 we have shown that (λ, a, b, d, w) is a bounded solution of (3.9) if and only if it is given by

(3.45)
$$(\lambda, a, b, d, w) := (\lambda, A_3(\lambda)\,d, B_1(\lambda)\,d, d, W_3(\lambda)\,d).$$

With (λ, a, b, d, w) at hand, the equations

$$w_i^+(0) = w_i^-(0) \qquad i = 1, ..., N$$

are fulfilled if and only if the jumps

$$\xi_i = \langle \psi_1(0), w_i^+(0) - w_i^-(0) \rangle$$

vanish identically. Indeed, then $w_i^+(0) = w_i^-(0)$ holds due to (3.9)(iii). In the next step, we derive formulae for the jumps ξ_i.

Lemma 3.6 *Let (λ, a, b, d, w) be given by (3.45). Then the relation $w_i^+(0) = w_i^-(0)$ holds for $i = 1, ..., N$ if and only if*

$$\xi_i = \langle \psi_1(0), w_i^+(0) - w_i^-(0) \rangle = 0.$$

Moreover, the jumps ξ_i can be written as

(3.46) $\xi_i = \langle \psi(T_i), P_0^u D_i\,d \rangle + \langle \psi(-T_{i-1}), P_0^s D_{i-1}\,d \rangle - \lambda \int_{-\infty}^{\infty} \langle \psi(t), H(t) \rangle\,dt\,d_i + (R(\lambda)\,d)_i$

for some function $R : V_\lambda \times \mathbb{C}^N \to \mathbb{C}^N$ analytic in λ and linear in d admitting the estimate

(3.47) $|R(\lambda)\,d| \leq C\big((e^{-\alpha T} + |G|)^2\,|D| + (e^{-\alpha T} + |G| + \|h_i - H\| + |\lambda|)\,|\lambda|\big)\,|d|.$

Proof. Throughout, we choose (λ, a, b, d, w) according to (3.45). Hence, ξ_i is linear in d and analytic in λ. Owing to Lemma 2.2 and (3.33) we obtain

$$(3.48) \qquad \langle \psi_1(0), w_i^+(0) - w_i^-(0) \rangle = \langle \psi(\mu)(0), w_i^+(0) - w_i^-(0) \rangle$$

$$= \Big\langle \psi(\mu)(0), \Phi_+^u(0, T_i)\, a_i^+ - \Phi_-^s(0, -T_{i-1})\, a_{i-1}^- +$$

$$\int_{T_i}^0 \Phi_+^u(0, s) \left((G_i^+(s) + \lambda B_i^+(s))\, w_i^+(s) + \lambda h_i^+(s)\, d_i \right) ds -$$

$$\int_{-T_{i-1}}^0 \Phi_-^s(0, s) \left((G_i^-(s) + \lambda B_i^-(s))\, w_i^-(s) + \lambda h_i^-(s)\, d_i \right) ds \Big\rangle$$

$$= \big\langle \psi(T_i), a_i^+ \big\rangle - \big\langle \psi(-T_{i-1}), a_{i-1}^- \big\rangle -$$

$$\int_0^{T_i} \big\langle \psi(t), (G_i^+(t) + \lambda B_i^+(t))\, w_i^+(t) + \lambda h_i^+(t)\, d_i \big\rangle dt -$$

$$\int_{-T_{i-1}}^0 \big\langle \psi(t), (G_i^-(t) + \lambda B_i^-(t))\, w_i^-(t) + \lambda h_i^-(t)\, d_i \big\rangle dt$$

by using Lemma 2.2. Next we estimate the expressions appearing in (3.48) term by term. Substituting (3.38) yields

$$(3.49)\ \ \langle \psi(T_i), a_i^+ \rangle - \langle \psi(-T_{i-1}), a_{i-1}^- \rangle =$$

$$\langle \psi(T_i), P_0^u D_i\, d \rangle + \langle \psi(-T_{i-1}), P_0^s D_i\, d \rangle + \mathcal{O}\Big(e^{-\alpha T} \left((e^{-\alpha T} + |G|)\, |D| + |\lambda| \right) |d| \Big)$$

using the estimate (3.39) and

$$(3.50) \qquad\qquad |\psi(t)| \le C e^{-\alpha |t|}.$$

Owing to (3.50) we also obtain

$$(3.51) \qquad \int_0^{T_i} \langle \psi(t), h_i^+(t)\, d_i \rangle dt + \int_{-T_{i-1}}^0 \langle \psi(t), h_i^-(t)\, d_i \rangle dt =$$

$$\int_{-\infty}^{\infty} \langle \psi(t), H(t) \rangle dt + \mathcal{O}\Big(e^{-\alpha T} \|H\| + \|h_i - H\| \Big).$$

Substituting (3.37) obtains

$$(3.52)\ \ \left| \int_0^{T_i} \langle \psi(t), B_i^+(t)\, w_i^+(t) \rangle dt + \int_{-T_{i-1}}^0 \langle \psi(t), B_i^-(t)\, w_i^-(t) \rangle dt \right| \le C(|D| + |\lambda|)\, |d|.$$

Therefore, it remains to estimate the expression

$$(3.53) \qquad \int_0^{T_i} \langle \psi(t), G_i^+(t)\, w_i^+(t) \rangle dt + \int_{-T_{i-1}}^0 \langle \psi(t), G_i^-(t)\, w_i^-(t) \rangle dt.$$

In order to get a sharper estimate, we have to exploit the inequalities

$$|w_i^-(t)| \le C \Big(e^{-\alpha^s(T_{i-1}-t)} |a_{i-1}^-| + |b_i^-| + (|G| + |\lambda|) \|w\| + |\lambda|\, |d| \Big)$$

$$\le C \Big((e^{-\alpha^s(T_{i-1}-t)} + e^{-\alpha T} + |G|)\, |D| + |\lambda| \Big) |d|$$

$$|w_i^+(t)| \le C \Big(e^{-\alpha^u(T_i-t)} |a_i^+| + |b_i^+| + (|G| + |\lambda|) \|w\| + |\lambda|\, |d| \Big)$$

$$\le C \Big((e^{-\alpha^u(T_i-t)} + e^{-\alpha T} + |G|)\, |D| + |\lambda| \Big) |d|,$$

which follow from (3.14), (3.37) and Lemma 3.2. Substituting these estimates into (3.53) yields

$$(3.54) \qquad \left| \int_0^{T_i} \langle \psi(t), G_i^+(t)\, w_i^+(t) \rangle \, dt \right| \leq \left| \int_0^{T_i} |\psi(t)|\, |G|\, |w_i^+(t)| \, dt \right|$$

$$\leq C \left| \int_0^{T_i} e^{-\alpha t}\, |G|\, |w_i^+(t)| \, dt \right|$$

$$\leq C\, |G| \left((e^{-\alpha T} + |G|)\, |D| + |\lambda| \right) |d|$$

and similarly

$$(3.55) \qquad \left| \int_{-T_{i-1}}^0 \langle \psi(t), G_i^-(t)\, w_i^-(t) \rangle \, dt \right| \leq C\, |G| \left((e^{-\alpha T} + |G|)\, |D| + |\lambda| \right) |d|.$$

Here, we made the constant α appearing in (3.50) a bit larger than $\min(\alpha^s, \alpha^u)$, which is possible due to (H1).

Summarizing the estimates (3.49), (3.51), (3.52) and (3.54), (3.55) yields

$$\xi_i = \langle \psi(T_i), P_0^u D_i\, d \rangle + \langle \psi(-T_{i-1}), P_0^s D_{i-1}\, d \rangle - \lambda \int_{-\infty}^\infty \langle \psi(t), H(t) \rangle \, dt\, d_i + (R(\lambda)\, d)_i$$

and

$$|R(\lambda)\, d| \leq C \left((e^{-\alpha T} + |G|)^2\, |D| + |\lambda|\, (e^{-\alpha T} + |G| + \|h_i - H\| + |\lambda|) \right) |d|.$$

This proves the lemma. $\qquad \square$

3.3 The substitution

Finally, we return to the original equations (3.6). Substituting the estimates obtained in Lemma 3.1 into the inequalities appearing in Lemma 3.6 shows that

$$(3.56) \quad \begin{aligned} \xi_i &= \langle \psi(T_i), \dot{q}^-(-T_i) \rangle\, (d_{i+1} - d_i) + \langle \psi(-T_{i-1}), \dot{q}^+(T_{i-1}) \rangle\, (d_i - d_{i-1}) - \\ & \int_{-\infty}^\infty \langle \psi_1(t), B(q_1(t), \mu_1)\, \dot{q}_1(t) \rangle \, dt\, \lambda\, d_i + (R(\lambda)\, d)_i \end{aligned}$$

and

$$(3.57) \quad |R(\lambda)\, d| \leq C \left(e^{-2\alpha T} \sup_{t \geq T}(|q^+(t)| + |q^-(-t)|) + |\lambda|\, (e^{-\alpha T} + |\mu_N - \mu_1| + |\lambda|) \right) |d|.$$

By Lemma 3.6, (3.6) is solved iff (3.7) and $\xi = 0$ are solved with the substitutions made in (3.8). Therefore, Theorem 2 is proved.

Remark By examining (3.8) and Lemma 3.6, we observe that the reduced function $E(\lambda)$ associated to an N-pulse is a small perturbation of $E_1^N(\lambda)$. Here, $E_1(\lambda)$ denotes the reduced function associated to the primary pulse.

Finally, for future reference, let us define the solution operator

$$(3.58) \qquad\qquad v = V(\lambda)\,d = \mathcal{U}\,d + W(\lambda)\,d$$

according to (3.4) and (3.5), where we substituted (3.45) for w.

4 A multiplicity lemma

In this section we consider the steady state equation associated to (1.2), that is, equation (1.3). We are going to subsume the term $c\,U_\xi$ into either the nonlinearity or the operator $\mathcal{A}$. This is to avoid difficulties arising if the wave speed $c = 0$ equals zero and the original equation (1.3) contains a coupled ODE. We are not considering this case.

Therefore, consider the equation

$$(4.1) \qquad\qquad \mathcal{A}\,U + F(U,\mu) = 0,$$

where $\mathcal{A}$ is given by

$$(4.2) \qquad \mathcal{A}\,U = \begin{pmatrix} D_0 \frac{d}{dx} & & \cdot & \\ & D_1\,\Delta & & \\ & & \ddots & \\ & & & (-1)^{l+1} D_l\,\Delta^l \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ \vdots \\ U_l \end{pmatrix}.$$

Here, $U = (U_0, ..., U_l) \in BU(\mathbb{R}, \mathbb{R}^k)$ is uniformly continuous with components satisfying $U_j \in BU(\mathbb{R}, \mathbb{R}^{k_j})$. The matrices D_j are positive definite. Therefore, $\mathcal{A}$ is a linear, densely defined and closed operator on $BU(\mathbb{R}, \mathbb{R}^k)$. Of course, $\mathcal{A}$ need not to contain all of the blocks appearing in (4.2).

