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Coercivity for elliptic operators and positivity of solutions on Lipschitz domains

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Abstract

We show that usual second order operators in divergence form satisfy coercivity on Lipschitz domains if they are either complemented with homogeneous Dirichlet boundary conditions on a set of non-zero boundary measure or if a suitable Robin boundary condition is posed. Moreover, we prove the positivity of solutions in a general, abstract setting, provided that the right hand side is a positive functional. Finally, positive elements from $W^{-1,2}$ are identified as positive measures.

1 Introduction

The aim of this paper is to show the following two theorems:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and D be a closed subset of $\partial\Omega$. Furthermore, ε is a bounded, nonnegative function on $\partial\Omega \setminus D$ that is measurable with respect to the boundary measure σ on $\partial\Omega$. Suppose that μ is a bounded, Lebesgue measurable function on Ω taking its values in the set of real, symmetric $d \times d$ matrices and that is elliptic. Let the sesquilinear form $\mathfrak{t} : W^{1,2}_D(\Omega) \times W^{1,2}_D(\Omega) \to \mathbb{R}$ be defined by*

$$\mathfrak{t}(\psi, \varphi) := \int_\Omega \mu \nabla\psi \cdot \nabla\overline{\varphi} \, \mathrm{dx} + \int_{\partial\Omega \setminus D} \varepsilon \psi \overline{\varphi} \, d\sigma. \tag{1.1}$$

Then $\mathfrak{t}$ is continuous.

If, additionally either

1. *D is of positive boundary measure or*
2. *$D = \emptyset$ and $\int_{\partial\Omega} \varepsilon \, d\sigma > 0$,*

then the form $\mathfrak{t}$ is coercive.

Theorem 1.2. *Let $\mathfrak{t}_\mathbb{R}$ be the restriction of $\mathfrak{t}$ to the real part $\check{W}^{1,2}_D(\Omega)$ of $W^{1,2}_D(\Omega)$ and define $-\nabla \cdot \mu \nabla : \check{W}^{1,2}_D(\Omega) \to \check{W}^{-1,2}_D(\Omega)$ by $\langle -\nabla \cdot \mu \nabla \psi, \varphi \rangle = \mathfrak{t}_\mathbb{R}(\psi, \varphi)$. If $f \in \check{W}^{-1,2}_D(\Omega)$ takes nonnegative values on all elements of $\check{W}^{1,2}_D(\Omega)$, which are nonnegative almost everywhere on Ω, then the solution ψ of $-\nabla \cdot \mu \nabla \psi = f$ is nonnegative almost everywhere on Ω.*

All notation used here will be properly defined in the next section, and the boundary measure σ in Section 3.1

It has been known since long that coercivity is required in many contexts. It is well established when the underlying domain is a strong Lipschitz domain (see [12, Ch. 1.1.9] for the definition), cf. [14, Ch. 3.4] and references therein. However, in recent years it turned out that for applied problems the more general class of Lipschitz domains often provides the adequate frame. This class is, on one hand, general enough to include geometrical objects as the two crossing beams – which is not a strong Lipschitz domain. On the other hand, Lipschitz domains admit required functional analytic properties – such as the extension property for suitable functions spaces – which are essential e.g. for the proof of Poincaré's inequality (see [20, Ch. 4]) or for Gaussian estimates of the induced semigroup (see [16, Ch. 6/7] and references therein). Since there seems to be no reference, concerning coercivity on Lipschitz domains, we give a comprehensive proof here. It rests on some nontrivial facts on boundary measures (see Section 3.2) and embedding in case of Lipschitz domains, mostly quoted in the appendix in Section 5.

The point is that the consideration of such geometries is by far not only academic, but relevant in many real world applications: the reader may think e.g. of the combination of railroad track and the underlying railroad tie in view of a corresponding heat conduction problem. Moreover, some years ago, highly promising photonic crystals have been developed which have a so-called woodpile structure, see ([3], [11], [8, p. 102]).

Concerning Theorem 1.2, it is our aim to show that it can be deduced from purely algebraic properties of the determining bilinear form alone, see the abstract criterion in Theorem 4.2. It turns out that besides the strict positivity of the form of type (1.1), one only needs to show a specific orthogonality relation (see (4.1)) that is then easily verified.

Finally, in Subsection 4.3 we show that continuous, positive functionals on the space $W^{1,2}_D$ may be identified as measures. On one hand this is interesting in itself, since this property is false in general if the positivity assumption is dropped. On the other hand this can help to overcome the following conflict that appears in several contexts: A conservation law demands the total mass of a quantity to be fixed as a given number N, while the adequate instrument for treating the problem are monotonicity arguments in the $W^{1,2}_D \leftrightarrow W^{-1,2}_D$ context. In particular, this is the case in the mathematical treatment of the Schrödinger-Poisson system, see [10], in particular Remark 2.19 therein.

It turns out that both views may be combined in the setting of positive elements from $W^{-1,2}_D(\Omega)$. If one has the a priori information that the positive functional attains a prescribed value N on the function which is identically 1 on Ω, then we show that the corresponding measure is bounded, cf. Theorem 4.4.

