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An anisotropic, inhomogeneous, elastically modified Gibbs–Thomson law as singular limit of a diffuse interface model

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Abstract

We consider the sharp interface limit of a diffuse phase field model with prescribed total mass taking into account a spatially inhomogeneous anisotropic interfacial energy and an elastic energy. The main aim is the derivation of a weak formulation of an anisotropic, inhomogeneous, elastically modified Gibbs–Thomson law in the sharp interface limit. To this end we show that one can pass to the limit in the weak formulation of the Euler–Lagrange equation of the diffuse phase field energy.

1 Introduction

Phase transition phenomena are described mainly by two types of models: namely *sharp interface* and *diffuse phase field* models. In a sharp interface model interfaces, separating coexisting phases or structural domains, are modeled as hypersurfaces at which certain quantities fulfill jump conditions. Interfacial energy can be accounted for by integrating a surface energy density over the hypersurface. In general the surface energy density will be inhomogeneous and anisotropic, i. e. the density will depend on the position in space and on the local orientation of the interface.

In a diffuse interface model interfacial energy is modeled within the context of the van der Waals–Cahn–Hilliard theory of phase transitions. Classically, the interfacial energy is then given as

$$\int_\Omega \Big(\frac{\varepsilon}{2}\big|\nabla c\big|^2 + \frac{1}{\varepsilon}\psi(c)\Big)dx, \tag{1.1}$$

where $\varepsilon > 0$ is small, c is an order parameter, ψ is a double well potential and Ω is a domain in $\mathbb{R}^n$. The diffuse phase–field model describes the interface between different phases as a thin transition region, where the order parameter, representing the phases, changes its state smoothly.

It has been shown that the phase field energy (1.1) converges under suitable assumptions to a sharp interface energy measuring the total surface area, see [27] and [25]. In addition, it is also possible to pass to the limit in the Euler–Lagrange equation to (1.1) recovering the mean curvature of the interface in the sharp interface limit. More precisely, Luckhaus and Modica [24] showed that if one minimizes (1.1) subject to a mass constraint then the Lagrange multiplier related to the mass constraint converges to a Lagrange multiplier associated to the analogue constraint in the sharp interface limit. In fact, the Lagrange multiplier in the sharp interface limit is the mean curvature of the hypersurface. Also several results for the sharp interface limit of phase field models are known for time dependent situations, see [9], [22], [2] and [30].

In this paper we study situations in which the energy of the system consists of an inhomogeneous, anisotropic interfacial energy and an elastic energy resulting from stresses caused by different elastic properties of the phases. The consideration of these types of energies is for instance important for a realistic modeling of phase separation and coarsening processes in alloys, cf. [10, 7].

In a phase field framework it is possible to include anisotropic and inhomogeneous effects in the energy through the gradient term, see Subsection 2 and [8, 29, 12, 1]. Also in this case the sharp interface Γ–limit has been identified, see [8, 29]. It is also possible to pass to the limit in time dependent problems, e. g. for the anisotropic Allen–Cahn equation which is the L^2–gradient flow of the phase field energy, see [12, 1]. To the knowledge of the authors all results on the sharp interface limit of anisotropic phase field equations are based on maximum and comparison principles, see e.g. [12, 1].

We plan to study the sharp interface limit for an inhomogeneous anisotropic interfacial energy supplemented by elastic energy contributions in a situation where we prescribe the total mass. In such situations no maximum and comparison principles are available. Instead we will use approaches of Luckhaus and Modica [24] (with respect to the surface energy) and Garcke [18] (with respect to the elastic energy) together with new ideas to handle the anisotropic and inhomogeneous nature of the surface energy in order to pass to the sharp interface limit of the Lagrange multipliers. In the sharp interface limit we will derive an anisotropic, inhomogeneous, elastically modified Gibbs–Thomson law as the singular limit of the phase field model.

1.1 Phase–field and sharp interface energy functionals and the Gibbs–Thomson law

We consider a *phase–field energy functional* for a two phase system which is in a normalized form of the following structure

$$E_\varepsilon(c,u) = \int_\Omega \Big(\varepsilon\sigma^2(x,c,\nabla c) + \frac{1}{\varepsilon}\psi(c) + W\big(c,\mathcal{E}(u)\big)\Big)dx, \qquad \varepsilon > 0, \tag{1.2}$$

where c denotes a scalar phase–field function, i. e. a scaled concentration difference, and u is the displacement field. The first term in E_ε is a generalization of the classical Dirichlet energy and allows for anisotropy. The function ψ is the chemical free energy density, which has for a two phase system usually the form of a double–well potential, and W is the elastic free energy density, where $\mathcal{E}$ is the strain tensor. Note, this energy functional contains the classical Ginzburg–Landau energy functional as the special case $\sigma(x,c,\nabla c) = |\nabla c|$ and $W = 0$.

The width of the transition layer scales with order ε and the diffuse interface becomes sharp as $\varepsilon \to 0$. In the asymptotic limit, the first two volume integrals of E_ε reduce to an area integral. The corresponding *sharp energy functional* is given by

$$E_0(c,u) = \int_I \sigma_0(x,\nu_-)d\mathcal{H}^{n-1} + \int_\Omega W\big(c,\mathcal{E}(u)\big)\,dx, \tag{1.3}$$

where I denotes the interface, $\sigma_0(x,\nu_-) = 2\int_{c_-}^{c_+}\sqrt{\psi(s)}\sigma(x,s,\nu_-)ds$, c_- and c_+ represent the two phases and ν_- is the unit normal vector of the interface pointing into the phase with concentration c_+.

One naturally associated quantity to the generalized Ginzburg–Landau free energy is its functional derivative under mass conservation:

$$\lambda_\varepsilon = \frac{\delta E_\varepsilon}{\delta c} = -2\varepsilon\nabla\cdot\big(\sigma(x,c,\nabla c)\,\sigma_{,p}(x,c,\nabla c)\big) + 2\varepsilon\sigma(x,c,\nabla c)\,\sigma_{,c}(x,c,\nabla c) + \frac{1}{\varepsilon}\psi_{,c}(c) + W_{,c}(c,\mathcal{E}(u)),$$

where λ_ε is the Lagrange multiplier (for the mass constraint) and the indices c and p stand for the partial derivatives with respect to c and the n–dimensional variable p. The first variation of the sharp interface functional at a smooth hypersurface I subject to mass constraint leads to

$$\lambda = \big(- \sigma_{0,x}(x, \nu_-) \cdot \nu_- - \nabla_I \cdot \sigma_{0,p}(x, \nu_-) + \nu_-[WId - (\nabla u)^T W_{,\mathcal{E}}]_-^+ \nu_- \big)/(c_+ - c_-), \tag{1.4}$$

where λ is the Lagrange multiplier (for the mass constraint), $W_{,\mathcal{E}}$ is the partial derivative of W with respect to $\mathcal{E}$ and $[\cdot]_-^+$ denotes the jump of the quantity in brackets across the interface. The relation in (1.4) is an extended *Gibbs–Thomson law* which connects local geometric quantities at the phase boundary to functions in the bulk, where the Lagrange multiplier λ often corresponds to the chemical potential. The Gibbs-Thomson law states that the system is in local thermodynamical equilibrium.

1.2 Main results

The main aim of this work is the study of the limiting behavior of the weak formulation of the Euler–Lagrange equation for minimizers of the generalized Ginzburg–Landau energy in (1.2) as the interfacial thickness ε tends to zero. We show that the Lagrange multipliers (associated to a volume constraint) in the weak formulation of the Euler–Lagrange equation converge and achieve a weak formulation of a modified Gibbs–Thomson law in the sharp interface limit. The proof is based on the notion of a generalized total variation for BV–functions, on anisotropic energies and their geometric properties and on weak convergence theorems for homogeneous functions of measures. The crucial step is to obtain suitable approximations for certain phase–field quantities and the Cahn–Hoffman vector at the same time. Our main result is under suitable assumptions as follows.

Let $\Omega \subset \mathbb{R}^n$ be a domain with C^1–boundary and let the Assumptions A 2.1 – A 2.4, see Section 2, be satisfied. Further, let $(c_\varepsilon, u_\varepsilon)$, $\varepsilon > 0$, be a minimizer of E_ε subject to the mass constraint $⨍_\Omega c_\varepsilon dx = m \in (c_-, c_+)$. Then for each sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} \varepsilon_k = 0$, with

$$c_{\varepsilon_k} \to c \quad in \quad L^1(\Omega),$$

and

$$u_{\varepsilon_k} \to u \quad in \quad L^2(\Omega, \mathbb{R}^n),$$

the corresponding sequence of Lagrange multipliers $\{\lambda_{\varepsilon_k}\}_{k\in\mathbb{N}}$ (associated to the mass constraint) converges, i. e.

$$\lambda_{\varepsilon_k} \to \lambda,$$

and λ is a Lagrange multiplier of the minimum problem for E_0 with $⨍_\Omega c\, dx = m$, i. e.

$$\int_I \Big(\sigma_0(x, \nu_-) \nabla \cdot \xi + \sigma_{0,x}(x, \nu_-) \cdot \xi - \nu_- \cdot \nabla \xi \sigma_{0,p}(x, \nu_-) \Big) d\mathcal{H}^{n-1}$$
$$+ \int_\Omega \Big(W\big(c, \mathcal{E}(u)\big) Id - \big(\nabla u\big)^T W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big) \Big) : \nabla \xi \; dx = \lambda \int_\Omega c \, \nabla \cdot \xi dx$$

for all $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$, where ν_Ω is the outer unit normal of $\partial\Omega$.

Organization of the paper: In Section 2 we introduce some notation and state the assumptions. Section 3 treats the Γ–convergence of the phase–field energy functional. Then, in Section

4, we determine the weak formulation of the Euler–Lagrange equations for the phase–field model and the corresponding sharp interface model. Section 5 is devoted to the convergence of the Lagrange multipliers and the weak formulation of the Gibbs–Thomson law. We provide several asymptotic properties for minimizing sequences and construct approximations for the Cahn–Hoffman vector. Finally, we deduce in Section 6 the strong Euler–Lagrange equation for E_0 under certain regularity assumptions which establishes equilibrium conditions in the bulk, at the interface and at the boundary of the domain.

2 Notation and Assumptions

We begin with stating the hypotheses for the anisotropy function σ, the chemical free density ψ and the elastic free energy density W.

If not otherwise mentioned we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1–boundary. The first and second partial derivatives of a function f with respect to a variable s are denoted by $f_{,s}$ and $f_{,ss}$.

Assumption A 2.1
The anisotropy function $\sigma : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ satisfies the following properties:

(i) $\sigma \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$,
$\sigma_{,x}, \sigma_{,p} \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \setminus \{0\})$,
$\sigma_{,pp} \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \setminus \{0\})$.

(ii) σ is 1-homogeneous in the third variable such that $\sigma(x, s, \lambda p) = \lambda \sigma(x, s, p)$ for all $p \in \mathbb{R}^n$ and any $\lambda > 0$.

(iii) There exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\lambda_1 |p| \leq \sigma(x, s, p) \leq \lambda_2 |p| \quad \text{for all } x \in \overline{\Omega}, \text{ all } s \in \mathbb{R} \text{ and all } p \in \mathbb{R}^n.$$

(iv) σ is strictly convex as a 1-homogeneous function, i.e. there exists a constant $d_0 > 0$ such that

$$\sigma_{,pp}(x, s, p)\, q \cdot q \geq d_0 |q|^2$$

for all $x \in \Omega$, all $s \in \mathbb{R}$ and all $p, q \in \mathbb{R}^n$ with $p \cdot q = 0$, $|p| = 1$.

We like to mention that we consider several times the expressions $\sigma\, \sigma_{,p}$ and $g\, \sigma_{,p}$ with $g \in C^1(\Omega)$ and $g = 0$ in some neighborhood of $0 \in \mathbb{R}^n$. Clearly, σ_p is not differentiable at $0 \in \mathbb{R}^n$. However, if we set $\sigma\, \sigma_{,p} = 0$ and $g\, \sigma_{,p} = 0$ at $0 \in \mathbb{R}^n$ then the above combinations are well defined at 0.

To handle the anisotropy we need the notion of the generalized total variation of BV–functions:

Let $\sigma : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ be a continuous anisotropy function satisfying (ii) and (iii) of Assumption A 2.1. Then the dual function $\sigma^* : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ is given by

$$\sigma^*(x, s, q) = \sup \big\{ q \cdot p : p \in \mathbb{R}^n,\ \sigma(x, s, p) \leq 1 \big\} = \sup \left\{ \frac{q \cdot p}{\sigma(x, s, p)} : p \in \mathbb{R}^n \setminus \{0\} \right\}. \tag{2.1}$$

Moreover, let

$$K_{\sigma(s)}(\Omega) = \big\{ \eta \in C^1_c(\Omega, \mathbb{R}^n) : \sigma^*(x, s, \eta(x)) \leq 1 \text{ for a.e. } x \in \Omega \big\}.$$

For any $f \in BV(\Omega)$ the *generalized total variation* of f (with respect to $\sigma(s)$) in Ω is defined by

$$\int_\Omega |\nabla f|_{\sigma(s)} = \sup\left\{ \int_\Omega f \,\mathrm{div}\eta\, dx : \eta \in K_{\sigma(s)}(\Omega) \right\}.$$

The generalized total variation can be represented by an integral formula in terms of the measure $|\nabla f|$, cf. [3, 4]:

$$\int_\Omega |\nabla f|_{\sigma(s)} = \int_\Omega \sigma(x, s, \nu_f)\, |\nabla f|, \tag{2.2}$$

where $\nu_f(x) = -\frac{\nabla f}{|\nabla f|}(x)$ for $|\nabla f|$–almost all $x \in \Omega$.

We notice, $\int_\Omega |\nabla f|_{\sigma(s)}$ is $L^1(\Omega)$–lower semicontinuous on $BV(\Omega)$.

In the sequel we make use of the following properties for anisotropy functions.