Next we define the operators Γ_j and Γ by

$$\begin{array}{llll} \Gamma_j & : & BU(\mathbb{R}, \mathbb{R}^{k_j}) \to BU(\mathbb{R}, \mathbb{R}^{n_j}) & \Gamma_j U_j = (U_j, \frac{d}{dx} U_j, ..., \frac{d^{2j-1}}{dx^{2j-1}} U_j) \\ \Gamma & : & BU(\mathbb{R}, \mathbb{R}^k) \to BU(\mathbb{R}, \mathbb{R}^n) & \Gamma U = (\Gamma_0 U_0, ..., \Gamma_l U_l), \end{array}$$

for $n_j = 2j\,k_j$ and $n = \sum_{j=0}^{l} n_j$. Then Γ is densely defined and closed. Moreover, let

$$\begin{array}{llll} B_j & : & \mathbb{R}^{k_j} \to \prod_{i=1}^{2j} \mathbb{R}^{k_j} = \mathbb{R}^{n_j} & B_j U_j = (0, ..., 0, U_j) \\ B & : & \mathbb{R}^k \to \mathbb{R}^n & BU = (B_0 U_0, ..., B_l U_l). \end{array}$$

Now let $u = (u_0, ..., u_l) \in \mathbb{R}^n$ such that $u_j \in \mathbb{R}^{n_j}$ for $j = 0, ..., l$. We extend the matrices B_j and B to $\mathbb{R}^{n_j}$ and $\mathbb{R}^n$, respectively, by projecting

$$\mathbb{R}^{n_j} = \prod_{i=1}^{2j} \mathbb{R}^{k_j} \to \mathbb{R}^{k_j}$$

onto the first component of the product along the others. Finally define the Jordan-block matrices

$$C_j : \mathbb{R}^{n_j} \to \mathbb{R}^{n_j} \qquad (C_j u_j)_i \quad := \quad u_{j,i+1} \quad i = 1, \ldots, n_j - 1$$
$$(C_j u_j)_{n_j} \quad = \quad 0$$

and let $Cu = (C_0 \, u_0, \ldots, C_l \, u_l)$.

The nonlinearity $F(U, \mu)$ is a Nemitskii operator defined by

$$(4.3) \qquad F(U, \mu)(x) = G\Big(\Big(\frac{d^i}{dx^i} U_j\Big)_{i=0,\ldots,2j-1; j=0,\ldots,l}(x), \mu\Big) = G((\Gamma U)(x), \mu),$$

where $G : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^k$ is smooth. Here, $\mu \in \mathbb{R}^p$.

Owing to (4.2) and (4.3), the partial differential equation (4.1) is equivalent to a first order ordinary differential equation. Indeed, $u = \Gamma U$ solves

$$(4.4) \qquad\qquad u' = Cu - BG(u, \mu) =: f(u, \mu)$$

if and only if U solves

$$\mathcal{A} U + F(U, \mu) = 0.$$

Suppose that U_1 is a primary pulse solution of (4.1) for $\mu = \mu_1$ with corresponding steady-state $\bar{U}$, see section 2. Without loss of generality, we assume that the homogeneous equilibrium $\bar{U} = 0$ is zero. Let U_N denote an N-pulse relative to U_1. Then the linearizations of (4.1) at U_1 and U_N are given by the densely defined closed operators

$$L_1 \quad = \quad \mathcal{A} + D_U F(U_1, \mu_1)$$
$$L_N \quad = \quad \mathcal{A} + D_U F(U_N, \mu_N),$$

on $BU(\mathbb{R}, \mathbb{R}^k)$, respectively. The eigenvalue problem for L_N is given by

$$(4.5) \qquad\qquad L_N V = (\mathcal{A} + D_U F(U_N, \mu_N)) V = \lambda V.$$

Let $q_N = \Gamma U_N$ be the N-homoclinic solution of (4.4) corresponding to U_N. Then $v = \Gamma V$ is a bounded solution of

$$(4.6) \qquad v' = Cv + B(\lambda - D_u G(q_N, \mu_N)) v + BH = (D_u f(q_N, \mu_N) + \lambda B) v + BH$$

for $H \in BU(\mathbb{R}, \mathbb{R}^k)$ if and only if $V \in BU(\mathbb{R}, \mathbb{R}^k) \cap D(L)$ solves

$$L_N V = H + \lambda V.$$

Therefore, the approach given in section 2 is applicable to equation (4.5) provided the hypotheses (H1) and (H2) stated there are satisfied for (4.4). In particular, Jordan chains

$$(4.7) \qquad\qquad \begin{aligned} (L_N - \lambda_0) P_{1,i} &= 0 \\ (L_N - \lambda_0) P_{k,i} &= P_{k-1,i} \end{aligned}$$

associated to the operator L_N are given by solutions $v_{k,i}$

$$(4.8) \qquad \begin{aligned} v'_{1,i} &= (D_u f(q_N, \mu_N) + \lambda_0 B) v_{1,i} \\ v'_{k,i} &= (D_u f(q_N, \mu_N) + \lambda_0 B) v_{k,i} + B v_{k-1,i} \end{aligned}$$

for $i = 1, ..., m$ and $1 < k \le p(i)$. We suppose here that the geometric multiplicity of λ_0 is equal to m.

Owing to (H1), zero is not contained in the essential spectrum of the linearization of (4.1) evaluated at the trivial equilibrium $\bar{U} = 0$. By [Hen81, Thm. A.1], the following alternative holds: either zero is not contained in the essential spectrum of the linearization L_1 of (4.1) evaluated at the primary pulse U_1 or there exists a $\delta > 0$ such that any $\lambda \in U_\delta(0)$ is an eigenvalue of L_1. Indeed, L_1 is a relatively compact perturbation of the linearization at $\bar{U}$. This follows easily from Arzela-Ascoli's theorem and the fact that $U_1(x)$ converges to zero for $|x|$ tending to infinity. The eigenvalues close to zero associated with U_1 coincide with zeroes of $E_1(\lambda)$ by Theorem 2 and the above discussion. Here, $E_1(\lambda)$ is the function described in Theorem 2 when applied to the primary pulse itself. Therefore, zero is contained in the essential spectrum of L_1 iff $E_1(\lambda) = 0$ for $\lambda \in U_\delta(0)$. We will assume that zero is a normal point of the spectrum.

(H3) Zero is not contained in $\sigma_{ess}(L_1)$.

Then $E_1(\lambda)$ is not identically equal to zero. By the remark in section 3.3, we conclude that the function $E(\lambda)$ associated to the N-pulse U_N does not vanish neither. Hence, the essential spectrum of L_N is bounded away from zero once (H3) is fulfilled. Therefore, the isolated zeroes of $E(\lambda)$ close to zero correspond to eigenvalues of L_N with finite algebraic multiplicity and the remainder part of $\sigma(L_N)$ is bounded away from zero.

Lemma 4.1 *Assume that (H1) - (H3) are fulfilled. Then $\lambda_0 \in \sigma(L_N) \cap U_\delta(0)$ if and only if $E(\lambda_0) = 0$. Moreover, $mult(\lambda_0; L_N) = ord(\lambda_0; E)$, i.e., the algebraic multiplicity of the eigenvalue λ_0 of L_N coincides with the order of λ_0 as a zero of $E(\lambda)$.*

In [GJ90, Lemmata 5, 6] the multiplicity of the Evans function is related to the algebraic multiplicity of the eigenvalue. However, all attempts of the author to prove directly a relation between the Evans function investigated in [AGJ90, GJ90] to the function $E(\lambda)$ studied here failed.

Throughout, we fix an eigenvalue λ_0 of L_N close to zero. Owing to the estimates of $W(\lambda)$ in Lemma 3.5, the operator $V(\lambda) = \mathcal{U} + W(\lambda)$ appearing in (3.58) is injective. Using this fact and Theorem 2 we conclude that there exist precisely m linear independent vectors

$d_{1,1}, ..., d_{1,m} \in \mathbb{R}^N$ such that $S(\lambda_0) d_{1,i} = 0$ and

$$\Gamma P_{1,i} = V(\lambda_0) d_{1,i}$$

for $i = 1, ..., m$. Let $V(\lambda) d_{1,i} =: v_{1,i}(\lambda)$. In particular, we obtain $m = \dim \mathrm{Ke}\, S(\lambda_0)$. Next define

$$w_{k,i}(\lambda) := \sum_{j=1}^{k} \frac{1}{(k-j)!} W^{(k-j)}(\lambda) d_{j,i} = \sum_{j=0}^{k-1} \frac{1}{j!} W^{(j)}(\lambda) d_{k-j,i},$$

for $k \geq 2$, where we used the notation $W^{(j)} = \frac{d^j}{d\lambda^j} W$. Moreover, let $v_{k,i}(\lambda)$ be given according to

$$v_{k,i}(\lambda) = \mathcal{U} d_{k,i} + w_{k,i}(\lambda),$$

see (3.58). Owing to the analyticity of $W(\lambda)$, the functions $w_{k,i}(\lambda)$ are well-defined.

Lemma 4.2 *Let $p(i) \in \mathbb{N}$ for $i = 1, ..., m$. The functions $v_{k,i}(\lambda_0)$ solve (4.8) for $i = 1, ..., m$ and $1 < k \leq p(i)$ if and only if the vectors $d_{k,i}$ solve*

$$(4.9) \qquad \sum_{j=0}^{k-1} \frac{1}{j!} S^{(j)}(\lambda_0) d_{k-j,i} = 0$$

again for $i = 1, ..., m$ and $1 < k \leq p(i)$.