2 Notation

We will always suppose that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain in the spirit of [6], (see also [12, Ch. 1.1.9]). This means that for every $\mathrm{x} \in \partial\Omega$ there is an open neighbourhood $\Upsilon_\mathrm{x} \ni \mathrm{x}$ and a bi-Lipschitz mapping $\phi_\mathrm{x} : \Upsilon_\mathrm{x} \to \mathbb{R}^d$, such that $\Upsilon_\mathrm{x} \cap \Omega$ is mapped onto the open half unit cube and $\Upsilon_\mathrm{x} \cap \partial\Omega$ onto the equatorial plate $\mathcal{P}$ of the unit cube, i.e.

$$\phi_\mathrm{x}(\Upsilon_\mathrm{x} \cap \Omega) = \{\mathrm{x} \in \mathbb{R}^d : -1 < x_j < 1 \text{ for } j = 1, \dots, d-1,\ 0 < x_d < 1\}, \tag{2.1}$$

$$\phi_\mathrm{x}(\Upsilon_\mathrm{x} \cap \partial\Omega) = \{\mathrm{x} \in \mathbb{R}^d : -1 < x_j < 1 \text{ for } j = 1, \dots, d-1,\ x_d = 0\} =: \mathcal{P}, \quad \text{and} \tag{2.2}$$

$$\phi_\mathrm{x}(\mathrm{x}) = 0 \in \mathbb{R}^d.$$

If $\Pi \subset \mathbb{R}^d$ is a set and κ a positive measure on Π, then we denote by $L^p(\Pi;\kappa)$ the usual space of real-valued, p-integrable (equivalence classes of) functions on Π. If $\vartheta : \Pi \to \mathbb{R}$ is a bounded, κ-measurable function, then we denote the measure $f \mapsto \int_\Pi f\vartheta\, d\kappa$ by $\vartheta\kappa$. When κ is the d-dimensional Lebesgue measure, we write $L^p(\Omega)$ instead of $L^p(\Omega;\kappa)$.

For a closed set $D \subset \partial\Omega$, we denote by $W^{1,p}_D(\Omega)$ the closure of the set $\{\psi|_\Omega : \psi \in C^\infty(\mathbb{R}^d),\ \mathrm{supp}(\psi) \cap D = \emptyset\}$ in $W^{1,p}(\Omega)$. The symbol $W^{-1,p}_D(\Omega)$ stands for the space of continuous antilinear forms on $W^{1,p'}_D(\Omega)$. Finally, $\langle \cdot, \cdot \rangle$ indicates the sesquilinear pairing between a Banach space and its anti-dual.

3 Proof of Theorem 1.1

3.1 Boundary measure and traces

Let us first introduce a boundary measure on $\partial\Omega$ and point out some of its basic properties. Note that this is not canonic, because in our context Ω is not necessarily a strong Lipschitz domain (compare [14, Ch. 3.1.2]). Let, according to the definition of a Lipschitz domain, for every point $\mathrm{x} \in \partial\Omega$ an open neighborhood Υ_x of x and a bi-Lipschitz function $\phi_\mathrm{x} : \Upsilon_\mathrm{x} \to \mathbb{R}^d$ be given, which satisfy (2.1)/(2.2). Let $\Upsilon_{\mathrm{x}_1}, \dots, \Upsilon_{\mathrm{x}_l}$ be a finite subcovering of $\partial\Omega$ and $\eta_1, \dots, \eta_l$ be a continuous partition of unity over $\partial\Omega$, subordinated to this subcovering. We denote the inverse of ϕ_{x_j} restricted to $\mathcal{P}$ by ζ_j and define

$$\mathcal{J}_j := \Big(\sum_{k=1}^{d-1} \mathrm{Det}\Big(\frac{\partial(\zeta_j^1, \dots, \zeta_j^{k-1}, \zeta_j^{k+1}, \dots, \zeta_j^d)}{\partial(x_1, \dots, x_{d-1})}\Big)^2\Big)^{1/2}$$

as the Jacobian of $\zeta_j = (\zeta_j^1, \dots, \zeta_j^d) : \mathcal{P} \to \mathbb{R}^d$. In this notation, we define the measure σ on $\partial\Omega$ by

$$\int_{\partial\Omega} f\, d\sigma := \sum_{j=1}^{l} \int_{\mathcal{P}} (\eta_j f) \circ \zeta_j\ \mathcal{J}_j\ d\mathrm{x}, \quad f \in C(\partial\Omega), \tag{3.1}$$

where the integration on the right hand side is performed with respect to the $(d-1)$-dimensional Lebesgue measure on $\mathcal{P} = \phi_{\mathrm{x}_j}(\partial\Omega \cap \Upsilon_{\mathrm{x}_j})$. Note that ζ_j is Lipschitzian on $\mathcal{P}$; hence the partial derivatives are essentially bounded, and the integral on the right hand side of (3.1) exists for all continuous functions f.

Remark 3.1. i) In case of a smooth domain the thus defined measure σ is identical with the classical surface measure – which does not depend on the parametrization of $\partial\Omega$. The latter carries over to the case of only Lipschitzian mappings due to the rule for differentiation of the composition of Lipschitzian functions.

ii) In fact, one can show that the measure σ is nothing else but the restriction of the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$, restricted to $\partial\Omega$, compare [2, Ch. 3.3.4 C].

iii) In case of a strong Lipschitz domain the measure is locally defined by

$$f \mapsto \int_{\mathbb{R}^{d-1}} f \sqrt{1 + |\nabla g|^2} \, d\mathrm{x},$$

when the domain Ω lies locally under the graph of the Lipschitz function $\mathbb{R}^{d-1} \ni \mathrm{x} \mapsto g(x)$ and f is a continuous function with suitable support (see [2, Ch. 3.3] for more information).