Lemma 2.2
Let σ be an anisotropy function satisfying Assumption A 2.1. Then there exist constants $C_1 > 0$ and $C_2 > 0$, such that for all $x \in \Omega$, $s \in [c_-, c_+]$, $\nu_1, \nu_2 \in \mathbb{S}^{n-1}$ and $p, p_1, p_2 \in \mathbb{R}^n \backslash \{0\}$ the following properties are fulfilled:

(i)

$$\sigma_{,p}(x, s, p) \cdot p = \sigma(x, s, p), \qquad \sigma^*_{,p}(x, s, p) \cdot p = \sigma^*(x, s, p), \tag{2.3}$$

(ii)

$$\sigma(x, s, \nu_1) - \sigma_{,p}(x, s, \nu_2) \cdot \nu_1 \ge C_1 |\nu_1 - \nu_2|^2, \tag{2.4}$$

(iii)

$$|\sigma_{,p}(x, s, \nu_1) - \sigma_{,p}(x, s, \nu_2)| \le C_2 |\nu_1 - \nu_2|, \tag{2.5}$$

(iv)

$$\sigma_{,p}(x, s, \lambda p) = \sigma_{,p}(x, s, p), \qquad \sigma^*_{,p}(x, s, \lambda p) = \sigma^*_{,p}(x, s, p) \quad \textit{for } \lambda > 0, \tag{2.6}$$

(v)

$$\sigma\big(x, s, \sigma^*_{,p}(x, s, p_1)\big) = \sigma^*\big(x, s, \sigma_{,p}(x, s, p_2)\big) = 1. \tag{2.7}$$

(vi)

$$\sigma(x, s, p)\, \sigma^*_{,p}\big(x, s, \sigma_{,p}(x, s, p)\big) = p, \qquad \sigma^*(x, s, p)\, \sigma_{,p}\big(x, s, \sigma^*_{,p}(x, s, p)\big) = p, \tag{2.8}$$

A proof of Lemma 2.2 can be deduced from [6], [11] and [19] by slight modifications.

In the present work we consider exclusively variational problems under mass conservation. Thus the variational problem does not change if we subtract from the chemical energy density ψ an affine function. Therefore, throughout this paper, we restrict ourselves to normalized chemical energy densities with the form of a double well potential which attain the value 0 at their minima.

Assumption A 2.3
For the chemical energy density $\psi : \mathbb{R} \to [0, +\infty)$ we assume that

(i) $\psi \in C^1(\mathbb{R})$,

(ii) $\psi(c) = 0$ *if and only if* $c \in \{c_-, c_+\}$,

(iii) there exist constants $d_1, d_2 > 0$ *such that*

$$\psi(c) \geq d_1|c|^2 - d_2.$$

Elastic interactions are considered within the framework of linear elasticity. Since in phase separation processes of alloys the deformations are typically small we choose a theory based on the *linearized strain tensor* which is given by

$$\mathcal{E}(u) = \frac{1}{2}\Big(\nabla u + (\nabla u)^T\Big),$$

where $u : \Omega \to \mathbb{R}^n$ is the displacement field. The *elastic energy density* W is typically of quadratic form:

$$W(c, \mathcal{E}) = \frac{1}{2}\big(\mathcal{E} - \mathcal{E}^*(c)\big) : C(c)\big(\mathcal{E} - \mathcal{E}^*(c)\big). \tag{2.9}$$

Here, $\mathcal{E}^*(c)$ denotes the *eigenstrain* which is usually linear in c and $C(c)$ is the elasticity tensor which is symmetric and positive definite. If the elasticity tensor does not depend on the concentration, i. e. $C(c) = C$, we refer to *homogeneous* elasticity.

Rigorous results in the present work are obtained under certain growth conditions for the elastic energy density W. These conditions are, however, only satisfied for W as in (2.9) in the case of homogeneous elasticity or if $\mathcal{E}^*(c)$ does not depend linearly on c. To be more precise, we need the following hypotheses for W.

Assumption A 2.4
We suppose that $W : \mathbb{R} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ *satisfies the following properties:*

(i) $W \in C^1(\mathbb{R} \times \mathbb{R}^{n\times n})$,

(ii) $W(c, \mathcal{E}) = W(c, (\mathcal{E})^T)$ *for all* $c \in \mathbb{R}$ *and* $\mathcal{E} \in \mathbb{R}^{n\times n}$,

(iii) $W_{,\mathcal{E}}(c, \cdot)$ *is strongly monotone (uniformly in* c*), i. e. there exists a constant* $d_3 > 0$ *such that for all symmetric* $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{R}^{n\times n}$*:*

$$\Big(W_{,\mathcal{E}}(c, \mathcal{E}_2) - W_{,\mathcal{E}}(c, \mathcal{E}_1)\Big) : (\mathcal{E}_2 - \mathcal{E}_1) \geq d_3|\mathcal{E}_2 - \mathcal{E}_1|^2.$$

(iii) There exists a constant $d_4 \geq 0$ *such that*

- $|W(c, \mathcal{E})| \leq d_4\big(|\mathcal{E}|^2 + |c|^2 + 1\big)$,
- $|W_{,c}(c, \mathcal{E})| \leq d_4\big(|\mathcal{E}|^2 + |c| + 1\big)$,
- $|W_{,\mathcal{E}}(c, \mathcal{E})| \leq d_4\big(|\mathcal{E}| + |c| + 1\big)$.

The phase–field energy functional E_ε and the sharp interface functional E_0 depend on u only via $\mathcal{E}(u)$, cf. equations (1.2) and (1.3). Hence infinitesimal rigid displacements have no influence on E_ε and E_0. Therefore we may suppose that for critical points (c, u) of E_ε and E_0 the function u is chosen such that

$$u \in X_{\mathrm{ird}}^{\perp} = \big\{u \in H^1(\Omega, \mathbb{R}^n) : (u, v)_{H^1} = 0 \ \text{ for all } v \in X_{\mathrm{ird}}\big\},$$

where $X_{\mathrm{ird}} := \{u \in H^1(\Omega, \mathbb{R}^n) : u(x) = b + Kx$ with $b \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n\times n}$ skew symmetric$\}$.

Further, we assume that the mass is conserved with $\fint_\Omega c\,dx = m \in (c_-, c_+)$.

The space of functions of bounded variations is denoted by $BV(\Omega)$ and the symbol $\partial^* S$ stands for the reduced boundary of S, where $S \subset \Omega$ is a set of finite perimeter. We refer [20] and [5] for details.

To study variational solutions of E_ε and E_0 it is convenient to extend the energy functionals in the following way.

Definition 2.5
(i) Phase–field energy functional.
The phase–field energy functional $\hat{E}_\varepsilon : BV(\Omega) \times X_{\mathrm{ird}}^\perp \to \mathbb{R} \cup \{+\infty\}$ *is defined as*

$$\hat{E}_\varepsilon(c,u) = \begin{cases} \int_\Omega \big(\varepsilon\sigma^2(x,c,\nabla c) + \frac{1}{\varepsilon}\psi(c) + W(c,\mathcal{E}(u))\big)dx & \text{if } c \in H^1(\Omega) \text{ and} \\ & \fint_\Omega c\,dx = m, \\ \infty & \text{elsewhere.} \end{cases}$$

(ii) Sharp interface energy functional.
The sharp interface functional $\hat{E}_0 : BV(\Omega) \times X_{\mathrm{ird}}^\perp \to \mathbb{R} \cup \{+\infty\}$ *is given by*

$$\hat{E}_0(c,u) = \begin{cases} \int_I \sigma_0\big(x,\nu_-\big)d\mathcal{H}^{n-1} + \int_\Omega W(c,\mathcal{E}(u))dx & \text{if } c \in BV(\Omega,\{c_-,c_+\}) \text{ and} \\ & \fint_\Omega c\,dx = m, \\ \infty & \text{elsewhere,} \end{cases}$$

where $I := \partial^*\Omega_-$ *is the reduced boundary of* $\Omega_- := \{x \in \Omega : c(x) = c_-\}$ *and* ν_- *is the outer unit normal of* Ω_-*. The anisotropy function* $\sigma_0 : \Omega \times \mathbb{R}^n \to [0,+\infty)$ *has the form*

$$\sigma_0(x,p) = 2\int_{c_-}^{c_+} \sqrt{\psi(s)}\sigma(x,s,p)ds.$$

3 The Γ–limit of the Ginzburg–Landau energy

In this section we shortly discuss the Γ–limit of $\hat{E}_\varepsilon$ and the asymptotic behavior of its minimizers. Such kind of investigations originates in the works [27, 26] of Modica and Mortola, where the Γ–limit of

$$\int_\Omega \Big(\varepsilon|\nabla c|^2 + \frac{1}{\varepsilon}f(c)\Big)dx, \quad \varepsilon > 0,$$

was studied. In [25] Modica proved that the corresponding sequence of minimizers $\{c_\varepsilon\}$ (modulo a subsequence) converges in $L^1(\Omega)$ to a function which takes only the values $c = c_-$ and $c = c_+$. Moreover, he showed that the interface between $\{c = c_-\}$ and $\{c = c_+\}$ has minimal area.

Generalizations to *anisotropic* energies in a very general form but without elasticity were gained by Owen [28], Owen & Sternberg [29] and Bouchitté [8]. They considered the variational problem
Minimize

$$\tilde{E}_\varepsilon(c) = \int_\Omega \Big(\frac{1}{\varepsilon}f(x,c,\varepsilon\nabla c)\Big)dx, \quad \varepsilon > 0, \tag{3.1}$$

where $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n, [0,\infty))$ with some additional conditions. In particular the work [8] establishes the Γ–limit under very mild assumptions on f.

In the presence of *elasticity* rigorous results for the Γ–limit and the asymptotic behavior of minimizers were obtained by Garcke [17, 18]. He investigated the following variational problem for isotropic σ in the vectorial case:
Minimize

$$\tilde{E}_\varepsilon(c,u) = \int_\Omega \Big(\varepsilon|\nabla c|^2 + \frac{1}{\varepsilon}\psi(c) + W(c,\mathcal{E}(u))\Big)dx, \quad \varepsilon > 0, \tag{3.2}$$

subject to the constraint $\fint_\Omega c\,dx = m$.

The choice of the above scaling for the elastic part is motivated in [23] and [16] by formally matched asymptotic expansions.

For energy functionals of the form as in (1.2), with inhomogeneous anisotropic interfacial energy and elasticity, the results of [29], [8] and [17, 18] can be combined to characterize the asymptotic behavior of $\hat{E}_\varepsilon$ and its minimizers:

Theorem 3.1
Let the Assumptions A 2.1 (i) – (iii), A 2.3 and A 2.4 be satisfied and let $(c_\varepsilon, u_\varepsilon) \in H^1(\Omega) \times X^\perp_{\mathrm{ird}}$ *such that*

$$\hat{E}_\varepsilon(c_\varepsilon, u_\varepsilon) \quad \textit{is uniformly bounded as } \varepsilon \to 0.$$

Then there exists a sequence of minimizers $\{(c_{\varepsilon_k}, u_{\varepsilon_k})\}_{k\in\mathbb{N}} \subset H^1(\Omega) \times X^\perp_{\mathrm{ird}}$ *with*

$$c_{\varepsilon_k} \to c \quad \textit{in } L^2(\Omega), \quad c \in BV(\Omega, \{c_-, c_+\}),$$

and

$$u_{\varepsilon_k} \to u \quad \textit{in } H^1(\Omega, \mathbb{R}^n)$$

as $\varepsilon_k \to 0$.

In addition the following Γ–convergence result is true.

Theorem 3.2
Let the Assumptions A 2.1 (i) – (iii), A 2.3 and A 2.4 be satisfied. Then $\hat{E}_{\varepsilon_k}$ Γ*–converges to* $\hat{E}_0$ *in the following sense.*

(i) For every sequence $\{(c_{\varepsilon_k}, u_{\varepsilon_k})\}_{k\in\mathbb{N}} \subset BV(\Omega) \times X^\perp_{\mathrm{ird}}$, $\varepsilon_k > 0$ *and* $\lim_{k\to\infty}\varepsilon_k = 0$, *with*

$$c_{\varepsilon_k} \to c \ \textit{ in } L^1(\Omega), \qquad u_{\varepsilon_k} \to u \ \textit{ in } L^2(\Omega, \mathbb{R}^n),$$

it holds

$$\hat{E}_0(c,u) \le \liminf_{k\to\infty} \hat{E}_{\varepsilon_k}(c_{\varepsilon_k}, u_{\varepsilon_k}).$$

(ii) For any $(c,u) \in BV(\Omega) \times X^\perp_{\mathrm{ird}}$ *there exists a sequence* $\{(c_{\varepsilon_k}, u_{\varepsilon_k})\}_{k\in\mathbb{N}} \subset BV(\Omega) \times X^\perp_{\mathrm{ird}}$, $\varepsilon_k > 0$ *with* $\lim_{k\to\infty}\varepsilon_k = 0$, *satisfying*

$$c_{\varepsilon_k} \to c \ \textit{ in } L^1(\Omega), \qquad u_{\varepsilon_k} \to u \ \textit{ in } L^2(\Omega, \mathbb{R}^n)$$

such that

$$\hat{E}_0(c,u) \ge \limsup_{k\to\infty} \hat{E}_{\varepsilon_k}(c_{\varepsilon_k}, u_{\varepsilon_k}).$$

Remark 3.3
Let us comment on the hypotheses for Theorem 3.2. The quadratic growth condition on ψ ensures compactness on $\{c_{\varepsilon_k}\}_{k\in\mathbb{N}}$ in L^2 for sequences $\{(c_{\varepsilon_k}, u_{\varepsilon_k})\}_{k\in\mathbb{N}}$ with uniformly bounded energy $\hat{E}_{\varepsilon_k}(c_{\varepsilon_k}, u_{\varepsilon_k})$. This compactness property, the growth conditions on W and Korn's inequality establish uniform boundedness of $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ in $H^1(\Omega, \mathbb{R}^n)$. This, in turn, combined with the monotonicity and growth conditions for W enables to show the lower semicontinuity of $\int_\Omega W\,dx$.

The utility of the concept of Γ–convergence lies in the corollary below. Existence of absolute minimizers of $\hat{E}_\varepsilon$ with uniformly bounded energy as $\varepsilon \to 0$ results from the direct method in the calculus of variations and Theorem 3.2 (ii).

Corollary 3.4
Let $\{(c_{\varepsilon_k}, u_{\varepsilon_k})\}_{k\in\mathbb{N}} \subset BV(\Omega) \times X_{\mathrm{ird}}^{\perp}$ be any sequence of minimizers of $\hat{E}_{\varepsilon_k}$, $\varepsilon_k > 0$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Then there exists a subsequence $\{(c_{\varepsilon_{k_j}}, u_{\varepsilon_{k_j}})\}_{j\in\mathbb{N}}$ which converges in $L^2(\Omega) \times H^1(\Omega, \mathbb{R}^n)$ to a limit (c, u), where (c, u) is a minimizer of $\hat{E}_0$.

4 Weak formulation of the Euler–Lagrange equations

We are now going to establish the weak formulation of the Euler–Lagrange equations of the diffuse phase–field energy functional E_ε and the sharp energy functional E_0. To this end, we choose inner variations such that the mass constraint is satisfied as variations of $\hat{E}_0$ with respect to the dependent variable c are not possible since c attains only the values c_- and c_+.