Proof. It is straightforward to calculate that $w_{k,i}(\lambda)$ solve

$$\dot{w}_{k,i}^{\pm} = D_u f(q^{\pm}, \mu_N) w_{k,i}^{\pm} + (G^{\pm}(t) + \lambda B^{\pm}(t)) w_{k,i}^{\pm} + \lambda h^{\pm} d_{k,i} + B v_{k-1,i}(\lambda)$$

using (3.5) and the definition of $h^{\pm}$ in (3.8). The jumps $\xi_{k,i} = w_{k,i}^{+}(\lambda)(0) - w_{k,i}^{-}(\lambda)(0)$ are given by

$$(4.10) \qquad \xi_{k,i} = \sum_{j=0}^{k-1} \frac{1}{j!} S^{(j)}(\lambda_0) d_{k-j,i}.$$

Hence the functions $w_{k,i}(\lambda_0)$ are generalized eigenfunctions if and only if all jumps vanish. This proves the lemma. $\qquad \square$

Proof. (Lemma 4.1) By the equivalence of (4.7) and (4.8), Lemma 4.2 reformulates the existence of Jordan chains for the operator $(L_N - \lambda_0)$ in terms of solutions of the system

$$\sum_{j=0}^{k-1} \frac{1}{j!} S^{(j)}(\lambda_0) d_{k-j,i} = 0.$$

Defining

$$S_j := \tfrac{1}{j!} S^{(j)}(\lambda_0)$$

yields $S(\lambda) = \sum_{i=0}^{\infty} S_j (\lambda - \lambda_0)^j$. Computing solutions of (4.9) is therefore equivalent to the computation of the kernel of the matrix $\tilde{S}_M$ given by

$$(\tilde{S}_M)_{ij} = \begin{cases} 0 & i < j \leq M + 1 \\ S_{i-j} & j \leq i \leq M + 1. \end{cases}$$

for large M. An application of Lemma B.1 shows that the dimension of this kernel is equal to the order of $\det S(\lambda)$ at $\lambda = \lambda_0$. This proves the lemma. $\qquad\square$

5 Calculating zeroes of $E(\lambda)$

In this section we give some general strategies on solving the reduced equations (3.56)

$$(5.1) \qquad \xi = (A - M\lambda + R(\lambda))\, d = S(\lambda)\, d = 0$$

with respect to $\lambda \in \mathbb{C}$ close to zero and $d \in \mathbb{R}^N \setminus \{0\}$. Here, $A = (a_{ij})$ is given by

$$(5.2) \qquad a_{ij} = \begin{cases} \langle \psi(T_i), \dot{q}^-(-T_i) \rangle & j = i + 1 \\ -\langle \psi(-T_{i-1}), \dot{q}^+(T_{i-1}) \rangle & j = i - 1 \\ \langle \psi(-T_{i-1}), \dot{q}^+(T_{i-1}) \rangle - \langle \psi(T_i), \dot{q}^-(-T_i) \rangle & j = i \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j = 1, ..., N$ and

$$(5.3) \qquad M = \int_{-\infty}^{\infty} \langle \psi_1(t), B(q_1(t), \mu_1)\, \dot{q}_1(t) \rangle\, dt \ \in \mathbb{R}.$$

Remember that $T_0 = T_N = \infty$ by definition and $T = \min_{j=1,...,N-1} T_j$. Observe that the vector $d = (1, ..., 1)$ is contained in the kernel of A. The function $R(\lambda)$ defined in Lemma 3.6 is analytic in λ. Moreover, the estimate (3.57) holds, i.e., $R(\lambda)$ is small. Note that $a_{ij} \to 0$ as $T \to \infty$. We anticipate the expansion

$$A(r) = r A_0 + o(r)$$

for some matrix A_0, where r is related to T by $r = r^*(T)$ such that $r \to 0$ corresponds to $T \to \infty$. The matrix A_0 is independent of r and thus of T. Such a scaling will frequently be possible in (5.1), see section 6.

Motivated by these facts, we investigate a linear equation of the form

$$(5.4) \qquad (r A_0 - M\lambda + r A_1(r) + A_2(r, \lambda)\, \lambda)\, d = 0$$

for matrices A_0, $A_1(r)$ and $A_2(r, \lambda)$. Apparently, (5.4) admits a solution (λ, d) with $d \neq 0$ if and only if the function

$$E(r, \lambda) := \det(S(\lambda)) = \det(r\, A_0 - M\, \lambda + r\, A_1(r) + A_2(r, \lambda)\, \lambda)$$

possesses a zero at λ. Throughout, we assume the following hypotheses.

(EV1) The spectrum of A_0 is given by $\sigma(A_0) = \{\nu_1, ..., \nu_{N-1}\} \cup \{0\} =: \mathcal{N} \cup \{0\}$ counted with multiplicity: $\#\mathcal{N} = N-1$. Suppose that $\nu_j \neq 0$ for $j = 1, ..., N-1$.

(EV2) The matrices $A_1(r)$ and $A_2(r, \lambda)$ are continuous in $r \in [0, r_0]$ and fulfill $A_1(0) = A_2(0, 0) = 0$. Moreover, $A_2(r, \lambda)$ is analytic in λ.

(EV3) $M \neq 0$.

Scaling λ by

(5.5)
$$\lambda = r\, \nu$$

we see that (5.4) is equivalent to

(5.6)
$$(A_0 - M\, \nu + A_1(r) + A_2(r, r\, \nu)\, \nu)\, d =: (A_0 - M\, \nu + A_3(r, \nu))\, d = 0$$

for $r \neq 0$. The matrix $A_3(r, \nu)$ is analytic in ν and satisfies

$$A_3(r, \nu) \to 0 \text{ as } r \to 0$$

uniformly on compact subsets in ν. Of course, (5.6) is equivalent to

(5.7)
$$\tilde{E}(r, \nu) := \det(A_0 - M\, \nu + A_3(r, \nu)) = 0,$$

where $\tilde{E}(r, \nu)$ is analytic in ν. Applying the Theorem of Rouché, we obtain the following lemma.

Lemma 5.1 *Assume (EV1)-(EV3). Then, for any $\eta > 0$ with $\eta < \frac{1}{2} dist(0, \mathcal{N})$, there exists an $r_0(\eta) > 0$ with the following property. For any $r < r_0(\eta)$, $\tilde{E}(r, \nu)$ possesses precisely $N-1$ zeroes in $U_\eta(\mathcal{N})$ counted with multiplicity and one zero in $U_\eta(0)$.*

From the preceeding discussion the next lemma is an immediate consequence of Lemma 5.1.

Lemma 5.2 *Assume (EV1)-(EV3). Then there exist r_0, $\delta_0 > 0$ such that $E(r, \lambda)$ possesses for any $r \in [0, r_0]$ fixed precisely N zeroes in $U_{\delta_0}(0)$ counted with multiplicity. They are related to the zeroes of $\tilde{E}(r, \nu)$ given in the previous Lemma according to the transformation (5.5). In particular, their respective signs coincide.*

Therefore, it is often sufficient to investigate the spectrum of the tridiagonal matrix A_0. Indeed, if zero is a simple eigenvalue of A_0 and the eigenvalues in $\mathcal{N}$ do not lie on the imaginary axis, the stability of the N-pulse is determined due to the Lemmata 5.2 and 5.1. In applications, the matrix A_0 frequently possesses additional properties.

If the differential equation (3.2) is generic, the matrix A_0 often turns out to be triangular. In that case, the spectrum of A_0 is easily determined, because A_0 is already tridiagonal and hence the entries on the diagonal coincide with the eigenvalues.

Another interesting case arises, if the underlying differential equation (3.2) is time-reversible. Then A_0 is actually a symmetric tridiagonal matrix.

Lemma 5.3 *Assume that there exists a linear involution $R : \mathbb{R}^n \to \mathbb{R}^n$ such that f anticommutes with R, i.e., $f(Ru, \mu) = -Rf(u, \mu)$ for all $(u, \mu) \in \mathbb{R}^n \times \mathbb{R}^p$. Moreover, assume that $n = 2m$ is even and $\mathrm{Fix}(R) = \mathbb{R}^m \times \{0\}$. Let $q_1(0) \in \mathrm{Fix}(R)$. Then the matrix A defined in (5.2) is symmetric.*

Proof. Indeed, $q_1(0) \in \mathrm{Fix}(R)$ implies that $\dot{q}^-(-t) = -R\dot{q}^+(t)$ and $\psi(t) = R^*\psi(-t)$ for all $t \geq 0$. Therefore

$$
\begin{aligned}
a_{i,i-1} &= \langle \psi(-T_{i-1}), \dot{q}^+(T_{i-1}) \rangle = -\langle R^* \psi(T_{i-1}), R\dot{q}^-(-T_{i-1}) \rangle \\
&= -\langle \psi(T_{i-1}), \dot{q}^-(-T_{i-1}) \rangle = a_{i-1,i}
\end{aligned}
$$

and A is symmetric. $\qquad\square$

While it seems difficult to express the eigenvalues in terms of the coefficients of a tridiagonal and symmetric matrix, it is possible to determine their signs.

Lemma 5.4 *Suppose that A_0 is a tridiagonal and symmetric $N \times N$ matrix. Assume that $(1, ..., 1)$ is contained in the kernel of A_0. Then $A_0 = (a_{ij}^0)$ is determined by a vector $(a_1, ..., a_{N-1}) \in \mathbb{R}^{N-1}$ via*

$$
a_{ij}^0 = \begin{cases}
-a_{i-1} & j = i - 1 \\
-a_i & j = i + 1 \\
a_i + a_{i-1} & j = i \\
0 & otherwise
\end{cases}
$$

for $i, j = 1, ..., N$ and with $a_0 = a_N = 0$. We define $\sigma(A_0) =: \{0\} \cup \tilde{\sigma}(A_0)$ counting multiplicity. Then

$$
\begin{aligned}
\#(\tilde{\sigma}(A_0) \cap \mathbb{R}^-) &= \#\{j \mid a_j < 0\} \\
\#(\tilde{\sigma}(A_0) \cap \mathbb{R}^+) &= \#\{j \mid a_j > 0\} \\
\#(\tilde{\sigma}(A_0) \cap 0) &= \#\{j \mid a_j = 0\}
\end{aligned}
$$

obtains.

Lemma 5.4 is proved in Appendix C.

Next we give an interpretation of M in terms of the Evans function $D(\lambda)$ associated to the primary pulse $q_1(t)$. This function is defined in the following way, see [AJ94, section 2]. First, choose orientations $[E^u]$ and $[E^s]$ of the unstable and stable eigenspaces, such that the concatenation of these orients $\mathbb{R}^n$ in the canonical way. Then these orientations are transported to the tangent spaces of unstable and stable manifolds of

$$\dot{u} = f(u, \mu)$$

using the linearized flow. Define $E^u(\lambda)(t)$ and $E^s(\lambda)(t)$ as the unstable and stable manifolds of

$$\dot{v} = \left(D_u f(q_1, \mu_1) + \lambda B(q_1, \mu_1)\right) v,$$

respectively. Then there exists unique volume forms $V^u(\lambda)$ and $V^s(\lambda)$ on $E^u(\lambda)(0)$ and $E^s(\lambda)(0)$, respectively, assigning the volume one to an orthonormal basis of these spaces. The Evans function is now defined by

$$D(\lambda) = V^u(\lambda) \wedge V^s(\lambda) \in \mathbb{R}.$$

Next, choose vectors $v_1^u, ..., v_{n_u-1}^u$ and $v_1^s, ..., v_{n_s-1}^s$ such that

$$
\begin{aligned}
T_{q_1(0)} W^u(0, \mu_1) &= \text{span}(\dot{q}_1(0), v_1^u, ..., v_{n_u-1}^u) \\
T_{q_1(0)} W^s(0, \mu_1) &= \text{span}(\dot{q}_1(0), v_1^s, ..., v_{n_s-1}^s)
\end{aligned}
$$

and

$$
\begin{aligned}
{}[\dot{q}_1(0), v_1^u, ..., v_{n_u-1}^u] &= [E^u] \\
[\dot{q}_1(0), v_1^s, ..., v_{n_s-1}^s] &= [E^s].
\end{aligned}
$$