Lemma 3.2. *σ is a bounded, positive Radon measure on $\overline{\Omega}$, which additionally satisfies*

$$\sup_{\mathrm{x} \in \mathbb{R}^d} \sup_{r \in]0,1[} \sigma(B(\mathrm{x}, r) \cap \overline{\Omega}) r^{1-d} < \infty, \tag{3.2}$$

where, here and in the sequel, $B(\mathrm{x}, r)$ denotes the ball centered at x with radius r.

Proof. Since ζ_j is Lipschitzian on $\mathcal{P}$, the Jacobian $\mathcal{J}_j$ is essentially bounded with respect to the $(d-1)$-dimensional Lebesgue measure on $\mathcal{P}$, see [2, Ch. 4.2.3]. Additionally, we have $|\eta_j \circ \zeta_j| \le 1$. Hence, it suffices to prove (3.2), if σ is there replaced by any of the measures

$$\sigma_j : C(\partial\Omega) \ni f \mapsto \int_{\mathcal{P}} f \circ \zeta_j \, d\mathrm{x}.$$

The bi-Lipschitz property of ζ_j allows to estimate the distance between points and the distance of their images mutually from above and below by constants; thus, the proof of the assertion reduces to a proof for the $(d-1)$-dimensional Lebesgue measure on $\mathcal{P}$. For this the assertion is straightforward to check. □

Theorem 3.3. *There exists a trace operator* Tr *that maps any space $W^{1,p}(\Omega)$ continuously into $L^p(\partial\Omega; d\sigma)$, $p \in]1, \infty[$.*

Proof. The proof follows immediately from Lemma 3.2 and Proposition 5.3, since the set $\{\psi|_\Omega : \psi \in C^\infty(\mathbb{R}^d)\}$ is by definition dense in $W^{1,p}(\Omega)$. □

Corollary 3.4. *If $\Pi \subset \partial\Omega$ is measurable with respect to σ, then the measure $C(\partial\Omega) \ni \psi \mapsto \int_\Pi \psi \, d\sigma$ extends to a continuous linear form on $W^{1,p}(\Omega)$ for any $p > 1$.*

Proof. One has for $\psi \in W^{1,p}(\Omega)$

$$\Big| \int_\Pi \mathrm{Tr}\psi \, d\sigma \Big| \le \int_{\partial\Omega} |\mathrm{Tr}\psi| \, d\sigma \le \big(\sigma(\partial\Omega)\big)^{\frac{p-1}{p}} \|\mathrm{Tr}\psi\|_{L^p(\partial\Omega;\sigma)} \le c\|\psi\|_{W^{1,p}(\Omega)}.$$

□

Let here and in the sequel (as usual) $p^* := \frac{dp}{d-p}$ for $1 \le p < d$ and $p^* = \infty$ for $p \ge d$ denote the Sobolev conjugated index for p.

Theorem 3.5 (Poincaré's inequality)**.** *Let Ω be a Lipschitz domain and $\mathcal{M} \subset \partial\Omega$ a σ-measurable set which is not σ-negligible. Then, for every $p \in \,]1,\infty[$, there are constants $c_0(p), c_1(p)$ such that*

$$\|\psi\|_{L^p(\Omega)} \le c_0(p)\|\nabla\psi\|_{L^p(\Omega)} \quad \text{and} \quad \|\psi\|_{L^{p^*}(\Omega)} \le c_1(p)\|\nabla\psi\|_{L^p(\Omega)} \tag{3.3}$$

for all $\psi \in W^{1,p}(\Omega)$ which satisfy $\int_\mathcal{M} \psi \, d\sigma = 0$.

Proof. Due to Proposition 5.1, Ω – as a Lipschitz domain – is an extension domain. Thus, the proof results from Proposition 5.4 and Corollary 3.4. □

3.2 The proof

The continuity of $\mathfrak{t}$ follows from the continuity of the trace $\mathrm{Tr} : W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega;\sigma)$ and the boundedness of ε.

Concerning the coercivity, we consider the cases 1. and 2. in the supposition separately. Concerning the first, we focus on the case $\varepsilon \equiv 0$, which then implies the result for $\varepsilon \neq 0$ thanks to $\varepsilon \ge 0$. We put $\kappa := \chi_D\sigma$, where χ_D is the indicator function of D. Then for every $\psi \in W^{1,2}_D(\Omega)$ we have $\int_{\overline{\Omega}} \psi \, d\kappa = 0$. Hence, according to Theorem 3.5, one obtains $\|\psi\|_{L^2(\Omega)} \le c\|\nabla\psi\|_{L^2(\Omega)}$ for all $\psi \in W^{1,2}_D(\Omega)$, where c is independent from ψ. This yields the estimate

$$\mathfrak{t}(\psi,\psi) \ge m\|\nabla\psi\|^2_{L^2(\Omega)} \ge \frac{m}{2}\|\nabla\psi\|^2_{L^2(\Omega)} + \frac{m}{2c^2}\|\psi\|^2_{L^2(\Omega)} \ge \hat{c}\|\psi\|^2_{W^{1,2}_D(\Omega)},$$

where m is the ellipticity constant of μ.

