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Analytical and numerical aspects of time-dependent problems with internal variables

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Abstract

In this paper some analytical and numerical aspects of time-dependent models with internal variables are discussed. The focus lies on elasto/visco-plastic models of monotone type arising in the theory of inelastic behavior of materials. This class of problems includes the classical models of elasto-plasticity with hardening and viscous models of the Norton-Hoff type. We discuss the existence theory for different models of monotone type, give an overview on spatial regularity results for solutions to such models and illustrate a numerical solution algorithm at an example. Finally, the relation to the energetic formulation for rate-independent processes is explained and temporal regularity results based on different convexity assumptions are presented.

1 Introduction

In metallic materials various phenomena on the microscale induce macroscopically inelastic behavior: The hindering of the dislocation motion by other dislocations or grain boundaries cause hardening effects, which are observed on the macroscopic scale. The nucleation and growth of grain boundary cavities initiate the development of microcracks which may cause the failure the whole structure.

From the phenomenological point of view the macroscopic state of inelastic bodies is completely determined by the displacement or deformation field, the stress tensor and a finite number of internal variables representing internal processes on the microscale. The corresponding macroscopic models consist of the balance of forces, an evolution law for the internal variables and constitutive equations which relate the stresses with the displacement gradient and the internal variables. A thermodynamically consistent framework for such models is the class of generalized standard materials defined by Halphen and Nguyen Son and the more general class of models of monotone type introduced by Alber. From the mathematical point of view these models lead to coupled systems of linear hyperbolic/elliptic partial differential equations and nonlinear ordinary differential equations/inclusions. A typical application of such models is elasto(visco)-plasticity with hardening at small strains. In the rate-independent case an alternative energetic formulation for such models was proposed by Mielke et al. in the last years. This formulation provides a general tool to rigorously analyze effects like damage, fracture or hysteretic behavior in magnetic and ferroelectric bodies at both, small and finite strains. The aim of this paper is to review some recent analytical and numerical aspects of models of this type.

The starting point for the models discussed in this paper is the following: Given a time interval $[0,T]$ and a state space $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ let $u : [0,T] \to \mathcal{U}$ denote the generalized displacements and $z : [0,T] \to \mathcal{Z}$ the internal variables. It is assumed that $\mathcal{U}$ and $\mathcal{Z}$ are real, separable and reflexive Banach spaces. In the applications of plasticity, typical choices are $\mathcal{Z} = L^p(\Omega)$ and $\mathcal{U}$ is identified with a suitable subspace of the Sobolev space $W^{1,p}(\Omega)$. The set $\Omega \subset \mathbb{R}^d$ describes the physical body. In the first chapters of this presentation the associated elastic energy $\Psi : \mathcal{Q} \to \mathbb{R}$ is assumed to be quadratic and positive semidefinite, i.e. we have

$$\Psi(u,z) = \frac{1}{2}\langle \mathcal{A} \left(\begin{smallmatrix} u \\ z \end{smallmatrix} \right), \left(\begin{smallmatrix} u \\ z \end{smallmatrix} \right) \rangle$$

where $\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \mathcal{Q} = \mathcal{U} \times \mathcal{Z} \to \mathcal{Q}^*$ is a linear, bounded symmetric and positive semidefinite operator. In addition to the elastic energy Ψ we also consider the energy

$$\mathcal{E}(t, u, z) = \Psi(u, z) - \langle b(t), u \rangle$$

for given external loadings $b \in C^1([0, T]; \mathcal{U}^*)$. The evolution law for the internal variable z is characterized by a monotone, multivalued mapping $\mathcal{G} : \mathcal{Z} \to \mathcal{P}(\mathcal{Z}^*)$ with the property $0 \in \mathcal{G}(0)$. Thereby $\mathcal{U}^*$, $\mathcal{Z}^*$ and $\mathcal{Q}^*$ are the duals of the Banach spaces $\mathcal{U}$, $\mathcal{Z}$ and $\mathcal{Q}$ respectively and $\mathcal{P}(\mathcal{Z}^*)$ denotes the power set of $\mathcal{Z}^*$. The assumptions on $\mathcal{E}$ and $\mathcal{G}$ are motivated by thermodynamical considerations which are carried out in Section 2.1. There also the link to elasto-plasticity is explained more detailed. The evolution model associated with $\mathcal{E}$ and $\mathcal{G}$ consists of the force balance equation (1.1) which is coupled with the evolution law (1.2) for the internal variable: Find absolutely continuous functions $u \in AC([0, T]; \mathcal{U})$ and $z \in AC([0, T]; \mathcal{Z})$ with $z(0) = z_0 \in \mathcal{Z}$ such that for almost every $t \in [0, T]$ it holds

$$0 = \partial_u \mathcal{E}(t, u(t), z(t)) = A_{11} u(t) + A_{12} z(t) - b(t), \tag{1.1}$$

$$\partial_t z(t) \in \mathcal{G}(-\partial_z \mathcal{E}(t, u(t), z(t)) = \mathcal{G}(-(A_{21} u(t) + A_{22} z(t))). \tag{1.2}$$

Systems of this structure constitute the *class of models of monotone type* introduced by Alber [1]. The subclass of generalized standard materials is obtained if in addition to the above it is assumed that $\mathcal{G}$ is the convex subdifferential of a convex and proper function. The particular choice $\mathcal{G} = \partial \chi_{\mathcal{K}}$, where $0 \in \mathcal{K} \subset \mathcal{Z}$ is convex and closed, and where $\chi_{\mathcal{K}}$ denotes the characteristic function related to $\mathcal{K}$, finally leads to the subclass of rate-independent evolution models. Typical examples for these classes of models are elasto-plasticity in the small strain setting comprising for example linear kinematic hardening. An example for a rate-dependent model is the visco-plastic Norton-Hoff model.

The mathematical analysis of rate-independent elasto-plastic models has its roots in the fundamental contributions by Moreau, Duvaut/Lions and Johnson, [32, 53, 78]. More recent investigations, which also cover rate-dependent models, are due to Alber/Chelminski [2], see also [47]. If $\mathcal{A}$ and hence Ψ are positive definite, i.e. if $\Psi(u, z) \geq \frac{\alpha}{2}(\|u\|_{\mathcal{U}}^2 + \|z\|_{\mathcal{Z}}^2)$ for all $(u, z) \in \mathcal{Q}$, and if in addition $\mathcal{G}$ is maximal monotone, then classical results state the existence of a unique solution $(u, z) \in AC([0, T]; \mathcal{Q})$ for sufficiently regular given data b and z_0, which satisfy a certain compatibility condition.

In contrast to the positive definite case it is quite challenging to prove existence results for (1.1)–(1.2) if $\mathcal{A}$ is positive semidefinite, only. Typical examples for such models are the elastic-perfectly plastic Prandtl-Reuss model and models with linear isotropic hardening and we refer to [53, 28, 47, 23] for the discussion of existence questions. In Section 2.5 we present an existence proof for a model with a positive semi-definite energy Ψ under the assumption that a certain coupling condition is satisfied between the operators A_{12} and A_{22}. Here, we study the solvability for $u \in L^q(S; W^{1,q}(\Omega))$ and $z \in AC(S; L^q(\Omega))$ for suitable $q \in (1, \infty)$.

Apart from existence results it is of great interest to gain more insight into the qualitative properties of solutions, such as spatial or temporal regularity and stability. This knowledge is the basis for the construction of efficient and robust numerical algorithms. Section 3 is devoted to the discussion of spatial regularity results for solutions of models of monotone type. Depending on the positivity properties of the free energy Ψ different regularity results may be achieved.

In the positive semi-definite case one typically obtains the spatial regularity

$$\sigma \in L^\infty((0, T); H^1_{\mathrm{loc}}(\Omega))$$

for the stress tensor σ. The basic observation enabling this result is the fact that the complementary energy, which is the convex conjugate of the free energy, is positive definite with respect to the generalized stresses, although the energy Ψ might not be positive definite. In addition to the semidefinite case, for positive definite energies the following global spatial regularity results are available for domains with smooth boundary: For every $\delta > 0$ it holds

$$u \in L^\infty((0,T); H^{\frac{3}{2}-\delta}(\Omega)) \cap L^\infty((0,T); H^2_{\mathrm{loc}}(\Omega)), \tag{1.3}$$

$$\sigma, z \in L^\infty((0,T); H^{\frac{1}{2}-\delta}(\Omega)) \cap L^\infty((0,T); H^1_{\mathrm{loc}}(\Omega)). \tag{1.4}$$

The proof of the global results relies on stability estimates for the solutions of (1.1)–(1.2) and a reflection argument. A discussion concerning the optimality of (1.3)–(1.4) as well as an overview of the related literature is provided in Sections 3.2 and 3.3. Moreover, we discuss an example which shows that in spite of smooth data and a smooth geometry one should not expect a comparable spatial regularity result for the time derivatives $\partial_t u$ and $\partial_t z$.

In Section 4 we discuss and analyze a numerical algorithm for solving rate-independent elasto-plastic models. After a time discretization with an implicit Euler scheme the time incremental problem can be reformulated as a quasilinear elliptic system of partial differential equations to determine the displacements at time step t_k from the displacements and internal variables of the previous time step. The internal variable of the current time step then can be calculated via a straightforward update formula. Since the nonlinear elliptic operator is not Gâteaux-differentiable, classical Newton methods are not applicable for solving the PDE. Instead we discuss an approach where we use a so-called *slanting function* instead of the derivative resulting in a Slant Newton Method. The behavior of this algorithm is illustrated at some examples.

In the last section, Section 5, we focus on rate-independent models of the type (1.1)-(1.2) with $\mathcal{G} = \partial\chi_{\mathcal{K}}$. As already mentioned, in this case the model (1.1)–(1.2) can be reformulated in the global energetic framework for rate-independent evolution processes introduced by Mielke and Theil [70]. Indeed we will show in Section 5 that the model is equivalent to the following problem: Find a pair $(u, z) : [0, T] \to \mathcal{Q}$ with $(u(0), z(0)) = (u_0, z_0)$ which for every $t \in [0, T]$ satisfies stability relation (S) and the energy balance (E)

(S) for every $(v, \zeta) \in \mathcal{Q}$ we have $\mathcal{E}(t, u(t), z(t)) \le \mathcal{E}(t, v, \zeta) + \mathcal{R}(\zeta - z(t))$,

(E) $\mathcal{E}(t, u(t), z(t)) + \int_0^t \mathcal{R}(\partial_t z(\tau))\mathrm{d}\tau = \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_t\mathcal{E}(\tau, u(\tau), z(\tau))\mathrm{d}\tau$,

where $\mathcal{R} : \mathcal{Z} \to [0, \infty]$ is the convex conjugate of the characteristic function $\chi_{\mathcal{K}}$ and hence is convex and positively homogeneous of degree one.

The energetic framework allows for more general energies $\mathcal{E}$, which not necessarily have a quadratic structure or strict convexity properties, or which might not be Gâteaux differentiable with respect to u or z. The energetic formulation of rate-independent processes provides a general tool, which also applies to further physical phenomena like damage, fracture, shape memory effects or ferroelectric behavior. Since the energy $\mathcal{E}$ is not necessarily strictly convex, solutions may occur which are discontinuous in time. A general existence theorem is cited. Subsequent it is investigated to what extend different convexity assumptions on the energy yield solutions which are continuous, Hölder-continuous or even Lipschitz-continuous in time. These convexity assumptions are discussed for different examples modeling elasto-plasticity, shape memory effects and damage.

2 Elasto(visco)-plastic models of monotone type

2.1 Thermodynamic framework

In this subsection we show that the problem (1.1) - (1.2) is thermodynamically admissible. We start with a macroscopic model describing inelastic response of solids at small strains in the most general form, and then we extract a subclass of models, for which the Clausius-Duhem inequality is naturally satisfied. This subclass of models consists of problems of the type (1.1) - (1.2).

Setting of the problem

For the subsequent analysis we restrict ourselves only to the 3-dimensional case, although all of our results hold in any space-dimension. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $\mathcal{S}^3$ be the linear space of symmetric 3×3-matrices. Let T_e denote a positive number (time of existence). For $0 \le t \le T_e$ we introduce the space-time cylinder $\Omega_t = \Omega \times (0,t)$.

The initial boundary value problem for the unknown displacement $u(x,t) \in \mathbb{R}^3$, the Cauchy stress tensor $T(x,t) \in \mathcal{S}^3$ and the vector of internal variables $z(x,t) \in \mathbb{R}^N$ in a quasi-static setting is formed by the equations

$$
\begin{aligned}
-\operatorname{div}_x T(x,t) &= b(x,t), && (2.1)\\
T(x,t) &= A(\varepsilon(\nabla_x u(x,t)) - Bz(x,t)), && (2.2)\\
\frac{\partial}{\partial t} z(x,t) &\in f(\varepsilon(\nabla_x u(x,t)), z(x,t)), && (2.3)
\end{aligned}
$$

which must hold for all $x \in \Omega$ and all $t \in [0,\infty)$. The initial value for $z(x,t)$ and the Dirichlet boundary condition for $u(x,t)$ are given by

$$
\begin{aligned}
z(x,0) &= z^{(0)}(x), \quad \text{for } x \in \Omega, && (2.4)\\
u(x,t) &= \gamma(x,t), \quad \text{for } (x,t) \in \partial\Omega \times [0,\infty). && (2.5)
\end{aligned}
$$

Here $\nabla_x u(x,t)$ denotes the 3×3-matrix of first order derivatives of u, the deformation gradient, $(\nabla_x u(x,t))^T$ denotes the transposed matrix, and

$$
\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}(\nabla_x u(x,t) + (\nabla_x u(x,t))^T) \in \mathcal{S}^3,
$$

is the strain tensor. The linear mapping $B : \mathbb{R}^N \mapsto \mathcal{S}^3$ is a projector with $\varepsilon_p(x,t) = Bz(x,t)$, where $\varepsilon_p \in \mathcal{S}^3$ is a plastic strain tensor. We denote by $A : \mathcal{S}^3 \to \mathcal{S}^3$ a linear, symmetric, positive definite mapping, the elasticity tensor. The given data of the problem are the volume force $b : \Omega \times [0,\infty) \mapsto \mathbb{R}^3$, the boundary displacement $\gamma : \partial\Omega \times [0,\infty) \mapsto \mathbb{R}^3$, and the initial data for the vector of the internal variables $z^{(0)} : \Omega \mapsto \mathbb{R}^N$. The given function $f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \mapsto 2^{\mathbb{R}^N}$ is a constitutive function with the domain $D(f)$.

The differential inclusion (2.3) with a prescribed function f together with the equation (2.2) define the material behavior. They are the constitutive relations which model the elasto(visco)-plastic behavior of solid materials at small strains, whereas (2.1) is the force balance arising from the conservation law of linear momentum.

The initial boundary value problem (2.1) - (2.5) is written here in the most general form and, to the best of our knowledge, includes all elasto(visco)-plastic models at small strains used in the engineering. To guarantee that by equations (2.1) - (2.5) a thermodynamically admissible process is described, we claim the existence of a free energy density $\psi : D(f) \to [0, \infty)$ such that the Clausius-Duhem inequality

$$\rho \frac{\partial}{\partial t} \psi(\varepsilon(\nabla_x u), z) - \mathrm{div}_x(T u_t) - b \cdot u_t \leq 0 \tag{2.6}$$

holds in $\Omega \times (0, \infty)$ for all solutions (u, T, z) of (2.1) - (2.5). The function ρ denotes the mass density and it is assumed to be constant. The requirement (2.6) restricts the possible choices of f. Indeed, let (u, z) be a sufficiently smooth solution of (2.1) - (2.6). Firstly, we note that the symmetry of the stress tensor implies

$$T \cdot \varepsilon(\nabla_x u_t) = T \cdot \nabla_x u_t \ = \mathrm{div}_x(T^T u_t) - (\mathrm{div}_x T) \cdot u_t.$$

Then, as a direct consequence of the Clausius-Duhem inequality (2.6), one gets with the help of the previous relation and the symmetry of T the following inequality

$$\begin{aligned}
&\rho \nabla_\varepsilon \psi \cdot \varepsilon(\nabla_x u_t) + \rho \nabla_z \psi \cdot z_t - \mathrm{div}_x(T u_t) - b \cdot u_t \\
&= \rho \nabla_\varepsilon \psi \cdot \varepsilon(\nabla_x u_t) + \rho \nabla_z \psi \cdot z_t - T \cdot \varepsilon(\nabla_x u_t) = (\rho \nabla_\varepsilon \psi - T) \cdot \varepsilon(\nabla_x u_t) + \rho \nabla_z \psi \cdot z_t \\
&\leq 0.
\end{aligned}$$

Due to the arbitrariness of the strain rate $\dot{\varepsilon} = \varepsilon(\nabla_x u_t)$, we conclude that

$$\rho \nabla_\varepsilon \psi(\varepsilon, z) = T, \tag{2.7}$$

$$\rho \nabla_z \psi(\varepsilon, z) \cdot \zeta \leq 0 \tag{2.8}$$

for every $\zeta \in f(\varepsilon, z)$ and for all $(\varepsilon, z) \in D(f)$. Inequality (2.8) is called the dissipation inequality. Therefore, we call the constitutive equations (2.2) and (2.3) thermodynamically admissible if a free energy density ψ exists such that (2.7) and (2.8) are satisfied.

Now it is easy to extract a subclass of constitutive functions f, for which the dissipation inequality (2.8) is naturally fulfilled. This subclass consists of those functions f, which can be written in the form

$$f(\varepsilon, z) = g(-\rho \nabla_z \psi(\varepsilon, z)), \tag{2.9}$$

with a suitable free energy density $\psi : D(f) \to [0, \infty)$ satisfying (2.7), and with a suitable monotone function $g : D(g) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ with the property $0 \in g(0)$.

Relations (2.2) and (2.7) allow us to find the precise form of the free energy density: Integrating (2.7) with respect to ε we can easily obtain that

$$\rho \psi(\varepsilon, z) = \frac{1}{2} A(\varepsilon - Bz) \cdot (\varepsilon - Bz) + \psi_1(z)$$

with a suitable function $\psi_1 : D(\psi_1) \subseteq \mathbb{R}^N \to [0, \infty)$ as a constant of integration. For mathematical reasons we assume in this chapter that the free energy density ψ has a special form, namely it is a positive semi-definite quadratic form given by

$$\rho \psi(\varepsilon, z) = \frac{1}{2} A(\varepsilon - Bz) \cdot (\varepsilon - Bz) + \frac{1}{2}(Lz) \cdot z \tag{2.10}$$

with a symmetric, non-negative $N \times N$-matrix L. Differentiating (2.10) with respect to z yields

$$-\rho \nabla_z \psi(\varepsilon, z) = B^T A(\varepsilon - Bz) - Lz = B^T T - Lz.$$

In view of these considerations the initial boundary value problem (2.1) - (2.5) can be written as

$$
\begin{aligned}
-\operatorname{div}_x T(x,t) &= b(x,t), && (2.11)\\
T(x,t) &= A(\varepsilon(\nabla_x u(x,t)) - Bz(x,t)), && (2.12)\\
\frac{\partial}{\partial t} z(x,t) &\in g(B^T T(x,t) - Lz(x,t)) && (2.13)\\
z(x,0) &= z^{(0)}(x), && (2.14)
\end{aligned}
$$

for all $x \in \Omega$ and all $t \in [0,\infty)$, together with the Dirichlet boundary condition

$$u(x,t) = \gamma(x,t) \quad \text{for } x \in \partial\Omega, t \in [0,\infty). \tag{2.15}$$

The initial boundary value problem (2.11) - (2.15) is called the problem/model of monotone type. As we have already mentioned in the introduction, this class of models was introduced by Alber in [1] and it naturally generalizes the class of problems of generalized standard materials proposed by Halphen and Nguyen Quoc Son. We recall that the models of generalized standard materials are formed by equations (2.11) - (2.15) with the monotone function g given explicitly by the subdifferential of a proper convex function. Typical examples for models of monotone type are elasto-plastic models with linear or nonlinear hardening (for more details, consult the book [1, Chapter 3.3]).