Let Ω be a domain with C^1–boundary and let $\Phi : (-\tau_0, \tau_0) \times \overline{\Omega} \to \overline{\Omega}$ be a differentiable mapping such that $\Phi(\tau, \cdot)$, $\tau \in [-\tau_0, \tau_0]$, is a family of diffeomorphisms of $\overline{\Omega}$ onto itself given by the initial value problem

$$\Phi(0, x) = x \qquad \text{and} \qquad \Phi_{,\tau}(\tau, x) = \xi(\Phi(\tau, x)) \tag{4.1}$$

for $x \in \overline{\Omega}$ and $\tau \in [-\tau_0, \tau_0]$, where $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$. Then Φ fulfills the following properties:

(i) $\Phi(\tau, \cdot)$ is the inverse of $\Phi(-\tau, \cdot)$, i. e. $\Phi(\tau, \Phi(-\tau, x)) = x$.

In consequence,

$$Id = \Phi_{,x}\big(\tau, \Phi(-\tau, x)\big)\, \Phi_{,x}(-\tau, x),$$

(ii)

$$\frac{d}{d\tau}\big(\det \Phi_{,x}(\tau, x)\big)\Big|_{\tau=0} = \big(\nabla \cdot \xi\big)(x),$$

(iii)

$$\frac{d}{d\tau}\Big(\big(\Phi_{,x}(\tau, x)\big)^{-1}\Big)\Big|_{\tau=0} = -\nabla\xi(x).$$

To determine the weak formulation of the Euler–Lagrange equation we make use of the following variational property.

Lemma 4.1
Let $\Phi(\tau,x):[-\tau_0,\tau_0]\times\overline{\Omega}\to\overline{\Omega}$ be a family of diffeomorphisms of $\overline{\Omega}$ onto itself. Then

$$\frac{d}{d\tau}\int_\Omega \big|\nabla\chi_-\big(\Phi^{-1}(\tau,\cdot)\big)\big|_{\sigma_0}\Big|_{\tau=0}$$
$$=\int_\Omega\Bigg(\sigma_0\big(\Phi(\tau,x),H(\tau,x)\nu_-(x)\big)\,tr\left(\frac{\partial\Phi_{,\tau}(\tau,x)}{\partial x}\right)+\sigma_{0,x}\big(\Phi(\tau,x),H(\tau,x)\nu_-(x)\big)\cdot\frac{d}{d\tau}\Phi(\tau,x)$$
$$+\sigma_{0,p}\big(\Phi(\tau,x),H(\tau,x)\nu_-(x)\big)\cdot\frac{d}{d\tau}\big(\Phi_{,x}(\tau,x)\big)^{-T}\nu_-(x)\Bigg)\Bigg|_{\tau=0}\big|\nabla\chi_-(x)\big|,$$

where χ_- is the characteristic function of $\Omega_-=\{x\in\Omega : c(x)=c_-\}$ and $\nu_-=-\frac{\nabla\chi_-}{|\nabla\chi_-|}$ for $|\nabla\chi_-|$–almost all $x\in\Omega$.

Proof:
We have

$$\int_\Omega\big|\nabla\chi_-\big(\Phi^{-1}(\tau,\cdot)\big)\big|_{\sigma_0}=$$
$$=\sup\left\{\int_\Omega\chi_-\big(\Phi^{-1}(\tau,z)\big)\mathrm{div}g\big(\Phi^{-1}(\tau,z)\big)\,dz : g\in C^1_c(\Omega,\mathbb{R}^n),\sigma_0^*(z,g\big(\Phi^{-1}(\tau,z)\big)\le 1\ \text{ for } z\in\Omega\right\}$$
$$=\sup\left\{\int_\Omega g(x)\cdot H(\tau,x)\nu_-(x)\big|\nabla\chi_-\big| : g\in C^1_c(\Omega,\mathbb{R}^n),\sigma_0^*(\Phi(\tau,x),g(x))\le 1\ \text{ for } x\in\Omega\right\},$$

where the last equality follows from the change of area formula for BV–functions (see [20], proof of Lemma 10.1), i. e.

$$\int_\Omega\chi_-\big(\Phi^{-1}(\tau,z)\big)\,\mathrm{div}\,g\big(\Phi^{-1}(\tau,z)\big)dz=\int_\Omega g(x)\cdot H(\tau,x)\,\nu_-(x)\,|\nabla\chi_-(x)|$$

with $H(\tau,x)=|\det\Phi_{,x}(\tau,x)|\big(\Phi_{,x}(\tau,x)\big)^{-T}$. Since, compare (2.1),

$$g(x)\cdot H(\tau,x)\nu_-(x)\le\sigma_0^*\big(\Phi(\tau,x),g(x)\big)\,\sigma_0\big(\Phi(\tau,x),H(\tau,x)\nu_-(x)\big)$$

we obtain

$$\sup\left\{\int_\Omega g(x)\cdot H(\tau,x)\nu_-(x)\big|\nabla\chi_-\big| : g\in C^1_c(\Omega,\mathbb{R}^n),\sigma_0^*(\Phi(\tau,x),g(x))\le 1\ \text{ for } x\in\Omega\right\}$$
$$\le\int_\Omega\sigma_0\big(\Phi(\tau,x),H(\tau,x)\,\nu_-(x)\big)\big|\nabla\chi_-(x)\big|.$$

Next we verify the opposite inequality. As $\chi_-\in BV(\Omega)$ there exist approximate normals $\varphi^\delta\in C_c^\infty(\Omega,\mathbb{R}^n)$ with $|\varphi^\delta(x)|\le 1$ for $x\in\Omega$ such that

$$\int_\Omega\big(1-\varphi^\delta\cdot\nu_-\big)|\nabla\chi_-|\le\delta.$$

Utilizing the fact that φ^δ and ν_- have norm less or equal to one we obtain

$$|\nu_--\varphi^\delta|^2\le 2\,(1-\varphi^\delta\cdot\nu_-).$$

Therefore, we can choose a sequence of functions $\{\varphi_k\}_{k\in\mathbb{N}}$, $\varphi_k \in C_c^1(\Omega,\mathbb{R}^n)$, such that

$$\varphi_k \to \nu_- \quad \text{in } L^1(|\nabla\chi_-|) \qquad \text{and} \qquad \varphi_k \to \nu_- \quad \text{pointwise } |\nabla\chi_-|\text{–a. e.}.$$

We now choose a function $\eta \in C_c^1(\mathbb{R},\mathbb{R})$ with $0 \le |\eta| \le 1$, such that

$$\eta = 1 \quad \text{on } [1/4, 4] \qquad \text{and} \qquad \eta = 0 \quad \text{on } \mathbb{R}\backslash[1/8, 8].$$

Further, we define for $\tau \in [-\tau_0, \tau_0]$ fixed $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$F(x,p) = \eta(|p|)\, \sigma_{0,p}(\Phi(\tau,x),p).$$

For τ small enough we may assume

$$\frac{1}{2} \le |H\nu_-| \le 2 \quad \text{in } \Omega$$

and we may approximate H uniformly by functions $H_k \in C_c^1(\Omega,\mathbb{R}^n)$, $k \in \mathbb{N}$. Then

(i) $F \circ (H_k\varphi_k) \in C_c^1(\Omega,\mathbb{R}^n)$ with $\big(F \circ (H_k\varphi_k)\big)(x) := F\big(x, H_k(x)\varphi_k(x)\big)$,

(ii) $\sigma_0^*(\Phi, F\circ(H_k\varphi_k)) = \eta\big(|H_k\varphi_k|\big)\, \sigma_0^*(\Phi, \sigma_{0,p}(\Phi, H_k\varphi_k)) \le 1$,

(iii) $F \circ (H_k\varphi_k) \to F \circ (H\nu_-) \qquad |\nabla\chi_-|$–a. e.,

where (ii) follows from (2.7). Since F is bounded we obtain

$$\int_\Omega \big(F\circ(H_k\varphi_k)\big)\cdot H\nu_-|\nabla\chi_-| \to \int_\Omega \big(F\circ(H\nu_-)\big)\cdot H\nu_-|\nabla\chi_-| \qquad \text{for } k \to \infty.$$

Hence, as $1/2 \le |H\nu_-| \le 2$,

$$F\circ(H\nu_-) = \sigma_{0,p}(\Phi(\tau,x), H\nu_-).$$

Setting $g_k = F\circ(H_k\varphi_k)$, $k \in \mathbb{N}$, gives for $k \to \infty$ the reverse inequality:

$$\begin{aligned}\sup\Big\{\int_\Omega g(x)\cdot H(\tau,x)\nu_-(x)\big|\nabla\chi_-\big| : g \in C_c^1(\Omega,\mathbb{R}^n), \sigma_0^*(\Phi(\tau,x),g(x)) \le 1 \text{ for } x \in \Omega\Big\}&\\ \ge \int_\Omega \sigma_0\big(\Phi(\tau,x), H(\tau,x)\nu_-(x)\big)\big|\nabla\chi_-(x)\big|.&\end{aligned}$$

Next we compute the τ–derivative:

$$\begin{aligned}&\frac{d}{d\tau}\int_\Omega \big|\nabla\chi_-\big(\Phi^{-1}(\tau,\cdot)\big)\big|_{\sigma_0}\Big|_{\tau=0}\\ &= \int_\Omega \Bigg(\sigma_0\big(\Phi(\tau,x), H(\tau,x)\nu_-\big)\, tr\bigg(\frac{\partial\Phi_{,\tau}(\tau,x)}{\partial x}\bigg) + \sigma_{0,x}\big(\Phi(\tau,x), H(\tau,x)\nu_-(x)\big)\cdot\frac{d}{d\tau}\Phi(\tau,x)\\ &\qquad + \sigma_{0,p}\big(\Phi(\tau,x), H(\tau,x)\nu_-(x)\big)\cdot\frac{d}{d\tau}\big(\Phi_{,x}(\tau,x)\big)^{-T}\nu_-(x)(x)\Bigg)\Bigg|_{\tau=0}\big|\nabla\chi_-(x)\big|.\end{aligned}$$

■

We also recall that, if M is an $n \times n$–matrix then $Id + \eta M$, $\eta \in \mathbb{R}$, is invertible for $|\eta|$ sufficiently small and

$$\det(Id + \eta M) = 1 + \eta \mathrm{tr}(M) + \frac{1}{2}\eta^2\Big((\mathrm{tr}M)^2 - \mathrm{tr}(M^2)\Big) + O(\eta^3),$$

where $\mathrm{tr}(M)$ denotes the trace of M. Moreover,

$$(Id + \eta M)^{-1} = Id - \eta M + \eta^2 M^2 + O(\eta^3).$$

Theorem 4.2
Let Ω be a domain with C^1–boundary. Further, let assumptions A 2.1 – A 2.4 be satisfied. If $(c,u) \in BV(\Omega, \{c_-, c_+\}) \times X_{\mathrm{ird}}^{\perp}$ is a minimizer of $\hat{E}_0$ then there exists a real number λ such that

$$\int_I \Big(\sigma_0(x,\nu_-)\nabla\cdot\xi + \sigma_{0,x}(x,\nu_-)\cdot\xi - \nu_-\cdot\nabla\xi\,\sigma_{0,p}(x,\nu_-)\Big)d\mathcal{H}^{n-1} \\ + \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)Id - \big(\nabla u\big)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big)\Big) : \nabla\xi\; dx = \lambda \int_\Omega c\,\nabla\cdot\xi\, dx \quad (4.2)$$

for all $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$, where ν_Ω is the outer unit normal of $\partial\Omega$.

Proof:
We choose a family of diffeomorphisms $\Phi(\tau,\cdot)$, $\tau \in [-\tau_0, \tau_0]$, of $\overline{\Omega}$ onto itself defined by

$$\Phi(0,x) = x \qquad \text{and} \qquad \Phi_{,\tau}(\tau,x) = \xi\big(\Phi(\tau,x)\big)$$

for $x \in \overline{\Omega}$ and $\tau \in [-\tau_0, \tau_0]$. Let $h \in C_c^1(\Omega, \mathbb{R}^n)$ be any function with $\int_I h \cdot \nu_-\, d\mathcal{H}^{n-1} \neq 0$ and consider

$$\tilde{\Phi}(x\,;\tau,w) = \Phi(\tau,x) + wh(\Phi(\tau,x)) \quad (4.3)$$

for $x \in \Omega$, $\tau \in [-\tau_0, \tau_0]$ and $w \in \mathbb{R}$. Then $\tilde{\Phi}(\cdot\,;\tau,w)$, $\tau \in [-\tau_0, \tau_0]$, is also a diffeomorphism of $\overline{\Omega}$ onto itself if $|w|$ is sufficiently small. Via the above diffeomorphisms we define

$$\Omega_l^{\tau,w} = \{\tilde{\Phi}(x\,;\tau,w) : x \in \Omega_l\} \quad \text{for } \; l \in \{-,+\}. \quad (4.4)$$

Further, we consider

$$j(\tau,w) := \int_{\Omega_-^{\tau,w}} 1\, dy - |\Omega_-| \\ = \int_{\Omega_-} \big|\det\tilde{\Phi}_{,x}(x\,;\tau,w)\big|\, dx - |\Omega_-|,$$

where $|\Omega_-| = \frac{c_+ - m}{c_+ - c_-}|\Omega|$. Clearly, $j(0,0) = 0$. Moreover, $j \in C^1$ with

$$\frac{\partial j}{\partial \tau}(\tau,w)\Big|_{(\tau,w)=(0,0)} = \int_{\Omega_-} \nabla\cdot\xi\, dx$$

and

$$\frac{\partial j}{\partial w}(\tau,w)\Big|_{(\tau,w)=(0,0)} = \int_{\Omega_-} \nabla\cdot h\, dx = \int_I h\cdot\nu\, d\mathcal{H}^{n-1}$$

by the generalized divergence theorem. In consequence, $\frac{\partial j}{\partial w}(0,0) \neq 0$. Thus we may apply the implicit function theorem and obtain a C^1–function $\eta : \mathbb{R} \to \mathbb{R}$ with

$$\eta(0) = 0 \qquad \text{and} \qquad j(\tau, \eta(\tau)) = 0 \tag{4.5}$$

for τ sufficiently small. Without loss of generality we may assume that (4.5) holds for $\tau \in [-\tau_0, \tau_0]$. Differentiating, we get

$$\frac{\partial j}{\partial \tau}\big(\tau, \eta(\tau)\big) + \frac{\partial j}{\partial w}\big(\tau, \eta(\tau)\big)\eta'(\tau) = 0.$$