Concerning the case 2, it suffices to show that the identity on $W^{1,2}(\Omega)$, this space equipped once with the norm $\psi \mapsto \sqrt{\mathfrak{t}(\psi,\psi)}$ and once equipped with the usual norm, is continuous. Let in this spirit $\{\psi_n\}_n \subset W^{1,2}(\Omega)$ be a sequence with $\lim_{n\to\infty} \mathfrak{t}(\psi_n,\psi_n) = 0$. Since both integrals in $\mathfrak{t}(\psi_n, \psi_n)$ are non-negative, we obtain that $\{\mathrm{Tr}(\psi_n)\}_n$ approaches 0 in $L^2(\partial\Omega; \varepsilon\sigma)$ and $\{|\nabla\psi_n|\}_n$ goes to 0 in $L^2(\Omega)$. Clearly, one has

$$\nabla\psi_n = \nabla\Big(\psi_n - \frac{1}{|\Omega|}\int_\Omega \psi_n \, d\mathrm{x}\Big) \quad \text{and} \quad \int_\Omega \Big(\psi_n - \frac{1}{|\Omega|}\int_\Omega \psi_n \, d\mathrm{x}\Big)\, d\mathrm{y} = 0. \tag{3.4}$$

Thus, taking κ as the Lebesgue measure on Ω in Proposition 5.4, we get that $\{\psi_n - \frac{1}{|\Omega|}\int_\Omega \psi_n \, d\mathrm{x}\}_n$ goes to 0 in $L^2(\Omega)$. Hence, this sequence converges to 0 in $W^{1,2}(\Omega)$ equipped with the usual norm.

On the other hand this implies that, due to Theorem 3.3, the sequence $\{\mathrm{Tr}(\psi_n - \frac{1}{|\Omega|}\int_\Omega \psi_n \, d\mathrm{x})\}_n$ converges to 0 in $L^2(\partial\Omega; \sigma) \hookrightarrow L^2(\partial\Omega; \varepsilon\sigma)$. Since $\{\mathrm{Tr}(\psi_n)\}_n$ also converges to 0 in $L^2(\partial\Omega; \varepsilon\sigma)$, the sequence $\{\frac{1}{|\Omega|}\int_\Omega \psi_n \, d\mathrm{x}\}_n$ – viewed as elements from $L^2(\partial\Omega; \varepsilon\sigma)$ – converges in $L^2(\partial\Omega; \varepsilon\sigma)$ to 0. But this is nothing else than the convergence of the number sequence $\{\frac{1}{|\Omega|}\int_\Omega \psi_n \, d\mathrm{x}\}_n$, as $\int_{\partial\Omega} \varepsilon \, d\sigma > 0$. This implies the convergence to 0 of the sequence $\{\psi_n\}_n$ in $W^{1,2}(\Omega)$ and we are done.

4 Proof of Theorem 1.2

4.1 Positivity in an abstract setting

Let $\mathfrak{H}$ be a real Hilbert space. We assume the existence of a cone $\mathcal{K} \subset \mathfrak{H}$, such that any element $\psi \in \mathfrak{H}$ admits a unique representation $\psi = \psi_+ - \psi_-$ with $\psi_+, \psi_- \in \mathcal{K}$. Let $\mathfrak{r}$ be a continuous, symmetric bilinear form on $\mathfrak{H}$ which, additionally, is strictly positive. This means that $\mathfrak{r}(\psi, \psi) > 0$, whenever $\psi \neq 0$. We define a linear, continuous operator $A : \mathfrak{H} \to \mathfrak{H}^*$ by $\langle A\psi, \varphi\rangle := \mathfrak{r}(\psi, \varphi)$ for all $\varphi \in \mathfrak{H}$.

Definition 4.1. We call an element f from $\mathfrak{H}^*$ positive, if $\langle f, \psi\rangle \geq 0$ for every $\psi \in \mathcal{K}$.

Now we state the positivity result in the abstract setting of forms:

Theorem 4.2. *Let $\psi \in \mathfrak{H}$ with the decomposition $\psi = \psi_+ - \psi_-$ as above be given. If*

$$\mathfrak{r}(\psi_+, \psi_-) = 0 \tag{4.1}$$

and $A\psi \in \mathfrak{H}^*$ is positive, then $\psi \in \mathcal{K}$.

For the proof of Theorem 4.2 we need the following

Lemma 4.3. *An element $\psi \in \mathfrak{H}$ satisfies $A\psi = f$, iff it minimizes the functional $F : \mathfrak{H} \to \mathbb{R}$ given by*

$$F(\xi) = \frac{1}{2}\mathfrak{r}(\xi, \xi) - \langle f, \xi\rangle. \tag{4.2}$$

Proof. If ψ minimizes F, then for every $\varphi \in \mathfrak{H}$ the function

$$g_\varphi : \mathbb{R} \ni s \mapsto F(\psi + s\varphi) = F(\psi) + \frac{s^2}{2}\mathfrak{r}(\varphi,\varphi) + s\big[\mathfrak{r}(\psi,\varphi) - \langle f,\varphi\rangle\big]$$

has to take its minimum in $s = 0$. It is clear that this can only happen, if $\mathfrak{r}(\psi,\varphi) - \langle f,\varphi\rangle = \langle A\psi,\varphi\rangle - \langle f,\varphi\rangle = 0$.