Lemma 5.5 *The derivative of the Evans function associated to $q_1(t)$ is given by*

$$
\begin{aligned}
(5.8) \qquad D'(0) &= \left(\int_{-\infty}^{\infty} \langle \psi_1(t), B(q_1(t), \mu_1)\, \dot{q}_1(t) \rangle\, dt \right) \cdot \\
&\quad \left(\psi_1(0) \wedge v_1^u \wedge ... \wedge v_{n_u-1}^u \wedge \dot{q}_1(0) \wedge v_1^s \wedge ... \wedge v_{n_s-1}^s \right).
\end{aligned}
$$

Proof. The claim follows from [AGJ90, section 2, (2.15) and (2.17)] and the results given in section 3 applied to q_1 itself, i.e. for $N = 1$. Indeed, then the jump is given by

$$w_1^-(\lambda)(0) - w_1^+(\lambda)(0) = \left(\int_{-\infty}^{\infty} \langle \psi_1(t), B(q_1(t), \mu_1)\, \dot{q}_1(t) \rangle\, dt \right) \psi_1(0)\, \lambda$$

and the derivative with respect to λ yields the desired expression using [AGJ90, (2.17)] and the fact that volume forms are anti-symmetric. $\qquad\square$

Remark (i) If $\psi(0)$ is chosen such that

$$(5.9) \qquad \psi_1(0) \wedge v_1^u \wedge \ldots \wedge v_{n_u-1}^u \wedge \dot{q}_1(0) \wedge v_1^s \wedge \ldots \wedge v_{n_s-1}^s = 1 > 0$$

then

$$(5.10) \qquad D'(0) = \int_{-\infty}^{\infty} \langle \psi_1(t), B(q_1(t), \mu_1)\, \dot{q}_1(t) \rangle\, dt = M,$$

where M already appeared in Theorem 2. Here, an n-form in $\mathbb{R}^n$ is identified with $\mathbb{R}$ using the volume form distinguished by the orientation. For a one-dimensional unstable manifold, i.e. $n_u = 1$, this corresponds to a choice of ψ_1 such that

$$(5.11) \qquad \lim_{t \to \infty} e^{2\alpha^u t} \langle \psi_1(t), q_1(-t) \rangle > 0,$$

where α^u denotes the unique unstable eigenvalue of $D_u f(0, \mu_1)$. Note that this condition coincides with the sign convention in [PW92, (1.3)], who proved Lemma 5.5 under the assumption of one-dimensional unstable manifolds, see [PW92, Theorem 1.11].

(ii) It turns out that

$$D'(0) > 0$$

holds along stable primary pulses in several applications. In particular, it arises in systems of reaction-diffusion equations, see [AGJ90, Proposition 2.2] and [AJ94, Lemma 4.2 including proof], or for systems possessing one-dimensional unstable manifolds, see [PW92, Corollary 1.18].

In the next remark we indicate how a change of ψ influences equation (5.1).

Remark (i) The time-reversed system.
Consider the systems

$$\begin{aligned} \tfrac{d}{dt} u &= f(u, \mu) \\ \tfrac{d}{dt} \tilde{u} &= -f(\tilde{u}, \mu) \end{aligned}$$

with associated N-pulses $u(t) = \tilde{u}(-t)$ and matrices A and $\tilde{A}$, respectively. Then due to $\frac{d}{dt} u(t) = -\frac{d}{dt} \tilde{u}(-t)$ it follows that $A = -\tilde{A}$.

(ii) Change of ψ_1.
Equation (5.1) is linear in ψ_1, whence a change of ψ_1 in A_0 or M does not change the eigenvalues. However, changing ψ_1 to $-\psi_1$ in A_0 or exclusively in M corresponds to multiplying the solutions ν of $\det(A_0 - M\nu) = 0$ by (-1).

These simple remarks might be useful when applying the results in section 6 in different particular equations.

31

6 Applications

In this section we apply the theory developed above to a particular bifurcation scenario, namely the saddle-focus bifurcation. Consider

$$
\text{(6.1)} \qquad \dot{u} = f(u, \mu)
$$

$$
\text{(6.2)} \qquad \dot{v} = \left(D_u f(u, \mu) + \lambda B(u, \mu) \right) v
$$

for $u, v \in \mathbb{R}^n$ and $(\mu, \lambda) \in \mathbb{R} \times \mathbb{C}$ with $f, B \in C^2$. We assume that (q_1, μ_1) is a primary homoclinic solution of (6.1) converging to zero

$$
\lim_{t \to \pm \infty} q_1(t) = 0.
$$

Moreover, we assume the following hypothesis.

(A1) There are simple eigenvalues $-\alpha \pm i\beta_0$ for some $\alpha, \beta_0 > 0$ contained in $\sigma(D_u f(0, \mu_1))$. The modulus of the realpart of any other eigenvalue of $D_u f(0, \mu_1)$ is strictly larger than α.

The homoclinic orbit $q_1(t)$ is supposed to be non-degenerate:

(A2) $T_{q_1(0)} W^s(0, \mu_1) \cap T_{q_1(0)} W^u(0, \mu_1) = \mathbb{R} \dot{q}_1(0)$.

We denote by ψ_1 the bounded solution of

$$
\dot{w} = -D_u f(q_1, \mu_1)^* w,
$$

see section 2. The next assumption is concerned with the asymptotic behaviour of q_1 and ψ_1.

(A3) The limits $\lim_{t \to \infty} e^{\alpha t} |q_1(t)| \neq 0$ and $\lim_{t \to -\infty} e^{-\alpha t} |\psi_1(t)| \neq 0$ are non-zero.

The assumptions (A2) and (A3) are generically fulfilled. Finally, we assume that $q_1(t)$ is constructed transversally and that the Evans function $D(\lambda)$ associated with the primary pulse q_1 is regular at $\lambda = 0$, i.e. $D'(0) \neq 0$. In other words,

(A4) $\int_{-\infty}^{\infty} \langle \psi_1(t), D_\mu f(q_1(t), \mu_1) \rangle \, dt =: M_1 \neq 0$,

and

(A5) $\int_{-\infty}^{\infty} \langle \psi_1(t), B(q_1(t), \mu_1) \dot{q}_1(t) \rangle \, dt =: M_2 \neq 0$

are satisfied. We define the sets

(6.3)
$$\begin{aligned}
\mathcal{R} &= \{\exp(-\tfrac{2\pi\alpha}{\beta_0}\,n)\,|\,n \in \mathbb{N}_0\} \cup \{0\} \\
\mathcal{A} &= \{\exp(-\tfrac{\pi\alpha}{\beta_0}\,k)\,|\,k \in \mathbb{N}_0\}.
\end{aligned}$$

Note that $\mathcal{R}$ is a closed metric space. Then we can state the theorem.

Theorem 3 *Assume that (A1)-(A5) are satisfied. Then there exist a $\delta > 0$ such that for any $N \geq 2$ the following holds.*

For any sequence $a_j^0 \in \mathcal{A}$ for $j = 1, ..., N-1$ with $a_i^0 \in \{1, \exp(-\tfrac{\pi\alpha}{\beta_0})\}$ for some i, there exists an $r_0 \in \mathcal{R}$, $r_0 \neq 0$ with the following property.

(i)   *There are C^0-functions $a_j(r), \mu(r) \in \mathbb{R}$ for $r \in \mathcal{R}$, $r \leq r_0$ with $a_j(0) = a_j^0$ for $j = 1, ..., N-1$ and $\mu(0) = \mu_1$.*

(ii) *For any $r \in \mathcal{R}$, $0 < r \leq r_0$ there exists an N-pulse $q_N(r)$ for $\mu = \mu(r)$ with return times*
$$T_j(r) = -\frac{1}{2\alpha}\ln(a_j(r)\,r) + \tilde{T} \qquad\qquad j = 1, ..., N-1$$

(iii) *for some constant $\tilde{T}$. The N-pulses fulfilling (ii) are unique and there are no other N-pulses close to q_1.*

Denote by k_j^0 the natural numbers associated with the $a_j^0 \in \mathcal{A}$ chosen above. Then the stability of the N-pulses described above is determined as follows.

(iv) *For $M_2 > 0$ ($M_2 < 0$), we have*

$$\#\{\lambda \in \mathbb{C}\,|\,E(\lambda) = 0,\,|\lambda| < \delta,\,\mathrm{Re}\,\lambda < 0\} = \#\{j\,|\,1 \leq j \leq N - 1,\,k_j^0 \text{ is even (odd)}\}$$
$$\#\{\lambda \in \mathbb{C}\,|\,E(\lambda) = 0,\,|\lambda| < \delta,\,\mathrm{Re}\,\lambda > 0\} = \#\{j\,|\,1 \leq j \leq N - 1,\,k_j^0 \text{ is odd (even)}\},$$

counted with multiplicity.

(v)   *$\lambda = 0$ is a simple zero of $E(\lambda)$ and there are no other zeroes with norm less than δ on the imaginary axis.*

The geometric interpretation of k_j^0 even or odd is given in Lemma 6.1, (6.4)(i) and (6.5), see also figure 3.

We need some preparations before giving the proof of Theorem 3. First, we choose coordinates such that zero is an equilibrium of (6.1) for all μ close to μ_1. Moreover, time is rescaled such that the realpart of the simple eigenvalues $\alpha \pm i\beta(\mu)$ of $D_u f(0, \mu)$ is independent of μ. Here, $\beta(\mu_1) = \beta_0$ holds. Afterwards, we change coordinates according to Lemma 2.2 and 2.1. This does not change the stability properties of the N-pulses provided
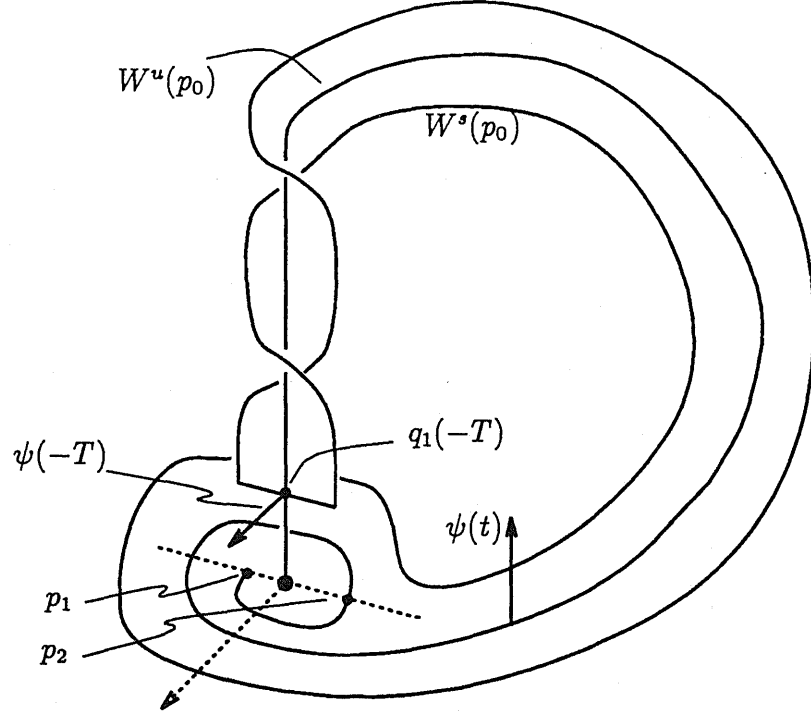
Figure 3: The geometry of the saddle-focus case: if T is chosen such that $q(T) = p_1$ or $q(T) = p_2$, a 2-pulse exists with return time close to T. If $q(T) = p_1$ the 2-pulse is stable, while it is unstable if $q(T) = p_2$ (provided $M > 0$).