Therefore,

$$\eta'(0) = -\frac{\frac{\partial j}{\partial \tau}(0,0)}{\frac{\partial j}{\partial w}(0,0)} = -\frac{\int_{\Omega_-} \nabla\cdot\xi(x)\,dx}{\int_I h(x)\cdot\nu_-(x)\;d\mathcal{H}^{n-1}}.$$

Note, $\hat{\Phi}(\cdot\,;\tau) := \tilde{\Phi}(\cdot\,;\tau,\eta(\tau))$ is for $|\tau|$ sufficiently small a diffeomorphism of $\overline{\Omega}$ onto itself. Via the variation $\hat{\Phi}$ we define

$$c^\tau(x) = c\big(\hat{\Phi}^{-1}(x\,;\tau)\big), \qquad \chi_-^\tau(x) = \chi_-\big(\hat{\Phi}^{-1}(x\,;\tau)\big) \qquad \text{and} \qquad u^\tau(x) = u\big(\hat{\Phi}^{-1}(x\,;\tau)\big),$$

where χ_- is the characteristic function of Ω_-. Furthermore, we set

$$\nu_-^\tau(x) = -\frac{\nabla\chi_-^\tau(x)}{\big|\nabla\chi_-^\tau(x)\big|}.$$

Observe, that the variation $\hat{\Phi}$ is mass preserving. Moreover, (c,u) stays a minimizer if we choose variations for u in the larger class $H^1(\Omega,\mathbb{R}^n)$. Thus we may take (c^τ, u^τ) as a comparison function and obtain $E_0(c,u) \leq E_0(c^\tau, u^\tau)$ for $\tau \in [-\tau_0, \tau_0]$. This implies

$$0 = \frac{d}{d\tau} E_0(c^\tau, u^\tau)\Big|_{\tau=0}.$$

Next we compute the above derivative. Here, we take advantage from the following properties of $\hat{\Phi}$:

(i) $\big|\det \hat{\Phi}_{,x}(x\,;0)\big| = 1$,

(ii) $\hat{\Phi}_{,x}^{-1}(\hat{\Phi}(x\,;\tau)\,;\tau) = \Big(\hat{\Phi}_{,x}(x\,;\tau)\Big)^{-1}$,

(iii) $\hat{\Phi}_{,x}(x\,;\tau) = \Big(Id + \eta(\tau)\nabla h(\Phi(\tau,x))\Big)\Phi_{,x}(\tau,x)$.
In particular,

$$\frac{d}{d\tau}\Big(\hat{\Phi}_{,x}(x\,;\tau)\Big)^{-1}\bigg|_{\tau=0} = -\nabla\xi(x) - \eta'(0)\nabla h(x).$$

Using the abbreviation $\hat{\xi}(x) := \xi(x) + \eta'(0)h(x)$ Lemma 4.1 gives

$$\begin{aligned}
&\frac{d}{d\tau}\int_\Omega \sigma_0\bigg(z, -\frac{\nabla_z\chi_-\big(\hat{\Phi}^{-1}(z\,;\tau)\big)}{\big|\nabla_z\chi_-\big(\hat{\Phi}^{-1}(z\,;\tau)\big)\big|}\bigg)\big|\nabla_z\chi_-\big(\hat{\Phi}^{-1}(z\,;\tau)\big)\big|\bigg|_{\tau=0} \\
&\quad = \int_I \Big(\sigma_0\big(x,\nu_-\big)\nabla\cdot\hat{\xi} + \sigma_{0,x}\big(x,\nu_-\big)\cdot\hat{\xi} - \nu_-\cdot\nabla\hat{\xi}\,\sigma_{0,p}\big(x,\nu_-\big)\Big)d\mathcal{H}^{n-1} \\
&\quad = \int_I \Big(\sigma_0\big(x,\nu_-\big)\nabla\cdot\xi + \sigma_{0,x}\big(x,\nu_-\big)\cdot\xi - \nu_-\cdot\nabla\xi\sigma_{0,p}\big(x,\nu_-\big)\Big)\;d\mathcal{H}^{n-1} \\
&\qquad\quad - \frac{\int_I\big(\sigma_0\big(x,\nu_-\big)\nabla\cdot h + \sigma_{0,x}\big(x,\nu_-\big)\cdot h - \nu_-\cdot\nabla h\sigma_{0,p}\big(x,\nu_-\big)\big)\,d\mathcal{H}^{n-1}}{\int_I h\cdot\nu_-\,d\mathcal{H}^{n-1}}\int_{\Omega_-}\nabla\cdot\xi\,dx.
\end{aligned}$$

For the elasticity part we have

$$\int_\Omega W\big(c^\tau(z), \mathcal{E}(u^\tau(z))\big)dz = \int_\Omega W\left(c\big(\hat\Phi^{-1}(z\,;\tau)\big), \frac{1}{2}\Big(\nabla_z\big(u(\hat\Phi^{-1}(z\,;\tau))\big) + \Big(\nabla_z\big(u(\hat\Phi^{-1}(z\,;\tau))\big)\Big)^T\Big)\right) dz\,.$$

We obtain

$$\int_\Omega W\big(c^\tau(z), \mathcal{E}(u^\tau(z))\big)dz = \int_\Omega W\left(c(x), \frac{1}{2}\Big(\nabla u(x)\big(\hat\Phi_{,x}(x\,;\tau)\big)^{-1} + \Big(\nabla u(x)\big(\hat\Phi_{,x}(x\,;\tau)\big)^{-1}\Big)^T\Big)\right) \big|\mathrm{det}\hat\Phi_{,x}(x\,;\tau)\big|dx.$$

The properties of $\hat\Phi$ and Φ, the symmetry of W and the growth conditions of W and $W_{,\mathcal{E}}$ gives

$$\begin{aligned}
&\frac{d}{d\tau}\Big(\int_\Omega W\big(c^\tau, \mathcal{E}(u^\tau)\big)\,dy\Big)\Big|_{\tau=0}\\
&= \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)\nabla\cdot\hat\xi - W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big) : \frac{1}{2}\Big(\nabla u\nabla\hat\xi + \big(\nabla u\nabla\hat\xi\big)^T\Big)\Big)dx\\
&= \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)\nabla\cdot\hat\xi - W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big) : \big(\nabla u\nabla\hat\xi\big)\Big)dx\\
&= \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)\nabla\cdot\hat\xi - (\nabla u)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big) : \nabla\hat\xi\Big)dx,
\end{aligned}$$

where $\hat\xi = \xi + \eta'(0)h$. Combining the above calculations shows

$$\int_I \Big(\sigma_0\big(x,\nu_-\big)\nabla\cdot\xi + \sigma_{0,x}\big(x,\nu_-\big)\cdot\xi - \nu_-\cdot\nabla\xi\sigma_{0,p}\big(x,\nu_-\big)\Big)\,d\mathcal{H}^{n-1} + \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)Id - (\nabla u)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big)\Big) : \nabla\xi\,dx = \lambda\int_\Omega c\,\nabla\cdot\xi\,dx$$

with

$$\lambda = -\Big(\int_I \Big(\sigma_0\big(x,\nu_-\big)\nabla\cdot h + \sigma_{0,x}\big(x,\nu_-\big)\cdot h - \nu_-\cdot\nabla h\sigma_{0,p}\big(x,\nu_-\big)\Big)\,d\mathcal{H}^{n-1} + \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)Id - (\nabla u)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big)\Big) : \nabla h\,dx\Big)\Big/\Big((c_+ - c_-)\int_I h\cdot\nu_-\,d\mathcal{H}^{n-1}\Big)$$

since $\int_\Omega c\,\nabla\cdot\xi\,dx = c_+\int_{\Omega_+}\nabla\cdot\xi + c_-\int_{\Omega_-}\nabla\cdot\xi = -(c_+ - c_-)\int_I \xi\cdot\nu_-\,d\mathcal{H}^{n-1}$ by the generalized divergence theorem. This completes the proof. ■

In the next theorem we state the first variation of $\hat E_\varepsilon$. In analogy to the limit problem $\hat E_0$, we choose inner variations.

Theorem 4.3
Let Ω be a domain with C^1–boundary. Further, let assumptions A 2.1 – A 2.4 be satisfied.

If $(c_\varepsilon, u_\varepsilon) \in H^1(\Omega) \times X_{\mathrm{ird}}^{\perp}$ *is a minimizer of* $\hat{E}_\varepsilon$ *then*

$$\int_\Omega \Big(\big(\varepsilon\sigma^2(x, c_\varepsilon, \nabla c_\varepsilon) + \frac{1}{\varepsilon}\psi(c_\varepsilon)\big)\nabla\cdot\xi + 2\,\varepsilon\,\sigma(x, c_\varepsilon, \nabla c_\varepsilon)\Big(\sigma_{,x}(x, c_\varepsilon, \nabla c_\varepsilon)\cdot\xi - \nabla c_\varepsilon\cdot\nabla\xi\;\sigma_{,p}(x, c_\varepsilon, \nabla c_\varepsilon)\Big)$$
$$+ \Big(W\big(c_\varepsilon, \mathcal{E}(u_\varepsilon)\big)Id - \big(\nabla u_\varepsilon\big)^T W_{,\mathcal{E}}\big(c_\varepsilon, \mathcal{E}(u_\varepsilon)\big)\Big) : \nabla\xi\Big)dx = \lambda_\varepsilon \int_\Omega c_\varepsilon \nabla\cdot\xi dx \qquad (4.6)$$

for all $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ *with* $\xi \cdot \nu_\Omega = 0$ *on* $\partial\Omega$*, where* ν_Ω *is the outer unit normal of* $\partial\Omega$*. The Lagrange–multiplier* λ_ε *has the following form:*

$$\lambda_\varepsilon = \fint_\Omega \Big(2\varepsilon\,\sigma(x, c_\varepsilon, \nabla c_\varepsilon)\,\sigma_{,c}(x, c_\varepsilon, \nabla c_\varepsilon) + \frac{1}{\varepsilon}\,\psi_{,c}(c_\varepsilon) + W_{,c}\big(c_\varepsilon, \mathcal{E}(u_\varepsilon)\big)\Big)dx. \qquad (4.7)$$

Proof:
The result established here is an anisotropic version of a computation in [17, 18]. For simplicity we omit the index ε in this proof. Let $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ be arbitrary up to $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$. We choose a one parametric family of diffeomorphisms $\Phi(\tau, \cdot)$, $\tau \in [-\tau_0, \tau_0]$, of $\overline{\Omega}$ onto itself defined by solutions of the initial value problems

$$\Phi(0, x) = x \qquad \text{and} \qquad \Phi_{,\tau}(\tau, x) = \xi\big(\Phi(\tau, x)\big),$$

for $x \in \overline{\Omega}$ and $\tau \in [-\tau_0, \tau_0]$. We consider (c^τ, u^τ) with

$$c^\tau(x) = c\big(\Phi(-\tau, x)\big) - \fint_\Omega c\big(\Phi(-\tau, y)\big)dy + m$$

and

$$u^\tau(x) = u\big(\Phi(-\tau, x)\big),$$

which is allowed as comparison function since the mass constraint is satisfied and (c, u) stays a minimizer if we choose variations for u in the larger class $H^1(\Omega, \mathbb{R}^n)$. This implies

$$0 = \frac{d}{d\tau}E(c^\tau, u^\tau)\Big|_{\tau=0}.$$

For the derivative of the σ–term we obtain

$$\frac{d}{d\tau}\Big(\int_\Omega \sigma^2\big(y, c^\tau(y), \nabla_y c^\tau(y)\big)dy\Big)\Big|_{\tau=0} = \int_\Omega \Big(\sigma^2(x, c, \nabla c)\nabla\cdot\xi$$
$$+ 2\sigma(x, c, \nabla c)\Big(\sigma_{,x}(x, c, \nabla c)\cdot\xi - \nabla c\cdot\nabla\xi\;\sigma_{,p}(x, c, \nabla c)\Big)\Big)dx$$
$$- \Big(\fint_\Omega 2\,\sigma(x, c, \nabla c)\sigma_{,c}(x, c, \nabla c)dx\Big)\int_\Omega c\,\nabla\cdot\xi\,dx.$$

The derivative of the ψ–expression and the elasticity part is computed as in [17, 18], i. e.

$$\frac{d}{d\tau}\Big(\int_\Omega \psi\big(c^\tau(y)\big)dy\Big)\Big|_{\tau=0} = \int_\Omega \Big(\psi(c)\nabla\cdot\xi\Big)dx - \Big(\fint_\Omega \psi_{,c}(c)dx\Big)\int_\Omega c\,\nabla\cdot\xi\,dx$$

and

$$\frac{d}{d\tau}\Big(\int_\Omega W\big(c^\tau(y),\mathcal{E}(u^\tau(y))\big)dy\Big)\Big|_{\tau=0} = \int_\Omega\Big(W\big(c,\mathcal{E}(u)\big)\nabla\cdot\xi - \Big((\nabla u)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big)\Big) : \nabla\xi\Big)dx$$
$$-\Big(\fint_\Omega W_{,c}\big(c,\mathcal{E}(u)\big)dx\Big)\int_\Omega c\,\nabla\cdot\xi\,dx.$$

Putting all together shows the claim. ■

Remark 1 *Let us point out that λ_ε is also the Lagrange multiplier of the Euler–Lagrange equation*

$$\int_\Omega\Big(2\varepsilon\sigma(x,c_\varepsilon,\nabla c_\varepsilon)\sigma_{,p}(x,c_\varepsilon,\nabla c_\varepsilon)\cdot\nabla\zeta + 2\varepsilon\sigma(x,c_\varepsilon,\nabla c_\varepsilon)\sigma_{,c}(x,c_\varepsilon,\nabla c_\varepsilon)\zeta + \frac{1}{\varepsilon}\psi_{,c}(c_\varepsilon)\zeta$$
$$+ W_{,c}\big(c_\varepsilon,\mathcal{E}(u_\varepsilon)\big)\zeta\Big)\,dx = \int_\Omega \lambda_\varepsilon\,\zeta \quad (4.8)$$

for all $\zeta \in L^\infty(\Omega)\cap H^1(\Omega)$, which are obtained by variations with respect to the dependent variable c (set $\zeta\equiv 1$ in (4.8) to obtain (4.7)). Luckhaus and Modica [24], who studied the case for isotropic surface tension and without elasticity, started with equation (4.8) and set $\zeta = \nabla c_\varepsilon\xi$ to deduce (4.6). Formally, it is also possible to derive (4.6) from (4.8). However we did not choose this approach because due to the elasticity there is not enough regularity known to make the formal calculations rigorous.