Conversely, assume $A\psi = f$. Then, by definition, $\mathfrak{r}(\psi,\varphi) - \langle f,\varphi\rangle = 0$ for all $\varphi \in \mathfrak{H}$. Thus, writing

$$F(\psi + \varphi) = F(\psi) + \frac{1}{2}\mathfrak{r}(\varphi,\varphi) + \mathfrak{r}(\psi,\varphi) - \langle f,\varphi\rangle = F(\psi) + \frac{1}{2}\mathfrak{r}(\varphi,\varphi),$$

this attains its minimum with respect to φ, if $\mathfrak{r}(\varphi,\varphi) = 0$. This implies $\varphi = 0$ by the strict positivity of $\mathfrak{r}$. $\square$

Proof of Theorem 4.2. Assume $A\psi = A(\psi_+ - \psi_-) = f$. We consider the function

$$\theta(s) := F(s\psi + (1-s)\psi_+) = F(\psi_+ - s\psi_-),$$

which has to take its minimum in $s = 1$, according to Lemma 4.3. In view of (4.1) one easily calculates

$$\theta(s) = F(\psi_+) - s\mathfrak{r}(\psi_+,\psi_-) + \frac{s^2}{2}\mathfrak{r}(\psi_-,\psi_-) + s\langle f,\psi_-\rangle = F(\psi_+) + \frac{s^2}{2}\mathfrak{r}(\psi_-,\psi_-) + s\langle f,\psi_-\rangle.$$

Since $f \in \mathfrak{H}^*$ is positive, we have $\langle f,\psi_-\rangle \geq 0$. Assume $\langle f,\psi_-\rangle > 0$. Then for negative s with sufficiently small absolute value $\frac{1}{2}s^2\mathfrak{r}(\psi_-,\psi_-) + s\langle f,\psi_-\rangle$ becomes strictly negative. This, however, contradicts the minimum property for $s = 1$. Hence, $\langle f,\psi_-\rangle = 0$. But then the minimum can be attained in $s = 1$ only if $\mathfrak{r}(\psi_-,\psi_-)$ also vanishes, what implies $\psi_- = 0$ by the strict positivity of $\mathfrak{r}$. $\square$

4.2 The proof

In order to apply Theorem 4.2 for the proof of Theorem 1.2, one must first define the cone $\mathcal{K}$ in $\mathfrak{H} := \breve{W}^{1,2}_D$ and afterwards show that the form $\mathfrak{t}_\mathbb{R}$ fulfils Condition (4.1). In the notation of Proposition 5.5 we define $\mathcal{K} = \{\psi : \psi = \psi_+\}$. Using (5.4), one gets that the function $\mu\nabla\psi_+ \cdot \nabla\psi_-$ is Lebesgue-negligible on Ω, and, hence, $\int_\Omega \mu\nabla\psi_+ \cdot \nabla\psi_- \, d\mathrm{x} = 0$ for every $\psi \in \breve{W}^{1,2}_D(\Omega)$. Moreover, the product $\psi_+\psi_-$ vanishes Lebesgue-almost everywhere on Ω. On the other hand, this product belongs to $\breve{W}^{1,\frac{d}{d-1}}(\Omega)$ by Proposition 5.6. Hence, it represents 0 in the space $\breve{W}^{1,\frac{d}{d-1}}(\Omega)$. Consequently, according to Theorem 3.3, the trace of $\psi_+\psi_-$ vanishes σ-almost everywhere on $\partial\Omega$, what implies $\int_{\partial\Omega} \varepsilon\psi_+\psi_- \, d\sigma = 0$ and, so $\mathfrak{t}_\mathbb{R}(\psi_+,\psi_-) = 0$. Since Theorem 4.2 then yields $\psi_- = 0$, the proof is finished.

4.3 Identification of positive functionals as measures

Our final aim is to show that every positive element of $\check{W}_D^{-1,2}(\Omega)$ may be identified as a positive Radon measure on $\Omega \cup (\partial\Omega \setminus D)$ and that this measure is bounded, whenever the positive functional attains a finite value on the function that is identically 1 on Ω.

In this spirit, we denote the dual space to $\check{W}_D^{1,2}(\Omega) \oplus \mathbf{1}$ by $\tilde{W}_D^{-1,2}(\Omega)$ and formulate our next result.

Theorem 4.4. *Assume that D is a closed subset of $\partial\Omega$ with positive boundary measure or $D = \emptyset$.*

i) Then every positive element w from $\tilde{W}_D^{-1,2}(\Omega)$ may be represented by a positive Radon measure ϖ on $\Omega \cup (\partial\Omega \setminus D)$ in the sense of

$$\langle w, \psi \rangle = \int_{\Omega \cup (\partial\Omega \setminus D)} \psi \, d\varpi. \tag{4.3}$$

ii) If w is a positive element from $\tilde{W}_D^{-1,2}(\Omega)$, then the total mass of the associated Radon measure ϖ (cf. i)) is finite and there exists a positive Radon measure ω on $\overline{\Omega}$ that coincides with ϖ on $\check{W}_D^{1,2}(\Omega)$ and satisfies $\int_{\overline{\Omega}} \mathbf{1} d\omega = \langle w, \mathbf{1} \rangle$.