(6.2) is transformed according to Lemma 2.3. Throughout, we assume that (A1)-(A5) are fulfilled.

Lemma 6.1 *For t sufficiently large,*

$$
\begin{array}{ll}
(i) & \langle \psi(\mu)(-t), q^+(\mu)(t) \rangle = s(\mu)\, e^{-2\alpha t}\, \sin(2\beta(\mu)t + \varphi(\mu)) + \mathcal{O}(e^{-(2\alpha+\gamma)t}) \\[2mm]
(6.4) \quad (ii) & \langle \psi(\mu)(-t), \dot{q}^+(\mu)(t) \rangle = s(\mu)\, e^{-2\alpha t}\, \big(\beta(\mu)\, \cos(2\beta(\mu)t + \varphi(\mu)) - \\[1mm]
& \qquad\qquad\qquad\qquad \alpha\, \sin(2\beta(\mu)t + \varphi(\mu))\big) + \mathcal{O}(e^{-(2\alpha+\gamma)t}) \\[2mm]
(iii) & |\langle \psi(\mu)(t), q^-(\mu)(-t) \rangle| + |\langle \psi(\mu)(t), \dot{q}^-(\mu)(-t) \rangle| \le C\, e^{-(2\alpha+\gamma)t}
\end{array}
$$

hold uniformly in μ. Here, $\gamma > 0$ is a constant. The functions $\varphi(\mu)$ and $s(\mu)$ are in C^2 and

$$(6.5) \qquad\qquad\qquad\qquad s_0 := s(\mu_1) > 0.$$

Proof. The estimate (iii) follows from (A1). By [San93, Lemma 1.5] and (A1) there exist vectors $v(\mu)$ and $w(\mu) \in \mathbb{R}^n$ contained in the eigenspaces of the eigenvalues $(-\alpha \pm i\beta(\mu))$

34

of $D_u f(0, \mu)$ and $D_u f(0, \mu)^*$, respectively, such that

$$
\begin{aligned}
\text{(6.6)} \qquad q^+(\mu)(t) &= \exp(D_u f(0, \mu)\, t)\, v(\mu) + \mathcal{O}(e^{-(\alpha+\gamma)t}) \\
\psi(\mu)(-t) &= \exp(D_u f(0, \mu)^*\, t)\, w(\mu) + \mathcal{O}(e^{-(\alpha+\gamma)t})
\end{aligned}
$$

for $t \to \infty$. Moreover, $v(\mu)$ and $w(\mu)$ are smooth in μ and according to (A3) we conclude $v(\mu_1), w(\mu_1) \neq 0$. Using the expansion (6.6) and the fact that v and w are therefore non-zero eigenvectors of $(-\alpha \pm i \beta(\mu))$, it is straightforward - but lengthy - to obtain (6.4)(i) and (ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Note that due to Lemma 6.1 we avoid a linearization of the flow near the equilibrium. In the next lemma, we formulate the equations for the return times T_j, solutions of which correspond to N-homoclinic solutions of (6.1) close to q_1. It turns out that the following definitions simplify the equations a lot.

For given large real numbers T and T_j, $j = 1, ..., N-1$, such that $T_j \geq T$ for all j we define

$$
\text{(6.7)} \qquad
\begin{aligned}
r &= e^{-\alpha(2T + \varphi_0/\beta_0)} & T &= -\tfrac{1}{2\alpha} \ln r - \tfrac{\alpha \varphi_0}{\beta_0} \\
a_j &= e^{-2\alpha(T_j - T)} & T_j &= -\tfrac{1}{2\alpha} \ln(a_j r) - \tfrac{\varphi_0}{2\beta_0}.
\end{aligned}
$$

Here, $\varphi_0 := \varphi(\mu_1)$. Thus $r \in (0, \delta_0)$, $a_j \in (0, 1]$ for all j.

Lemma 6.2 *Let $N \geq 2$ be fixed. Then there exists an N-pulse (q_N, μ) of (6.1) close to (q_1, μ_1) in phase space with associated return times $(T_j)_{j=1,...,N-1}$ if and only if*

$$
\text{(6.8)} \qquad
\begin{aligned}
a_j\, r\, e^{\frac{\varphi_0}{2\beta_0}} \sin\left(-\tfrac{\beta(\mu)}{\alpha} \ln(a_j r) + \tilde{\varphi}(\mu)\right) + M_1 (\mu - \mu_1) &= \mathcal{O}(r^{1+\gamma}) \\
M_1 (\mu - \mu_1) &= \mathcal{O}(r^{1+\gamma})
\end{aligned}
$$

for $j = 1, ..., N-1$. The variables a_j and r are related to T_j via (6.7). The function $\tilde{\varphi}$ is given by

$$
\tilde{\varphi}(\mu) = \tfrac{\varphi_0}{\beta_0} \beta(\mu) - \varphi(\mu) = \mathcal{O}(\mu - \mu_1).
$$

The remainder terms are smooth in $a_j \in (0, 1]$ and μ up to $r = 0$ and

$$
\text{(6.9)} \qquad \frac{d}{da_j} \mathcal{O} = \mathcal{O}(r^{1+\gamma}), \qquad \frac{d}{d\mu} \mathcal{O} = \mathcal{O}(r^{1+\gamma})
$$

holds.

Proof. The jump equations for N-homoclinic solutions read

$$
\langle \psi(T_i), q^-(-T_i) \rangle - \langle \psi(-T_{i-1}), q^+(T_{i-1}) \rangle + M_1 (\mu - \mu_1) + R((T_j), \mu) = 0,
$$

35

see [San93, Satz 3]. The claims follow now from the estimates of the remainder term given in [San93, Satz 3], Lemma 6.1 and (A4). For $a_j \in (0, 1]$ the remainder terms are differentiable in a_j and satisfy

$$\frac{d}{da_j} \mathcal{O} = \frac{d}{dT_j} \mathcal{O} \frac{dT_j}{da_j} = \mathcal{O}(e^{-2\alpha T_j} e^{-\gamma T}) \frac{1}{a_j r} r = \mathcal{O}(a_j r^{1+\gamma}) \frac{1}{a_j} = \mathcal{O}(r^{1+\gamma})$$

uniformly in $r > 0$ by using [San93, Satz 3]. Therefore, the remainder terms can be extended by zero to $r = 0$, $a_i \in (0, 1]$ in a differentiable way. This proves the lemma. $\square$

Now the reason for introducing the coordinates (6.7) is apparent. They allow for extending the bifurcation equations differentiably to $T_j = \infty$ - at least if all of them tend to zero at the same order - and get rid of the trivial solution $r = 0$ corresponding to the primary pulse q_1. In a certain sense, $r \to 0$ corresponds to a blow-up of the original bifurcation equation stated in [San93, Satz 3] by introducing the polar coordinate r.

Now we give the proof of the theorem.

Proof of Theorem 3. We will first prove the statements (i)-(iii) that is the existence part. By Lemma 6.2 we have to solve the system (6.8)

$$a_j r e^{\frac{\varphi_0}{2\beta_0}} \sin\left(-\frac{\beta(\mu)}{\alpha} \ln(a_j r) + \tilde{\varphi}(\mu)\right) + M_1 (\mu - \mu_1) = \mathcal{O}(r^{1+\gamma})$$
$$M_1 (\mu - \mu_1) = \mathcal{O}(r^{1+\gamma}),$$

for $j = 1, ..., N-1$, where the remainder term satisfies (6.9). Owing to (A4) we have $M_1 \neq 0$, whence the last equation can be solved with respect to μ yielding

(6.10) $$\mu = \mu^*(a_1, ..., a_{N-1}, r) = \mu(a, r) = \mu_1 + \mathcal{O}(r^{1+\gamma}).$$

Substituting μ into the remaining equations we obtain

(6.11) $$a_j r \sin\left(-\frac{\beta(\mu^*(a,r))}{\alpha} \ln(a_j r) + \tilde{\varphi}(\mu^*(a,r))\right) = \mathcal{O}(r^{1+\gamma}) \qquad j = 1, ..., N-1.$$

Dividing by r yields

(6.12) $$a_j \sin\left(-\frac{\beta(\mu^*(a,r))}{\alpha} \ln(a_j r) + \tilde{\varphi}(\mu^*(a,r))\right) = \mathcal{O}(r^{\gamma}) \qquad j = 1, ..., N-1.$$

Now, we have

(6.13)
$$\tilde{\varphi}(\mu^*) = \mathcal{O}(\mu^* - \mu_1) = \mathcal{O}(r^{1+\gamma})$$
$$\beta(\mu^*) - \beta_0 = \mathcal{O}(\mu^* - \mu_1) = \mathcal{O}(r^{1+\gamma})$$

in a C^1-sense with respect to a by (6.10). Substituting this into the argument of the sine function, we obtain

(6.14)
$$\sin\left(-\frac{\beta(\mu^*)}{\alpha} \ln(a_j r) + \tilde{\varphi}(\mu^*)\right)$$
$$= \sin\left(-\frac{\beta_0}{\alpha} \ln(a_j r) - \frac{\beta(\mu^*) - \beta_0}{\alpha} \ln(a_j r) + \tilde{\varphi}(\mu^*)\right)$$
$$= \sin\left(-\frac{\beta_0}{\alpha} \ln(a_j r)\right) + \mathcal{O}(r^{1+\gamma}).$$

The estimate is valid for derivatives with respect to a, too. We restrict to $r \in \mathcal{R}$, whence $r = \exp(-\frac{2\pi\alpha}{\beta_0} n)$ for some $n \in \mathbb{N}$ or $r = 0$. Hence, (6.12) reads

$$(6.15) \qquad a_j \sin(-\frac{\beta_0}{\alpha} \ln a_j) = \mathcal{O}(e^{-\gamma n})$$

for $j = 1, ..., N-1$. The remainder term appearing in (6.15) is continuous in (a, r) up to $r = 0$ and partially differentiable with respect to a_i again up to $r = 0$ for $a_j \in (0, 1]$ by (6.9) and the arguments given above. Setting $r = 0$ ($n = \infty$) in (6.15) we obtain the equations

$$(6.16) \qquad a_j \sin(-\frac{\beta_0}{\alpha} \ln a_j) = 0.$$

Therefore, the chosen sequence $(a_j^0)_{j=1,...,N-1}$ with $a_j^0 \in \mathcal{A}$ solves (6.16). In addition, the partial derivatives of (6.16) with respect to a_j evaluated at a_j^0 are given by

$$(6.17) \qquad \left(\sin(-\tfrac{\beta_0}{\alpha} \ln a_j) - \tfrac{\beta_0}{\alpha} \cos(-\tfrac{\beta_0}{\alpha} \ln a_j) \right) \Big|_{a_j = a_j^0 = \exp(-\frac{\pi\alpha}{\beta_0} k_j^0)} = (-1)^{k_j^0 + 1} \tfrac{\beta_0}{\alpha},$$

and thus are non-zero. All other entries in the Jacobian except the ones on the diagonal computed in (6.17) vanish.