5 Convergence of the Lagrange–multipliers λ_ε

5.1 Properties of anisotropy functions

In the sequel we discuss some properties of anisotropy functions which are utilized to pass to the limit in the weak formulation of the Euler–Lagrange equation.

Anisotropy can be visualized by the Wulff shape W which varies in our case with $x\in\Omega$ and $s\in\mathbb{R}$:

$$W(x,s) = \{q\in\mathbb{R}^n : \sigma^*(x,s,q)\le 1\}.$$

The Wulff shape $W(x,s)$ is convex and its boundary can be expressed as follows:

$$\partial W(x,s) = \big\{\sigma_{,p}(x,s,\tilde\nu) : \tilde\nu\in\mathbb{S}^{n-1}\big\}.$$

The outer unit normal at the point $\sigma_{,p}(x,s,\nu)$ on $\partial W(x,s)$ is ν. For more details on this matter we refer [21] and [19].

The following lemma is an essential tool for constructing suitable approximations of the Cahn-Hoffman vector $\sigma_{,p}$.

Lemma 5.1
Let σ be an anisotropy function satisfying Assumption A 2.1. Then there exists a constant $C>0$ such that

$$C\,|\sigma_{,p}(x,s,\nu)-p|^2 \le \sigma(x,s,\nu) - p\cdot\nu$$

for all $x\in\Omega$, $s\in[c_-,c_+]$, $\nu\in\mathbb{S}^{n-1}$ and all $p\in\mathbb{R}^n\backslash\{0\}$ with $\sigma^*(x,s,p)\le 1$.

Proof:
Let $x \in \Omega$, $s \in [c_-, c_+]$, $\nu \in \mathbb{S}^{n-1}$ and define $B_\varepsilon(\sigma_{,p}(x,s,\nu)) = \{\zeta \in \mathbb{R}^n : |\zeta - \sigma_{,p}(x,s,\nu))| < \varepsilon\}$ for $\varepsilon > 0$.

(i) We first assume that $p \in B_\varepsilon(\sigma_{,p}(x,s,\nu))$ and $\sigma^*(x,s,p) \leq 1$, where $\varepsilon > 0$ is chosen so small that $0 \notin B_\varepsilon(\sigma_{,p}(x,s,\nu))$. Since, compare (2.7) and (2.8),

$$\begin{aligned}\sigma^*(x,s,p+t\nu) &\geq 1 + \sigma^*_{,p}\big(x,s,\sigma_{,p}(x,s,\nu)\big)(t\nu + p - \sigma_{,p}(x,s,\nu)) \\ &= 1 + \frac{t}{\sigma(x,s,\nu)} + \sigma^*_{,p}\big(x,s,\sigma_{,p}(x,s,\nu)\big)(p - \sigma_{,p}(x,s,\nu))\end{aligned}$$

we may also assume for $\varepsilon > 0$ small enough that for all $p \in B_\varepsilon(\sigma_{,p}(x,s,\nu))$

$$1 > \tau(x,s) := \sup\{t \geq 0 : \sigma^*(x,s,p+t\nu) \leq 1\}.$$

We set

$$q(x,s) = p + \tau(x,s)\nu.$$

Due to the the continuity of σ^* we obtain $\sigma^*(x,s,q(x,s)) = 1$. Note,

$$|q(x,s) - p| = \tau(x,s) \qquad \text{and} \qquad q(x,s)\cdot\nu = p\cdot\nu + \tau(x,s).$$

As $\sigma^*(x,s,q(x,s)) = 1$ there exists some $\overline{\nu} \in \mathbb{S}^{n-1}$ such that

$$\sigma_{,p}(x,s,\overline{\nu}) = q(x,s).$$

From (2.4) and (2.5) we derive

$$\sigma(x,s,\nu) - \sigma_{,p}(x,s,\overline{\nu})\cdot\nu \geq C_3\,|\sigma_{,p}(x,s,\nu) - \sigma_{,p}(x,s,\overline{\nu})|^2$$

for $C_3 \leq C_1/C_2^2$. This yields

$$\begin{aligned}\sigma(x,s,\nu) - p\cdot\nu &= \sigma(x,s,\nu) - q(x,s)\cdot\nu + \tau(x,s) \\ &\geq C_3\,\big|\sigma_{,p}(x,s,\nu) - q(x,s)\big|^2 + \tau(x,s).\end{aligned}$$

Further, we have

$$\begin{aligned}\big|\sigma_{,p}(x,s,\nu) - p\big|^2 &= \big|\sigma_{,p}(x,s,\nu) + q(x,s) - q(x,s) - p\big|^2 \\ &\leq 2\big(\big|\sigma_{,p}(x,s,\nu) - q(x,s)\big|^2 + |q(x,s) - p|^2\big) \\ &\leq 2\big(\big|\sigma_{,p}(x,s,\nu) - q(x,s)\big|^2 + \tau(x,s)\big).\end{aligned}$$

This shows that there exists some $C > 0$ (independent of x, s and ν) such that

$$\sigma(x,s,\nu) - p\cdot\nu \geq C\,\big|\sigma_{,p}(x,s,\nu) - p\big|^2.$$

(ii) Now we assume that the claim is false for the general case. Then there exist sequences $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \in \Omega$, $\{s_k\}_{k\in\mathbb{N}}$ with $s_k \in [c_-, c_+]$, $\{\nu_k\}_{k\in\mathbb{N}}$ with $\nu_k \in \mathbb{S}^{n-1}$, $\{p_k\}_{k\in\mathbb{N}}$ with $p_k \in \mathbb{R}^n \backslash B_\varepsilon(\sigma_{,p}(x,s,\nu))$ and $\sigma^*(x_k,s_k,p_k) \leq 1$ such that

$$\sigma(x_k,s_k,\nu_k) - p_k\cdot\nu_k \leq \frac{1}{k}\big|\sigma_{,p}(x_k,s_k,\nu_k) - p_k\big|^2.$$

Furthermore, there exist convergent subsequences,

$$x_{k_j} \to x \in \overline{\Omega}, \quad s_{k_j} \to s \in [c_-, c_+], \quad \nu_{k_j} \to \nu \in \mathbb{S}^{n-1}, \quad p_{k_j} \to \hat{p} \in \mathbb{R}^n \backslash B_\varepsilon(\sigma_{,p}(x,s,\nu)).$$

In particular, we have

$$\sigma^*(x,s,\hat{p}) \leq 1 \qquad \text{and} \qquad \sigma(x,s,\nu) - \hat{p}\cdot\nu = 0$$

due to the continuity of σ, σ^* and the uniform boundedness of $\sigma_{,p}$. It follows from (2.1) that $\hat{p} \in \partial W(x,s)$. Thus there exists some $\tilde{\nu} \in \mathbb{S}^{n-1}$ such that

$$\hat{p} = \sigma_{,p}(x,s,\tilde{\nu}).$$

But this implies $\tilde{\nu} = \nu$ by (2.4) which is a contradiction to $\hat{p} \notin B_\varepsilon(\sigma_{,p}(x,s,\nu))$. ■

5.2 Slicing and indicator measures

For convenience, we summarize some results on slicing and indicator measures. We refer [5], [13], [14] and [15] for details. Let Θ be a finite, nonnegative Radon measure on $\Omega \times \mathbb{R}^n$. The canonical projection onto Ω is denoted by π, i. e.

$$\pi(E) := \Theta(E \times \mathbb{R}^n)$$

for each Borel set $E \subset \Omega$.

Proposition 5.2 (cf. [5])
For π–a. e. point $x \in \Omega$ there exists a Radon probability measure λ_x on $\mathbb{R}^n$ such that

(i) the mapping $x \to \int_{\mathbb{R}^n} f(x,y)\, d\lambda_x(y)$ is π measurable,

(ii) $\int_{\Omega\times\mathbb{R}^n} f(x,y)\, d\Theta(x,y) = \int_\Omega \big(\int_{\mathbb{R}^n} f(x,y) d\lambda_x(y)\big)\, d\pi(x)$ (Fubini's decomposition)

for every continuous and bounded function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$.

Let $\hat{\mu}$ be an $\mathbb{R}^n$–valued measure on Ω with polar decomposition $d\hat{\mu} = \alpha\, d\mu$. Then the *indicator measure* of $\hat{\mu}$ is the finite, nonnegative Radon measure Θ on $\Omega \times \mathbb{S}^{n-1}$ defined by

$$\langle \Theta, f \rangle = \int_\Omega f\big(x, \alpha(x)\big) d\mu(x)$$

for every continuous and bounded function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$. If $E \subset \Omega$ is a set with finite perimeter, i. e.

$$\mathrm{per}(E) = \int_\Omega |\nabla \chi_E| < \infty, \qquad \chi_E : \text{ characteristic function of } E,$$

then the indicator measure of $\nabla \chi_E$ has the form

$$\langle \Theta, f \rangle = \int_{\partial^* E} f\big(x, -\nu_E(x)\big)\, d\mathcal{H}^{n-1}(x)\,, \qquad \nu_E : \text{ unit outer normal of } E.$$

Proposition 5.3 (cf. [5], [15])
Let $\{\hat{\mu}_k\}_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$–valued measures on Ω with polar decompositions $d\hat{\mu}_k = \alpha_k\, d\mu_k$ and suppose that $\hat{\mu}_k \to \hat{\mu}$ weakly* with $\hat{\mu} = \alpha\mu$. Then there exists a subsequence $\{k_j\}_{j\in\mathbb{N}}$ and a nonnegative Radon measure $\Theta_\infty \equiv \pi_\infty \otimes \lambda_x^\infty$ on $\Omega \times \mathbb{S}^{n-1}$, λ_x^∞ being probability measures, such that

(i) $\Theta_{k_j} \equiv \mu_{k_j} \otimes \delta_{\alpha_{k_j}(x)} \to \Theta_\infty \equiv \pi_\infty \otimes \lambda_x^\infty$ *weakly* ,* $\qquad \delta_y$ *Dirac mass,*

(ii) $\mu_{k_j} \to \pi_\infty$ *weakly* ,*

(iii) $\pi_\infty \geq \mu$.

Moreover, for every $f \in C_c(\Omega \times \mathbb{R}^n)$

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega f\big(x, \alpha_{k_j}(x)\big) d\mu_{k_j} &= \int_{\Omega\times\mathbb{S}^{n-1}} f\big(x,y\big) d\Theta_\infty(x,y) \\ &= \int_\Omega \bigg(\int_{\mathbb{S}^{n-1}} f\big(x,y\big) d\lambda_x^\infty(y) \bigg) d\pi_\infty(x).\end{aligned}$$

5.3 Conclusions for minimizing sequences

For our further considerations it is convenient to introduce some abbreviations.

Notation:
Let $(c_\varepsilon, u_\varepsilon)$, $\varepsilon > 0$, be a minimizer of $\hat{E}_\varepsilon$ and let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence with $\lim_{k\to\infty} \varepsilon_k = 0$ such that $c_{\varepsilon_k} \to c$ in $L^1(\Omega)$. Then $c \in BV(\Omega, \{c_-, c_+\})$ and we will use the following notation:

- $\chi_{s,k}$ stands for the characteristic function of $\Omega_{s,k} = \{x \in \Omega : c_{\varepsilon_k}(x) < s\}$.
- χ_- denotes the characteristic function of $\Omega_- = \{x \in \Omega : c(x) = c_-\}$.
- $\nu_{s,k}(x) := -\dfrac{\nabla\chi_{s,k}}{|\nabla\chi_{s,k}|}(x)$ for $|\nabla\chi_{s,k}|$–almost all $x \in \Omega$.
- $\nu_-(x) := -\frac{\nabla\chi_-}{|\nabla\chi_-|}(x)$ for $|\nabla\chi_-|$–almost all $x \in \Omega$.

Lemma 5.4
Suppose Assumptions A 2.1 – A 2.4 are fulfilled. Let $(c_\varepsilon, u_\varepsilon)$, $\varepsilon > 0$, be a minimizer of $\hat{E}_\varepsilon$. If $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a sequence with $\lim_{k\to\infty} \varepsilon_k = 0$ such that

$$c_{\varepsilon_k} \to c \quad \text{in } L^1(\Omega) \quad \text{as } k \to \infty$$

then the following conditions are satisfied:

(i) There exists a subsequence $\{\varepsilon_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\chi_{s,k_j} \to \chi_- \quad \text{in } L^1(\Omega) \text{ as } j \to \infty \text{ for a. e. } s \in [c_-, c_+].$$

(ii) For every open set $U \subseteq \Omega$

$$\int_U \sigma(x, s, \nu_-)|\nabla\chi_-| \leq \liminf_{k\to\infty} \int_U \sigma(x, s, \nu_{s,k})|\nabla\chi_{s,k}| \quad \text{for a. e. } s \in [c_-, c_+].$$

(iii) $\lim_{k\to\infty}\int_{(-\infty,c_-)\cup(c_+,\infty)}\sqrt{\psi(s)}\int_\Omega \sigma(x,s,\nu_{s,k})|\nabla\chi_{k,s}|\,ds=0.$

In particular,

$$\lim_{k\to\infty}\int_{(-\infty,c_-)\cup(c_+,\infty)}\sqrt{\psi(s)}\int_\Omega |\nabla\chi_{k,s}|\,ds=0.$$

(iv) $\lim_{k\to\infty}\int_{c_-}^{c_+}2\sqrt{\psi(s)}\int_\Omega \sigma(x,s,\nu_{s,k})|\nabla\chi_{s,k}|\,ds=\int_\Omega\sigma_0(x,\nu_-)|\nabla\chi_-|.$

(v) There exists a subsequence $\{\varepsilon_{k_j}\}_{j\in\mathbb{N}}$ *such that*

$$\int_\Omega \sigma(x,s,\nu_{s,k_j})|\nabla\chi_{s,k_j}|\to\int_\Omega\sigma(x,s,\nu_-)|\nabla\chi_-|$$

as $j\to\infty$ *for* a.e. $s\in[c_-,c_+]$.

(vi) There exists a subsequence $\{\varepsilon_{k_j}\}_{j\in\mathbb{N}}$ *and some constant* $M>0$ *such that*

$$\limsup_{j\to\infty}\int_\Omega|\nabla\chi_{s,k_j}|<M$$

for a.e. $s\in[c_-,c_+]$.