First existence results for the classical model of perfect plasticity (Prandtl-Reuss-model) were derived in [76, 32, 53]. Since the elastic energy in this case is positive semidefinite, only, the displacements in general belong to the space of bounded deformations, only, [102, 104, 105]. The existence theory for elasto-plastic models with a positive definite energy (like elasto-plasticity with linear kinematic hardening) was initiated by Johnson [54], we refer to the monographs [47, 39] for a historical survey on the subject. In the late 90ies these results were extended to models of monotone type with general maximal monotone functions g, still assuming that the energy is positive-definite, [1, 2]. In [3, 23, 22, 24, 82, 84, 85] an approach for the derivation of the existence of solutions to the problem (2.11) - (2.15) initiated in [1] was continued and extended to particular models of monotone type with a positive semi-definite energy. In the present paper, we briefly discuss the existence result in [2] for models with a positive definite energy in order to point out the main differences and difficulties which arise in the treatment of monotone problems with a positive semi-definite energy. An existence proof for a special class with a positive semi-definite energy is discussed afterwards.

2.2 Function spaces and notation

For $m \in \mathbb{N}$, $q \in [1,\infty]$, we denote by $W^{m,q}(\Omega,\mathbb{R}^k)$ the Banach space of Lebesgue integrable functions having q-integrable weak derivatives up to order m. This space is equipped with the norm $\|\cdot\|_{m,q,\Omega}$. If $m = 0$ we also write $\|\cdot\|_{q,\Omega}$. If m is not integer, then the corresponding Sobolev-Slobodeckij space is denoted by $W^{m,q}(\Omega,\mathbb{R}^k)$. We set $H^m(\Omega) = W^{m,2}(\Omega)$, cf. [42].

We choose the numbers p, q satisfying $1 < p, q < \infty$ and $1/p + 1/q = 1$. For such p and q one can define the bilinear form on the product space $L^p(\Omega,\mathbb{R}^k) \times L^q(\Omega,\mathbb{R}^k)$ by

$$(\xi,\zeta)_\Omega = \int_\Omega \xi(x)\cdot\zeta(x)dx.$$

If (X, H, X^*) is an evolution triple (known also as a "Gelfand triple" or "spaces in normal position"), then

$$W_{p,q}(0, T_e; X) = \{u \in L^p(0, T_e; X) \mid \dot{u} \in L^q(0, T_e; X^*)\}$$

is a separable reflexive Banach space furnished with the norm

$$\|u\|^2_{W_{p,q}} = \|u\|^2_{L^p(0,T_e;X)} + \|\dot{u}\|^2_{L^q(0,T_e;X^*)},$$

where the time derivative $\dot{u}$ of u is understood in the sense of vector-valued distributions. We recall that the embedding $W_{p,q}(0, T_e; X) \subset C([0, T_e], H)$ is continuous ([50, p. 4], for instance). Finally we frequently use the spaces $W^{k,p}(0, T_e; X)$, which consist of Bochner measurable functions with a p-integrable weak derivatives up to order k. Observe that $W_{2,2}(0, T_e; X) = W^{1,2}(0, T_e; X)$.

2.3 Basic properties of the operator of linear elasticity

Here, we state the assumptions on the coefficient matrices in (2.11) - (2.13):

$$\begin{aligned} &A \in L^\infty(\Omega, \mathrm{Lin}(\mathcal{S}^3, \mathcal{S}^3)) \text{ is symmetric and uniformly positive definite,}\\ &\text{i.e. there exists } \alpha > 0 \text{ such that } A(x)\varepsilon \cdot \varepsilon \geq \alpha \|\varepsilon\|^2 \text{ for all } \varepsilon \in \mathcal{S}^3 \text{ and a.e. } x \in \Omega,\\ &L \in L^\infty(\Omega; \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^N)) \text{ is symmetric and positive semi-definite.}\end{aligned} \tag{2.16}$$

Since the linear mapping $A(x) : \mathcal{S}^3 \to \mathcal{S}^3$ is uniformly positive definite, a new bilinear form on $L^p(\Omega, \mathcal{S}^3) \times L^q(\Omega, \mathcal{S}^3)$ can be defined by

$$[\xi, \zeta]_\Omega = (A\xi, \zeta)_\Omega.$$

From [108, Theorem 4.2] we recall an existence theorem for the following boundary value problem describing linear elasticity:

$$-\mathrm{div}_x T(x) = \hat{b}(x), \qquad \text{for } x \in \Omega, \tag{2.17}$$

$$T(x) = A(x)(\varepsilon(\nabla_x u(x)) - \hat{\varepsilon}_p(x)), \qquad \text{for } x \in \Omega, \tag{2.18}$$

$$u(x) = \hat{\gamma}(x), \qquad \text{for } x \in \partial\Omega. \tag{2.19}$$

To given $\hat{b} \in W^{-1,q}(\Omega, \mathbb{R}^3)$, $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ and $\hat{\gamma} \in W^{1,p}(\Omega, \mathbb{R}^3)$ the problem (2.17) - (2.19) has a unique weak solution $(u, T) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathcal{S}^3)$ with $1 < p < \infty$ and $1/p + 1/q = 1$ provided $A \in C(\overline{\Omega}, \mathrm{Lin}(\mathcal{S}^3, \mathcal{S}^3))$ and Ω is of class C^1. For $p = 2$ this result for the problem (2.17) - (2.19) holds provided that A satisfies condition (2.16) and that Ω is a Lipschitz domain. For $\hat{b}$=$\hat{\gamma}$=0 there is a constant $C > 0$ such that the solution of (2.17) - (2.19) satisfies the inequality

$$\|\varepsilon(\nabla_x u)\|_{p,\Omega} \leq C\|\hat{\varepsilon}_p\|_{p,\Omega}.$$

Definition 2.1. For every $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ we define a linear operator $P_p : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$ by $P_p\hat{\varepsilon}_p = \varepsilon(\nabla_x u)$, where $u \in W_0^{1,p}(\Omega, \mathbb{R}^3)$ is the unique weak solution of (2.17) - (2.19) for the given function $\hat{\varepsilon}_p$ and $\hat{b} = \hat{\gamma} = 0$.

Let the subset $\mathcal{G}^p \subset L^p(\Omega, \mathcal{S}^3)$ be defined by

$$\mathcal{G}^p = \{\varepsilon(\nabla_x u) \mid u \in W_0^{1,p}(\Omega, \mathbb{R}^3)\}.$$

The following lemma states the main properties of P_p.

Lemma 2.2. *For every $1 < p < \infty$ the operator P_p is a bounded projector onto the subset $\mathcal{G}^p$ of $L^p(\Omega, \mathcal{S}^3)$. The projector $(P_p)^*$, which is the adjoint with respect to the bilinear form $[\xi, \zeta]_\Omega$ on $L^p(\Omega, \mathcal{S}^3) \times L^q(\Omega, \mathcal{S}^3)$, satisfies*

$$(P_p)^* = P_q, \quad \textit{where} \ \ \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

This implies $\ker(P_p) = H^p_{sol}$ *with* $H^p_{sol} = \{\xi \in L^p(\Omega, \mathcal{S}^3) \mid [\xi, \zeta]_\Omega = 0 \ \ \textit{for all}\ \zeta \in \mathcal{G}^q\}$.

The projection operator

$$Q_p = (I - P_p) : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$$

with $Q_p(L^p(\Omega, \mathcal{S}^3)) = H^p_{sol}$ is a generalization of the classical Helmholtz projection.

Corollary 2.3. *Let $(B^T A Q_p B + L)^T$ be the adjoint operator of*

$$B^T A Q_p B + L : L^p(\Omega, \mathbb{R}^N) \to L^p(\Omega, \mathbb{R}^N)$$

with respect to the bilinear form $(\xi, \zeta)_\Omega$ on the product space $L^p(\Omega, \mathbb{R}^N) \times L^q(\Omega, \mathbb{R}^N)$. Then

$$(B^T A Q_p B + L)^T = B^T A Q_q B + L : L^q(\Omega, \mathbb{R}^N) \to L^q(\Omega, \mathbb{R}^N).$$

Moreover, the operator $B^T A Q_2 B + L$ is non-negative and self-adjoint.

The last result in this corollary is proved in [2].

Remark 2.4. If the matrix L is uniformly positive definite, then the operator $B^T A Q_2 B + L$ is positive definite.

Remark 2.5. H^p_{sol} is a reflexive Banach space with dual space H^q_{sol}.

Finally we cite an existence result for the following Cauchy problem in a Hilbert space H with a maximal monotone operator $A : D(A) \subset H \to 2^H$:

$$\frac{d}{dt} u(t) + A(u(t)) \ \ni \ f(t), \tag{2.20}$$
$$u(0) \ = \ u_0. \tag{2.21}$$

Theorem 2.6. *[11, 97] Assume that $u_0 \in D(A)$. If $f \in W^{1,1}(0, T_e; H)$, then the Cauchy problem (2.20) - (2.21) has a unique solution $u \in W^{1,\infty}(0, T_e; H)$. If $A = \partial\phi$, where $\partial\phi$ is the subdifferential of a proper convex lower-semi-continuous function, then for every $f \in L^2(0, T_e; H)$ the problem (2.20) - (2.21) has a unique solution $u \in W^{1,2}(0, T_e; H)$.*

2.4 Existence of solutions in the case of positive definite energy

It is already known (see [2, Theorem 1.3]) that the initial boundary value problem (2.11) - (2.15) has a unique solution provided the mapping $z \mapsto g(z)$ is maximal monotone and the matrix L is uniformly positive definite. We now state the existence result due to Alber and Chelminski [2].

Theorem 2.7. *Assume that the coefficient matrices satisfy* (2.16), *that in addition L in (2.13) is uniformly positive definite and that the mapping $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is maximal monotone with*

$0 \in g(0)$. *Suppose that* $b \in W^{2,1}(0, T_e; L^2(\Omega, \mathbb{R}^3))$ *and* $\gamma \in W^{2,1}(0, T_e; H^1(\Omega, \mathbb{R}^3))$. *Finally, assume that* $z^{(0)} \in L^2(\Omega, \mathbb{R}^N)$ *and that there exists* $\zeta \in L^2(\Omega, \mathbb{R}^N)$ *such that*

$$\zeta(x) \in g(B^T T^{(0)}(x) - L(x) z^{(0)}(x)), \quad \textit{a.e. in } \Omega, \tag{2.22}$$

where $(u^{(0)}, T^{(0)})$ *is a weak solution of the elasticity problem (2.17)-(2.19) to the data* $\hat{b} = b(0)$, $\hat{\varepsilon}_p = B z^{(0)}$, $\hat{\gamma} = \gamma(0)$.

Then for every $T_e > 0$ *there is a unique solution of the initial boundary value problem (2.11) - (2.15)*

$$(u, T, z) \in W^{1,2}(0, T_e; H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3) \times L^2(\Omega, \mathbb{R}^N)).$$

If $g = \partial \chi_K$, *where* $\partial \chi_K$ *is the subdifferential of the characteristic function associated with the convex, closed set* $0 \in K \subset \mathbb{R}^N$, *then it is sufficient to require* $b \in W^{1,2}(0, T_e; L^2(\Omega, \mathbb{R}^3))$ *and* $\gamma \in W^{1,2}(0, T_e; H^1(\Omega, \mathbb{R}^3))$.

Remark 2.8. We note that L is uniformly positive definite if and only if the free energy density ψ is a positive definite quadratic form on $\mathcal{S}^3 \times \mathbb{R}^N$. The constitutive equations for linear kinematic hardening satisfy this requirement, while models for linear isotropic hardening are not covered.

The main idea of the proof of Theorem 2.7 consists in the reduction of the equations (2.11) - (2.15) to an autonomous evolution inclusion in a Hilbert space governed by a maximal monotone operator. To this evolution inclusion Theorem 2.6 is applied, which allows to conclude that the initial boundary value problem (2.11) - (2.15) has a (unique!) solution. For the reduction it is crucial that the coefficient function L is uniformly positive definite. To indicate the main differences between the case of a positive definite free energy density compared to a positive semi-definite density we briefly sketch the proof of Theorem 2.7. Details can be found in [2].

Proof. We note that equations (2.11) - (2.12), (2.15) form a boundary value problem for the components $(u(t), T(t))$ of the solution. Obviously one has an additive decomposition

$$(u(t), T(t)) = (\tilde{u}(t), \tilde{T}(t)) + (v(t), \sigma(t)),$$

with the solution $(v(t), \sigma(t))$ of the Dirichlet boundary value problem (2.17) - (2.19) to the data $\hat{b} = b(t)$, $\hat{\gamma} = \gamma(t)$, $\hat{\varepsilon}_p = 0$, and with the solution $(\tilde{u}(t), \tilde{T}(t))$ of the problem (2.17) - (2.19) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = Bz(t)$. We thus obtain

$$\varepsilon(\nabla_x u) - Bz = (P_2 - I)Bz + \varepsilon(\nabla_x v).$$

Inserting this into (2.12) we receive that (2.13) can be rewritten in the form

$$z_t(t) \in \mathcal{G}\big(-(B^T A Q_2 B + L) z(t) + B^T \sigma(t) \big), \tag{2.23}$$

where $\mathcal{G} : D(G) \subset L^2(\Omega, \mathbb{R}^N) \to 2^{L^2(\Omega, \mathbb{R}^N)}$ defined by $\mathcal{G}(\xi) = \{\hat{\xi} \in L^2(\Omega, \mathbb{R}^N) \mid \hat{\xi}(x) \in g(\xi(x)) \ a.e.\}$. The function σ, as a solution of the problem (2.17) - (2.19) to the given data b, γ, is considered as known.

According to Remark 2.4 the operator $B^T A Q_2 B + L$ is positive definite, therefore the equation (2.23) can be reduced to an autonomous evolution equation in $L^2(\Omega, \mathbb{R}^N)$ using the transformation $h(t) = -(B^T A Q_2 B + L) z(t) + B^T \sigma(t)$. It then reads as

$$h_t(t) + C(h(t)) \ni B^T \sigma_t(t) \ \text{ with } C(\xi) = (B^T A Q_2 B + L)\mathcal{G}(\xi) \text{ for } \xi \in L^2(\Omega, \mathbb{R}^N). \tag{2.24}$$

The *crucial step* in the proof is that the operator C is maximal monotone with respect to the new scalar product $[[\hat{\xi}, \xi]] := ((B^T AQ_2B + L)^{-1}\hat{\xi}, \xi)$ (see [2]). This scalar product is well defined, since the operator $B^T AQ_2B + L$ is positive definite due to the uniform positivity of L. Therefore, Theorem 2.6 can be applied to (2.24) in $L^2(\Omega, \mathbb{R}^N)$ equipped with the scalar product $[[\hat{\xi}, \xi]]$ to derive the existence and uniqueness of solutions. The assumption (2.22) guarantees that the initial value $h(0)$ belongs to the domain of the operator C. Substituting the solution of (2.23), which exists due to the equivalence of (2.23) and (2.24), into the boundary value problem formed by equations (2.11) - (2.12) and (2.15) yields the existence of (u, T) by the existence theory for linear elliptic problems. □

2.5 Existence of solutions in the case of a positive semi-definite energy

As we saw in the proof of Theorem 2.7 the positivity of L plays the essential role: It allowed to define a new scalar product in $L^2(\Omega, \mathbb{R}^N)$, with respect to which the operator C from (2.24) is maximal monotone so that Theorem 2.6 is applicable. Obviously, this strategy cannot be applied if L is only positive semi-definite and one has to overcome this difficulty. In the following we restrict ourselves to a subclass of problems of monotone type with a positive semi-definite free energy density, for which the existence of solutions can be verified. Existence theorems for the entire class of models of monotone type are still an open problem. For simplicity, we assume that the coefficient matrices in (2.11) - (2.13) are independent of x.

Under the assumption that g is single-valued and that $\operatorname{Ker} B + \operatorname{Ker} L = \mathbb{R}^N$, the authors of [3] showed that the initial boundary value problem (2.11) - (2.15) is equivalent to the following problem: for all $t \in [0, \infty)$ and $x \in \Omega$

$$
\begin{aligned}
-\operatorname{div}_x T(x,t) &= b(x,t), && (2.25)\\
T(x,t) &= A\big(\varepsilon(\nabla_x u(x,t)) - \varepsilon_p(x,t)\big), && (2.26)\\
\partial_t \varepsilon_p(x,t) &= g_1\Big(T(x,t), -\tilde{z}(x,t)\Big), && (2.27)\\
\partial_t \tilde{z}(x,t) &= g_2\Big(T(x,t), -\tilde{z}(x,t)\Big), && (2.28)\\
u(x,t) &= \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty), && (2.29)\\
\varepsilon_p(x,0) &= \varepsilon_p^{(0)}(x), \;\; \tilde{z}(x,0) = \tilde{z}^{(0)}(x). && (2.30)
\end{aligned}
$$

Here the vector of internal variables $z(x,t)$ is split into two parts, i.e. $z(x,t) = (\varepsilon_p(x,t), \tilde{z}(x,t)) \in \mathcal{S}^3 \times \mathbb{R}^{N-6}$. We assume for simplicity that $\varepsilon_p^{(0)}(x) = 0$. The functions $g_1 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathcal{S}^3$ and $g_2 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathbb{R}^{N-6}$ are given such that $(T, y) \to (g_1(T, y), g_2(T, y)) : \mathbb{R}^N \to \mathbb{R}^N$ is a monotone mapping.