Proof. It is clear that $\Omega \cup (\partial\Omega \setminus D)$ is a locally compact space (recall that D is a closed subset of $\partial\Omega$). Let for all what follows K be an arbitrary compact subset of $\Omega \cup (\partial\Omega \setminus D)$. Assume $\eta \in C^\infty(\mathbb{R}^d)$ to be a nonnegative function which equals 1 on K and 0 on a neighbourhood of D. Then one gets for every

$$\psi \in X_K := \{\varphi|_{\overline{\Omega}} : \varphi \in C^\infty(\mathbb{R}^d),\ \varphi \equiv 0 \text{ on } \overline{\Omega} \setminus K\} \subset \check{W}_D^{1,2}$$

the estimate

$$-\sup_{\mathrm{x} \in K} |\psi(\mathrm{x})| \, \eta \le \psi \le \sup_{\mathrm{x} \in K} |\psi(\mathrm{x})| \, \eta.$$

Thus, the positivity of w implies $|\langle w, \psi \rangle| \le \sup_{\mathrm{x} \in K} |\psi(\mathrm{x})| \langle w, \eta \rangle$. This means, that w is continuous on X_K, when this space is equipped with the sup norm. Let Y_K be the space of continuous functions on $\Omega \cup (\partial\Omega \setminus D)$ having their support in K. We will show that X_K is dense in Y_K. Let $\psi \in C(\Omega \cup (\partial\Omega \setminus D))$ with support in K and $\epsilon > 0$ be given. Since K is compact, it must also be a closed subset of $\mathbb{R}^d$. Hence, ψ possesses a continuous extenxion $\hat{\psi}$ to the whole $\mathbb{R}^d$, which, additionally, has compact support. Smoothing $\hat{\psi}$ with a suitable convolution kernel, one obtains a compactly supported function $\tilde{\psi} \in C^\infty(\mathbb{R}^d)$ which satisfies

$$\sup_{\mathrm{x} \in \Omega \cup (\partial\Omega \setminus D)} |\psi(\mathrm{x}) - \tilde{\psi}(\mathrm{x})| \le \sup_{\mathrm{x} \in \mathbb{R}^d} |\hat{\psi}(\mathrm{x}) - \tilde{\psi}(\mathrm{x})| \le \epsilon. \tag{4.4}$$

Of course, $\tilde{\psi}|_{\Omega \cup (\partial\Omega \setminus D)}$ need not belong to X_K, since the support property of ψ is not maintained under convolution. In the following step we modify $\tilde{\psi}$ in a manner that this will hold true for the resulting function. For this, let ∂_K denote the boundary of K within the set $\Omega \cup (\partial\Omega \setminus D)$. It is not hard to see that ∂_K is closed in $\mathbb{R}^d$. Note that, according to (4.4), $|\tilde{\psi}| \leq \epsilon$ on ∂_K. Let $\mathcal{U}$ be a neighbourhood of ∂_K in $\mathbb{R}^d$ such that $|\tilde{\psi}| \leq 2\epsilon$ on $\mathcal{U}$. Furthermore, let $\varrho : \mathbb{R}^d \to [0, \infty[$ be a regularizing distance function for ∂_K (see [19, Ch. VI.2]) and δ a positive number, such that $\{\mathrm{x} : \varrho(\mathrm{x}) \leq \delta\} \subset \mathcal{U}$. Next, we introduce a monotonously increasing function $\varsigma \in C^\infty(\mathbb{R})$ that is identically 0 on $]-\infty, \frac{\delta}{2}]$ and identically 1 on $[\delta, \infty[$. Finally, we define $\hat{\varrho}$ as the composition of ϱ and ς and put $\check{\psi} := \hat{\varrho}\tilde{\psi}$ on K and 0 elsewhere on $\Omega \cup (\partial\Omega \setminus D)$.

Obviously, then $\check{\psi}|_{\overline{\Omega}} \in X_K$. Moreover, $\check{\psi}$ differs on K from ψ by construction not more than 3ϵ in the sup norm. Hence, any function from Y_K may be approximated in the sup norm arbitrarily good by functions from X_K. This means, that the linear form w extends uniquely to a continuous linear form on Y_K. Since this is true for all compact subsets K, w induces a – also uniquely determined – positive Radon measure ϖ on $\Omega \cup (\partial\Omega \setminus D)$. One easily verifies from the definitions that $\bigcup_K X_K$ is dense in $\breve{W}^{1,2}_D(\Omega)$, thus w acts as the positive Radon measure ϖ on the whole of $\breve{W}^{1,2}_D(\Omega)$. This proves i).

If $D = \emptyset$, then $\mathbf{1} \in \breve{W}^{1,2}_D(\Omega)$, and ii) is also proved. Assume now $\sigma(D) \neq 0$. Let $\{K_n\}_n$ be an increasing sequence of compact subsets of $\Omega \cup (\partial\Omega \setminus D)$ which exhausts $\Omega \cup (\partial\Omega \setminus D)$, i.e. $\bigcup_n K_n = \Omega \cup (\partial\Omega \setminus D)$. If ψ_n is a C^∞ function which satisfies $0 \leq \psi_n \leq 1$ and which equals 1 on K_n and 0 on D, then one can estimate, due to the positivity of w,

$$\langle w, \mathbf{1}\rangle \geq \langle w, \psi_n\rangle = \int \psi_n \, d\varpi \geq \varpi(K_n).$$