Proof:
To (i): Since $c_{\varepsilon_k}\to c$ in $L^1(\Omega)$ and

$$\int_\Omega|c_{\varepsilon_k}-c|dx=\int_{-\infty}^{+\infty}\Big(\int_\Omega\big|\chi_{s,k}-\chi_{\{c<s\}}\big|dx\Big)ds,$$

where $\chi_{\{c<s\}}$ is the characteristic function of $\{x\in\Omega : c(x)<s\}$, we can choose a subsequence $\{\chi_{s,k_j}\}_{j\in\mathbb{N}}$ such that

$$\chi_{s,k_j}\to\chi_-\quad\text{in } L^1(\Omega)\ \text{ as } j\to\infty\quad\text{for a.e. } s\in[c_-,c_+].$$

To (ii): This property follows immediately since $\int_\Omega|\nabla\chi_-|_{\sigma(s)}$ is $L^1(\Omega)$–lower semicontinuous on $BV(\Omega)$.
To (iii) and (iv): From Theorem 3.2 we conclude

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega\Big(\varepsilon_k\,\sigma^2(x,c,\nabla c_{\varepsilon_k})+\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})\Big)\,dx&=\lim_{k\to\infty}\int_\Omega 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c,\nabla c_{\varepsilon_k})\,dx\\&=\int_{c_-}^{c_+}2\sqrt{\psi(s)}\Big(\int_\Omega\sigma(x,s,\nu_-)\,|\nabla\chi_-|\Big)\,ds.\end{aligned}$$

Using the coarea formula and (ii) we deduce

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c,\nabla c_{\varepsilon_k})\,dx&=\lim_{k\to\infty}\int_{-\infty}^{+\infty}2\sqrt{\psi(s)}\Big(\int_\Omega\sigma(x,s,\nu_{s,k})\,|\nabla\chi_{k,s}|\Big)\,ds\\&\geq\lim_{k\to\infty}\int_{-c}^{+c}2\sqrt{\psi(s)}\Big(\int_\Omega\sigma(x,s,\nu_{s,k})\,|\nabla\chi_{k,s}|\Big)\,ds\\&=\int_{c_-}^{c_+}2\sqrt{\psi(s)}\Big(\int_\Omega\sigma(x,s,\nu_-)\,|\nabla\chi_-|\Big)\,ds.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\int_{-c}^{+c} 2\sqrt{\psi(s)}\left(\int_\Omega \sigma(x,s,\nu_{s,k})\,|\nabla\chi_{k,s}|\right)ds = \int_{c_-}^{c_+} 2\sqrt{\psi(s)}\left(\int_\Omega \sigma(x,s,\nu_-)\,|\nabla\chi_-|\right)ds.$$

This shows assertion (i) and (iii).
To (v): Because of (ii) and (iv) we get

$$\lim_{k\to\infty}\int_{c_-}^{c_+}\sqrt{\psi(s)}\left|\int_\Omega \sigma(x,s,\nu_{s,k})|\nabla\chi_{s,k}| - \int_\Omega \sigma(x,s,\nu_-)|\nabla\chi_-|\right| ds = 0.$$

Thus there exists a subsequence $\{\varepsilon_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\int_\Omega \sigma(x,s,\nu_{s,k_j})|\nabla\chi_{s,k_j}| \to \int_\Omega \sigma(x,s,\nu_-)|\nabla\chi_-| \quad \text{as } j\to\infty \quad \text{for a.e. } s\in[c_-,c_+].$$

To (vi): By (v) we know that there exists some constant $M>0$ such that

$$\limsup_{j\to\infty}\int_\Omega |\nabla\chi_{s,k_j}| < M \qquad \text{for a.e. } s\in[c_-,c_+].$$

■

Theorem 5.5
Suppose assumptions A 2.1 – A 2.4 are fulfilled. Further, let $(c_\varepsilon,u_\varepsilon)$, $\varepsilon>0$, be a minimizer of $\hat{E}_\varepsilon$. If $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a sequence with $\lim_{k\to\infty}\varepsilon_k=0$ and

$$c_{\varepsilon_k}\to c \quad \text{in } L^1(\Omega) \quad \text{as } k\to\infty$$

then there exists a subsequence $\{\varepsilon_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\int_\Omega f\big(x,s,\nu_{s,k_j}(x)\big)\big|\nabla\chi_{s,k_j}(x)\big| \to \int_\Omega f\big(x,s,\nu_-(x)\big)\big|\nabla\chi_-(x)\big| \tag{5.1}$$

for all continuous and bounded functions $f:\Omega\times[c_-,c_+]\times\mathbb{R}^n\to\mathbb{R}$ and a.e. $s\in[c_-,c_+]$.

Proof:
Items (i) and (vi) of Lemma 5.4 assure that there exist a set $S\subset[c_-,c_+]$ of Lebesgue measure zero and a sequence $\{\varepsilon_{k_j}\}_{j\in\mathbb{N}}$ such that $\chi_{s,k_j}\to\chi_-$ in $L^1(\Omega)$ and $\{\mathrm{per}(\Omega_{s,k_j})\}_{j\in\mathbb{N}}$ is bounded for $s\in[c_-,c_+]\backslash S$. This, in turn, is equivalent to $\chi_{s,k_j}\to\chi_-$ in $L^1(\Omega)$ and $\nabla\chi_{s,k_j}\to\nabla\chi_-$ weakly* for $s\in[c_-,c_+]\backslash S$.

For any $s\in[c_-,c_+]\backslash S$ we conclude from Proposition 5.3 (modulo a subsequence) that there exists a nonnegative Radon measure $\Theta_\infty\equiv\pi_\infty\otimes\lambda_x^\infty$ on $\Omega\times\mathbb{S}^{n-1}$ such that

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega F\big(x,\nu_{s,k_j}\big)|\nabla\chi_{s,k_j}| &= \int_{\Omega\times\mathbb{S}^{n-1}} F\big(x,y\big)d\Theta_\infty(x,y)\\ &= \int_\Omega\left(\int_{\mathbb{S}^{n-1}} F\big(x,y\big)d\lambda_x^\infty(y)\right)d\pi_\infty(x)\end{aligned}$$

for any $F\in C_c(\Omega\times\mathbb{R}^n)$.

We now claim that λ_x^∞ is a Dirac mass for $|\nabla\chi_-|$–a. e. $x \in \Omega$. For any $\hat{x} \in \Omega$ we choose $r > 0$ such that $B(\hat{x}, r) = \{x \in \mathbb{R}^n : ||x - \hat{x}|| < r\} \Subset \Omega$, and set

$$F_g(x, y; s) = \Phi_1(x)\Phi_2(y)|\sigma_{,p}(x, s, y) - g_s(x)|^2,$$

where $g_s \in K_{\sigma(s)}(\Omega)$, $\Phi_1 \in C_c(\Omega)$ with $0 \leq \Phi_1 \leq 1$ in Ω and $\Phi_1 \equiv 1$ in $B(\hat{x}, r)$ and $\Phi_2 \in C_c(\mathbb{R}^n)$ with $\Phi_2(y) = 0$ in $\{y \in \mathbb{R}^n : ||y|| < h\}$ for some $h > 0$ and $\Phi_2(y) = 1$ on $\mathbb{S}^{n-1}$. Thus $F_g(\cdot, \cdot; s) \in C_c(\Omega \times \mathbb{R}^n)$ and from Proposition 5.3 we attain (modulo a subsequence)

$$\begin{aligned}\int_\Omega \Phi_1(x)\Big(\int_{\mathbb{S}^{n-1}} \Phi_2(y)|\sigma_{,p}(x, s, y) - g_s(x)|^2 d\lambda_x^\infty(y)\Big)\big|\nabla\chi_-(x)\big| \\ \leq \int_\Omega \Phi_1(x)\Big(\int_{\mathbb{S}^{n-1}} \Phi_2(y)|\sigma_{,p}(x, s, y) - g_s(x)|^2 d\lambda_x^\infty(y)\Big) d\pi_\infty(x) \\ = \lim_{j\to\infty} \int_\Omega \Phi_1(x)\Phi_2(\nu_{s,k_j})|\sigma_{,p}(x, s, \nu_{s,k_j}) - g_s(x)|^2|\nabla\chi_{s,k_j}| \\ \leq \lim_{j\to\infty} \int_\Omega |\sigma_{,p}(x, s, \nu_{s,k_j}) - g_s(x)|^2|\nabla\chi_{s,k_j}|.\end{aligned} \tag{5.2}$$

Applying Lemma 5.1 and 5.4 (v) we estimate for a. e. $s \in [c_-, c_+]$

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega C|\sigma_{,p}(x, s, \nu_{s,k_j}) - g_s(x)|^2|\nabla\chi_{s,k_j}| &\leq \lim_{j\to\infty} \int_\Omega \big(\sigma(x, s, \nu_{s,k_j}) - g_s(x) \cdot \nu_{s,k_j}\big)|\nabla\chi_{s,k_j}| \\ &= \int_\Omega \Big(\sigma(x, s, \nu_-) - g_s(x) \cdot \nu_-\Big)|\nabla\chi_-| \\ &\leq \int_\Omega \big|\sigma_{,p}(x, s, \nu_-) - g_s(x)\big||\nabla\chi_-|,\end{aligned} \tag{5.3}$$

where $C > 0$ is some constant.

Next we construct smooth approximations for the Cahn–Hoffman vector $\sigma_{,p}$. Due to (2.2) there exists for every $\delta > 0$ and $s \in [c_-, c_+]$ approximative functions $g_s^\delta \in K_{\sigma(s)}$ such that

$$\int_\Omega \big(\sigma(x, s, \nu_-) - g_s^\delta(x) \cdot \nu_-\big)|\nabla\chi_-| \leq \delta^2.$$

Thus, by Lemma 5.1,

$$\int_{\partial^*\Omega_-} |\sigma_{,p}(x, s, \nu_-) - g_s^\delta(x)| \, d\mathcal{H}^{n-1}(x) \leq C_1 \, \delta$$

for some constant $C_1 > 0$ and for every $s \in [c_-, c_+]$. This implies the existence of a sequence $\{g_s^l\}_{l\in\mathbb{N}}$, $g_s^l \in C_c^1(\Omega, \mathbb{R}^n)$, with $g_s^l(\cdot) \to \sigma_{,p}(\cdot, s, \nu_-(\cdot))$ in $L^1(\mathcal{H}^{n-1}\lfloor\partial^*\Omega_-)$ for $s \in [c_-, c_+]$ since $\delta > 0$ may be chosen arbitrarily small. Hence, using (5.2) and (5.3),

$$\int_\Omega \Phi_1(x)\Big(\int_{\mathbb{S}^{n-1}} |\sigma_{,p}(x, s, y) - \sigma_{,p}(x, s, \nu_-(x))|^2 d\lambda_x^\infty(y)\Big)\big|\nabla\chi_-(x)\big| = 0.$$

In particular

$$\int_\Omega \Phi_1(x)\Big(\int_{\mathbb{S}^{n-1}} |\sigma_{,p}(x, s, y) \cdot y - \sigma_{,p}(x, s, \nu_-(x)) \cdot y|^2 d\lambda_x^\infty(y)\Big)\big|\nabla\chi_-(x)\big| = 0.$$

This implies, according to Lemma 2.2 (ii),

$$\int_{\mathbb{S}^{n-1}} \left|\nu_-(x) - y\right|^4 d\lambda_x^\infty(y) = 0 \quad \text{for } \left|\nabla\chi_-\right|\text{–a. e. } x \in B(\hat{x}, r).$$

Hence we obtain that λ_x^∞ is a Dirac mass, i. e. $\lambda_x^\infty = \delta_{y=\nu_-(x)}$ for $\left|\nabla\chi_-\right|$–a. e. $x \in B(\hat{x}, r)$ and the claim follows as $\hat{x} \in \Omega$ was arbitrary.

Since $\lambda_x^\infty = \delta_{y=\nu_-(x)}$ for $\left|\nabla\chi_-\right|$–a. e. $x \in \Omega$ we infer from Lemma 5.3

$$\begin{aligned}\int_\Omega \sigma(x, s, \nu_-(x))|\nabla\chi_-(x)| &= \int_\Omega \left(\int_{\mathbb{S}^{n-1}} \sigma(x, s, y)\, d\lambda_x^\infty(y)\right)|\nabla\chi_-(x)| \\ &= \int_\Omega \left(\int_{\mathbb{S}^{n-1}} \sigma(x, s, y)\, d\lambda_x^\infty(y)\right) g(x)\, d\pi_\infty(x) \\ &\leq \int_{\Omega\times\mathbb{S}^{n-1}} \sigma(x, s, y)\, d\Theta_\infty(x, y),\end{aligned}$$

where g is the density of $|\nabla\chi_-|$ with respect to π_∞ and $0 \leq g(x) \leq 1$ for π_∞–a. e. $x \in \Omega$.
On the other hand, by means of Lemma 5.4 (v) we get

$$\begin{aligned}\int_{\Omega\times\mathbb{S}^{n-1}} \sigma(x, s, y)\, d\Theta_\infty(x, y) &\leq \liminf_{j\to\infty} \int_{\Omega\times\mathbb{S}^{n-1}} \sigma(x, s, y)\, d\,\Theta_{k_j}(x, y) \\ &= \liminf_{j\to\infty} \int_\Omega \sigma(x, s, \nu_{s,k_j}(x))|\nabla\chi_{s,k_j}(x)| \\ &= \int_\Omega \sigma(x, s, \nu_-(x))|\nabla\chi_-(x)|.\end{aligned}$$

Consequently, as $\int_{\mathbb{S}^{n-1}} \sigma(x, s, y)\, d\lambda_x^\infty(y) > 0$ for π_∞–a. e. $x \in \Omega$ we deduce

$$g \equiv 1 \quad \text{and} \quad |\nabla\chi_-| = \pi_\infty \quad \text{for } \pi_\infty\text{–a. e. } x \in \Omega.$$

Moreover, $\Theta_{k_j}(\Omega \times \mathbb{S}^{n-1}) = |\nabla\chi_{s,k_j}|(\Omega)$ converges to $|\nabla\chi_-|(\Omega) = \Theta_\infty(\Omega \times \mathbb{S}^{n-1})$ for a. e. $s \in [c_-, c_+]$.
Next we take advantage from the property that $\lim_{j\to\infty} \Theta_{k_j}(\Omega \times \mathbb{S}^{n-1}) = \Theta_\infty(\Omega \times \mathbb{S}^{n-1})$ and $\Theta_{k_j} \to \Theta_\infty$ weakly* implies

$$\lim_{j\to\infty} \int_{\Omega\times\mathbb{S}^{n-1}} u(x, y)\, d\Theta_{k_j}(x, y) = \int_{\Omega\times\mathbb{S}^{n-1}} u(x, y)\, \Theta_\infty(x, y)$$

for every continuous and bounded function $u : \Omega \times \mathbb{R}^n \to \mathbb{R}$. We conclude

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega f(x, s, \nu_{s,k_j})|\nabla\chi_{s,k_j}| &= \lim_{j\to\infty} \int_{\Omega\times\mathbb{S}^{n-1}} f(x, s, y)\, d\Theta_{k_j}(x, y) \\ = \int_{\Omega\times\mathbb{S}^{n-1}} f(x, s, y)\, \Theta_\infty(x, y) &= \int_\Omega f(x, s, \nu_-)|\nabla\chi_-|\end{aligned}$$

for every continuous and bounded function $f : \Omega \times [c_-, c_+] \times \mathbb{R}^n \to \mathbb{R}$ and a. e. $s \in [c_-, c_+]$ as required. ■

5.4 Proof of the main result

Now we are in a position to prove the main result of this paper.