Following [3] we rewrite the problem (2.25) - (2.29) in terms of an operator $\mathcal{H} : F(\Omega_{T_e}, \mathcal{S}^3) \to F(\Omega_{T_e}, \mathcal{S}^3)$, where $F(\Omega_{T_e}, \mathcal{S}^3)$ denotes the set of all functions mapping Ω_{T_e} to $\mathcal{S}^3$. The operator $\mathcal{H}$ is defined by the following rule: For given T and $\tilde{z}^{(0)}$ let $(h, \tilde{z})$ be a solution of the problem

$$
\begin{aligned}
h(x,t) &= g_1\big(T(x,t), -\tilde{z}(x,t)\big) && \text{for } (x,t) \in \Omega_{T_e}, && (2.31)\\
\partial_t \tilde{z}(x,t) &= g_2\big(T(x,t), -\tilde{z}(x,t)\big) && \text{for } (x,t) \in \Omega_{T_e}, && (2.32)\\
\tilde{z}(x,0) &= \tilde{z}^{(0)}(x) && \text{for } x \in \Omega, && (2.33)
\end{aligned}
$$

Then the operator $\mathcal{H}$ on $F(\Omega_{T_e}, \mathcal{S}^3)$ is given by $\mathcal{H}(T) = h$. In terms of the operator $\mathcal{H}$ the problem (2.25) - (2.29) reads as follows: for all $(x,t) \in \Omega_{T_e}$

$$\begin{aligned}
-\mathrm{div}_x T(x,t) &= b(x,t), && (2.34)\\
T(x,t) &= A\big(\varepsilon(\nabla_x u(x,t)) - \varepsilon_p(x,t)\big), && (2.35)\\
\partial_t \varepsilon_p(x,t) &= \mathcal{H}(T), && (2.36)\\
\varepsilon_p(x,0) &= 0, && (2.37)\\
u(x,t) &= \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). && (2.38)
\end{aligned}$$

Now we can state the existence result of [82] for the problem (2.34) - (2.38).

Theorem 2.9. *Let $2 \le p < \infty$ and $1 < q \le 2$ be numbers with $1/p + 1/q = 1$. Assume that $\mathcal{H} : L^p(\Omega_{T_e}, \mathcal{S}^3) \to L^q(\Omega_{T_e}, \mathcal{S}^3)$ is maximal monotone and that the inverse $\mathcal{H}^{-1}$ is locally bounded at 0* [1] *and strongly coercive, i.e. either $D(\mathcal{H}^{-1})$ is bounded or $D(\mathcal{H}^{-1})$ is unbounded and*

$$\frac{\langle v^*, v\rangle}{\|v\|_{q,\Omega_{T_e}}} \to +\infty \quad as \ \|v\|_{q,\Omega_{T_e}} \to \infty, \quad v^* \in \mathcal{H}^{-1}(v).$$

Suppose that $b \in L^p(\Omega_{T_e}, \mathbb{R}^3)$ and $\gamma \in L^p(0, T_e, W^{1,p}(\Omega, \mathbb{R}^3))$. Then there exists a solution of the problem (2.34) - (2.38)

$$u \in L^q(0,T_e; W^{1,q}(\Omega,\mathbb{R}^3)), \quad T \in L^p(\Omega_{T_e}\mathcal{S}^3), \quad \varepsilon_p \in W^{1,q}(0,T_e, L^q(\Omega,\mathcal{S}^3)).$$

Remark 2.10. The monotonicity of $\mathcal{H}$ is implied by the monotonicity of the mapping $(T,y) \to (g_1(T,y), g_2(T,y))$ (see [3, Lemma 4.1]).

Remark 2.11. To gain the existence of solutions to (2.25) - (2.29) one has to check first whether the operator $\mathcal{H} : L^p(\Omega_{T_e}, \mathcal{S}^3) \to L^q(\Omega_{T_e}, \mathcal{S}^3)$ is well defined, i.e. whether the problem (2.31) -(2.33) has a solution (not necessary unique). Then apply Theorem 2.9.

Remark 2.12. The proof of Theorem 2.9 in [82] contains a gap, although the result remains true. The operator defined in Lemma 4.1 of [82] is not maximal monotone as it is stated there. The proof of this is given in the end of this section.

In [3] Theorem 2.9 is proved for $\mathcal{H}$ with polynomial growth and under the additional assumption that $\mathcal{H}$ is coercive. The last assumption causes there difficulties in the derivation of the existence of the solutions to the model of nonlinear kinematic hardening (see the next section for more details). In order to show the coercivity of the operator $\mathcal{H}$ defined by the constitutive relations (specific choice of the functions g_1 and g_2) of nonlinear kinematic hardening, the authors of [3] had to impose a restriction on the exponents in the constitutive relations for the different internal variables. The approach initiated in [82] is actually based on the constructions in [3] and repeats the main steps of that work with the major difference that the general duality principle for the sum of two operators from [9] is used to obtain the existence of the solutions to the problem (2.34) - (2.38). The application of this duality principle allows to avoid the coercivity assumption on $\mathcal{H}$. Here we present the improved version of the proof of Theorem 2.9 presented in [82].

Proof. Let us denote

$$W = L^p(\Omega, \mathcal{S}^3), \quad \mathcal{W} = L^p(0,T_e; W), \quad X = H^p_{sol}(\Omega, \mathcal{S}^3), \quad \mathcal{X} = L^p(0,T_e; X).$$

[1] An operator $A : V \to 2^{V^*}$ is called locally bounded at a point $v_0 \in V$ if there exists a neighborhood U of v_0 such that the set
$A(U) = \{Av \mid v \in D(A) \cap U\}$ is bounded in V^*.

Repeating word by word the proof of Theorem 2.7 one can reduce the initial-boundary value problem (2.34) - (2.38) to the following abstract equation

$$\mathcal{L}\varepsilon_p \quad = \quad \mathcal{H}\big(- AQ_p\varepsilon_p + \sigma\big), \tag{2.39}$$

where the linear operator $\mathcal{L} : \mathcal{W} \to \mathcal{W}^*$ is defined by

$$\mathcal{L}\eta = \partial_t\eta \;\text{ with } D(\mathcal{L}) = \{\eta \in W_{p,q}(0, T_e; W) \mid \eta(0) = 0\}.$$

The function σ in (2.39) is given as in the proof of Theorem 2.7. Applying the operator Q_q to (2.39) from the left formally and denoting $\tau = Q_q\varepsilon_p$ we arrive at the equation

$$\mathcal{L}\tau \quad = \quad Q_q\mathcal{H}\big(- A\tau + \sigma\big), \tag{2.40}$$

where now $\mathcal{L} : \mathcal{X} \to \mathcal{X}^*$ denotes the operator

$$\mathcal{L}\eta = \partial_t\eta \;\text{ with } D(\mathcal{L}) = \{\eta \in W_{p,q}(0, T_e; X) \mid \eta(0) = 0\}.$$

The strategy of Theorem 2.7 is not applicable here, since the composition of two operators, one of them being monotone, $\xi \to Q_q\mathcal{H}\big(- A\xi + \sigma\big)$ is not monotone in general. It turns out that applying the general duality principle (see [9]) it is possible to “release” the monotone operator from another operator preserving its monotonicity property and use the classical theory of monotone operators. This is the main idea of the proof of Theorem 2.9.
By the general duality principle [9], the inclusion (2.40) in $\mathcal{X}$ is equivalent to the following inclusion in $\mathcal{X}^*$

$$\mathcal{L}^{-1}AQ_qw + \mathcal{H}^{-1}w \ni \sigma, \quad w \in \mathcal{X}^*. \tag{2.41}$$

Indeed, (2.40) holds iff there exists $v \in \mathcal{L}\tau \cap Q_qw$ with $w = \mathcal{H}(-A\tau + \sigma)$. Taking the inverse of the operators $\mathcal{L}$ and $\mathcal{H}$ gives (2.41). Thus, if we can solve (2.41), by the equivalence we obtain that the problem (2.40) has a solution as well.

Due to Lemma 2.13 here below the operator $\mathcal{L}^{-1}AQ_q : D(\mathcal{L}^{-1}AQ_q) \subset \mathcal{X}^* \to \mathcal{X}$ is linear and maximal monotone.

Now we can show that (2.41) has a solution. Note first that the operator $\mathcal{H}^{-1}$ is maximal monotone as the inverse of a maximal monotone operator. Since $\mathcal{H}^{-1}$ is locally bounded at 0, by Lemma III.24 [2] in [48] the point 0 belongs to the interior of $D(\mathcal{H}^{-1}) = R(\mathcal{H})$. Therefore, the operators $\mathcal{L}^{-1}AQ_q$ and $\mathcal{H}^{-1}$ satisfy the condition

$$D(\mathcal{L}^{-1}AQ_q) \cap \operatorname{int}D(\mathcal{H}^{-1}) \neq \emptyset,$$

yielding that the sum $\mathcal{L}^{-1}AQ_q + \mathcal{H}^{-1}$ is maximal monotone (by Theorem II.1.7 in [11]). The coercivity of $\mathcal{H}^{-1}$ implies the coercivity of the sum, i.e.

$$\frac{\big\langle \mathcal{L}^{-1}AQ_qv + v^*, v\big\rangle}{\|v\|} \geq \frac{\langle v^*, v\rangle}{\|v\|} \to +\infty \;\text{ as } \|v\| \to \infty, \quad v^* \in \mathcal{H}^{-1}(v).$$

Theorem III.2.10 in [83] guarantees that the maximal monotone and coercive operator $\mathcal{L}^{-1}AQ_q + \mathcal{H}^{-1}$ is surjective. Thus, equation (2.41) is solvable and, as consequence, problem (2.40) has a solution.

[2] This result is proved in a Hilbert space, but it can be easily generalized to reflexive Banach spaces.

The construction of the solution of the problem (2.34) - (2.38) can be now performed as in [3]: Let $(v(t), \sigma(t))$ be the solution of the Dirichlet boundary value problem (2.17) - (2.19) to the data $\hat{b} = b(t)$, $\hat{\gamma} = \gamma(t)$, $\hat{\varepsilon}_p = 0$ and let $\tau \in \mathcal{X}$ be the unique solution of (2.40). With the function τ let $\varepsilon_p \in W^{1,q}(0, T_e, L^q(\Omega, \mathcal{S}^3))$ be the solution of

$$\begin{aligned} \partial_t \varepsilon_p(t) &= \mathcal{H}\big(- A\tau(t) + \sigma(t)\big), \quad \text{for a.e. } t \in (0, T_e) && (2.42) \\ \varepsilon_p(0) &= 0. && (2.43) \end{aligned}$$

Moreover, by the linear elliptic theory, there is a unique solution $(\tilde{u}(t), \tilde{T}(t))$ of problem (2.17) - (2.19) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = \varepsilon_p(t)$. The solution of (2.34) - (2.38) is now given as follows

$$(u, T, \varepsilon_p) = (\tilde{u} + v, \tilde{T} + \sigma, \varepsilon_p) \in L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3)) \times L^p(\Omega_{T_e} \mathcal{S}^3) \times W^{1,q}(0, T_e, L^q(\Omega, \mathcal{S}^3)).$$

To see that (u, T, ε_p) satisfies (2.36), we apply the operator Q_q to (2.42) - (2.43) from the left and obtain

$$\partial_t (Q_q \varepsilon_p) = Q_q \mathcal{H}\big(- A\tau(t) + \sigma(t)\big) = \partial_t \tau, \quad Q_q \varepsilon_p(0) = \tau(0) = 0.$$

The last line implies that $Q_q \varepsilon_p = \tau$. Thus

$$T = \tilde{T} + \sigma = -AQ_q \varepsilon_p + \sigma = -A\tau + \sigma \in L^p(\Omega_{T_e} \mathcal{S}^3).$$

The last observation completes the proof. □

Lemma 2.13. *The operator $\mathcal{L}^{-1} AQ_q : D(\mathcal{L}^{-1} AQ_q) \subset \mathcal{X}^* \to \mathcal{X}$ is linear and maximal monotone.*

Proof. According to Theorem 2.7 in [83], the operator $\mathcal{L}^{-1} AQ_q$ is maximal monotone, if it is a densely defined closed monotone operator such that its adjoint $(\mathcal{L}^{-1} AQ_q)^*$ is monotone. Since all these properties of $\mathcal{L}^{-1} AQ_q$ can be easily established, we leave their verification to the reader. More details can be also found in [81]. □

Now we prove the result announced in Remark 2.12.

Lemma 2.14. *The operator $Q_p \mathcal{L}^{-1} : \mathcal{W}^* \to \mathcal{W}$ is not maximal monotone (we use the notations introduced above).*

Proof. Note first of all that the following identity

$$Q_p \mathcal{L}^{-1} v = \mathcal{L}^{-1} Q_q v \tag{2.44}$$

holds for all $v \in D(Q_p \mathcal{L}^{-1}) = D(\mathcal{L}^{-1})$ [3]. The previous identity (2.44) follows easily from

$$P_p \mathcal{L}^{-1} v = \mathcal{L}^{-1} P_q v, \tag{2.45}$$

which holds for $v \in D(\mathcal{L}^{-1})$. Relation (2.45) can be proved as follows: Choose $v \in D(\mathcal{L}^{-1})$. Then, according to the definition of P_p, the boundary value problem

$$\begin{aligned} -\operatorname{div} A\varepsilon(\nabla u(x,t)) &= -\operatorname{div} Av(x,t) \quad \text{for } x \in \Omega, && (2.46) \\ u(x,t) &= 0 \quad \text{for } x \in \partial\Omega, && (2.47) \end{aligned}$$

[3] Recall that $D(\mathcal{L}^{-1}) = \{z \in \mathcal{W}^* \mid \int_0^t z(s) ds \in \mathcal{W}\}$

has a unique solution $u(t) \in W_0^{1,q}(\Omega, \mathbb{R}^3)$, i.e. the function u satisfies the equation

$$(A\varepsilon(\nabla u(t)), \varepsilon(\nabla\phi))_\Omega = (Av(t), \varepsilon(\nabla\phi))_\Omega, \quad \text{for all } \phi \in W_0^{1,p}(\Omega, \mathbb{R}^3).$$

Similarly, we obtain that the problem

$$\begin{aligned} -\operatorname{div} A\varepsilon(\nabla w(x,t)) &= -\operatorname{div} A\Big(\int_0^t v(x,s)ds\Big) && \text{for } x \in \Omega, \\ w(x,t) &= 0 && \text{for } x \in \partial\Omega \end{aligned}$$

has a unique solution $w(t) \in W_0^{1,p}(\Omega, \mathbb{R}^3)$. Integrating (2.46) we get that the identity

$$\Big(A\varepsilon\Big(\nabla \int_0^t u(s)ds\Big), \varepsilon(\nabla\phi)\Big)_\Omega = \Big(A\Big(\int_0^t v(s)ds\Big), \varepsilon(\nabla\phi)\Big)_\Omega$$

holds for all $\phi \in W_0^{1,p}(\Omega, \mathbb{R}^3)$. Thus, by the definition of P_p, we have that $w(t) = \int_0^t u(s)ds$. This proves (2.45).

Next we show that the operator $Q_p\mathcal{L}^{-1}$ is not maximal monotone. To this end, consider a function $\psi \in W^*$ such that $\psi = \varepsilon(\nabla u)$ with $u \in W_0^{1,q}(\Omega, \mathbb{R}^3)$ and $\varepsilon(\nabla u) \notin W$ for any $p > q$ (since $\varepsilon(\nabla u) \notin D(\mathcal{L}^{-1})$). Obviously, such a function u is the solution of the problem

$$-\operatorname{div} A\varepsilon(\nabla \hat{u}) = -\operatorname{div} A\psi, \qquad \hat{u} \in W_0^{1,q}(\Omega, \mathbb{R}^3).$$

The last relation implies that $\psi \in R(P_q)$ and consequently that $\psi \in \ker Q_q$.
To show that $Q_p\mathcal{L}^{-1}$ is not maximal monotone, we need to find a pair $(y^*, y) \in \mathcal{W} \times \mathcal{W}^*$ such that the inequality

$$(Q_p\mathcal{L}^{-1}v - y^*, v - y)_\Omega \geq 0 \tag{2.48}$$

holds for all $v \in D(\mathcal{L}^{-1})$, but $(y^*, y) \notin \operatorname{Graph}(Q_p\mathcal{L}^{-1})$. Take any $v \in D(\mathcal{L}^{-1})$. Set $y = v + \psi$ with ψ from above and $y^* = \mathcal{L}^{-1}Q_q y$, i.e. $y^* = \mathcal{L}^{-1}Q_q v = Q_p\mathcal{L}^{-1}v$. Then $(Q_p\mathcal{L}^{-1}v - y^*, v - y)_\Omega = 0$. Therefore (2.48) is fulfilled for all $v \in D(\mathcal{L}^{-1})$, but $v + \psi \notin D(Q_p\mathcal{L}^{-1})$. Thus, the proof is complete. □

2.6 Model of nonlinear kinematic hardening

We apply Theorem 2.9 to the model of nonlinear kinematic hardening. It consists of the equations (cf. [1, 3])

$$-\operatorname{div}_x T = b, \tag{2.49}$$

$$T = A\big(\varepsilon(\nabla_x u) - \varepsilon_p\big), \tag{2.50}$$

$$\partial_t \varepsilon_p = c_1 |T - k(\varepsilon_p - \varepsilon_n)|^r \frac{T - k(\varepsilon_p - \varepsilon_n)}{|T - k(\varepsilon_p - \varepsilon_n)|}, \tag{2.51}$$

$$\partial_t \varepsilon_n = c_2 |k(\varepsilon_p - \varepsilon_n)|^m \frac{k(\varepsilon_p - \varepsilon_n)}{|k(\varepsilon_p - \varepsilon_n)|}, \tag{2.52}$$

$$\varepsilon_n(0) = \varepsilon_n^0, \quad \varepsilon_p(0) = 0, \tag{2.53}$$

$$u = \gamma, \quad x \in \partial\Omega, \tag{2.54}$$

where $c_1, c_2, \kappa > 0$ are given constants and $\varepsilon_p, \varepsilon_n \in \mathcal{S}^3$. The equations (2.49) - (2.53) can be written in the general form (2.25) - (2.29) with $g = (g_1, g_2) : \mathcal{S}^3 \times \mathcal{S}^3 \to \mathcal{S}^3 \times \mathcal{S}^3$ defined by

$$(g_1, g_2)(T, \tilde{z}) = \Big(c_1 |T + k^{1/2}\tilde{z}|^r \frac{T + k^{1/2}\tilde{z}}{|T + k^{1/2}\tilde{z}|},\ c_1 k^{1/2} |T + k^{1/2}\tilde{z}|^r \frac{T + k^{1/2}\tilde{z}}{|T + k^{1/2}\tilde{z}|} + c_2 k^{1/2} |k^{1/2}\tilde{z}|^m \frac{\tilde{z}}{|\tilde{z}|} \Big),$$

where $\tilde{z} = k^{1/2}(\varepsilon_p - \varepsilon_n)$. Maximal monotonicity of the mapping $(T, \tilde{z}) \to (g_1(T, \tilde{z}), g_2(T, \tilde{z}))$ follows from the fact that $g = (g_1, g_2)$ is the gradient of the continuous convex function

$$\phi(T, \tilde{z}) = \frac{c_1}{r+1} |T + k^{1/2}\tilde{z}|^{r+1} + \frac{c_2}{m+1} |k^{1/2}\tilde{z}|^{m+1}.$$

We have the following existence result for the problem (2.49) - (2.54) (see also [3]).