Taking the limit for $n \to \infty$, one obtains $\varpi(\Omega \cup (\partial\Omega \setminus D)) = \lim_{n\to\infty} \varpi(K_n) \leq \langle w, \mathbf{1}\rangle$. If σ is again the boundary measure on $\partial\Omega$, we define the measure ω_D on $\overline{\Omega}$ by $\omega_D := \frac{\langle w,\mathbf{1}\rangle - \varpi(\Omega\cup(\partial\Omega\setminus D))}{\sigma(D)} \chi_D \sigma$, where χ_D is the indicator function of the set D. ω_D annihilates the space $\breve{W}^{1,2}_D(\Omega)$, thus ϖ and $\omega := \varpi + \omega_D$ coincide on this space. Moreover, the measure ω attains on the function which is constantly 1 the prescribed value $\langle w, \mathbf{1}\rangle$. □

Remark 4.5. The above considerations show the following: Two measures ω and $\tilde{\omega}$ on $\overline{\Omega}$ represent the same positive linear form, iff they coincide on the set $\overline{\Omega} \setminus D$ and, additionally, satisfy $\omega(D) = \tilde{\omega}(D)$.

An intrinsic characterization of Radon measures which define continuous linear forms on $W^{1,2}$ is given in [20, Ch. 4].

Corollary 4.6. *Any set of positive functionals on $\breve{W}^{1,2}_D(\Omega) \oplus \mathbf{1}$, whose elements w satisfy the norming condition $\langle w, \mathbf{1}\rangle = N$, forms a bounded set in any space $(\breve{W}^{1,q}_D(\Omega) \oplus \mathbf{1})'$, if $q \in [1, \frac{d}{d-1}[$.*

Proof. To every such w corresponds a positive Radon measure on $\overline{\Omega}$ with total mass N. By the embedding $\breve{W}^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ for all $q > d$ and duality one obtains the assertion. □

Remark 4.7. For every $\varphi \in \breve{W}^{1,2}_D(\Omega)$, the element $|\varphi|$ also belongs to $\breve{W}^{1,2}_D(\Omega)$. Thus, for every element $\varphi \in \breve{W}^{1,2}_D(\Omega)$ and $\psi := |\varphi|$, (4.3) holds true. This means that every

$\varphi \in \breve{W}^{1,2}_D(\Omega)$ in fact belongs to $L^1(\Omega \cup (\partial\Omega \setminus D); d\varpi)$, and the integral in (4.3) is a classical Lebesgue integral (including all rules of this calculus).

5 Appendix

In this section we will recall some results needed in our above considerations.

Proposition 5.1 ([4, Thm. 7.25], see also [5, Thm. 3.10])**.** *If Ω is a Lipschitz domain, then it is an extension domain for all spaces $W^{1,p}(\Omega)$ ($p \in]1,\infty[$), that means; there exists a linear, continuous operator $\mathfrak{E}_p : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$, such that the composition of $\mathfrak{E}_p$ with the restriction operator gives the identity on $W^{1,p}(\Omega)$. Moreover, if $p < q$, then the restriction of $\mathfrak{E}_p$ to $W^{1,q}(\Omega)$ equals $\mathfrak{E}_q$.*

Remark 5.2. Inspecting the construction of the extension operator it becomes clear that it also acts as a continuous extension operator from $L^p(\Omega)$ into $L^p(\mathbb{R}^d)$, $p \in [1,\infty]$.

Proposition 5.3 ([12, Ch. 1.4.7])**.** *If ρ is a finite measure on $\overline{\Omega}$ which satisfies*

$$\sup_{\mathrm{x}\in\mathbb{R}^d} \sup_{r\in]0,1[} \rho(B(\mathrm{x},r)\cap\overline{\Omega})r^{1-d} < \infty, \tag{5.1}$$

then for every $p \in]1,\infty[$ there is a constant $c = c(p)$, such that

$$\|\psi|_{\overline{\Omega}}\|_{L^p(\overline{\Omega},\rho)} \le c\|\psi|_\Omega\|_{W^{1,p}(\Omega)} \text{ for every } \psi \in C^\infty(\mathbb{R}^d). \tag{5.2}$$

Proposition 5.4 ([20, Thm. 4.8.1])**.** *Assume $p \in]1,\infty[$. Let Ω be a bounded extension domain. Let $\kappa \neq 0$ be a non-negative measure on $\overline{\Omega}$ which defines a continuous linear form on $W^{1,p}(\Omega)$. Then there are constants $c_0(p), c_1(p)$ such that*

$$\|\psi\|_{L^p(\Omega)} \le c_0(p)\|\nabla\psi\|_{L^p(\Omega)} \quad \text{and} \quad \|\psi\|_{L^{p^*}(\Omega)} \le c_1(p)\|\nabla\psi\|_{L^p(\Omega)} \tag{5.3}$$

holds for all $\psi \in W^{1,p}(\Omega)$ with $\int_{\overline{\Omega}} \psi \, d\kappa = 0$.

Proposition 5.5 ([4, Ch. 7.4])**.** *For every $\psi \in W^{1,2}(\Omega)$ the elements $\psi_+ := \max(0,\psi)$, $\psi_- := -\min(0,\psi)$ also belong to $W^{1,2}(\Omega)$ and satisfy $\psi = \psi_+ - \psi_-$. Moreover, one has*

$$\nabla\psi_+ = \begin{cases} \nabla\psi, & \text{if } \psi > 0 \\ 0, & \text{if } \psi \le 0, \end{cases} \qquad \nabla\psi_- = \begin{cases} 0, & \text{if } \psi \ge 0 \\ -\nabla\psi, & \text{if } \psi < 0. \end{cases} \tag{5.4}$$

Proposition 5.6 ([6, Thm. 1.4.4.2])**.** *Forming the product of two functions maps the space $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ continuously into the space $W^{1,\frac{d}{d-1}}(\Omega)$.*

Remark 5.7. Proposition 5.6 is formulated in [6] only for the case $\Omega = \mathbb{R}^d$. But it is clear that it also holds, if Ω is an extension domain, cf. Proposition 5.1.