Theorem 5.6
Let $\Omega \subset \mathbb{R}^n$ be a domain with C^1–boundary and let assumptions A 2.1 – A 2.4 be satisfied. Further, let $(c_\varepsilon, u_\varepsilon)$, $\varepsilon > 0$, be a minimizer of $\hat{E}_\varepsilon$ with mass constraint $⨍_\Omega c_\varepsilon dx = m$. Then for each sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} \varepsilon_k = 0$, with

$$c_{\varepsilon_k} \to c \quad \textit{in} \quad L^1(\Omega) \tag{5.4}$$

and

$$u_{\varepsilon_k} \to u \quad \textit{in} \quad L^2(\Omega, \mathbb{R}^n) \tag{5.5}$$

the corresponding sequence of Lagrange multipliers $\{\lambda_{\varepsilon_k}\}_{k\in\mathbb{N}}$ (associated to the mass constraint) converges, i. e.

$$\lambda_{\varepsilon_k} \to \lambda,$$

where λ is a Lagrange multiplier of the minimum problem for E_0 with $⨍_\Omega c\, dx = m$, i. e.

$$\int_I \Big(\sigma_0(x,\nu_-)\nabla\cdot\xi + \sigma_{0,x}(x,\nu_-)\cdot\xi - \nu_-\cdot\nabla\xi\sigma_{0,p}(x,\nu_-)\Big)d\mathcal{H}^{n-1} + \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)Id - \big(\nabla u\big)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big)\Big) : \nabla\xi \; dx = \lambda \int_\Omega c\,\nabla\cdot\xi dx$$

for all $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi\cdot\nu_\Omega = 0$ on $\partial\Omega$, where ν_Ω is the outer unit normal of $\partial\Omega$.

Proof:
The goal is to pass to the limit in the first variation formula (4.6) for the phase field energy. First we notice that as in the proof of Theorem 3.2, cf. [17, 18], we can extract even stronger convergence properties than in (5.4) and (5.5). More precisely, we have

$$\begin{aligned} c_{\varepsilon_k} &\to c && \text{in } L^2(\Omega), \\ u_{\varepsilon_k} &\to u && \text{in } H^1(\Omega,\mathbb{R}^n), \\ E_{\varepsilon_k}(c_{\varepsilon_k}, u_{\varepsilon_k}) &\to E_0(c_0, u_0). && \end{aligned}$$

In addition, (c,u) is a global minimizer of E_0 and $c \in \{c_-, c_+\}$ a. e.. We divide the proof into several steps.
Claim 1

$$\int_\Omega \Big(W(c_{\varepsilon_k}, \mathcal{E}(u_{\varepsilon_k}))Id - (\nabla u_{\varepsilon_k})^T W_{,\mathcal{E}}(c_{\varepsilon_k}, \mathcal{E}(u_{\varepsilon_k}))\Big) : \nabla\xi \to \int_\Omega \Big(W(c,\mathcal{E}(u))Id - (\nabla u)^T W_{,\mathcal{E}}(c,\mathcal{E}(u))\Big) : \nabla\xi.$$

Proof of Claim 1:
The proof of the claim can be found in [17, 18]. The key for this convergence result is the strong convergence of $\{\nabla u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ in $L^2(\Omega, \mathbb{R}^n \times \mathbb{R}^n)$. With this property, the convergence of $\{c_{\varepsilon_k}\}_{k\in\mathbb{N}}$ in $L^2(\Omega)$, the growth conditions on W and $W_{,\mathcal{E}}$ one can pass to the limit.

In the following step we show

Claim 2

$$\int_\Omega \Big(\varepsilon_k\sigma^2(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})+\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})\Big)\nabla\cdot\xi dx \to \int_\Omega \sigma_0(x,\nu_-)\nabla\cdot\xi\,|\nabla\chi_-|.$$

Proof of Claim 2:

Here, the first aim is to prove that

$$\mu_k := 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k}) \quad\rightarrow\quad \sigma_0(x,\nu_-)|\nabla\chi_-| \quad \text{weakly}^* \text{ in } \Omega.$$

In order to verify this assertion we show that μ_k is lower semicontinuous on each open set $U\subseteq\Omega$ and upper semicontinuous on each compact set $K\subset\Omega$.

Applying the coarea formula, Fatou's lemma and (ii) of Lemma 5.4 shows that μ_k is lower semicontinuous on each open set $U\subseteq\Omega$, i. e.

$$\begin{aligned}
&\liminf_{k\to\infty}\int_U 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})dx\\
&\quad=\liminf_{k\to\infty}\int_{-\infty}^{\infty}2\sqrt{\psi(s)}\int_U\sigma(x,s,\nu_{s,k})|\nabla\chi_{s,k}|ds\\
&\quad\geq 2\int_{c_-}^{c_+}\sqrt{\psi(s)}\liminf_{k\to\infty}\int_U\sigma(x,s,\nu_{s,k})|\nabla\chi_{s,k}|ds\\
&\quad\geq 2\int_{c_-}^{c_+}\sqrt{\psi(s)}\int_U\sigma(x,s,\nu_-)|\nabla\chi_-|ds=\int_U\sigma_0(x,\nu_-)\,|\nabla\chi_-|.
\end{aligned}\tag{5.6}$$

Note, by Theorem 3.2 we have

$$\begin{aligned}
\lim_{k\to\infty}\int_\Omega\Big(\varepsilon_k\,\sigma^2(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})+\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})\Big)dx &= \lim_{k\to\infty}\int_\Omega 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})dx\\
&=\int_\Omega\sigma_0(x,\nu_-)\,|\nabla\chi_-|.
\end{aligned}\tag{5.7}$$

Hence the upper semicontinuity for compact sets $K\subset\Omega$ follows from equation (5.7) and the lower semicontinuity property on open sets:

$$\begin{aligned}
&\limsup_{k\to\infty}\int_K 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\,dx\\
&\quad\leq\limsup_{k\to\infty}\int_\Omega 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\,dx-\liminf_{k\to\infty}\int_{\Omega\setminus K}2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\,dx\\
&\quad\leq\int_\Omega\sigma_0(x,\nu_-)\,|\nabla\chi_-|-\int_{\Omega\setminus K}\sigma_0(x,\nu_-)\,|\nabla\chi_-|\\
&\quad=\int_K\sigma_0(x,\nu_-)\,|\nabla\chi_-|.
\end{aligned}$$

Thus $\mu_k\to\sigma_0(x,\nu_-)|\nabla\chi_-|$ weakly* in Ω. Due to (5.7) and Young's inequality we obtain

$$\lim_{k\to\infty}\int_\Omega\Big|\varepsilon_k\,\sigma^2(x,c,\nabla c_{\varepsilon_k})+\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})-2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c,\nabla c_{\varepsilon_k})\Big|dx=0.\tag{5.8}$$

Consequently, by the Reshetnyak continuity theorem, see [5] Theorem 2.39,

$$\lim_{k\to\infty}\int_\Omega\Big(\varepsilon_k\,\sigma^2(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})+\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})\Big)\nabla\cdot\xi\,dx$$
$$=\lim_{k\to\infty}\int_\Omega 2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\nabla\cdot\xi\,dx=\int_\Omega\sigma_0(x,\nu_-)\nabla\cdot\xi\,|\nabla\chi_-|$$

and the claim follows.

In the next step we verify
Claim 3

$$\lim_{k\to\infty}\int_\Omega 2\varepsilon_k\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\nabla c_{\varepsilon_k}\cdot\nabla\xi\,\sigma_{,p}(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\,dx=\int_I\nu_-\cdot\nabla\xi\,\sigma_{0,p}\big(x,\nu_-\big),$$

Proof of Claim 3:
From equation (5.8) we deduce

$$0=\lim_{k\to\infty}\int_\Omega\Big|\varepsilon_k\,\sigma^2(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})+\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})-2\sqrt{\psi(c_{\varepsilon_k})}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\Big|dx$$
$$=\lim_{k\to\infty}\int_\Omega\left(\sqrt{\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})}-\sqrt{\varepsilon_k}\sigma\left(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k}\right)\right)^2dx.$$

This leads to

$$\lim_{k\to\infty}\int_\Omega\Big|\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})-\varepsilon_k\,\sigma^2(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\Big|dx$$
$$=\lim_{k\to\infty}\int_\Omega\left(\sqrt{\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})}-\sqrt{\varepsilon_k}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\right)\left(\sqrt{\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})}+\sqrt{\varepsilon_k}\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\right)dx$$
$$=0$$

since $\int_\Omega\big(\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k})+\varepsilon_k\,\sigma^2\left(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k}\right)\big)dx$ is uniformly bounded. Further, by the uniform boundedness of σ on $\Omega\times\mathbb{R}\times\mathbb{S}^{n-1}$ we derive

$$\lim_{k\to\infty}\int_\Omega\Big|\sqrt{\psi(c_{\varepsilon_k})}-\varepsilon_k\,\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\Big||\nabla c_{\varepsilon_k}|dx=0.\tag{5.9}$$

Due to (5.9) and the coarea formula we obtain

$$\lim_{k\to\infty}\int_\Omega\varepsilon_k\sigma(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})\nabla c_{\varepsilon_k}\cdot\nabla\xi\sigma_{,p}(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})dx$$
$$=\lim_{k\to\infty}\int_\Omega\sqrt{\psi(c_{\varepsilon_k})}\frac{\nabla c_{\varepsilon_k}}{|\nabla c_{\varepsilon_k}|}\cdot\nabla\xi\sigma_{,p}(x,c_{\varepsilon_k},\nabla c_{\varepsilon_k})|\nabla c_{\varepsilon_k}|\,dx$$
$$=\lim_{k\to\infty}\int_{\mathbb{R}}\sqrt{\psi(s)}\bigg(\int_\Omega\nu_{s,k}\cdot\nabla\xi\sigma_{,p}(x,s,\nu_{s,k})|\nabla\chi_{s,k}|\bigg)\,ds.$$

According to Theorem 5.5 there exists a subsequence with

$$\lim_{j\to\infty}\int_\Omega\nu_{s,k_j}\cdot\nabla\xi\sigma_{,p}(x,s,\nu_{s,k_j})|\nabla\chi_{s,k_j}|=\int_\Omega\nu_-\cdot\nabla\xi\sigma_{,p}(x,s,\nu_-)|\nabla\chi_-|$$

for a. e. $s \in [c_-, c_+]$. Therefore, we finally derive from Lemma 5.4 (iii), (iv) and the generalized Lebesgue convergence theorem

$$\lim_{k\to\infty} \int_\Omega 2\varepsilon_k \sigma(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k}) \nabla c_{\varepsilon_k} \cdot \nabla\xi \, \sigma_{,p}(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\, dx = \int_I \nu_- \cdot \nabla \xi \;\sigma_{0,p}(x, \nu_-)\, d\mathcal{H}^{n-1}$$

as required.

Claim 4

$$\lim_{k\to\infty} \int_\Omega 2\varepsilon_k \sigma(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\, \xi \cdot \sigma_{,x}(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\, dx = \int_I \xi \cdot \sigma_{0,x}\big(x, \nu_-\big)\, dx\,.$$

Proof of Claim 4:
Using equation (5.9), the coarea formula and the fact that $\sigma_{,x}$ is one-homogeneous with respect to the third argument we obtain

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega \varepsilon_k \sigma(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\, \xi \cdot \sigma_{,x}(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\, dx \\
&\qquad\qquad = \lim_{k\to\infty} \int_\Omega \sqrt{\psi(c_{\varepsilon_k})}\, \xi \cdot \sigma_{,x}(x, c_{\varepsilon_k}, \nu_{c_{\varepsilon_k}})|\nabla c_{\varepsilon_k}|\, dx \\
&\qquad\qquad = \lim_{k\to\infty} \int_{\mathbb{R}} \sqrt{\psi(s)} \bigg(\int_\Omega \xi \cdot \sigma_{,x}(x, s, \nu_{s,k}) |\nabla \chi_{s,k}| \bigg)\, ds.
\end{aligned}$$

From Theorem 5.5 we conclude that there exists a subsequence with

$$\lim_{j\to\infty} \int_\Omega \xi \cdot \sigma_{,x}(x, s, \nu_{s,k_j}) |\nabla \chi_{s,k_j}| = \int_\Omega \xi \cdot \sigma_{,x}(x, s, \nu_-)|\nabla\chi_-|$$

for a. e. $s \in [c_-, c_+]$. Hence, due to Lemma 5.4 (iii), (iv) and the generalized Lebesgue convergence theorem,

$$\lim_{k\to\infty} \int_\Omega 2\varepsilon_k \sigma(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\, \xi \cdot \sigma_{,x}(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\, dx = \int_I \xi \cdot \sigma_{0,x}(x, \nu_-)\, d\mathcal{H}^{n-1}.$$

We still have to prove convergence of the Lagrange multipliers λ_{ε_k}. Here, we take $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$ such that

$$\int_\Omega c\, \nabla{\cdot}\xi\, dx > 0.$$

This is possible since $c \in \{c_-, c_+\}$ a. e. and $⨍_\Omega c\, dx \in (c_-, c_+)$. Now the convergence of the Lagrange multipliers follows from the convergence of the left hand side in (4.6) and the fact that

$$\lim_{\varepsilon_k\to 0} \int_\Omega c_{\varepsilon_k}\, \nabla{\cdot}\xi\, dx = \int_\Omega c\, \nabla{\cdot}\xi\, dx > 0.$$

■

6 Strong formulation of the Euler–Lagrange equation for E_0

In this section we determine necessary conditions in the bulk, on the interface and on the boundary of the interface for minimizers of E_0 which satisfy certain regularity properties. These properties are chosen in order to apply the divergence theorem on manifolds and to ensure the existence of traces of ∇u on $\partial^*\Omega_l$ that lie in $L^2(\partial^*\Omega_l, \mathbb{R}^n)$, $l \in \{-,+\}$.