Theorem 2.15. *Let c_1, c_2, k be positive constants and let r and m satisfy $r, m > 1$. Let us define $p = 1 + r$, $q = 1 + 1/r$, $\hat{p} = \max\{p, 1 + m\}$ and $\hat{q} = \min\{q, 1 + 1/m\}$. Suppose that $b \in L^p(\Omega_{T_e}, \mathbb{R}^3)$, $\gamma \in L^p(0, T_e, W^{1,p}(\Omega, \mathbb{R}^3))$ and $\varepsilon_n^{(0)} \in L^2(\Omega, \mathcal{S}^3)$. Then there exists a solution*

$$u \in L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3)), \quad T \in L^p(\Omega_{T_e}, \mathcal{S}^3),$$
$$\varepsilon_p \in W^{1,q}(0, T_e, L^q(\Omega, \mathcal{S}^3)), \quad \varepsilon_n \in W^{1,\hat{q}}(0, T_e, L^{\hat{q}}(\Omega, \mathcal{S}^3))$$

of the problem (2.49) - (2.54). Moreover, $\varepsilon_p - \varepsilon_n \in W_{\hat{p},\hat{q}}(0, T_e, L^{\hat{p}}(\Omega, \mathcal{S}^3))$.

Remark 2.16. In [3] Theorem 2.15 is proved provided m and r satisfy the inequality $m > r$. This condition the authors of [3] use to show that the operator $\mathcal{H}$ defined by the equations (2.51) - (2.53) according to the rule given above is coercive.

Remark 2.17. Using the theory of Orlic spaces and the monotone operator method similar results are obtained in [85] with the same restrictions on m and r as in Theorem 2.15.

Proof. To apply Theorem 2.9 one has to show that the operator $\mathcal{H}$ defined by (2.51) - (2.53) is well-defined, the (multivalued) inverse $\mathcal{H}^{-1}$ is locally bounded at 0 and coercive . The coercivity of $\mathcal{H}^{-1}$ as well as the fact that the well-posedness of $\mathcal{H}$ are shown in [82]. Therefore, it remains to verify that $\mathcal{H}^{-1}$ is locally bounded at 0. Here we show that $\mathcal{H}^{-1}$ is not only locally bounded at 0, but has even a polynomical growth.

For the function $y = \varepsilon_p - \varepsilon_n$ we have

$$\partial_t \frac{k}{2} |y(x,t)|^2 = ky \cdot c_1 |T - ky|^r \frac{T - ky}{|T - ky|} - ky \cdot c_2 |ky|^m \frac{ky}{|ky|} \le c_1 \Big(\frac{1}{p\alpha^p} |ky|^p + \frac{\alpha^q}{q} |T - ky|^{qr} \Big) - c_2 |ky|^{m+1}.$$

Here we used Young's inequality with $\alpha > 0$. Therefore,

$$\frac{k}{2} \|y(T_e)\|_{2,\Omega}^2 + c_2 \|ky\|_{m+1,\Omega_{T_e}}^{m+1} \le c_1 \Big(\frac{1}{p\alpha^p} \|ky\|_{p,\Omega_{T_e}}^p + \frac{\alpha^q}{q} \|T - ky\|_{p,\Omega_{T_e}}^p \Big) + \frac{k}{2} \|y(0)\|_{2,\Omega}^2$$

and consequently

$$c_2 \|ky\|_{m+1,\Omega_{T_e}}^{m+1} \le c_1 \Big(\frac{1}{p\alpha^p} \|ky\|_{p,\Omega_{T_e}}^p + \frac{\alpha^q}{q} \|T - ky\|_{p,\Omega_{T_e}}^p \Big) + \frac{k}{2} \|y(0)\|_{2,\Omega}^2. \tag{2.55}$$

On the other hand we have

$$\|T\|_{p,\Omega_{T_e}}^p \le \|ky\|_{p,\Omega_{T_e}}^p + \|T - ky\|_{p,\Omega_{T_e}}^p. \tag{2.56}$$

Multiplying (2.56) by $\frac{1}{p\alpha^p}$ and then subtracting (2.55) we get the estimate

$$\frac{1}{p\alpha^p}\|T\|^p_{p,\Omega_{T_e}} - \frac{c_2}{c_1}\|ky\|^{m+1}_{m+1,\Omega_{T_e}} \leq \Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)\|T-ky\|^p_{p,\Omega_{T_e}} - \frac{k}{2c_1}\|y(0)\|^2_{2,\Omega}$$
$$\leq \Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)\|T-ky\|^p_{p,\Omega_{T_e}}. \tag{2.57}$$

For sufficiently small α the constant $\Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)$ is positive. More precisely, $\alpha \in (0, \alpha_0)$ with $\alpha_0 := (q/p)^{1/(p+q)}$. Later we give more precisely the upper bound for α.

Now we derive the estimate for $\|ky\|_{m+1,\Omega_{T_e}}$ in terms of $\|T\|_{p,\Omega_{T_e}}$:

$$\partial_t \frac{k}{2}|y(x,t)|^2 = -(T-ky)\cdot c_1|T-ky|^r\frac{T-ky}{|T-ky|} - ky\cdot c_2|ky|^m\frac{ky}{|ky|} + T\cdot c_1|T-ky|^r\frac{T-ky}{|T-ky|}$$

$$\leq -c_1|T-ky|^p - c_2|ky|^{m+1} + c_1|T||T-ky|^r \leq -c_1|T-ky|^p - c_2|ky|^{m+1} + c_1\Big(\frac{1}{p\delta^p}|T|^p + \frac{\delta^q}{q}|T-ky|^{qr}\Big).$$

Here we used Young's inequality with δ. Choosing $\delta = (q/2)^{1/q}$ we arrive at the estimate

$$\frac{k}{2}\|y(T_e)\|^2_{2,\Omega} + \frac{c_1}{2}\|T-ky\|^p_{p,\Omega_{T_e}} + c_2\|ky\|^{m+1}_{m+1,\Omega_{T_e}} \leq \frac{k}{2}\|y(0)\|^2_{2,\Omega} + c_1\frac{1}{p\delta^p}\|T\|^p_{p,\Omega_{T_e}}$$

and consequently

$$c_2\|ky\|^{m+1}_{m+1,\Omega_{T_e}} \leq \frac{k}{2}\|y(0)\|^2_{2,\Omega} + c_1\frac{1}{p\delta^p}\|T\|^p_{p,\Omega_{T_e}}. \tag{2.58}$$

Thus from (2.57) and (2.58) we obtain

$$\Big(\frac{1}{p\alpha^p} - \frac{1}{p\delta^p}\Big)\|T\|^p_{p,\Omega_{T_e}} - \frac{k}{2c_1}\|y(0)\|^2_{2,\Omega} \leq \Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)\|T-ky\|^p_{p,\Omega_{T_e}}. \tag{2.59}$$

Choosing $\alpha = \min\{\delta/2, \alpha_0/2\}$ in (2.59) we obtain

$$C_1\|T\|^p_{p,\Omega_{T_e}} - C_2 \leq C_3\|T-ky\|^p_{p,\Omega_{T_e}} \tag{2.60}$$

with some positive constants C_1, C_2 and C_3. Recalling that $\|\mathcal{H}(T)\|^q_{q,\Omega_{T_e}} = c_1^q\|T-ky\|^p_{p,\Omega_{T_e}}$, the inequality (2.60) implies

$$C_1\|T\|^p_{p,\Omega_{T_e}} - C_2 \leq C_3 c_1^q\|\mathcal{H}(T)\|^q_{q,\Omega_{T_e}},$$

which yields the polynomial growth for the inverse of $\mathcal{H}(T)$, i.e.

$$\|\mathcal{H}^{-1}(v)\|_{p,\Omega_{T_e}} \leq C_4(1 + \|v\|^{q/p}_{q,\Omega_{T_e}}) \tag{2.61}$$

with some positive constant C_4. Thus $\mathcal{H}^{-1}$ is coercive and bounded. Hence, Theorem 2.9 yields the existence of u, T and ε_p. The existence of ε_n is shown in [82] (see also [3]). Therefore, the proof of Theorem 2.15 is complete. □

3 Spatial regularity for elasto-(visco)plastic models of monotone type

In order to predict convergence rates of numerical schemes, more information about higher spatial regularity of solutions is needed. Depending on the properties of the constitutive function g in (2.9) different results can be obtained.

While local regularity properties were derived in the recent years for a quite large class of models of monotone type, only very few results are known concerning the global regularity. In Section 3.1 we present in detail global regularity results and discuss their optimality in Section 3.2 . An overview on the literature on spatial regularity results for models of monotone type, for viscous regularizations of these models and for models which appear as a time discretized version of the evolution models is given in Section 3.3. By $S = [0, T]$ we denote the time interval.

3.1 Regularity for maximal monotone g and positive definite elastic energy

Historically, local spatial regularity results were first deduced by Seregin [93] for elasto-plasticity with linear kinematic or isotropic hardening and with a von Mises flow rule. The proof is done by carrying over local regularity properties of a time-discretized version to the time-continuous problem. Here we follow a different approach working directly with the time-continuous model.

The model of monotone type formulated in (2.11)–(2.15), consists of an elliptic system of partial differential equations, which is strongly coupled with an evolutionary variational inequality describing the evolution of the displacements u and the internal variable z subjected to external loadings. There exist various powerful analytic tools to characterize the spatial regularity of systems of elliptic PDEs both on smooth and nonsmooth domains. The problem in the elasto-plastic case is to maintain the regularity properties of the elliptic system in spite of the strong coupling between the elliptic system and the evolutionary variational inequality.

Let $\mathcal{Q} \subset H^1(\Omega) \times L^2(\Omega) \ni (u(t), z(t))$ denote the state space and assume for the moment that the initial datum $z^0 = 0$. The intrinsic difficulty of proving spatial regularity results for plasticity problems stems from the fact that the flow rule (2.12) is non smooth and has no regularizing terms. As a consequence the data-to-solution-map is not Lipschitz from $W^{1,1}(S; \mathcal{Q}^*) \to W^{1,1}(S; \mathcal{Q})$, but only as a map from $W^{1,1}(S; \mathcal{Q}^*) \to L^\infty(S; \mathcal{Q})$. The latter Lipschitz property is the basis for proving the local and tangential regularity results in Sobolev spaces. Roughly spoken, the local regularity of (u, z) follows from the Lipschitz estimate

$$\|(u_h - u, z_h - z)\|_{L^\infty(S;\mathcal{Q})} \leq c_{\mathrm{Lip}} \|f_h - f\|_{W^{1,1}(S;\mathcal{Q}^*)}\,, \tag{3.1}$$

where the index h indicates a local shift of the functions u and z by a (small) vector $h \in \mathbb{R}^d$. The function f_h contains the shifted datum f and further corrections due to the shift, so that (u_h, z_h) is a solution to (2.11)–(2.13) with respect to the datum f_h. If f is smooth enough such that the estimate

$$\sup_{|h|<h_0} |h|^{-1} \|f_h - f\|_{W^{1,1}(S;\mathcal{Q}^*)} \leq c_f \tag{3.2}$$

is valid, then it follows that $(u, z) \in L^\infty(S; H^2_{\mathrm{loc}}(\Omega) \times H^1_{\mathrm{loc}}(\Omega))$. Since a similar Lipschitz estimate is not known for the time derivatives $(\partial_t u, \partial_t z)$, we cannot show that e.g. $\partial_t z \in L^1(S; H^1_{\mathrm{loc}}(\Omega))$. Indeed, the example in Section 3.2 reveals that the latter regularity is not valid in spite of smooth data. Similar arguments can be applied in order to derive tangential regularity properties at the boundary of smooth domains.

In order to obtain information on the regularity in the normal direction, the problem is reflected at $\partial\Omega$. The reflected functions $(\tilde{u}, \tilde{z})$ solve an evolution system of similar type with new datum $\tilde{f}$, which consists of the reflected datum f and the tangential derivatives of ∇u and z: $\tilde{f} = (f_{\text{refl}}, \partial_{\text{tang}}\nabla u, \partial_{\text{tang}} z)$. Due to the terms $\partial_{\text{tang}}\nabla u$ and $\partial_{\text{tang}} z$ the new datum does not have the temporal regularity allowing for an estimate like (3.2). In view of the tangential regularity results, we can guarantee at least that

$$\sup_{|h|<h_0} |h|^{-1} \|\tilde{f}_h - \tilde{f}\|_{L^\infty(S;\mathcal{Q}^*)} \leq c.$$

Hence, the Lipschitz estimate (3.1) has to be replaced with the following weaker version for the extended functions $(\tilde{u}, \tilde{z})$:

$$\|(\tilde{u}_h - \tilde{u}, \tilde{z}_h - \tilde{z})\|_{L^\infty(S;\mathcal{Q})} \leq c \|\tilde{f}_h - \tilde{f}\|^{\frac{1}{2}}_{L^\infty(S;\mathcal{Q}^*)} \leq c\,|h|^{\frac{1}{2}}\,, \tag{3.3}$$

see Theorem 3.2. From the latter estimate we finally deduce that $(u, z) \in L^\infty(S; H^{\frac{3}{2}-\delta}(\Omega) \times H^{\frac{1}{2}-\delta}(\Omega))$ for every $\delta > 0$. These steps are explained in detail in Sections 3.1.1-3.1.3.

3.1.1 Basic assumptions and stability estimates

The arguments explained above are not restricted to the operator of linear elasticity occuring in (2.11)–(2.12). We consider here the case with general displacements $u : S \times \Omega \to \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain, and replace the operator of linear elasticity by a more general linear elliptic operator. For $\theta \in \mathbb{R}^{m\times d}$ and $z \in \mathbb{R}^N$ the energy density ψ is assumed to be of the form

$$\psi(x, \theta, z) = \frac{1}{2}\langle A(x) \left(\begin{smallmatrix}\theta\\ z\end{smallmatrix}\right), \left(\begin{smallmatrix}\theta\\ z\end{smallmatrix}\right)\rangle \equiv \frac{1}{2}\left(\langle A_{11}(x)\theta, \theta\rangle + \langle A_{12}(x)z, \theta\rangle + \langle A_{21}(x)\theta, z\rangle + \langle A_{22}(x)z, z\rangle\right) \tag{3.4}$$

where $A \in L^\infty(\Omega; \mathrm{Lin}(\mathbb{R}^{m\times d} \times \mathbb{R}^N, \mathbb{R}^{m\times d} \times \mathbb{R}^N))$ is a given coefficient matrix and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^s$. For $u \in H^1(\Omega, \mathbb{R}^m)$ and $z \in L^2(\Omega, \mathbb{R}^N)$ the corresponding elastic energy is defined as

$$\Psi(u, z) = \int_\Omega \psi(x, \nabla u(x), z(x))\,\mathrm{d}x. \tag{3.5}$$

The basic assumptions in this section are the following

R1 $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^{1,1}$-smooth boundary, see e.g. [42].

R2 The coefficient matrix A belongs to $C^{0,1}(\overline{\Omega}, \mathrm{Lin}(\mathbb{R}^{m\times d} \times \mathbb{R}^N, \mathbb{R}^{m\times d} \times \mathbb{R}^N))$, is symmetric and there exists a constant $\alpha > 0$ such that $\Psi(v, z) \geq \frac{\alpha}{2}\big(\|v\|^2_{H^1(\Omega)} + \|z\|^2_{L^2(\Omega)}\big)$ for all $v \in H^1_0(\Omega)$ and $z \in L^2(\Omega)$.

R3 The function $g{:}\mathbb{R}^N \to 2^{\mathbb{R}^N}$ is maximal monotone with $0 \in g(0)$ and $\mathcal{G} : D(\mathcal{G}) \subset L^2(\Omega, \mathbb{R}^N) \to \mathcal{P}(L^2(\Omega, \mathbb{R}^N))$ is defined as $\mathcal{G}(\eta)=\{\, z \in L^2(\Omega, \mathbb{R}^N)\,;\; z(x) \in g(\eta(x))$ a.e. in $\Omega\,\}$.

Observe that $\mathcal{G}$ is a maximal monotone operator. The energy density ψ introduced in (2.10) is contained as a special case and further examples are given in Section 3.1.3.

In order to shorten the presentation, the discussion is restricted to the case with vanishing Dirichlet conditions on $\partial\Omega$. Hence, with $V = H_0^1(\Omega, \mathbb{R}^m)$ and $Z = L^2(\Omega, \mathbb{R}^N)$ the state space $\mathcal{Q}$ takes the form $\mathcal{Q} = V \times Z$. We investigate the spatial regularity properties of functions $(u, z) : [0, T] \to \mathcal{Q}$ which for all $v \in V$ and almost every $t \in S$ satisfy

$$D_u\Psi(u(t), z(t))[v] = \int_\Omega \langle A \begin{pmatrix} \nabla u(t) \\ z(t) \end{pmatrix}, \begin{pmatrix} \nabla v \\ 0 \end{pmatrix} \rangle \, \mathrm{d}x = \langle b(t), v \rangle, \tag{3.6}$$

$$\partial_t z(t) \in \mathcal{G}(-D_z\Psi(u(t), z(t)) + F(t)), \tag{3.7}$$

$$z(0) = z^0, \; u(t)\big|_{\partial\Omega} = 0. \tag{3.8}$$

Here, $D_u\Psi$ and $D_z\Psi$ denote the variational derivatives of Ψ with respect to u and z, and F is a further forcing term not present in (2.11)-(2.13). The data b, F are comprised in the function $(b, F) = f : S \to V^* \times Z \equiv \mathcal{Q}^*$. We call the initial value z_0 and the forces f compatible if there exists $u_0 \in V$ with $D_u\Psi(u_0, z_0) = b(0)$ and $-D_z\Psi(u_0, z_0) + F(0) \in D(\mathcal{G})$, where $D(\mathcal{G})$ denotes the domain of $\mathcal{G}$. The compatibility assumption is equivalent to the assumption in Theorem 5.8, where the initial data shall belong to the set of stable states.

Since the elastic energy Ψ is assumed to be positive definite on $\mathcal{Q}$, see **R2**, similar arguments as pointed out in Section 2.4 lead to the following existence theorem:

Theorem 3.1. *Assume that* **R2** *and* **R3** *are satisfied and that the data* $z_0 \in L^2(\Omega, \mathbb{R}^N)$ *and* $f = (b, F) \in W^{2,1}(S; \mathcal{Q}^*)$ *are compatible. Then there exists a unique pair* $(u, z) \in W^{1,1}(S; \mathcal{Q})$ *satisfying* (3.6)–(3.8)*. If* $\mathcal{G} = \partial\chi_{\mathcal{K}}$*, where* $\mathcal{K} \subset L^2(\Omega, \mathbb{R}^N)$ *is convex, closed and with* $0 \in \mathcal{K}$ *and* $\chi_{\mathcal{K}}$ *is the characteristic function of the convex set* $\mathcal{K}$*, then it is sufficient to assume that* $f = (b, F) \in W^{1,1}(S; \mathcal{Q}^*)$.

The next stability estimates rely on the positivity of the energy Ψ and are the basis for our regularity results.