By (6.16), (6.17) and the differentiability properties of (6.15) stated above, we can use the implicit function theorem to conclude that statements (i)-(iii) of the theorem are fulfilled.

It remains to prove items (iv) and (v), which are concerned with the stability properties of the N-pulses. By Theorem 2 we have to find zeroes of

$$(6.18) \qquad E(\lambda) = \det(A - M_2 \lambda + R(\lambda)).$$

We have the estimate

$$\|R(\lambda)\| = \mathcal{O}(r^{1+\gamma} + |\lambda| (r^\gamma + |\lambda|))$$

owing to Theorem 2, (A1) and (6.10). Moreover, by Lemma 6.1, we obtain

$$A = r A_1(r) + \mathcal{O}(r^{1+\gamma}),$$

where $A_1(r)$ is the lower triangular, tridiagonal matrix

$$(6.19) \qquad (A_1)_{ij} = \begin{cases} 0 & i = 1 \\ s_0 \, e^{\frac{\varphi_0}{2\beta_0}} \beta_0 \cos\left(-\frac{\beta_0}{\alpha} \ln a_i(r)\right) & j = i, \, i > 1 \\ -s_0 \, e^{\frac{\varphi_0}{2\beta_0}} \beta_0 \cos\left(-\frac{\beta_0}{\alpha} \ln a_i(r)\right) & j = i-1, \, i > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, the scalar products in the definition of A given in Theorem 2 can be estimated by (6.4)(i) and (ii). The sine term appearing in (6.4)(ii) is $\mathcal{O}(r^{1+\gamma})$ by (6.11), while the cosine is handled similarly to (6.14). Therefore, (6.19) obtains. Observe that for $r \to 0$ the limit

$$\lim_{r \to 0} \cos\left(-\tfrac{\beta_0}{\alpha} \ln a_i(r)\right) = \cos\left(-\tfrac{\beta_0}{\alpha} \ln a_i^0\right) = \cos(-\pi k_i^0) = (-1)^{k_i^0}$$

exists. Thus, the limit $\lim_{r \to 0} A_1(r) =: A_0$ exists, too, and is given by

$$(A_0)_{ij} = \begin{cases} 0 & i = 1 \\ (-1)^{k_i^0} s_0 \beta_0 e^{\frac{\varphi_0}{2\beta_0}} & j = i, i > 1 \\ (-1)^{k_i^0 + 1} s_0 \beta_0 e^{\frac{\varphi_0}{2\beta_0}} & j = i - 1, i > 1 \\ 0 & \text{otherwise.} \end{cases}$$

A_0 admits the eigenvalues

$$\nu_j = (-1)^{k_i^0} s_0 \beta_0 e^{\frac{\varphi_0}{2\beta_0}} \qquad j = 1, ..., N - 1$$
$$\nu_N = 0.$$

Owing to (6.5) and (A1) the product $s_0 \beta_0 > 0$ is positive.

Therefore, (6.18) reads

$$E(\lambda) = \det\left(r A_0 - M_2 \lambda + r (A_1(r) - A_0) + \mathcal{O}(r^{1+\gamma}) + \mathcal{O}(|\lambda| (r^\gamma + |\lambda|)) \right)$$

with $(A_1(r) - A_0) = o(1)$ and

$$\sigma(A_0) = \{0\} \cup \{s_0 \beta_0 (-1)^{k_j^0} \mid j = 1, ..., N - 1\}.$$

The statements (iv) and (v) follow now directly from Lemma 5.1 for all $r \in \mathcal{R}$ sufficiently small by observing that (EV1) - (EV3) are fulfilled.

The proof of the theorem is therefore complete. $\qquad \qquad \square$

7 Discussion

First, let us compare our results to the ones obtained in [AJ93] for triple pulses. The statement of Theorem 3 is valid for the time-reversed system, too, if one takes the remark at the end of section 5 into account. This clarifies a comment in [AJ93, p. 192].

We would like to emphasize that Theorem 2 (in combination with Lemma A.1) should be applicable to other bifurcations producing multiple pulses, too. In particular, the Hamiltonian and reversible saddle-focus bifurcations studied by [Dev76] and [Cha94, Här93], respectively, can be handled. Here, the Hamiltonian saddle-focus arises in the localized buckling of struts on a nonlinear elastic foundation describing their equilibrium states, see [BCT94]. The stability question is important (and unsolved) in this application. Moreover, the technique should work for some of the bifurcations to double pulses investigated first by [Yan87]. These are the non-orientable resonant bifurcation [CDF90], the inclination-flip

[KKO93] and the orbit-flip [San93]. In the flip bifurcations, N-pulses for any N occur if certain conditions on the eigenvalues are satisfied, see [HKK94] and [San93]. However, in these cases, further improvements of the estimates stated in Theorem 2 and Lemma A.1 seem to be necessary in order to apply the technique presented here. A detailed investigation of the stability properties of pulses occuring in the bifurcations mentioned above is in progress.

The whole theory should carry over to front and back solutions. Indeed, the existence results obtained in [Lin90] or [San93] are formulated for general heteroclinic cycles. It should be possible to extend the stability results to this setting, too.

The reduced system

$$E(\lambda) = \det S(\lambda) = \det(A - M\lambda\, id + R(\lambda)) = 0$$

given in Theorem 2 describes only the eigenvalues close to zero. Actually, $E(\lambda)$ and the Evans function $D(\lambda)$ do not coincide in general. Indeed, by construction, $E(\lambda)$ takes only changes of the N-pulses in the direction of $\psi(0)$ into account, but neglects the changes of the vectors v_i^u and v_i^s with λ, see Lemma 5.5 and [AJ94, section 2]. However, using the technique presented here, it is possible to describe the splitting of other eigenvalues which are not necessarily close to zero. Instead of parametrizing the solutions of the eigenvalue equation by

$$v^\pm = (\dot{q}^\pm + \dot{u}^\pm)\, d + w^\pm$$

as in (3.5), one has to use coordinates according to

$$v^\pm = \varphi\, d + w^\pm,$$

where $\varphi(t)$ is an eigenfunction with eigenvalue λ_0 of the primary pulse. Then one obtains a reduced system of the form

$$\det\bigl(A - D'(\lambda_0)\,(\lambda - \lambda_0)\,id + R(\lambda - \lambda_0)\bigr) = 0$$

for λ close to λ_0. Provided λ_0 is a simple zero of the Evans function, i.e. a simple eigenvalue, the reduced equation describes the splitting of precisely N eigenvalues of the N-pulse close to λ_0. In fact, the principal part of the Evans function associated to an N-pulse can be computed nearby any eigenvalue of the primary pulse.

Frequently, travelling waves are investigated in the context of singular perturbed equations like the FitzHugh-Nagumo equation. It would be interesting to compare the matrix $S(\lambda)$ appearing in the reduced equation

$$E(\lambda) = \det S(\lambda) = 0$$

39

to the SLEP-matrix, see [Nis94] and the references therein. A relation between the SLEP matrix and the Evans function was detected in [SNI94]. While the method studied here is described in terms of ordinary differential equations, the SLEP method takes a partial-differential-equation-point of view.

A   Improved estimates

In this appendix, we give improved estimates for the jumps ξ. They are needed when investigating N-pulses bifurcations of codimension two. Suppose that the following hypothesis is fulfilled.

(P)   There exist projections $Q_+^u(t)$ and $Q_+^{uu}(t)$ for $t > 0$ such that

$$P_+^u(t) = Q_+^u(t) + Q_+^{uu}(t) \qquad\qquad |\Phi_+(s,t)\,Q_+^u(t)| \leq Ce^{-\alpha^u(t-s)}$$
$$Q_+^u(t)\,Q_+^{uu}(t) = Q_+^{uu}(t)\,Q_+^u(t) = 0 \qquad |\Phi_+(s,t)\,Q_+^{uu}(t)| \leq Ce^{-\alpha^{uu}(t-s)}$$

is satisfied for some $0 < \alpha^u < \alpha^{uu}$ and $0 < s < t$. In addition, $P_0^u = Q_0^u + Q_0^{uu}$ for spectral projections Q_0^u and Q_0^{uu}. We assume that $\lim_{t\to\infty} Q_+^u(t) = Q_0^u$ and $\lim_{t\to\infty} Q_+^{uu}(t) = Q_0^{uu}$. Moreover, the same holds for the stable projection $P_-^u(t)$ for $t < 0$.

We define

$$(A.1) \qquad p_1(T) := \sup_{t \geq T} \Big(|P_+^s(t) + Q_+^u(t) - P_0^s - Q_0^u| + |Q_+^{uu}(t) - Q_0^{uu}| +$$
$$|P_-^u(-t) + Q_-^s(-t) - P_0^u - Q_0^s| + |Q_-^{ss}(-t) - Q_0^{ss}| \Big).$$

Although hypothesis (P) might sound exotic it occurs in the investigation of the homoclinic flip-bifurcations, see [San93].