Theorem 6.1
Let Ω be a domain with C^1–boundary and let $W \in C^2$. Assume $(c,u) \in BV(\Omega, \{c_-, c_+\}) \times X_{\text{ird}}^{\perp}$ is a minimizer of E_0 such that

(i) $I = \partial^*\Omega_-$ is a C^2–hypersurface.

(ii) ∂I consists of a finite number of C^1–(n–2)–dimensional surfaces.

(iii) $u\,|_{\Omega_-} \in H^2(\Omega_-, \mathbb{R}^n)$ and $u\,|_{\Omega_+} \in H^2(\Omega_+, \mathbb{R}^n)$.

Then the following conditions are satisfied:

(i) In Ω:

$$\nabla \cdot W_{,\mathcal{E}}(c_-, \mathcal{E}(u)) = 0 \quad \text{in } \Omega_- \text{ a.e.} \quad \text{and} \quad \nabla \cdot W_{,\mathcal{E}}(c_+, \mathcal{E}(u)) = 0 \quad \text{in } \Omega_+ \text{ a.e.}.$$

(ii) On $\partial\Omega$:

$$W_{,\mathcal{E}}\nu_\Omega = 0 \quad \text{on } \partial\Omega \;\; \mathcal{H}^{(n-1)}\text{–a.e.}.$$

(iii) On the interface I:

- $[W_{,\mathcal{E}}\nu_-]_-^+ = 0 \quad \text{on } I \;\; \mathcal{H}^{(n-1)}\text{–a.e.}, \qquad [u]_-^+ = 0 \quad \text{on } I \;\; \mathcal{H}^{(n-1)}\text{–a.e.}.$
- *anisotropic, inhomogeneous Gibbs–Thomson law:*
  $-\sigma_{0,x} \cdot \nu_- - \nabla_I \cdot \sigma_{0,p}(x, \nu_-) + \nu_-[W Id - (\nabla u)^T W_{,\mathcal{E}}]_-^+ \nu_- = \lambda [c]_-^+ \quad$ *on* $I \;\; \mathcal{H}^{(n-1)}$–a.e.,
  where ∇_I denotes the tangential gradient of I and λ is the constant Lagrange multiplier of equation (4.2).

(iv) On $\partial I \cap \partial\Omega$ the boundary condition $\sigma_{0,p} \cdot \nu_\Omega = 0$ holds.

Proof:
To (i), (ii) and (iii): The first variation with respect to u gives

$$\begin{aligned}\int_\Omega W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big) : \nabla\theta \, dx = &- \int_{\Omega_-} \nabla \cdot W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big) \cdot \theta dx - \int_{\Omega_+} \nabla \cdot W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big) \cdot \theta \, dx \\ &+ \int_{\partial\Omega} \theta \cdot W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big)\nu_\Omega \, d\mathcal{H}^{n-1} - \int_I \theta \cdot \Big[W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big)\Big]_-^+ \nu_- \, d\mathcal{H}^{n-1} = 0\end{aligned}$$

for all $\theta \in H^1(\Omega, \mathbb{R}^n)$. This implies

$$\nabla \cdot W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big) = 0 \quad \text{in } \Omega_l \text{ a.e.}, \quad l \in \{-,+\},$$

and

$$\Big[W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big)\Big]_-^+ \nu_- = 0 \quad \text{on } I \;\; \mathcal{H}^{n-1}\text{–a.e.} \quad \text{and} \quad W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big)\nu_\Omega = 0 \quad \text{on } \partial\Omega \;\; \mathcal{H}^{n-1}\text{–a.e.}.$$

The condition $[u]_-^+ = 0$ on I $\mathcal{H}^{n-1}$–a. e. follows since $u \in H^1(\Omega, \mathbb{R}^n)$.
Next we deduce the Gibbs–Thomson equation. We consider equation (4.2) and choose test functions which are of the form $\xi = \eta\nu_-$ on I, where η is an arbitrary function of $C_c^1(\Omega, \mathbb{R})$. For the first and third summand of the area part of equation (4.2) we compute

$$\int_I \nu_- \cdot \nabla\xi \, \sigma_{0,p}(x, \nu_-) \, d\mathcal{H}^{n-1} = \int_I \nabla\eta \cdot \sigma_{0,p}(x, \nu_-) \, d\mathcal{H}^{n-1}$$

and

$$\begin{aligned}\int_I \sigma_0(x, \nu_-) \nabla\cdot\xi \, d\mathcal{H}^{n-1} &= \int_I \nu_- \cdot \left(\nabla\eta \cdot \nu_- + \eta \, \nabla\cdot\nu_- \right) \sigma_{0,p}(x, \nu_-) \, d\mathcal{H}^{n-1} \\ &= \int_I \left(\nabla\eta - \nabla_I\eta\right) \cdot \sigma_{0,p}(x, \nu_-) \, d\mathcal{H}^{n-1} + \int_I \eta \, \kappa \left(\sigma_{0,p}(x, \nu_-) \cdot \nu_-\right) d\mathcal{H}^{n-1},\end{aligned}$$

where $\kappa = \nabla_I\cdot\nu_-$ is the mean curvature. The divergence theorem on manifolds gives

$$\begin{aligned}\int_I \nabla_I\eta \cdot \sigma_{0,p}(x, \nu_-) \, d\mathcal{H}^{n-1} + \int_I \eta \, \nabla_I\cdot\sigma_{0,p}(x, \nu_-) \, d\mathcal{H}^{n-1} &= \int_I \nabla_I\cdot\left(\eta \, \sigma_{0,p}(x, \nu_-)\right) d\mathcal{H}^{n-1} \\ &= \int_I \kappa \, \eta \left(\sigma_{0,p}(x, \nu_-) \cdot \nu_-\right) d\mathcal{H}^{n-1}.\end{aligned}$$

In consequence,

$$\int_I \Big(\sigma_0\big(x, \nu_-\big) \nabla\cdot\xi - \nu_- \cdot \nabla \, \xi \sigma_{0,p}\big(x, \nu_-\big)\Big) \, d\mathcal{H}^{n-1} = \int_I \eta \, \nabla_I\cdot\sigma_{0,p}(x, \nu_-) \, d\mathcal{H}^{n-1}.$$

Now we evaluate the elastic part of equation (4.2). Since $W_{,\mathcal{E}}$ is symmetric in the second variable we obtain

$$\nabla \cdot \left(WId - (\nabla u)^T W_{,\mathcal{E}}\right) = \nabla W - \left(\partial_i \nabla u : W_{,\mathcal{E}}\right)_{i=1,\dots,n} - \left(\nabla u\right)^T \nabla\cdot W_{,\mathcal{E}} = 0$$

in Ω_l a. e., $l \in \{-, +\}$. Hence

$$\begin{aligned}\int_{\Omega_l} \left(WId - (\nabla u)^T W_{,\mathcal{E}}\right) : \nabla\xi \, dx &= -\int_{\Omega_l} \Big(\nabla \cdot \left(WId - (\nabla u)^T W_{,\mathcal{E}}\right)\Big) \cdot \xi \, dx \\ &\qquad + \int_{\partial\Omega_l} \xi \cdot \left(WId - (\nabla u)^T W_{,\mathcal{E}}\right)\nu_l \, dx \\ &= \int_{\partial\Omega_l} \xi \cdot \left(WId - (\nabla u)^T W_{,\mathcal{E}}\right)\nu_l \, dx\end{aligned} \tag{6.1}$$

for $l \in \{+, -\}$, where ν_l is the outer normal of $\partial\Omega_l$. In consequence, equation (4.2) takes the form

$$\begin{aligned}\int_I \eta \, \Big(\sigma_{0,x}(x, \nu_-) \cdot \nu_- + \nabla_I\cdot\sigma_{0,p}(x, \nu_-) - \nu_- \cdot \Big[W\big(c, \mathcal{E}(u)\big)Id - \big(\nabla u\big)^T W_{,\mathcal{E}}\big(c, \mathcal{E}(u)\big)\Big]_-^+ \nu_-\Big) \, d\mathcal{H}^{n-1} \\ = -\lambda(c_+ - c_-) \int_I \eta \, d\mathcal{H}^{n-1}.\end{aligned}$$

Since $\eta \in C_c^1(\Omega)$ was arbitrary we end up with

$$-\sigma_{0,x}\big(x, \nu_-\big) \cdot \nu_- - \nabla_I\cdot\sigma_{0,p}(x, \nu_-) + \nu_-[WId - (\nabla u)^T W_{,\mathcal{E}}]_-^+ \nu_- = \lambda[c]_-^+ \quad \text{on } I \quad \mathcal{H}^{(n-1)}\text{–a. e..}$$

To (iv): We take arbitrary functions $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$ and choose an orthonormal basis $\tau_1 = \tau_I, \tau_2, \dots, \tau_{n-1}$ of the tangent space $T\,I$, where τ_I is the outer unit normal of ∂I. Then, using the Einstein sum convention, ξ can be written in the form $\xi = \eta_\nu \nu_- + \eta_{\tau_j}\tau_j$. We compute

$$\begin{aligned}\int_I \sigma_0(x,\nu_-)\nabla\cdot\big(\eta_{\tau_j}\tau_j\big)\,d\mathcal{H}^{n-1} = \int_{\partial I} \sigma_0(x,\nu_-)\eta_{\tau_I}\,d\mathcal{H}^{n-2} - \int_I \nabla_I\sigma_0(x,\nu_-)\cdot\eta_{\tau_j}\tau_j\,d\mathcal{H}^{n-1}&\\ + \int_I \sigma_0(x,\nu_-)\eta_{\tau_j}\nu_-\nabla\tau_j\nu_-\,d\mathcal{H}^{n-1}.&\end{aligned}$$

Since $\big(\nabla(\eta_{\tau_j}\tau_j)\big)^T\nu_- = -(\nabla\nu_-)^T\big(\eta_{\tau_j}\tau_j\big)$ we obtain

$$\int_I \nu_-\cdot\nabla\big(\eta_{\tau_j}\tau_j\big)\sigma_{0,p}(x,\nu_-)\,d\mathcal{H}^{n-1} = -\int_I \big(\eta_{\tau_j}\tau_j\big)\cdot\nabla\nu_-\sigma_{0,p}(x,\nu_-)\,d\mathcal{H}^{n-1}.$$

Note, equation (6.1) is valid for arbitrary test functions $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi\cdot\nu_\Omega = 0$ on $\partial\Omega$. Thus we get for (4.2) the following representation

$$\begin{aligned}&\int_I \Big(\sigma_0\big(x,\nu_-\big)\nabla\cdot\xi + \sigma_{0,x}\big(x,\nu_-\big)\cdot\xi - \nu_-\cdot\nabla\xi\,\sigma_{0,p}\big(x,\nu_-\big)\Big)d\mathcal{H}^{n-1}\\ &\qquad\qquad + \int_\Omega \Big(W\big(c,\mathcal{E}(u)\big)Id - \big(\nabla u\big)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big)\Big) : \nabla\xi\ dx\\ &= \int_{\partial I}\Big(- \eta_\nu\sigma_{0,p}(x,\nu_-)\cdot\tau_I + \sigma_0(x,\nu_-)\ \eta_{\tau_I}\Big)\,d\mathcal{H}^{n-2} + \int_I \eta_\nu\nabla_I\cdot\sigma_{0,p}(x,\nu_-)\,d\mathcal{H}^{n-1}\\ &\quad - \int_I\Big(\big(\nabla_I\sigma_0(x,\nu_-) - \nabla\nu_-\sigma_{0,p}(x,\nu_-)\big)\cdot\big(\eta_{\tau_j}\tau_j\big) + \sigma_0(x,\nu_-)\eta_{\tau_j}\nu_-\nabla\tau_j\nu_-\Big)\,d\mathcal{H}^{n-1}\\ &\quad + \int_I \sigma_{0,x}\big(x,\nu_-\big)\cdot\xi\,d\mathcal{H}^{n-1} - \int_I \xi\cdot\Big[W\big(c,\mathcal{E}(u)\big)Id - \big(\nabla u\big)^T W_{,\mathcal{E}}\big(c,\mathcal{E}(u)\big)\Big]_-^+\nu_-\,d\mathcal{H}^{n-1}\\ &= -\lambda(c_+ - c_-)\int_I \eta_\nu\,d\mathcal{H}^{n-1}.\end{aligned}$$

Since

$$\begin{aligned}&\int_{\partial I}\Big(\sigma_0(x,\nu_-)\ \eta_{\tau_I} - \eta_\nu\sigma_{0,p}(x,\nu_-)\cdot\tau_I\Big)\,d\mathcal{H}^{n-2}\\ &\qquad\qquad = \int_{\partial I}\xi\Big(\big(\sigma_{0,p}(x,\nu_-)\cdot\nu_-\big)\tau_I - \big(\sigma_{0,p}(x,\nu_-)\cdot\tau_I\big)\nu_-\Big)\,d\mathcal{H}^{n-2}\end{aligned}$$

we derive by choosing suitable variations in the neighborhood of points of ∂I

$$(\sigma_{0,p}(x,\nu_-)\cdot\nu_-)\tau_I - (\sigma_{0,p}(x,\nu_-)\cdot\tau_I)\,\nu_- = l\,\nu_\Omega$$

with $l = |(\sigma_{0,p}(x,\nu_-)\cdot\nu_-)\tau_I - (\sigma_{0,p}(x,\nu_-)\cdot\tau_I)\,\nu_-|$. In consequence,

$$l\,\nu_\Omega\cdot\tau_I = \sigma_{0,p}(x,\nu_-)\cdot\nu_-,\qquad l\,\nu_\Omega\cdot\nu_- = -\sigma_{0,p}(x,\nu_-)\cdot\tau_I,\qquad \nu_\Omega\cdot\tau_j = 0\quad \text{for } j\in\{2,\dots,n-1\}.$$

This yields

$$\begin{aligned}\sigma_{0,p}(x,\nu_-)\cdot\nu_\Omega &= (\sigma_{0,p}(x,\nu_-)\cdot\nu_-)(\nu_-\cdot\nu_\Omega) + (\sigma_{0,p}(x,\nu_-)\cdot\tau_j)\,(\tau_j\cdot\nu_\Omega)\\ &= \big(- (\sigma_{0,p}(x,\nu_-)\cdot\nu_-)(\sigma_{0,p}(x,\nu_-)\cdot\tau_I) + (\sigma_{0,p}(x,\nu_-)\cdot\tau_I)\,(\sigma_{0,p}(x,\nu_-)\cdot\nu_-)\big)/l\\ &= 0.\end{aligned}$$

■

We like to mention that the dependence of σ on x has no influence on the boundary condition at intersections of the interface with the outer boundary.

7 References

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