Theorem 3.2. *Assume that* **R2** *and* **R3** *are satisfied.*

(a) There exists a constant $\kappa > 0$ *such that for all* $u_i \in W^{1,1}(S; H^1(\Omega))$*,* $z_i \in W^{1,1}(S; L^2(\Omega))$*,* $i \in \{1, 2\}$*, which satisfy* (3.6)–(3.8) *with* $f_i \in W^{1,1}(S; \mathcal{Q}^*)$ *and* $z_i^0 \in L^2(\Omega, \mathbb{R}^N)$*, it holds*

$$\|u_1 - u_2\|_{L^\infty(S;H^1(\Omega))} + \|z_1 - z_2\|_{L^\infty(S;L^2(\Omega))} \le \kappa\big(\|z_1^0 - z_2^0\|_{L^2(\Omega)} + \|f_1 - f_2\|_{W^{1,1}(S;\mathcal{Q}^*)}\big). \tag{3.9}$$

(b) There exists a constant $\kappa > 0$ *such that for all* $u_i \in L^\infty(S; H^1(\Omega))$*,* $z_i \in W^{1,1}(S; L^2(\Omega))$*,* $i \in \{1, 2\}$*, which satisfy* (3.6)–(3.8) *with* $f_i \in L^\infty(S; \mathcal{Q}^*)$ *and* $z_i^0 \in L^2(\Omega, \mathbb{R}^N)$*, it holds*

$$\begin{aligned} &\|u_1 - u_2\|_{L^\infty(S;H^1(\Omega))} + \|z_1 - z_2\|_{L^\infty(S;L^2(\Omega))} \\ &\le \kappa\big(\|z_1^0 - z_2^0\|_{L^2(\Omega)} + \|f_1 - f_2\|_{L^\infty(S;\mathcal{Q}^*)} + \|z_1 - z_2\|^{\frac{1}{2}}_{W^{1,1}(S;L^2(\Omega))} \|f_1 - f_2\|^{\frac{1}{2}}_{L^\infty(S;\mathcal{Q}^*)}\big). \end{aligned} \tag{3.10}$$

Part (a) of the theorem gives the Lipschitz continuity of the data-to-solution mapping $\mathcal{T} : Z \times W^{1,1}(S; \mathcal{Q}^*) \to L^\infty(S; \mathcal{Q}); (z^0, f) \mapsto (u, z)$, while part (b) describes Hölder-like continuity of the data-to-solution mapping in the case where the data have less temporal regularity. We refer to [62, 58] and the references therein for a proof of the estimates.

3.1.2 Local spatial regularity and tangential regularity

Local and tangential regularity results are derived with a difference quotient argument in combination with the stability estimates of Theorem 3.2. Concerning the data it is assumed that

R4$_a$ $z^0 \in H^1(\Omega)$, $f = (b, F) \in W^{1,1}(S; \mathcal{Y}_1)$ with $\mathcal{Y}_1 = L^2(\Omega, \mathbb{R}^m) \times H^1(\Omega, \mathbb{R}^N)$.

R4$_b$ $z^0 \in H^1(\Omega)$, $f = (b, F) \in L^\infty(S; \mathcal{Y}_i)$ with $\mathcal{Y}_i = L^2(\Omega, \mathbb{R}^m) \times \{\, \theta \in L^2(\Omega, \mathbb{R}^N)\,;\, \partial_i \theta \in L^2(\Omega, \mathbb{R}^N)\,\}$ for a fixed $i \in \{1, \dots, d\}$.

Let $x_0 \in \Omega$ and choose $\varphi \in C_0^\infty(\Omega, \mathbb{R})$ with $\varphi \equiv 1$ in a ball $B_\rho(x_0)$. For $h \in \mathbb{R}^d$, the inner variation $\tau_h : \Omega \to \mathbb{R}^d$ is defined as $\tau_h(x) = x + \varphi(x)h$. There exists a constant $h_0 > 0$ such that the mappings $\tau_h : \Omega \to \Omega$ are diffeomorphisms for every $h \in \mathbb{R}^d$ with $|h| \le h_0$. Let the pair $u \in L^\infty(S; V)$ and $z \in W^{1,1}(S; Z)$ be a solution of (3.6)–(3.8). We define $u_h(t, x) = u(t, \tau_h(x))$, $z_h(t, x) = z(t, \tau_h(x))$. Straightforward calculations show that the shifted pair (u_h, z_h) solves (3.6)–(3.8) with respect to the shifted initial condition z_h^0 and modified data $\tilde{f}_h$ having the property

$$\big\|\tilde{f}_h - f\big\|_{W^{1,1}(S;\mathcal{Q}^*)} \le c\,|h|\,\|(f, u, z)\|_{W^{1,1}(S;\mathcal{Y}_1 \times V \times L^2(\Omega))} \tag{3.11}$$

if f satisfies **R4**$_a$, and

$$\big\|\tilde{f}_h - f\big\|_{L^\infty(S;\mathcal{Q}^*)} \le c\,|h|\,\|(f, u, z)\|_{L^\infty(S;\mathcal{Y}_i \times V \times L^2(\Omega))} \tag{3.12}$$

if f is given according to **R4**$_b$. The local regularity Theorem 3.3 here below is now an immediate consequence of the stability estimates in Theorem 3.2.

Theorem 3.3. *Let conditions* **R2** *and* **R3** *be satisfied.*

(a) Let $(u, z) \in W^{1,1}(S; V \times Z)$ *be a solution of* (3.6)–(3.8) *with data satisfying* **R4**$_a$*. Then* $u \in L^\infty(S; H^2_{loc}(\Omega))$ *and* $z \in L^\infty(S; H^1_{loc}(\Omega))$*.*

(b) Let $u \in L^\infty(S; V)$ *and* $z \in W^{1,1}(S; Z)$ *be a solution of* (3.6)–(3.8) *with data according to* **R4**$_b$*. Then there exists* $h_0 > 0$ *such that*

$$\sup_{0<h<h_0} h^{-\frac{1}{2}}\,\|\nabla u_{he_i} - \nabla u\|_{L^\infty(S;L^2(B_\rho(x_0)))} < \infty,$$
$$\sup_{0<h<h_0} h^{-\frac{1}{2}}\,\|z_{he_i} - z\|_{L^\infty(S;L^2(B_\rho(x_0)))} < \infty.$$

Proof. Estimate (3.11) in combination with Theorem 3.2, part (a), yields

$$\sup_{|h|\le h_0} |h|^{-1}\Big(\|u - u_h\|_{L^\infty(S;H^1(B_\rho(x_0)))} + \|z - z_h\|_{L^\infty(S;L^2(B_\rho(x_0)))}\Big) \le \|(f, u, z)\|_{W^{1,1}(S;\mathcal{Y}_1,V,Z)}$$

from which we conclude with Lemma 7.24 in [41] that $u \in L^\infty(S; H^2_{\mathrm{loc}}(\Omega))$ and $z \in L^\infty(S; H^1_{\mathrm{loc}}(\Omega))$. The results in part (b) of the theorem are obtained in a similar way. $\square$

If **R4**$_b$ is satisfied for all basis vectors e_i, $1 \le i \le d$, and all $x_0 \in \Omega$, then $u(t)$ and $z(t)$ belong to the Besov spaces $B^{\frac{3}{2}}_{2,\infty}(\Omega')$ and $B^{\frac{1}{2}}_{2,\infty}(\Omega')$ for every $\Omega' \Subset \Omega$. Via the embedding theorems for Besov spaces into Sobolev-Slobodeckij spaces we conclude that $u \in L^\infty(S; H^{\frac{3}{2}-\delta}_{\mathrm{loc}}(\Omega))$ and $z \in L^\infty(S; H^{\frac{1}{2}-\delta}_{\mathrm{loc}}(\Omega))$ for every $\delta > 0$.

In a similar way, tangential regularity properties can be deduced after a suitable local transformation of the boundary to a subset of a hyperplane. Here, the assumption **R1** on the smoothness of $\partial\Omega$ is essential.

Part (a) of Theorem 3.2 with a general maximal monotone function g and with ψ as in (2.10) was proved by Alber and Nesenenko in [4, 5] and extended in [25] to an elasto-plastic model including Cosserat effects. In the paper [58] the result was extended to the slightly more general situation, where the operator of linear elasticity and the Cosserat operators are replaced by a more general linear elliptic system, part (b) was added and more general boundary conditions allowing for different kinds of boundary conditions in the different components of u were investigated. We refer to Section 3.3 for a more detailed discussion of the related literature.

3.1.3 Global spatial regularity

The first global spatial regularity result for problems of the type (3.6)–(3.8) was proved by Alber and Nesenenko [4, 5]. The authors showed that the local and tangential regularity properties in Theorem 3.3, part (a), already imply that the solution belongs to the spaces $u \in L^\infty(S; H^{1+\frac{1}{4}}(\Omega))$, $z \in L^\infty(S; H^{\frac{1}{4}}(\Omega))$. By an iteration procedure the final regularity $u \in L^\infty(S; H^{1+\frac{1}{3}}(\Omega))$ and $z \in L^\infty(S; H^{\frac{1}{3}}(\Omega))$ was obtained. With a completely different argument, a reflection argument, the result can be improved. This will be explained in detail in this section.

To shorten the presentation we assume that there is a point $x_0 \in \partial\Omega$ such that $\partial\Omega$ locally coincides with a hyperplane and that Ω lies above the hyperplane. The general case can be reduced to this situation by a suitable local transformation of coordinates. Moreover it is assumed that the data are given according to $\mathbf{R4}_a$.

Let $C_+ = (-1,1)^{d-1} \times (0,1)$ be the upper half cube, $C_- = (-1,1)^{d-1} \times (-1,0)$ the lower half cube and assume that $\Gamma = (-1,1)^{d-1} \times \{0\} \subset \partial\Omega$ and that $C_+ \cap \Omega = C_+$ and $C_- \cap \Omega = \emptyset$, see Figure 1. By $C = (-1,1)^d$ we denote the unit cube in $\mathbb{R}^d$. Let $R = \mathbb{I} - 2e_d \otimes e_d$ be the orthogonal reflection at Γ. The elasto-plastic model is extended from C_+ to C by means of an odd extension for the displacements and an even extension for the internal variable and the initial datum:

$$u_e(t,x) = \begin{cases} u(t,x) & x \in C_+ \\ -u(t,Rx) & x \in C_- \end{cases}, \quad z_e(t,x) = \begin{cases} z(t,x) & x \in C_+ \\ z(t,Rx) & x \in C_- \end{cases}, \quad z_e^0 = \begin{cases} z^0 & \text{in } C_+ \\ z^0 \circ R & \text{in } C_- \end{cases}. \tag{3.13}$$

Moreover, the extended coefficient matrix A_e and the extended elastic energy are defined as

$$A_e = \begin{cases} A & \text{in } C_+ \\ A \circ R & \text{in } C_- \end{cases}, \qquad \Psi_e(v,z) = \frac{1}{2} \int_{\Omega \cup C} \langle A_e \left(\begin{smallmatrix} \nabla v \\ z \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ z \end{smallmatrix} \right) \rangle \, \mathrm{d}x \tag{3.14}$$

for $v \in H^1(\Omega \cup C)$ and $z \in L^2(\Omega \cup C)$. Technical calculations show that the extended functions satisfy for all $v \in H^1_0(C)$

$$\int_C \langle A_e \left(\begin{smallmatrix} \nabla u_e(t) \\ z_e(t) \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x = \int_C b_e(t) \cdot v \, \mathrm{d}x,$$
$$\partial_t z_e(t) \in \mathcal{G}(-D_z \Psi_e(\nabla u_e(t), z_e(t)) + F_e(t)),$$

where

$$b_e(t,x) = \begin{cases} b(t,x) & x \in C_+ \\ -b(t,Rx) - \operatorname{div}\Big(\big(A_{11}\nabla u(t) + A_{12}z(t)\big)\big|_{Rx}(R+\mathbb{I})\Big) & x \in C_- \end{cases}, \tag{3.15}$$

$$F_e(t,x) = \begin{cases} F(t,x) & x \in C_+ \\ F(t,Rx) - A_{21,e}(\nabla u(t)\big|_{Rx}(R+\mathbb{I})) & x \in C_- \end{cases}. \tag{3.16}$$

The tangential regularity results from the previous section guarantee that $b_e\big|_{C_-} \in L^\infty(S;L^2(C_-))$. Indeed, due to the factor $(R+\mathbb{I})$ terms like $\partial_d^2 u$ and $\partial_d z$ do not appear in the definition of b_e and hence, tangential derivatives of ∇u and z enter in the definition of b_e, only, which, by Theorem 3.3, belong to $L^\infty(S;L^2(C_-))$. Again from the regularity results in the previous section we obtain that $\partial_d F_e\big|_{C_\pm} \in L^\infty(S;L^2(C_\pm))$. Taking into account that $u\big|_\Gamma = 0$, it follows that $\nabla u(R+\mathbb{I})\big|_\Gamma = 0$ and hence the traces of $F_e\big|_{C_+}$ and $F_e\big|_{C_-}$ coincide on Γ. This implies that $\partial_d F_e \in L^\infty(S;L^2(C))$. The local regularity result described in Theorem 3.3, part (b), is therefore applicable and leads to the following theorem:

Theorem 3.4. *Assume that* **R1**–**R3** *and* $\mathbf{R4}_a$ *are satisfied. Then the unique solution* (u,z) *of problem* (3.6)–(3.8) *satisfies: For every* $\delta > 0$

$$u \in L^\infty(S;H^{\frac{3}{2}-\delta}(\Omega)) \cap L^\infty(S;H^2_{loc}(\Omega)), \qquad z \in L^\infty(S;H^{\frac{1}{2}-\delta}(\Omega)) \cap L^\infty(S;H^1_{loc}(\Omega)). \tag{3.17}$$

Moreover, for every $\delta > 0$ *there exists a constant* $c_\delta > 0$ *such that*

$$\|u\|_{L^\infty(S;H^{\frac{3}{2}-\delta}(\Omega))} + \|z\|_{L^\infty(S;H^{\frac{1}{2}-\delta}(\Omega))} \le c_\delta(\|z^0\|_{H^1(\Omega)} + \|f\|_{W^{1,1}(S;\mathcal{Y}_1)}). \tag{3.18}$$

We refer to [58] for a detailed proof of the global results and a slightly more general variant of Theorem 3.4, where also further types of boundary conditions are discussed.

Estimates (3.9) and (3.18) allow to apply Tartar's nonlinear interpolation theorem showing that for data with less spatial regularity than required in Theorem 3.4, one obtains the corresponding spatial regularity of the solution in a natural way. We assume here that $g = \partial\chi_K$, where $K \subset \mathbb{R}^N$ is convex, closed and $0 \in K$. $\partial\chi_K$ denotes the convex subdifferential of the characteristic function χ_K associated with K. Let $\mathcal{Y}_0 := \mathcal{Q}^*$, $\mathcal{Y}_1 := L^2(\Omega,\mathbb{R}^m) \times H^1(\Omega,\mathbb{R}^N)$ and $\mathcal{Q}_1^\delta := (H_0^1(\Omega,\mathbb{R}^m) \cap H^{\frac{3}{2}-\delta}(\Omega,\mathbb{R}^m)) \times H^{\frac{1}{2}-\delta}(\Omega,\mathbb{R}^N)$ for $\delta > 0$. Due to Theorem 3.1 and the stability estimate (3.9) for all $r,q \in [1,\infty]$ the solution operator $\mathcal{T}$ defined by

$$\mathcal{T} : L^2(\Omega,\mathbb{R}^N) \times W^{1,r}(S;\mathcal{Y}_0) \to L^q(S;\mathcal{Q}), \quad (z^0,f) \mapsto \mathcal{T}(z^0,f) = (u,z),$$

where $(u,z) \in W^{1,1}(S;\mathcal{Q})$ is the unique solution of (3.6)–(3.8) with data $f = (b,F)$ and initial condition z^0, is well defined and Lipschitz-continuous. Moreover, for all $\delta > 0$ the solution operator

$$\mathcal{T} : H^1(\Omega,\mathbb{R}^N) \times W^{1,r}(S;\mathcal{Y}_1) \to L^q(S;\mathcal{Q}_1^\delta)$$

is a bounded operator according to Theorem 3.4. Hence, Tartar's interpolation Theorem [103, Thm. 1] guarantees that for all $\theta \in (0,1)$ and all $p \in [1,\infty]$ the following implication holds true:

$$\begin{aligned} &z^0 \in (H^1(\Omega);L^2(\Omega))_{\theta,p},\ f \in (W^{1,r}(S;\mathcal{Y}_1);W^{1,r}(S;\mathcal{Y}_0))_{\theta,p} \\ &\qquad\qquad \implies \mathcal{T}(z^0,f) = (u,z) \in (L^q(S;\mathcal{Q}_1^\delta);L^q(S;\mathcal{Q}))_{\theta,p}. \end{aligned}$$

Here, $(\cdot\,;\,\cdot)_{\theta,p}$ stands for real interpolation, see e.g. [107]. If for example $r = q = p = 2$ and $\theta \in (0,1)$, then given $z^0 \in H^\theta(\Omega)$, $b \in W^{1,2}(S;(\widetilde{H}^{1-\theta}(\Omega))^*)$, where $\widetilde{H}^s(\Omega) = \{\,\eta \in H^s(\Omega)\,;\,\exists \tilde\eta \in H^s(\mathbb{R}^m)$ with $\operatorname{supp}\tilde\eta \subset \overline{\Omega}, \tilde\eta\big|_\Omega = \eta\,\}$, and $F \in W^{1,2}(S;H^\theta(\Omega))$ we obtain that $u \in L^2(S;H^{1+\theta(\frac12-\delta)}(\Omega))$ and $z \in L^2(S;H^{\theta(\frac12-\delta)}(\Omega))$.