First we have

$$(A.2) \qquad |\Phi_+^u(0,T_j)\,P_0^u D_j\,d - \Phi_+^u(0,T_j)\,Q_0^u D_j\,d|$$
$$= |\Phi_+^u(0,T_j)\,(Q_+^u(T_j) + Q_+^{uu}(T_j))\,P_0^u D_j\,d - \Phi_+^u(0,T_j)\,Q_0^u D_j\,d|$$
$$\leq |\Phi_+^u(0,T_j)\,(Q_+^u(T_j) - Q_0^u) + \Phi_+^u(0,T_j)\,Q_+^{uu}(T_j)|\,|P_0^u D_j\,d|$$
$$\leq C(e^{-\alpha T}p_1(T) + e^{-\alpha^{uu}T})|D|\,|d|$$

by using (P). Indeed, we have

$$\Phi_+^u(0,T_j)\,(Q_+^u(T_j) - Q_0^u)P_0^u\,x$$
$$= \Phi_+^u(0,T_j)\,(Q_+^u(T_j) - Q_0^u)P_0^u\,x + \Phi_+^u(0,T_j)\,(P_+^s(t) - P_0^s)P_0^u\,x$$

40

owing to $\Phi_+^u(0, T_j) = \Phi_+^u(0, T_j) P_+^u(t)$ and Lemma 3.2. By the estimate (3.48),

$$
\begin{aligned}
|\Phi_+^u(0, T_j) a_j^+| &= |\Phi_+^u(0, T_j) (P_0^u D_j\, d + (A_2(\lambda)(b, d))_j^+)| \\
&\leq |\Phi_+^u(0, T_j) P_0^u D_j\, d| + C(e^{-\alpha T} + |G| + |\lambda|)\, |b| + \\
&\quad\; C(p_1(T) + |G|)\, |D|\, |d| + C|\lambda|\, |d| \\
&\leq C\big(e^{-\alpha T}|Q_0^u D| + (p_1(T) + |G| + e^{-\alpha^{uu} T})\, |D| + |\lambda|\big)\, |d| + \\
&\quad\; C(e^{-\alpha T} + |G| + |\lambda|)\, |b|
\end{aligned}
$$

obtains. Of course, the corresponding estimate holds for $|\Phi_-^s(0, -T_j)\, a_j^-|$. Therefore, owing to (3.40), we conclude the improved estimate

(A.3)
$$
\begin{aligned}
|L_4(\lambda)| &\leq \delta\, |b| + |\Phi_+^u(0, T_j)\, a_j^+| + |\Phi_-^s(0, -T_{j-1})\, a_{j-1}^-| + \\
&\quad\; C\big((|G| + |\lambda|)\, |w| + |\lambda|\, |d|\big) \\
&\leq \delta\, |b| + C\big(e^{-\alpha T}(|Q_0^u D| + |Q_0^s D|) + \\
&\quad\; (p_1(T) + |G| + e^{-\alpha^{uu} T} + e^{-\alpha^{ss} T})\, |D| + |\lambda|\big)\, |d|
\end{aligned}
$$

instead of (3.42). Thus, the estimate (3.44) can be sharpened yielding

(A.4)
$$
\begin{aligned}
|B_1(\lambda)\, d| \leq C\big(&e^{-\alpha T}(|Q_0^u D| + |Q_0^s D|) \\
&+ (p_1(T) + |G| + e^{-\alpha^{uu} T} + e^{-\alpha^{ss} T})\, |D| + |\lambda|\big)\, |d|.
\end{aligned}
$$

Next we reformulate equation (3.27)

$$
\begin{aligned}
P_-^s(-T_j)\, a_j^- - P_+^u(T_j)\, a_j^+ &= D_j\, d + \Phi_+^s(T_j, 0)\, b_j^+ - \Phi_-^u(-T_j, 0)\, b_{j+1}^- + \\
&\quad \int_0^{T_j} \Phi_+^s(T_j, s)\big((G_j^+(s) + \lambda B_j^+(s))\, w_j^+(s) + \lambda h_j^+(s)\, d_j\big)\, ds\; + \\
&\quad \int_{-T_j}^0 \Phi_-^u(-T_j, s)\big((G_{j+1}^-(s) + \lambda B_{j+1}^-(s))\, w_{j+1}^-(s) + \lambda h_{j+1}^-(s)\, d_{j+1}\big)\, ds \\
&=: D_j\, d + (L_5(\lambda)(b, d, w))_j
\end{aligned}
$$

with

(A.5) $\quad |L_5(\lambda)(b, d, w)| \leq C\big((e^{-\alpha T} + |G| + |\lambda|)\, |b| + (|G| + |\lambda|)\, |a| + |\lambda|\, |d|\big),$

see (3.29). Applying the projections P_0^u and P_0^s yields

$$
\begin{aligned}
P_-^s(-T_j)\, a_j^- &= P_0^s D_j\, d + P_0^s(P_+^u(T_j) - P_0^u)\, a_j^+ + P_0^u(P_-^s(-T_j) - P_0^s)\, a_j^- + \\
&\quad P_0^s\, (L_5(\lambda)(b, d, w))_j \\
P_+^u(T_j)\, a_j^+ &= -P_0^u D_j\, d - P_0^u(P_-^s(-T_j) - P_0^s)\, a_j^- + P_0^s(P_+^u(T_j) - P_0^u)\, a_j^+ - \\
&\quad P_0^u\, (L_5(\lambda)(b, d, w))_j.
\end{aligned}
$$

Using the definition

(A.6)
$$p_2(T) := \sup_{t \geq T} \left(|P_0^s(P_+^u(t) - P_0^u)| + |P_0^u(P_-^s(-t) - P_0^s)| \right),$$

we obtain

(A.7)
$$|P_-^s(-T_j)\, a_j^- - P_0^s\, D_j\, d| + |P_+^u(T_j)\, a_j^+ + P_0^u\, D_j\, d|$$

$$\leq \; p_2(T)\, |a| + |L_5(\lambda)(b, d, w)|$$

$$\leq \; C\Big(p_2(T)|a| + (e^{-\alpha T} + |G| + |\lambda|)\, |b| + (|G| + |\lambda|)\, |a| + |\lambda|\, |d| \Big)$$

$$\leq \; C\Big((p_1(T) + p_2(T) + |G| + e^{-\alpha^{uu} T} + e^{-\alpha^{ss} T})\, |D| +$$
$$e^{-2\alpha T}(|Q_0^u D| + |Q_0^s D|) + |\lambda| \Big)\, |d|$$

$$\leq \; C(|\lambda| + S)\, |d|$$

using (3.37) and (A.4) for estimating $|a|$ and $|b|$, respectively. Here, S is defined by

(A.8) $S := \Big((p_1(T) + p_2(T) + |G| + e^{-\alpha^{uu} T} + e^{-\alpha^{ss} T})\, |D| + e^{-2\alpha T}(|Q_0^u D| + |Q_0^s D|) \Big),$

where p_1 and p_2 are defined in (A.1) and (A.6), respectively.

Lemma A.1 *Assume hypothesis (P). Then the jumps ξ satisfy*

$$\xi_i = \langle \psi(T_i), P_0^u D_i\, d \rangle + \langle \psi(-T_{i-1}), P_0^s D_{i-1}\, d \rangle - \lambda \int_{-\infty}^{\infty} \langle \psi(t), H(t) \rangle\, dt\, d_i + (R(\lambda)\, d)_i$$

and $R(\lambda)$ satisfies the improved estimate

(A.9) $|R(\lambda)\, d| \leq C\Big(e^{-\alpha T} S + |G|^2\, |D| + (e^{-\alpha T} + |G| + \|h_i - H\| + |\lambda|)\, |\lambda| \Big)\, |d|,$

where S is defined in (A.8).

Proof. We refer here to Lemma 3.6. The estimate (3.49) can be replaced by

$$\langle \psi(T_i), a_i^+ \rangle - \langle \psi(-T_{i-1}), a_{i-1}^- \rangle$$
$$= \; \langle \psi(T_i), P_0^u D_i\, d \rangle + \langle \psi(-T_{i-1}), P_0^s D_{i-1}\, d \rangle +$$
$$\langle \psi(0), \Phi_+^u(0, T_i)\, (P_+^u(T_i)\, a_i^+ + P_0^u\, D_i\, d) - \Phi_-^s(0, -T_{i-1})\, (P_-^s(-T_{i-1})\, a_{i-1}^- - P_0^s\, D_{i-1}\, d) \rangle$$
$$= \; \langle \psi(T_i), P_0^u D_i\, d \rangle + \langle \psi(-T_{i-1}), P_0^s D_{i-1}\, d \rangle + \mathcal{O}\Big(e^{-\alpha T}\, (|\lambda| + S)\, |d| \Big)$$

due to (3.48) and (A.7). The other estimates in the proof of (3.47) remain unchanged and (A.9) obtains. $\qquad\square$

Now the estimates from Lemma 3.1 can be substituted as in section 3.3.

B The multiplicity lemma

For any analytic function $V(\lambda)$ with values in the vector space of $N \times N$ matrices, we associate matrices V_j for $j \geq 0$ by

$$V(\lambda) = \sum_{j=0}^{\infty} \lambda^j V_j.$$

Using the Taylor coeffcients V_j, we define the matrix $\tilde{V}$

$$\tilde{V}_{ij} := \begin{cases} 0 & i < j \\ V_{i-j} & i \geq j \end{cases}$$

for $i, j \in \mathbb{N}_0$. Hence, the entries on each band of the matrix $\tilde{V}$ coincide and the upper right triangle consists of zeroes

$$(\text{B.1}) \qquad \tilde{V} = \begin{pmatrix} V_0 & & & 0 \\ V_1 & V_0 & & \\ V_2 & V_1 & V_0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

It is straightforward to check that the mapping $V(\lambda) \mapsto \tilde{V}(\lambda)$ is an embedding of the algebra of analytic matrix-valued maps to the algebra of matrices of the form (B.1). In particular, $\widetilde{VW} = \tilde{V} \cdot \tilde{W}$ holds.

The truncated matrices $\tilde{V}_M$ for any number $M \in \mathbb{N}$ are given by the $(M+1) \times (M+1)$ matrix

$$(\tilde{V}_M)_{ij} = \begin{cases} 0 & i < j \leq M+1 \\ V_{i-j} & j \leq i \leq M+1. \end{cases}$$

We denote the order of a zero of a analytic complex-valued function $f(\lambda)$ at $\lambda = \lambda_0$ by $\text{ord}(\lambda_0, f(\lambda))$. Moreover, $\text{Ke} \, \tilde{V}_M$ denotes the kernel of $\tilde{V}_M$.

Lemma B.1 *Assume that $S(\lambda)$ is an analytic matrix-valued function and suppose that $\text{ord}(0, \det S(\lambda)) < \infty$. Then there exists an M_0 such that*

$$\dim \text{Ke} \, \tilde{S}_M = \text{ord}(0, \det S(\lambda))$$

for all $M \geq M_0$.

Proof. Choose any analytic matrix-valued functions $V(\lambda)$ and $W(\lambda)$ such that $V(0)$ and $W(0)$ are invertible. Then we note that neither $\text{ord}(0, \det S(\lambda))$ nor the dimension of the kernel of $\tilde{S}_M$ change if we replace $S(\lambda)$ by

$$(\text{B.2}) \qquad V(\lambda) \, S(\lambda) \, W(\lambda)$$

and accordingly $\tilde{S}_M$ by

(B.3)
$$\tilde{V}_M \, \tilde{S}_M \, \tilde{W}_M.$$

Indeed, the matrices $\tilde{V}_M$ and $\tilde{W}_M$ are invertible, because $V_0 = V(0)$ and W_0 are.