6 Concluding remarks

- The operator $-\nabla \cdot \mu \nabla$, restricted to any space $L^p(\Omega)$ ($p \in]1, \infty[$) is complemented by the following boundary conditions:

$$\psi|_D = 0 \quad \text{and} \quad \nu \cdot \mu \nabla \psi = \varepsilon \psi \quad \text{on } \partial\Omega \setminus D \quad \text{for } \psi \in \operatorname{dom}_{L^p(\Omega)}(-\nabla \cdot \mu \nabla), \tag{6.1}$$

  where ν is the outer unit normal, and the second condition in (6.1) is to be understood in the distributional sense.

- An explicit construction of the mappings ϕ_{x} which flatten the two crossing beams near the four critical crossing points is presented in [7, Ch. 7.3].

- It is known (see [2, Ch. 5.8]) that the measure σ is the adequate one to formulate the Gauss-Green theorem on general sets (recall Remark 3.1), compare also [15] and [1] for recent results.

- It has been known since long that the additional property for a linear form on $W^{1,2}(\Omega)$ of being positive implies things that are false for general linear forms, see the classical paper of Murat [13].

- A systematic approach to function spaces on subsets of $\mathbb{R}^d$ is contained in [9], compare in particular Example 1 in Ch. II.1 there.

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References

[1] G.-Q. Chen, W. P. Ziemer and M. Torres, Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws, Commun. Pure Appl. Math. **62** (2009), no. 2, 242–304.

[2] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions. Studies in advanced mathematics, CRC Press, Boca Raton-New York-London-Tokyo, 1992.

[3] J. G. Fleming and S.-Y. Lin, Three-dimensional photonic crystal with a stop band from 1.35 to 1.95 μm. Optics Letters **24** (1999), no. 1, 49–51.

[4] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order. Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, Berlin, 1983.

[5] E. Giusti, Metodi diretti del calcolo delle variazioni. Unione Matematica Italiana, Bologna, 1994.

[6] P. Grisvard, Elliptic problems in nonsmooth domains. Monographs and Studies in Mathematics, 24, Pitman Publishing Inc., Boston-London-Melbourne, 1985.

[7] R. Haller-Dintelmann and J. Rehberg, Maximal parabolic regularity for divergence operators including mixed boundary conditions, J. Differential Equations **247** (2009), no. 5, 1354–1396.

[8] D. Joannopoulos, S. G. Johnson, J. N. Winn, R. D. Meade, Photonic Crystals: Molding the Flow of Light, Princeton University Press, Princeton, N.J., 2008.

[9] A. Jonsson and H. Wallin, Function spaces on subsets of $\mathbb{R}^n$, Math. Rep. **2** (1984), no. 1.

[10] H.-C. Kaiser and J. Rehberg, About a stationary Schödinger-Poisson system with Kohn-Sham potential in a bounded two- or three-dimensional domain, Nonlinear Anal. **41** (2000), no. 1–2(A), 33–72.

[11] S. Y. Lin, J. G. Fleming, D. L. Hetherington, B. K. Smith, R. Biwas, K. M. Ho, M. M. Sigalas, W. Zubrycki, S. R. Kurtz, J. Bur, A three-dimensional photonic crystal operating at infrared wavelenghts, Nature **394** (1999), 251–253.

[12] V. G. Maz'ya, Sobolev spaces. Springer, Berlin, 1985.

[13] F. Murat, L'injection du cone positif de H^{-1} dans $W^{1,q}$ est compacte pour tout $q < 2$ (avec remarque par Haim Brezis), J. Math. Pures Appl. (9) **60** (1981), 309–322.

[14] J. Nečas, Les méthodes directes en théorie des équations elliptiques. Masson, Paris, 1967.

[15] D. J. F. Nonnenmacher, Sets of finite perimeter and the Gauss-Green theorem with singularities, J. London Math. Soc. (2) **52** (1995), no. 2, 335–344.

[16] E. M. Ouhabaz, Analysis of heat equations on domains. London Mathematical Society Monographs Series, 31, Princeton University Press, Princeton, N.J., 2005.

[17] S. Satpathy, Z. Zhang, M. R. Salehpour, Theory of Photon Bands in Three-Dimensional Periodic Dielectric Structures, Physical Review Letters **11** (1990), no. 11, 1239–1242.

[18] R. E. Showalter, Monotone operators in Banach spaces and nonlinear partial differential equations. Mathematical Surveys and Monographs, 49, American Mathematical Society, Providence, RI, 1997.

[19] E. M. Stein, Singular integrals and differentiability properties of functions. Princeton Mathematical Series, 30, Princeton University Press, Princeton, N.J., 1970.

[20] W. P. Ziemer, Weakly differentiable functions. Sobolev spaces and functions of bounded variation. Graduate Texts in Mathematics, 120, Springer-Verlag, Berlin, 1989.