Example 3.5. Theorem 3.4 and the interpolation result are applicable to rate-independent elasto-plasticity with linear kinematic hardening and with a von Mises or a Tresca flow rule. Here, the vector of internal variables is identified with the plastic strains $\varepsilon_p \in \mathbb{R}^{d\times d}_{\text{sym,dev}}$ (i.e. $\operatorname{tr}\varepsilon_p = 0$) and the elastic energy takes the form

$$\Psi(u,\varepsilon_p) = \int_\Omega \psi(\varepsilon(\nabla u),\varepsilon_p)\,\mathrm{d}x \quad \text{with} \quad \psi(\varepsilon,\varepsilon_p) = \tfrac12 A(\varepsilon-\varepsilon_p)\cdot(\varepsilon-\varepsilon_p) + \tfrac12 L\varepsilon_p\cdot\varepsilon_p, \tag{3.19}$$

for $(\varepsilon,\varepsilon_p) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym,dev}}$. The coefficient tensors $A \in C^{0,1}(\overline{\Omega},\operatorname{Lin}(\mathbb{R}^{d\times d}_{\text{sym}},\mathbb{R}^{d\times d}_{\text{sym}}))$ and $L \in C^{0,1}(\overline{\Omega},\operatorname{Lin}(\mathbb{R}^{d\times d}_{\text{sym,dev}},\mathbb{R}^{d\times d}_{\text{sym,dev}}))$ are assumed to be symmetric and uniformly positive definite. Hence, due to Korn's inequality, assumption **R2** is satisfied. Let $K \subset \mathbb{R}^{d\times d}_{\text{sym,dev}}$ be convex, closed and with $0 \in K$. The set K describes the set of admissible stress states. Choosing $g = \partial\chi_K$ as the convex subdifferential of the characteristic function χ_K associated with K, we obtain classical rate-independent models for elasto-plastic material behavior. In particular, the von Mises flow rule is associated with the set $K_{\text{vM}} = \{\,\tau \in \mathbb{R}^{d\times d}_{\text{sym,dev}}\,;\,(\tau\cdot\tau)^{\frac12} \le c_0\,\}$, whereas the Tresca flow rule is based on the set $K_{\text{T}} = \{\,\tau \in \mathbb{R}^{d\times d}_{\text{sym,dev}}\,;\,\max_{i\neq j}|\tau_i - \tau_j| \le c_0\,\}$. Here, $\{\,\tau_i\,;\,1 \le i \le d\}$ are the eigenvalues (principle stresses) of $\tau \in \mathbb{R}^{d\times d}_{\text{sym,dev}}$. The regularity Theorem 3.4 and the interpolation result are applicable to these models.

Example 3.6. In [80] an elastic-plastic model was introduced which incorporates Cosserat micropolar effects. This model is analyzed in [80, 25] with respect to existence and local regularity and in [59] with respect to global regularity of a time discretized version. In this model, not only the displacements u but also linearized micro-rotations Q are taken into account. The generalized displacements are given by the pair $(u,Q) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{skew}} \cong \mathbb{R}^m$, whereas the internal variable z is identified with the plastic strain tensor $z = \varepsilon_p \in \mathbb{R}^{d\times d}_{\text{sym, dev}}$. For $u \in H^1(\Omega,\mathbb{R}^d)$, $Q \in H^1(\Omega,\mathbb{R}^{d\times d}_{\text{skew}})$ and $\varepsilon_p \in L^2(\Omega,\mathbb{R}^{d\times d}_{\text{sym,dev}})$ the elastic energy reads

$$\Psi_C((u,Q),\varepsilon_p) = \int_\Omega \mu\,|\varepsilon(\nabla u)-\varepsilon_p|^2 + \mu_c\,|\text{skew}\,(\nabla u - Q)|^2 + \frac{\lambda}{2}\,|\text{tr}\,\nabla u|^2 + \gamma\,|\nabla Q|^2\,\mathrm{d}x.$$

Here, $\lambda,\mu > 0$ are the Lamé constants, $\mu_c > 0$ is the Cosserat couple modulus and $\gamma > 0$ depends on the Lamé constants and a further internal length parameter. It is shown in [80] that Ψ_C satisfies condition **R2**. If $\mathcal{G}$ is chosen according to **R3**, then solutions to (3.6)–(3.8) with Ψ_C have the global regularity properties described in Theorem 3.4. In addition, $Q \in L^\infty(S;H^2(\Omega))$, since Q is coupled with $\varepsilon(\nabla u)$ and ε_p through lower order terms, only, see [25].

3.2 Discussion of the regularity results

It is an unsolved problem whether the result in Theorem 3.4 is optimal or whether one should expect the regularity $u \in L^\infty(S;H^2(\Omega))$, $z \in L^\infty(S;H^1(\Omega))$ for domains with smooth boundaries. This would extend the local regularity results described in Theorem 3.3 in a natural way. If u is a scalar function, then under certain coupling conditions on the coefficients the spatial regularity $u \in L^\infty(S;H^2(\Omega))$ can be achieved for the evolution model (see Section 3.2.1). In Section 3.2.2 we give an example which shows that in spite of smooth data a similar regularity result is not valid for the time derivatives $\partial_t u$ and $\partial_t z$.

3.2.1 Improved regularity for scalar u

The regularity results in Theorem 3.4 can be improved if u is scalar and if certain compatibility conditions between the submatrices A_{ij} of A and the constitutive function g are satisfied. Here the idea is to construct a reflection operator R, which is adapted to the structure of the the coefficient matrix A_{11}. In contrast to Section 3.1.3 the problem is not reflected perpendicular to the boundary but with respect to the vector $A_{11}\nu$, where $\nu : \partial\Omega \to \partial B_1(0) \subset \mathbb{R}^d$ is the interior normal vector to $\partial\Omega$. Due to the compatibility conditions between the coefficients and the constitutive function g the reflected data do not contain second spatial derivatives of u or first derivatives of z. Hence the reflected data have the regularity $(b_e, F_e) \in W^{1,1}(S; \mathcal{Y}_1)$ instead of $(b_e, F_e) \in L^\infty(S; \mathcal{Y}_1)$ with $\mathcal{Y}_1 = L^2(\Omega_e) \times H^1(\Omega_e)$. Thus, we may apply part (a) of Theorem 3.3 and obtain the improved global regularity described in Theorem 3.7 here below.

To be more precise, the problem under consideration reads: Find $u : S \times \Omega \to \mathbb{R}$, $z : S \times \Omega \to \mathbb{R}^N$ such that for given $A_{11} \in C^{0,1}(\overline{\Omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, $A_{12} = A_{21}^\top \in C^{0,1}(\overline{\Omega}, \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^d))$ and $A_{22} \in C^{0,1}(\overline{\Omega}, \mathbb{R}^{N\times N}_{\mathrm{sym}})$ we have

$$\begin{aligned} D_u\Psi(u(t))[v] = \int_\Omega (A_{11}\nabla u(t) + A_{12}z(t)) \cdot \nabla v \,\mathrm{d}x = \int_\Omega b(t) \cdot v \,\mathrm{d}x \qquad \forall v \in V, \\ \partial_t z(t) \in \mathcal{G}(-(A_{21}\nabla u(t) + A_{22}z(t)) + F(t)), \\ z(0) = z_0. \end{aligned}$$

It is assumed that $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{(d+N)\times(d+N)})$ is uniformly positive definite. Let $\nu : \partial\Omega \to \partial B_1(0)$ be the interior normal vector on $\partial\Omega$. In order to formulate the compatibility conditions, we define for $x \in \partial\Omega$

$$R_{\nu(x)} = \mathbb{I} - \frac{2}{A_{11}(x)\nu(x) \cdot \nu(x)} A_{11}(x)\nu(x) \otimes \nu(x). \tag{3.20}$$

The matrix R_ν locally determines the reflection at $\partial\Omega$. Simple calculations show that $R^2_{\nu(x)} = \mathbb{I}$ and $R_{\nu(x)}A_{11}(x)R^\top_{\nu(x)} = A_{11}(x)$. The basic assumptions and compatibility conditions are:

- **R5** $\Omega \subset \mathbb{R}^d$ is a bounded domain with a $C^{2,1}$-smooth boundary (it is used that $\nu \in C^{1,1}(\partial\Omega)$).
- **R6** $(b, F) \in W^{1,1}(S; \mathcal{Y}_1)$ with $\mathcal{Y}_1$ from $\mathbf{R4}_a$, $z_0 = 0$.
- **R7** There exists a mapping $P \in C^{0,1}(\partial\Omega, \mathbb{R}^{N\times N})$ such that for every $x \in \partial\Omega$ the inverse matrix $(P(x))^{-1}$ exists and the following conditions hold for all $\eta \in \mathbb{R}^N$

$$R_{\nu(x)}A_{12}(x)P(x) = A_{12}(x), \;\; P(x)^\top A_{22}(x)P(x) = A_{22}(x), \;\; -P(x)^{-1}g(-P(x)^{-\top}\eta) = g(\eta).$$

Theorem 3.7. *[58] Let* **R5**-**R7** *be satisfied and assume that the pair* $(u, z) \in W^{1,1}(S; H^1_0(\Omega) \times L^2(\Omega))$ *solves* (3.6)–(3.8). *Then* $u \in L^\infty(S; H^2(\Omega))$ *and* $z \in L^\infty(S; H^1(\Omega))$.

We refer to [58] for a detailed proof.

Example 3.8. Assume that the coefficient matrix A is constant, that $N = d$, $A_{12} = -A_{11}$ and $A_{22} = A_{11} + L$ with $L \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ positive definite. Hence, $\Psi(u, z) = \frac{1}{2}\int_\Omega A_{11}(\nabla u - z)\cdot(\nabla u - z) + Lz \cdot z \mathrm{d}x$. Moreover we assume that $A_{11} = \mathbb{I}$, which can always be achieved after a suitable change of coordinates and a suitable transformation in the state space of z. The mapping R_ν now takes the form $R_\nu = \mathbb{I} - 2\nu \otimes \nu$ for $\nu \in \partial B_1(0)$ and the compatibility conditions reduce to

R7' $P_\nu = R_\nu$, $R_\nu^\top L R_\nu = L$ and $-R_\nu^\top g(-R_\nu \eta) = g(\eta)$ for all $\eta \in \mathbb{R}^d$.

It is shown in [58] that **R7'** is satisfied if and only if there exists $\alpha > 0$ such that $L = \alpha\mathbb{I}$. Moreover, if $g = \partial\chi_K$ with $K \subset \mathbb{R}^d$ convex, closed and $0 \in K$, then **R7'** holds if and only if $K = -R_\nu K$ for all $\nu \in \mathbb{R}^d$. In this situation, Theorem 3.7 yields the improved regularity result.

This example shows that if the "anisotropy" in Hooke's law given by the matrix A_{11} is correlated with the anisotropy in the hardening coefficients A_{22} and L and the constitutive function g, then the displacements $u(t)$ have full H^2-regularity up to the boundary $\partial\Omega$. It is an open question whether this regularity is still valid if the compatibility condition **R7** is violated. Moreover it is not known, whether a similar result is true for real elasto-plastic models, where u is not a scalar function.

3.2.2 Example: $\partial_t z(t) \notin H^1(\Omega)$

The following example shows that in spite of smooth data there might exist a time interval (t_1, t_2) such that $\partial_t z(t) \notin H^1(\Omega)$ for all $t \in (t_1, t_2)$. Hence, one should not expect $z \in W^{1,1}(S; H^1(\Omega))$. The example is inspired by Seregin's paper [95].

Let $0 < R_1 < R_2$. We set $\Omega = B_{R_2}(0) \backslash B_{R_1}(0)$ and choose the following energy for $u, z : \Omega \to \mathbb{R}$:

$$\Psi(u,z) = \tfrac{1}{2} \int_\Omega \left|\nabla u - \tfrac{x}{|x|} z\right|^2 + z^2 \, \mathrm{d}x.$$

Moreover, $g(\eta) := \partial\chi_{[-1,1]}(\eta)$ for $\eta \in \mathbb{R}$. It is assumed that $u(t)\big|_{\partial B_{R_1}} = 0$, $u(t)\big|_{\partial B_{R_2}} = t$, $z_0 = 0$ and that the remaining data (F, b) vanish. It is easily checked that the assumptions of Theorem 3.7 are satisfied and hence the problem has a unique solution with the regularity $\nabla u, z \in W^{1,1}(S; L^2(\Omega)) \cap L^\infty(S; H^1(\Omega))$. Due to the rotational symmetry of the problem the solution does not depend on the angle and can be calculated explicitly. Introducing polar-coordinates, the solution $u, z : S \times (R_1, R_2) \to \mathbb{R}$ has to satisfy for $r \in (R_1, R_2)$ and $t \in S$

$$\begin{aligned} \partial_r^2 u + r^{-1}\partial_r u - \partial_r z - r^{-1} z = 0 \quad &\text{in } S \times (R_1, R_2), \\ \partial_t z \in \partial\chi_{[-1,1]}(\partial_r u - 2z) \quad &\text{in } S \times (R_1, R_2), \\ z(0,\cdot) = 0, \quad u(t, R_1) = 0, \quad &u(t, R_2) = t. \end{aligned}$$

For $t \leq t_1 := R_1 \ln(R_2/R_1)$ it follows that $u(t,r) = \frac{t \ln(r/R_1)}{\ln(R_2/R_1)}$, $z(t,r) = 0$. In this regime, no plastic strains are present. For $t > t_1$ the plastic variable z starts to grow and there exists $r_*(t)$ such that $z(t,r) > 0$ for $r < r_*$ and $z(r,t) = 0$ for $r > r_*$, i.e. $r_*(t)$ separates the plastic region from the elastic region. The dependence of r_* on t is given implicitly by the relation

$$t(r_*) = R_1 - r_* + r_*(\ln R_2 r_* - \ln R_1^2).$$

Simple calculations show that $t(r_*)$ is strictly increasing, and hence $r_*(t) \geq R_1$ is strictly growing, as well. Moreover, for $t \geq t_1$ we have

$$u(t,r) = \begin{cases} b(t) - r + 2r_*(t)\ln r & \text{if } r \leq r_*(t) \\ c(t) + r_*(t)\ln r & \text{else} \end{cases}, \quad z(t,r) = \begin{cases} -1 + r_*(t) r^{-1} & \text{if } r \leq r_*(t), \\ 0 & \text{else} \end{cases},$$

with functions $b(t) = R_1 - 2r_*(t)\ln R_1$ and $c(t) = t - r_*(t)\ln R_2$. Since $\partial_t r_*(t) > 0$ for $t \geq t_1$ it follows that $\partial_t z(t,\cdot) \notin H^1(R_1, R_2)$ for $t > t_1$, see also Figure 1.

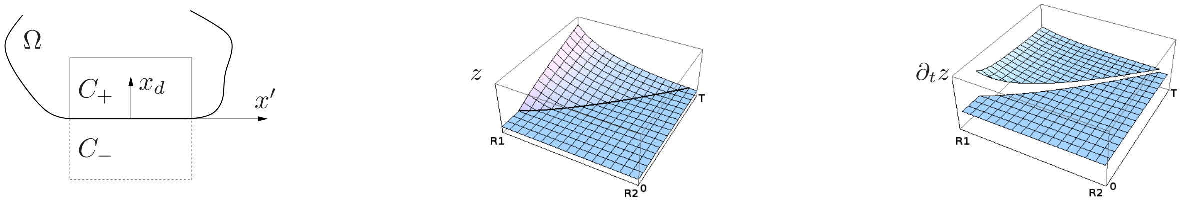

Figure 1: Example for the notation in Section 3.1.3 (left); Graph of the solution $z : (0,T) \times (R_1, R_2) \to \mathbb{R}$ (middle) and of the time derivative $\partial_t z$ (right).

3.3 Regularity for variants of the elasto-plastic model and overview on the corresponding literature

The starting point for the review of the literature on spatial regularity properties of elasto-plastic models is the system introduced in (3.6)–(3.8) with the particular energy density

$$\psi(\varepsilon, z) = \tfrac{1}{2}\big(A(\varepsilon - Bz)\cdot(\varepsilon - Bz) + Lz \cdot z\big) \tag{3.21}$$

for $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $z \in \mathbb{R}^N$. It is assumed that $A \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is symmetric and positive definite, $L \in \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^N)$ is symmetric and positive semi-definite and $B \in \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^{d\times d}_{\mathrm{sym}})$. The corresponding evolution model reads

$$\operatorname{div}\sigma(t) + b(t) = 0, \qquad \sigma(t) = A(\varepsilon(\nabla u(t)) - Bz(t)), \tag{3.22}$$

$$\partial_t z(t) \in \mathcal{G}(-\partial_z\psi(\varepsilon(\nabla u(t)), z(t)) + F(t)). \tag{3.23}$$

together with initial and boundary conditions. Depending on the properties of L and $\mathcal{G}$ different spatial regularity results were derived in the literature.

3.3.1 Regularity for models with positive semi-definite elastic energy and monotone, multivalued g

Only very few regularity results are available for models where the elastic energy density ψ in (3.21) is positive semi-definite but not positive definite. The corresponding elastic energy is convex but not strictly convex on the full state space $\mathcal{Q}$. As a consequence, a-priori estimates like those provided in Theorem 3.2 cannot be obtained in general. In contrast, the complementary energy, which is expressed via the generalized stresses, is still coercive. The regularity investigations therefore typically take a stress based version of (3.22)–(3.23) as a starting point. In this framework to the authors' knowledge only the Prandtl-Reuss model and models with linear isotropic hardening are discussed in the literature with regard to regularity questions.

The Prandtl-Reuss model describes elastic, perfectly plastic material behavior without hardening. The internal variable z is identified with the plastic strain tensor $\varepsilon_p \in \mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$, $B = \mathbb{I}$ and $L = 0$. Moreover, the constitutive function g is typically identified with $\partial\chi_K$, where K is a convex set given according to the von Mises or the Tresca flow rule, see Example 3.5.

The existence theorems provide stresses with $\sigma(t) \in L^2(\Omega)$ and $u(t) \in BD(\Omega)$, where $BD(\Omega)$ denotes the space of bounded deformations, see e.g. [53, 67, 102, 8, 105, 28]. Higher spatial regularity is derived by Bensoussan and Frehse [13] and Demyanov [31] for the case that K is defined by the von Mises yield condition. They obtain $\sigma \in L^\infty([0,T]; H^1_{\mathrm{loc}}(\Omega))$, which coincides with the local results in Theorem 3.4. The stress regularity is proved by approximating the Prandtl-Reuss model with the viscous power-law like Norton-Hoff model [13] and by time discretization [31]. Tangential properties are discussed in [18]. To the author's knowledge these are the only known spatial regularity results for the Prandtl-Reuss model. In particular there is no information about higher global regularity. In the dynamical case, Shi proved a local spatial result for σ and u [96].

If $z(0) = 0$, then the first step in the time discretization of the Prandtl-Reuss model leads to the stationary, elastic, perfectly plastic Hencky model. Here, it is proved for the von Mises case that $\sigma \in H^1_{\mathrm{loc}}(\Omega) \cap H^{\frac{1}{2}-\delta}(\Omega)$ for every $\delta > 0$, where Ω is a bounded Lipschitz domain which satisfies an additional geometrical condition near those points, where the Dirichlet and Neumann boundary intersect. We refer to [12] and [92, 39] together with the references therein for the local result and to [56, 15] for the global and a tangential result. The key of the proofs is to approximate the Hencky model with nonlinear elastic models and to derive uniform regularity estimates for the approximating models. In addition, the authors in [39] obtain a result concerning partial regularity of the solutions. It is an open problem whether the global result can be improved in the case of a smooth boundary with pure Dirichlet or pure Neumann conditions, see the discussion in [95].