This allows for transforming $S(\lambda)$ into a simpler form. First transform $S(0)$ such that

$$S(0) = S_0 = \begin{pmatrix} 0 & 0 \\ 0 & \underbrace{1}_{k} \end{pmatrix}$$

for some $0 \leq k \leq N$ afterwards. Here we denote by k the number of rows and columns of the identity appearing in the matrix. Let $n_0^+ := N - k$ and $n_0^0 := k$. Then

$$S(\lambda) = \begin{pmatrix} \lambda\,A(\lambda) & \lambda\,B(\lambda) \\ \lambda\,C(\lambda) & 1 + \lambda\,D(\lambda) \end{pmatrix}$$

for some matrix-valued analytic functions A, B, C ad D. Next let

(B.4)
$$
\begin{aligned}
V(\lambda) &= \begin{pmatrix} 1 & -\lambda\,B(\lambda)\,(1 + \lambda\,D(\lambda))^{-1} \\ 0 & (1 + \lambda\,D(\lambda))^{-1} \end{pmatrix}, \\
W(\lambda) &= \begin{pmatrix} 1 & 0 \\ -\lambda\,(1 + \lambda\,D(\lambda))^{-1}\,C(\lambda) & 1 \end{pmatrix}
\end{aligned}
$$

and transform $S(\lambda)$ according to (B.2). This yields the matrix function (which we again refer to as $S(\lambda)$)

$$S(\lambda) = \begin{pmatrix} \lambda\,(A(\lambda) - \lambda\,B(\lambda)\,(1 + \lambda\,D(\lambda))^{-1}\,C(\lambda)) & 0 \\ 0 & 1 \end{pmatrix} =: \begin{pmatrix} \lambda\,T(\lambda) & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we can inductively perform the same procedure on $T(\lambda)$ as before on $S(\lambda)$. Therefore, after the mth step, $S(\lambda)$ can be written according to

$$S(\lambda) = \sum_{j=0}^{m} \lambda^j\,S_j + \lambda^{m+1}\,T_{m+1}(\lambda).$$

Here

(B.5)
$$S_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \underbrace{0}_{n_j^+} & \underbrace{0}_{n_j^0} & \underbrace{0}_{n_j^-} \end{pmatrix}$$

for $j = 0, ..., m$ with

(B.6)
$$n_j^- = \sum_{i=1}^{j-1} n_i^0 \qquad n_j^+ + n_j^0 + n_j^- = N$$

44

and

$$T_{m+1}(\lambda) = \left(\begin{array}{cc} t_{m+1}(\lambda) & 0 \\ \underbrace{0}_{n_m^+} & \underbrace{0}_{N-n_m^+} \end{array} \right).$$

Thus the matrix $S(\lambda)$ looks like

$$S(\lambda) = \left(\begin{array}{cccc} \lambda^{m+1} t_{m+1}(\lambda) & & & 0 \\ & \lambda^m \cdot 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{array} \right),$$

where some of the blocks might be of zero size. The determinant of $S(\lambda)$ up to products with functions $f(\lambda)$ such that $f(0) \neq 0$ is therefore given by

$$\det S(\lambda) = \lambda^{m+1} \det(t_{m+1}(\lambda)) \cdot \prod_{j=1}^{m} \lambda^{j n_j^0}.$$

In particular, the procedure described above has to stop at the Mth step, given by

(B.7)
$$\sum_{i=1}^{M} n_i^0 = N.$$

Otherwise, $\det t_{M+1}(\lambda)$ and therefore $\det S(\lambda)$ would be of infinite order at $\lambda = 0$ contradicting the assumption. Hence

(B.8)
$$\begin{aligned} S(\lambda) &= \sum_{j=0}^{M} \lambda^j S_j \\ \det S(\lambda) &= \prod_{j=1}^{M} \lambda^{j n_j^0} \end{aligned}$$

for matrices S_j given by (B.5), whence

(B.9)
$$\mathrm{ord}(0, \det S(\lambda)) = \sum_{j=1}^{M} j\, n_j^0.$$

Next we compute the dimension of the kernel of the matrices $\tilde{S}_m$ for $m \geq M$. Remember that the matrices $V(\lambda)$ and $W(\lambda)$ transform $S(\lambda)$ into the normal form (B.8) associated to the family S_j defined in (B.5). The matrices $\tilde{V}$ and $\tilde{W}$ will transform the matrix $\tilde{S}$ such that the transformed matrix is again given by the family S_j. Indeed, as mentioned above, the mapping $S \to \tilde{S}$ is an injective algebra-homomorphism. Therefore, we can compute the dimension of the kernel of $\tilde{S}_m$ using the matrices S_j as defined in (B.5). Hence

$$\tilde{S}_m = \left(\begin{array}{cccc} S_0 & & & 0 \\ S_1 & S_0 & & \\ \vdots & \ddots & \ddots & \\ S_m & \cdots & S_1 & S_0 \end{array} \right)$$

for $m \geq M$ with $S_j = 0$ for $j > M$ and

(B.10)
$$\dim \text{Ke } \tilde{S}_m = \sum_{j=0}^{M} n_j^+.$$

It remains to prove that the expressions in (B.9) and (B.10) coincide. Owing to (B.6) and (B.7) we obtain

$$\sum_{j=0}^{M} n_j^+ = \sum_{j=0}^{M} (N - n_j^0 - n_j^-) = \sum_{j=0}^{M} (N - n_j^0 - \sum_{i=0}^{j-1} n_i^0)$$

$$= \sum_{j=0}^{M} \left(N - \sum_{i=0}^{j} n_i^0 \right) = \sum_{j=0}^{M} \sum_{i=j+1}^{M} n_i^0 = \sum_{j=0}^{M} j \, n_i^0$$

and the lemma is proved. □

C   Proof of Lemma 5.4

The matrix A_0 is symmetric, whence it leaves $V = (\mathbb{R}(1, ..., 1))^{\perp}$ invariant. The subspace V admits the basis

$$v_j = (0, .., 0, 1, -1, 0, .., 0) \quad \text{for} \quad j = 1, ..., N - 1,$$

where the one is placed at the jth component. Moreover, it is easy to see that

$$A_0 v_j = -a_{j-1} v_{j-1} + 2a_j v_j - a_{j+1} v_{j+1}$$

for $j = 1, ..., N-1$, where $a_0 = a_N = 0$ by definition. Hence, $A_0 : V \to V$ is represented by the matrix

(C.1)
$$B_{N-1} = \begin{pmatrix} 2a_1 & -a_1 & & & \\ -a_2 & 2a_2 & -a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{N-2} & 2a_{N-2} & -a_{N-2} \\ & & & -a_{N-1} & 2a_{N-1} \end{pmatrix}$$

with respect to the basis (v_j) of V. We denote the matrix representation of $A|_V$ by $B_{N-1} : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$. The determinant of B_N is given by

(C.2)
$$\det B_N = \det \begin{pmatrix} 2a_1 & 0 & & & \\ -a_2 & \frac{3}{2}a_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{N-1} & \frac{N}{N-1}a_{N-1} & 0 \\ & & & -a_N & \frac{N+1}{N}a_N \end{pmatrix} = (N+1) \prod_{i=1}^{N} a_i.$$

Indeed, this can be proved by induction. First add one half of the first column to the second one, yielding a diagonal entry $\frac{3}{2}$ in the second column. Then notice that adding $\frac{i}{i+1}$ times the ith column to the $(i+1)$th removes the element $b_{i,i+1}$ but changes the diagonal element $b_{i+1,i+1}$ to $(2 - \frac{i}{i+1}) = \frac{i+2}{i+1}$. This proves the claim.

After this preparation, we prove the statements of Lemma 5.4 by induction on N. Therefore, suppose that Lemma 5.4 is proved for the matrices B_N with $n \leq N-1$ and consider B_N next. There are two different cases.

(1) Suppose that $a_k = 0$ for some k with $1 \leq k \leq N$. Then one row of B_N vanishes identically, see (C.1). Moreover,

$$
B_N = \begin{pmatrix}
2a_1 & -a_1 & & & & & & \\
-a_2 & 2a_2 & -a_2 & & & & & \\
& \ddots & \ddots & \ddots & & & & \\
& & -a_{k-1} & 2a_{k-1} & -a_{k-1} & & & \\
& & & 0 & 0 & 0 & & \\
& & & & -a_{k+1} & 2a_{k+1} & -a_{k+1} & \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & -a_N & 2a_N
\end{pmatrix}.
$$

Hence, there is one eigenvalue equal to zero, because the kth row vanishes, while the other eigenvalues are determined by the two matrices $B_{k-1}(a_1, ..., a_{k-1})$ and $B_{N-k}(a_{k+1}, ..., a_N)$ of size smaller than N. Using the induction step proves the claim in this case.

(2) Owing to (1), we can assume that $a_i \neq 0$ for $i = 1, ..., N$. Define $\tilde{a} = (a_1, ..., a_{N-1})$ and consider $\hat{a} = (\tilde{a}, \epsilon)$ for $\epsilon \in \mathbb{R}$. Then the eigenvalues $\nu_1(\epsilon), ..., \nu_N(\epsilon)$ of $B_N(\tilde{a}, \epsilon)$ are C^1 in ϵ for fixed $\tilde{a}$. Indeed, they coincide with the eigenvalues of the symmetric matrix $A_0(\tilde{a}, \epsilon)$ by definition, to which we can apply [Kat66, Theorem II.6.8].

Now, we take $\epsilon = 0$ first. Then due to $a_i \neq 0$ for $i = 1, ..., N-1$ we know that

$$
\nu_1(0), ..., \nu_{N-1}(0) \neq 0
$$

using (1). In particular, the claim follows for the eigenvalues $\nu_1(0), ..., \nu_{N-1}(0)$. Moreover,

$$
\prod_{i=1}^{N-1} \nu_i(0) = N \prod_{i=1}^{N-1} a_i \neq 0
$$

owing to (C.2), and

$$
\nu_N(0) = 0,
$$

because the last row of $B_N(\tilde{a}, 0)$ vanishes. Therefore,

$$
(C.3) \qquad \operatorname{sign}\left(\prod_{i=1}^{N-1} \nu_i(\epsilon)\right) = \operatorname{sign}\left(\prod_{i=1}^{N-1} a_i\right) \neq 0
$$

for all small ϵ. Applying (C.2) to $B_N(\tilde{a}, \epsilon)$ yields

$$\prod_{i=1}^{N} \nu_i(\epsilon) = (N+1)\,\epsilon \prod_{j=1}^{N-1} a_j \neq 0$$

and (C.3) implies that

$$\mathrm{sign}(\nu_N(\epsilon)) = \mathrm{sign}(\epsilon)$$

for all small ϵ. On the other hand,

$$\det B_N(\tilde{a}, \epsilon) = (N+1)\,\epsilon \prod_{i=1}^{N-1} a_i \neq 0$$

is valid for all $\epsilon \neq 0$, whence no eigenvalue can change sign as long as ϵ is non-zero. This proves the lemma.

It is possible to prove the lemma using [Wil65, paragraphs 36-38].

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