A further typical elasto-plastic model with a positive semidefinite energy density ψ describes linear isotropic hardening. Here, the internal variable z consists of the plastic strains ε_p and a scalar hardening variable γ characterizing the radius of the set of admissible stress states. The quadratic elastic energy is given by $\psi(\varepsilon, \varepsilon_p, \gamma) = \frac{1}{2}\big(A(\varepsilon - \varepsilon_p) \cdot (\varepsilon - \varepsilon_p) + \alpha\gamma^2\big)$ for $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\varepsilon_p \in \mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$ and fixed $\alpha > 0$. The constitutive function is defined as $g = \partial\chi_K$ with $K = \{\,(\tau, \mu) \in \mathbb{R}^{d\times d}_{\mathrm{sym,dev}} \times \mathbb{R}\,;\, \mu \geq 0,\, |\tau| \leq \sigma_0 + \sigma_1\mu\,\}$ and constants $\sigma_i > 0$. The first investigations concerning spatial regularity in the isotropic case were carried out by Seregin [93]. Here, the results $\sigma \in L^\infty(S; H^1_{\mathrm{loc}}(\Omega))$, $\gamma \in L^\infty(S; H^1_{\mathrm{loc}}(\Omega))$, $\nabla u \in L^\infty(S; BD_{\mathrm{loc}}(\Omega))$ were obtained by studying the regularity properties of a time-discretized version and proving uniform bounds. Hölder properties of the solutions were investigated in [37].

3.3.2 Spatial regularity for regularized models

Replacing the maximal monotone constitutive function $\mathcal{G} : L^2(\Omega, \mathbb{R}^N) \to \mathcal{P}(L^2(\Omega, \mathbb{R}^N))$ from (3.23) with its Yosida approximation leads to regularized elasto-visco-plastic models with a Lipschitz-continuous nonlinearity in the evolution law. The therewith obtained models are a subclass of the elasto-visco-plastic models studied e.g. by Sofonea et al., see [52, 35]. Given an energy $\Psi : \mathcal{Q} \to \mathbb{R}$ as defined in (3.4)–(3.5) with a coefficient matrix $A \in L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^{m\times d} \times \mathbb{R}^N, \mathbb{R}^{m\times d} \times \mathbb{R}^N))$ and given a Lipschitz-continuous operator $\mathcal{F} : \mathcal{Q} \to L^2(\Omega, \mathbb{R}^N)$ these models read as follows:

$$D_u\Psi(u(t), z(t)) = b(t), \quad \partial_t z(t) = \mathcal{F}(u(t), z(t)), \quad z(0) = z_0 \tag{3.24}$$

together with boundary conditions on $\partial\Omega$. If the submatrix $A_{11} \in L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^{m\times d}, \mathbb{R}^{m\times d}))$ of A is symmetric and if the induced bilinear form $a(u, v) = \int_\Omega A_{11}\nabla u \cdot \nabla v \,\mathrm{d}x$ is coercive on V, then a standard application of Banach's fixed point theorem implies the existence of a unique solution $(u, z) \in W^{1,\infty}(S; \mathcal{Q})$ provided that $b \in W^{1,\infty}(S; V^*)$.

For these models the local spatial regularity was investigated in [75] with a difference quotient argument and in [61], while the global regularity was studied in [19]. The global regularity theorem in [19] states that if the linear elliptic operator induced by A_{11} is an isomorphism between the spaces $H^1_{\Gamma_{\text{Dir}}}(\Omega) \cap H^{1+s}(\Omega)$ and Y_s for some $s \in (0,1]$, where Y_s is a suitable subspace of $H^{s-1}(\Omega)$, then for every $b \in W^{1,\infty}(S;Y_s)$ the solution of (3.24) satisfies $u \in W^{1,\infty}(S;H^{1+s}(\Omega))$ and $z \in W^{1,\infty}(S;H^s(\Omega))$. In this way, global regularity properties of elliptic operators on possibly nonsmooth domains and with mixed boundary conditions directly influence the regularity properties of the viscous evolution model (3.24). The proof is carried out by deriving uniform regularity bounds for the sequence of approximating solutions generated via the Banach fixed point theorem. Here it is not needed that the elastic energy Ψ is coercive on $\mathcal{Q}$, the coercivity of $a(u,v) := \int_\Omega A_{11}\nabla u \cdot \nabla v \, dx$ on V is sufficient.

While for elasto-plasticity models (with a multivalued monotone constitutive function g) *local* regularity results can be deduced by proving uniform regularity bounds for the sequence of the approximating Yosida-regularized models, see e.g. [5], it is an unsolved problem, how to obtain uniform bounds in order to carry over *global* spatial regularity results from the viscous model to the elasto-plastic limit problem.

A further possibility to regularize elasto-plastic models is to replace the constitutive function $\mathcal{G} = \partial\chi_{\mathcal{K}}$ with a power-law like ansatz. This approach is used in [105] in order to regularize the Prandtl-Reuss model. Assume again that $z = \varepsilon_p \in \mathbb{R}^{d\times d}_{\text{sym,dev}}$, $B = \mathbb{I}$, $L = 0$ and replace $\partial\chi_{K_{\text{vM}}}$ (cf. Example 3.5) with

$$g_N(\sigma) = c_0^{1-N} \left|\sigma^D\right|^{N-2} \sigma_D,$$

for $\sigma \in \mathbb{R}^{d\times d}_{\text{sym}}$. Here, $\sigma^D = \sigma - \frac{1}{d}\operatorname{tr}\sigma\,\mathbb{I}$ denotes the deviatoric part of the tensor σ. The parameter $N > 1$ is a strain hardening exponent, whereas c_0 can be interpreted as a yield stress. The resulting viscous model is the so called Norton/Hoff model and consists of the relation (3.22) which is completed by the evolution law $\partial_t\varepsilon_p(t) = g_N(\sigma(t))$. For $N \to \infty$, the Norton/Hoff model approximates the Prandtl/Reuss model [105]. After eliminating the plastic strains ε_p one obtains the usual form of the Norton/Hoff model:

$$\operatorname{div}\sigma(t) + b(t) = 0, \quad A^{-1}\partial_t\sigma(t) + c_0^{1-N}\left|\sigma^D(t)\right|^{N-2}\sigma^D(t) = \partial_t\varepsilon(\nabla u(t)).$$

Bensoussan/Frehse [12] proved the local spatial regularity result $\sigma \in L^\infty((0,T);H^1_{\text{loc}}(\Omega))$ for the stress tensor via a difference quotient argument. A global result seems not to be available in the literature.

A time discretization of the Norton/Hoff model leads to the stationary Norton/Hoff or Ramberg/Osgood model, which is given by equation (3.22) in combination with the relation $\varepsilon(\nabla u) = A^{-1}\sigma + c_0^{1-N}\left|\sigma^D\right|^{N-2}\sigma^D$. Several authors studied local and global regularity and the Hölder properties of the stresses and displacements of this model for domains with smooth boundaries as well as for domains with nonsmooth boundaries [12, 101, 55, 56, 14, 33].

3.3.3 Spatial regularity for time incremental versions

A further way to prove regularity properties of elasto-viscoplastic models is to study the smoothness of solutions to time-discretized versions and to derive regularity bounds which are uniform with respect to the time step size. This method was applied e.g. in [93] to obtain local results, while for global results uniform bounds are not known. We discuss here global regularity properties for the time discretized version under the assumption that the elastic energy Ψ is coercive and that $g = \partial\chi_K$ with a convex and closed set K. The different equivalent formulations of the discretized equations, which we present here below, are commonly used in a computational context of elasto-plasticity, [99, 98].

Let **R1** and **R2** be satisfied and assume that $g = \partial\chi_K$, where $K \subset \mathbb{R}^N$ is convex, closed and with $0 \in K$. Let further $\mathcal{K} = \{\,\eta \in L^2(\Omega)\,;\,\eta(x) \in K$ a.e. in $\Omega\,\}$. A time discretization via an implicit Euler scheme leads to the following problem with $\Delta t = T/n$, $0 = t_0^n < t_1^n < \ldots < t_n^n = T$: Find $(u_k^n, z_k^n) \in V \times L^2(\Omega)$, $1 \le k \le n$, which satisfy

$$D_u\Psi(u_k^n, z_k^n) - b(t_k^n) = 0, \qquad \tfrac{1}{\Delta t}(z_k^n - z_{k-1}^n) \in \partial\chi_{\mathcal{K}}(-D_z\Psi(u_k^n, z_k^n)). \tag{3.25}$$

Observe that z_k^n solves (3.25) if and only if

$$z_k^n = \operatorname{argmin}\{\,F(u_k^n, \eta, z_{k-1}^n, \Delta t)\,;\,\eta \in L^2(\Omega)\,\}, \tag{3.26}$$

$$F(u_k^n, \eta, z_{k-1}^n, \Delta t) = \frac{1}{2}\int_\Omega A_{22}(\eta - z_{k-1})\cdot(\eta - z_{k-1})\,\mathrm{d}x + \Delta t\,\chi_{\mathcal{K}}(-(A_{21}\nabla u_k^n + A_{22}z_k^n)). \tag{3.27}$$

In terms of the new variables $\Sigma_k^{\text{trial}} = -(A_{21}\nabla u_k^n + A_{22}z_{k-1}^n)$ and $\Sigma_k = -(A_{21}\nabla u_k^n + A_{22}z_k^n)$, it follows that z_k satisfies (3.26) if and only if

$$z_k^n = z_{k-1}^n + A_{22}^{-1}(\Sigma_k^{\text{trial}} - \Sigma_k), \tag{3.28}$$

$$\Sigma_k = \operatorname{argmin}\{\,\widetilde{F}(\theta, \Sigma_k^{\text{trial}}, \Delta t)\,;\,\theta \in L^2(\Omega)\,\}, \tag{3.29}$$

$$\widetilde{F}(\theta, \Sigma_k^{\text{trial}}, \Delta t) = \frac{1}{2}\int_\Omega A_{22}^{-1}(\theta - \Sigma_k^{\text{trial}})\cdot(\theta - \Sigma_k^{\text{trial}})\,\mathrm{d}x + \Delta t\,\chi_{\mathcal{K}}(\theta). \tag{3.30}$$

Since the coefficient matrix A_{22}^{-1} induces a scalar product on $L^2(\Omega)$, Σ_k can be interpreted as the projection of Σ_k^{trial} onto the convex and closed set $\mathcal{K}$ with respect to this scalar product. Let $\mathcal{P}_{A_{22}^{-1},\mathcal{K}} : L^2(\Omega) \to L^2(\Omega)$ be the projection operator on $\mathcal{K}$. Hence, $\Sigma_k = \mathcal{P}_{A_{22}^{-1},\mathcal{K}}(\Sigma_k^{\text{trial}})$ and in addition, $\Sigma_k(x) = P_{A_{22}^{-1}(x),K}(\Sigma_k^{\text{trial}}(x))$ in Ω, where $P_{A_{22}^{-1}(x),K} : \mathbb{R}^N \to \mathbb{R}^N$ is the corresponding pointwise projection operator on K. With these notations, problem (3.25) is equivalent to the following problem: Find $u_k^n \in V$ and $z_k^n \in L^2(\Omega)$ such that for given $z_{k-1}^n \in L^2(\Omega)$ we have

$$\int_\Omega \mathcal{M}(x, \nabla u_k^n(x), z_{k-1}^n(x))\cdot\nabla v(x)\,\mathrm{d}x = \langle b(t_k^n), v\rangle \qquad \forall v \in V, \tag{3.31}$$

$$z_k^n = -A_{22}^{-1}\Big(A_{21}\nabla u_k^n + \mathcal{P}_{A_{22}^{-1},\mathcal{K}}\big(-A_{21}\nabla u_k^n - A_{22}z_{k-1}^n\big)\Big), \tag{3.32}$$

where the mapping $\mathcal{M} : \Omega \times \mathbb{R}^{m\times d} \times \mathbb{R}^N \to \mathbb{R}^{m\times d}$ is defined as

$$\mathcal{M}(x, F, z) = L_1(x)F - A_{12}(x)A_{22}(x)^{-1}P_{A_{22}^{-1}(x),K}\big(-A_{21}(x)F - A_{22}(x)z\big)$$

with the Schur complement matrix $L_1 = A_{11} - A_{12}A_{22}^{-1}A_{21} \in C^{0,1}(\Omega, \mathrm{Lin}(\mathbb{R}^{m\times d}, \mathbb{R}^{m\times d}))$. Observe that in general $\mathcal{M}$ is not differentiable with respect to F and z. The Lipschitz-continuity of the projection operator, assumption **R2** and the assumption $0 \in K$ imply that the mapping $\mathcal{M}$ has the following properties: there exist constants $c_1, c_2 > 0$ such that for every $x, x_i \in \Omega$, $F, F_i \in \mathbb{R}^{m\times d}$ and $z, z_i \in \mathbb{R}^N$ we have

$$|\mathcal{M}(x_1, F, z) - \mathcal{M}(x_2, F, z)| \le c_1(|F| + |z|)\,|x_1 - x_2|\,, \tag{3.33}$$

$$|\mathcal{M}(x, F_1, z_1) - \mathcal{M}(x, F_2, z_2)| \le c_2(|F_1 - F_2| + |z_1 - z_2|), \tag{3.34}$$

$$\mathcal{M}(x, 0, 0) = 0. \tag{3.35}$$

Moreover, $\mathcal{M}$ induces a strongly monotone operator on V, i.e. there exists a constant $\beta > 0$ such that for all $u_1, u_2 \in V$ and $z \in L^2(\Omega)$ we have:

$$\int_\Omega \big(\mathcal{M}(x, \nabla u_1, z) - \mathcal{M}(x, \nabla u_2, z)\big) : \nabla(u_1 - u_2)\,\mathrm{d}x \geq \beta \left\|u_1 - u_2\right\|^2_{H^1(\Omega)} .$$

This follows from the monotonicity of the projection operator and from the fact that due to assumption **R2**, the induced bilinear form $b(u, v) := \int_\Omega L_1 \nabla u \cdot \nabla v\,\mathrm{d}x$, $u, v \in V$, is symmetric and V-coercive. Finally, the mapping $\mathcal{M}$ is strongly rank-one monotone. That means that there exists a constant $c_{LH} > 0$ such that for every $x \in \overline{\Omega}$, $F \in \mathbb{R}^{m\times d}$, $z \in \mathbb{R}^N$, $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^d$ we have

$$\big(\mathcal{M}(x, F + \xi \otimes \eta, z) - \mathcal{M}(x, F, z)\big) : \xi \otimes \eta \geq c_{LH} \left|\xi\right|^2 \left|\eta\right|^2 . \tag{3.36}$$

This is a consequence of the monotonicity of the pointwise projection operator and the positivity properties of L_1, see e.g. [108, Th. 6.1]. Altogether it follows that $\mathcal{M}$ generates a quasilinear elliptic system of PDEs of second order for determining u^n_k. Standard existence results for equations involving Lipschitz-continuous, strongly monotone operators guarantee the existence of a unique element $u^n_k \in V$ solving (3.31) for arbitrary data $z^n_{k-1} \in L^2(\Omega)$ and $b \in V^*$, [110]. Moreover, u^n_k depends Lipschitz-continuously on the data. The regularity result in [59] guarantees that for given $b(t_k) \in L^2(\Omega)$ and $z^n_{k-1} \in H^1(\Omega)$ we have the global regularity $(u^n_k, z^n_k) \in H^2(\Omega) \times H^1(\Omega)$ provided that **R1** and **R2** are satisfied. Unfortunately it is not known how to derive estimates for $\|u^n_k\|_{H^2(\Omega)}$ which are uniform with respect to the time step Δt.

Quasilinear elliptic systems of a similar structure resulting from various regularizing ansatzes for elasto-plastic models were also studied with respect to regularity questions in the references [86, 20, 57, 91, 94, 38, 79, 89].

4 Numerical realization via a Slant Newton Method

As it is pointed out in Section 3.3.3 one possibility to numerically solve the system of elasto-plasticity is to solve the system of nonlinear elliptic equations which emerges after an (implicit) time discretization and an elimination of the internal variables. This system in general involves a nonlinearity which is not differentiable as an operator between function spaces. Hence, a standard Newton's method, which relies on the derivative of the nonlinear operator, is not appropriate to solve the nonlinear system. Instead we discuss a Newton-like method, where the derivative is replaced by a slanting function leading to a Slant Newton Method. This approach is explained for a rate-independent elasto-plastic model with linear isotropic hardening.

4.1 Problem Specification

Consider the *Prandtl-Reuß elastoplasticity problem with isotropic hardening*, which is a specialization of (2.1)–(2.5) in the following way: Define the internal variable with size $N = 7$ via $z(x, t) = (z_1(x, t), \dots, z_6(x, t), \gamma(x, t))$, and the projection

$$B : \mathbb{R}^N \to \mathcal{S}^3 , \; z \mapsto \varepsilon_p = \begin{pmatrix} z_1 & z_4 & z_5 \\ z_4 & z_2 & z_6 \\ z_5 & z_6 & z_3 \end{pmatrix} . \tag{4.1}$$

For easier notation let us, from now on, denote the plastic strain by p instead of ε_p. The associated free energy density is assumed to be of the form

$$\psi(\varepsilon, p, \gamma) = \frac{1}{2}\langle A(\varepsilon - p),\, \varepsilon - p\rangle_F + \frac{1}{2}\gamma^2\,,$$

where $\varepsilon \in \mathcal{S}^3$, $p \in \mathcal{S}^3$, $\gamma \in \mathbb{R}$, the Frobenius scalar product for matrices is defined $\langle B, C\rangle_F = \sum_{ij} B_{ij}\, C_{ij}$, and it is assumed that the elasticity tensor A characterizes isotropic material behavior and has the explicit form

$$A : \mathcal{S}^3 \to \mathcal{S}^3,\ \varepsilon \mapsto 2\mu\varepsilon + \lambda\, \mathrm{tr}\, \varepsilon\, \mathbb{I}\,.$$

Here, $\lambda, \mu > 0$ are the Lamé constants and describe the elastic behavior of the material. This choice of the elastic energy density induces the following relation between the generalized plastic strains $\Pi = (p, \gamma) \in \mathcal{S}^3 \times \mathbb{R}$ and the generalized stresses $\Sigma = (T, \alpha) \in \mathcal{S}^3 \times \mathbb{R}$:

$$\begin{aligned} T &= \partial_\varepsilon \psi(\varepsilon, p, \gamma) = -\partial_p \psi(\varepsilon, p, \gamma) = A(\varepsilon - p)\,, \\ \alpha &= -\partial_\gamma \psi(\varepsilon, p, \gamma) = -\gamma\,. \end{aligned}$$

The constitutive flow law (2.3) in the Prandtl-Reuss case with isotropic hardening reads

$$\partial_t \Pi(x, t) \in \partial\chi_K(\Sigma(x, t))\,, \tag{4.2}$$

where $\partial\chi_K$ denotes the subgradient of the indicator function regarding the convex set K of admissible generalized stresses, which is given by

$$K = \{\, \Sigma \in \mathcal{S}^3 \times \mathbb{R}\,;\, \phi(\Sigma) \le 0\,\} \tag{4.3}$$

with the yield function

$$\phi(\Sigma) = \|\mathrm{dev}\, T\|_F - T_y(1 + H\alpha) + \chi_{[0,\infty)}(\alpha). \tag{4.4}$$

The parameters *yield stress* $T_y > 0$ and *modulus of hardening* $H > 0$ describe the plastic behavior of the material, the deviator, a projection onto the trace-free subspace of $\mathcal{S}^3$, is calculated by $\mathrm{dev}\, T = T - (\mathrm{tr}\, T / \mathrm{tr}\, \mathbb{I})\, \mathbb{I}$, and the Frobenius norm reads $\|T\|_F^2 = \langle T, T\rangle_F$. Notice, that (4.2) is a specialization of (2.3). Geometrically spoken, the subgradient $\partial\chi_K$ describes the normal cone of the convex set of admissible stresses K at the point Σ. In other words, the prescription $\frac{\partial\Pi}{\partial t} \in \partial\chi_K(\Sigma)$ means that either there is no solution with respect to the generalized strain Π (if Σ is not in K), or Π remains constant (if Σ is in the interior of K), or $\frac{\partial\Pi}{\partial t}$ has to be chosen such that it is orthogonal to the boundary of the set of admissible stresses K at the point Σ (if Σ is on the boundary of K).

Summarizing, the problem of Prandtl-Reuß elastoplasticity with isotropic hardening reads: Find the displacement $u(x, t) \in \mathbb{R}^3$, the plastic strain $p(x, t) \in \mathcal{S}^3$, and the hardening parameter $\alpha(x, t) \in \mathbb{R}$, which solve

$$-\,\mathrm{div}_x\, T(x, t) = b(x, t)\,, \tag{4.5}$$

$$T(x, t) = A(\varepsilon(u(x, t)) - p(x, t))\,, \tag{4.6}$$

$$\frac{\partial\, \Pi}{\partial t}(x, t) \in \partial\chi_K(\Sigma(x, t))\,, \quad \text{where } \Pi = (p, -\alpha) \text{ and } \Sigma = (T, \alpha)\,, \tag{4.7}$$

$$\Pi(x, 0) = \Pi^{(0)}(x)\,, \tag{4.8}$$

$$u(x, t) = \gamma_D(x, t)\,, \quad \text{if } x \in \Gamma_D \subset \partial\Omega\,, \tag{4.9}$$

$$T(x, t)\, n(x, t) = \gamma_N(x, t)\,, \quad \text{if } x \in \Gamma_N \subset \partial\Omega\,. \tag{4.10}$$

We turn to the numerical solution of the problem (4.5)–(4.10). The algorithm described in this section is of Newton's type, enjoying the property of local super-linear convergence. It is an interesting question for future investigation, whether there is a more general class of problems covered by the laws (2.1)–(2.5), to which this algorithm is applicable.

We define $V := \left[H^1(\Omega)\right]^3$, $V_0 := \{\, v \in V\,;\, v = 0 \text{ on } \Gamma_D \,\}$, $V_D := \{\, v \in V\,;\, v = u_D \text{ on } \Gamma_D \,\}$ for $u_D \in \left[H^{1/2}(\Gamma_D)\right]^3$, $Q := \left[L^2(\Omega, \mathcal{S}^3)\right]$, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$.

Analogously to the discussion in Section 5 the problem (4.5)–(4.10) may equivalently be formulated in the global energetic framework based on the energy

$$\mathcal{E}(t, u, \Pi) = \int_\Omega \psi(\varepsilon(\nabla u), p, \gamma)\,\mathrm{d}x - \langle b(t), u\rangle$$

and the dissipation potential

$$\mathcal{R}(u, p, \gamma) = \int_\Omega \rho(p(x), \gamma(x))\,\mathrm{d}x$$

for $u \in V_D$, $p \in Q$ and $\gamma \in L^2(\Omega)$. The density ρ is given as the convex conjugate of χ_K and has the structure

$$\rho(p, \gamma) = \chi_K^*(p, \gamma) = \begin{cases} T_y\, \|p\|_F & \text{if } \operatorname{tr} p = 0 \text{ and } \|p\|_F \le -\frac{\gamma}{T_y H}\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

Using an implicit Euler-discretization for a partition $0 = t_0 < t_1 < \ldots < t_n = T$ and the sets

$$\begin{aligned} L^2_+(\Omega) &= \{\, f \in L^2(\Omega)\,;\, f \ge 0 \text{ almost everywhere} \,\}\,, \\ L^2_-(\Omega) &= \{\, f \in L^2(\Omega)\,;\, f \le 0 \text{ almost everywhere} \,\}\,, \end{aligned}$$

the time discretized problem reads:

Problem 4.1. Given $(u_{k-1}, p_{k-1}, \gamma_{k-1}) \in V_D \times Q \times L^2_-(\Omega)$ find $(u_k, p_k, \gamma_k) \in V_D \times Q \times L^2_-(\Omega)$ such that

$$(u_k, p_k, \gamma_k) \in \operatorname{argmin}\{\, \mathcal{E}(t_k, v, q, \xi) + \mathcal{R}(v - u_{k-1}, q - p_{k-1}, \xi - \gamma_{k-1})\,;\, (v, q, \xi) \in V_D \times Q \times L^2_-(\Omega) \,\}\,.$$

It is shown in [21, 6] that the hardening variable $\alpha_k = -\gamma_k$ can be eliminated from the minimization problem in such a way that for determining $(u_k, p_k, -\alpha_k)$ one can equivalently solve the following problem:

Problem 4.2. Given $(u_{k-1}, p_{k-1}, \alpha_{k-1}) \in V_D \times Q \times L^2_+(\Omega)$ find $(u_k, p_k, \alpha_k) \in V_D \times Q \times L^2_+(\Omega)$ such that

$$(u_k, p_k) \in \operatorname{argmin}\{\, \bar{J}_k(v, \theta)\,;\, (v, \theta) \in V_D \times Q \,\}\,, \tag{4.11}$$

$$\alpha_k = \alpha_{k-1} + T_y H\, \|p_k - p_{k-1}\|_F\,. \tag{4.12}$$

Here, the global energy functional $\bar{J}_k : V_D \times Q \to \overline{\mathbb{R}}$ is defined by

$$\bar{J}_k(v, q) := \frac{1}{2}\|\varepsilon(v) - q\|_A^2 + \psi_k(q) - l_k(v)\,, \tag{4.13}$$

with

$$
\begin{aligned}
\langle q_1, q_2\rangle_A &:= \int_\Omega \langle A\, q_1(x)\,,\, q_2(x)\rangle_F \,\mathrm{d}x\,, \qquad \|q\|_A \;:=\; \langle q, q\rangle_A^{1/2}\,, && (4.14)\\
\tilde{\alpha}_k(q) &:= \alpha_{k-1} + T_y H \|q - p_{k-1}\|_F\,, && (4.15)\\
\psi_k(q) &:= \begin{cases} \int_\Omega \left(\frac{1}{2}\tilde{\alpha}_k(q)^2 + T_y\|q - p_{k-1}\|_F\right)\,\mathrm{d}x & \text{if } \operatorname{tr} q = \operatorname{tr} p_{k-1}\,,\\ +\infty & \text{else}\,, \end{cases} && (4.16)\\
l_k(v) &:= \int_\Omega b_k \cdot v \,\mathrm{d}x + \int_{\Gamma_N} \gamma_{N,k}\cdot v\,\mathrm{d}s\,. && (4.17)
\end{aligned}
$$

The body force $b(t_k) = b_k \in \left[L^2(\Omega)\right]^3$ and the traction $\gamma_N(t_k) = \gamma_{N,k} \in \left[H^{-1/2}(\Gamma_N)\right]^3$ are given. The functional $\bar{J}_k$ expresses the mechanical energy of the deformed system at the kth time step. Notice, that $\bar{J}_k$ is smooth with respect to the displacements v, but not with respect to the plastic strains q.

4.2 Solver Analysis

In [21] a method of an alternate minimization regarding the displacement v and the plastic strain q was investigated to solve Problem 4.2. The global linear convergence of the resulting method was shown and a local super-linear convergence was conjectured. Another interesting technique is to reduce Problem 4.2 to a minimization problem with respect to the displacements v only. This can be achieved by substituting the known explicit minimizer of J_k with respect to the plastic strain field for some given displacement v, namely by $q = \tilde{p}_k(\varepsilon(v))$. We will observe that such a reduced minimization problem is smooth with respect to the displacements v and its derivative is explicitly computable.

The following theorem is formulated for functionals mapping from a Hilbert space $\mathbb{H}$ provided with the scalar product $\langle \circ, \diamond\rangle_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}^2 := \langle \cdot, \cdot\rangle_{\mathbb{H}}$. If a function F is Fréchet differentiable, we shall denote its derivative in a point x by $\mathrm{D}\,F(x)$ and its Gâteaux differential in the direction y by $\mathrm{D}\,F(x\,;\,y)$. We refer to [34] concerning the definitions of convex, proper, lower semi-continuous, and coercive.

Theorem 4.3. *Let the function $f : \mathbb{H}\times\mathbb{H} \to \overline{\mathbb{R}}$ be defined*

$$
f(x, y) = \frac{1}{2}\|x - y\|_{\mathbb{H}}^2 + \psi(x) \tag{4.18}
$$

where ψ is a convex, proper, lower semi-continuous, and coercive function of $\mathbb{H}$ into $\overline{\mathbb{R}}$. Then $F(y) := \inf_{x\in\mathbb{H}} f(x, y)$ maps into $\mathbb{R}$, and there exists a unique function $\tilde{x} : \mathbb{H} \to \mathbb{H}$ such that $F(y) = f(\tilde{x}(y), y)$ for all $y \in \mathbb{H}$. Moreover, it holds:

1. *F is strictly convex and continuous in $\mathbb{H}$.*
2. *F is Fréchet differentiable with the Fréchet derivative*

$$
\mathrm{D}\,F(y) = \langle y - \tilde{x}(y)\,,\,\cdot\rangle_{\mathbb{H}} \quad \textit{for all } y \in \mathbb{H}\,. \tag{4.19}
$$

Proof. See [77, 7.d. Proposition]. □

We apply Theorem 4.3 to Problem 4.2 and obtain the following proposition.

Proposition 4.4. *Let $k \in \{1, \dots, n\}$ denote the time step, and let $\bar{J}_k$ be defined as in (4.13). Then there exists a unique mapping $\tilde{p}_k : Q \to Q$ satisfying*

$$\bar{J}_k\left(v, \tilde{p}_k\left(\varepsilon\left(v\right)\right)\right) = \inf_{q \in Q} \bar{J}_k\left(v, q\right) \quad \forall v \in V_D\,. \tag{4.20}$$

Let J_k be a mapping of V_D into $\mathbb{R}$ defined as

$$J_k(v) := \bar{J}_k(v, \tilde{p}_k(\varepsilon(v))) \quad \forall v \in V_D\,. \tag{4.21}$$

Then, J_k is strictly convex and Fréchet differentiable. The associated Gâteaux differential reads

$$\mathrm{D}\, J_k(v\,;\,w) = \langle \varepsilon(v) - \tilde{p}_k(\varepsilon(v))\,,\, \varepsilon(w)\rangle_A - l_k(w) \quad \forall w \in V_0 \tag{4.22}$$

with the scalar product $\langle \circ, \diamond\rangle_A$ defined in (4.14) and l_k defined in (4.17).

Proof. The functional $\bar{J}_k : V \times Q \to \overline{\mathbb{R}}$ defined in (4.13) using (4.14), (4.16) and (4.17) can be decomposed in $\bar{J}_k(v, q) = f_k(\varepsilon(v), q) - l_k(v)$, where the functional $f_k : Q \times Q \to \overline{\mathbb{R}}$ reads $f_k(s, q) := \frac{1}{2}\|q - s\|_A^2 + \psi_k(q)$. Theorem 4.3 states the existence of a unique minimizer $\tilde{p}_k : Q \to Q$ which satisfies the condition $f_k(s, \tilde{p}_k(s)) = \inf_{q \in Q} f_k(s, q)$, where the functional $F_k(s) := f_k(s, \tilde{p}_k(s))$ is strictly convex and differentiable with respect to $s \in Q$. Since the strain $\varepsilon(v)$ is a Fréchet differentiable, linear and injective mapping from V_D into Q, the composed functional $F_k(\varepsilon(v))$ is Fréchet differentiable and strictly convex with respect to $v \in V_D$. Considering the Fréchet differentiability and linearity of l_k with respect to $v \in V_D$, we conclude the strict convexity and Fréchet differentiability of the functional J_k defined in (4.21). The explicit form of the Gâteaux differential $\mathrm{D}\, J_k(v\,;\,w)$ in (4.22) results from the linearity of the two mappings l_k and ε, and the Fréchet derivative $DF_k(\varepsilon(v)\,;\,\cdot) = \langle \varepsilon(v) - \tilde{p}_k(\varepsilon(v))\,,\,\cdot\rangle_A$ as in (4.19), combined with the chain rule. □

The minimizer $\tilde{p}_k$ can be calculated by hand (see [6, 43]) and it exactly recovers the classical return mapping algorithm [98]. Let the trial stress $\tilde{T}_k : Q \to Q$ at the kth time step and the yield function $\phi_{k-1} : Q \to \mathbb{R}$ at the $k-1$st time step be defined by

$$\tilde{T}_k(q) := A(q - p_{k-1}) \quad \text{and} \quad \phi_{k-1}(T) := \|\mathrm{dev}\, T\|_F - T_y(1 + H\,\alpha_{k-1})\,. \tag{4.23}$$

Then, the minimizer $\tilde{p}_k$ reads

$$\tilde{p}_k(\varepsilon(v)) = \frac{1}{2\mu + T_y^2 H^2}\, \max\{0, \phi_{k-1}(\tilde{T}_k(\varepsilon(v)))\}\, \frac{\mathrm{dev}\, \tilde{T}_k(\varepsilon(v))}{\|\mathrm{dev}\, \tilde{T}_k(\varepsilon(v))\|_F} + p_{k-1}\,. \tag{4.24}$$

We obtain a smooth minimization problem by using J_k as in (4.21) with $\tilde{p}_k$ as in (4.24):

Problem 4.5. Find $u_k \in V_D$ such that $J_k(u_k) = \inf_{v \in V_D} J_k(v)$.

Remark 4.6. Problem 4.5 is uniquely solvable. This is due to the fact that functional J_k is strictly convex, coercive, proper and lower semi-continuous (see, e. g., [34, Chapter II, Proposition 1.2]). Solving Problem 4.5 numerically might be realized by applying Newton's Method $v^{j+1} = v^j - \left(\mathrm{D}^2 J_k(v^j)\right)^{-1} \mathrm{D}\, J_k(v^j)$. Unfortunately, the second derivative of J_k does not exist since the max-function in (4.24) is not differentiable. Therefore, we apply a Newton-like method which uses slanting functions (see [26]) instead of the second derivative. We shall call such a method a Slant Newton Method.

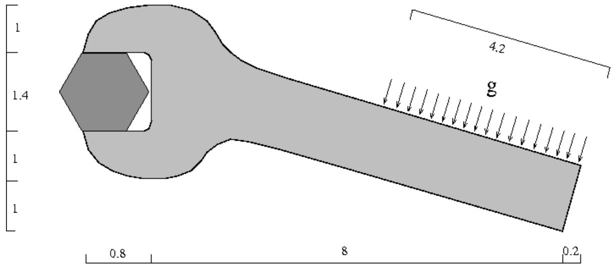

Figure 2: Problem setup.

Henceforth, let X and Y be Banach spaces, and $\mathcal{L}(X,Y)$ denote the set of all linear mappings of X into Y.

Definition 4.7. Let $U \subseteq X$ be an open subset and $x \in U$. A function $F : U \to Y$ is said to be *slantly differentiable at* x if there exists a mapping $F^o : U \to \mathcal{L}(X,Y)$ which is uniformly bounded in an open neighborhood of x, and a mapping $r : X \to Y$ with $\lim_{h\to 0}\|r(h)\|_Y \, \|h\|_X^{-1} = 0$ such that $F(x+h) = F(x) + F^o(x+h)\,h + r(h)$ holds for all $h \in X$ satisfying $(x+h) \in U$. We say, $F^o(x)$ is a *slanting function for* F *at* x. F is called *slantly differentiable in* U if there exists $F^o : U \to \mathcal{L}(X,Y)$ such that F^o is a slanting function for F for all $x \in U$. F^o is then called a *slanting function for* F *in* U.

Theorem 4.8. *Let $U \subseteq X$ be an open subset, and $F : U \to Y$ be a slantly differentiable function with a slanting function $F^o : U \to \mathcal{L}(X,Y)$. We suppose, that $x^* \in U$ is a solution to the nonlinear problem $F(x) = 0$. If $F^o(x)$ is non-singular for all $x \in U$ and $\{\|F^o(x)^{-1}\|_{\mathcal{L}(Y,X)} : x \in U\}$ is bounded, then the Newton-like iteration*

$$x^{j+1} = x^j - F^o(x^j)^{-1}F(x^j) \tag{4.25}$$

converges super-linearly to x^*, provided that $\|x^0 - x^*\|_X$ is sufficiently small.

The proof can be found in [26, Theorem 3.4] or [49, Theorem 1.1].

We apply the Slant Newton Method (4.25) to elastoplasticity by choosing $F = \mathrm{D}\,J_k$ as in (4.22). The max-function is slantly differentiable [49, Proposition 4.1] as a mapping of $L^p(\Omega)$ into $L^q(\Omega)$ if $p > q$ but not if $p \leq q$. Therefore, if it holds $\phi_{k-1}(\tilde{T}_k(\varepsilon(v))) \in L^{2+\delta}(\Omega)$ for some $\delta > 0$, then $\mathrm{D}\,J_k$ (cf. (4.22),(4.24)) has a slanting function which reads

$$(\mathrm{D}\,J_k)^o\,(v; w, \bar{w}) := \langle \varepsilon(w) - \tilde{p}_k^o(\varepsilon(v); \varepsilon(w))\,,\, \varepsilon(\bar{w})\rangle_A \tag{4.26}$$

with a slanting function for $\tilde{p}_k$, e. g.,

$$\tilde{p}_k^o(\varepsilon(v)\,;\,q) := \begin{cases} 0 & \text{if } \beta_k \leq 0\,, \\ \xi\left(\beta_k \operatorname{dev} q + (1-\beta_k)\,\frac{\langle \operatorname{dev}\tilde{T}_k\,,\,\operatorname{dev} q\rangle_F}{\|\operatorname{dev}\tilde{T}_k\|_F^2}\operatorname{dev}\tilde{T}_k\right) & \text{else}\,, \end{cases} \tag{4.27}$$

where the abbreviations $\xi := \frac{2\mu}{2\mu + T_y^2 H^2}$, $\tilde{T}_k := \tilde{T}_k(\varepsilon(v))$ and $\beta_k := \frac{\phi_{k-1}(\tilde{T}_k)}{\|\operatorname{dev}\tilde{T}_k\|_F}$ with ϕ_{k-1} and $\tilde{T}_k$ defined in (4.23) are used. $(\mathrm{D}\,J_k)^o$ in Equation (4.26) is commonly known as the *consistent tangent*, see [98]. For fixed $v \in V_D$, the bilinear form $(\mathrm{D}\,J_k)^o\,(v;\cdot,\cdot)$ in (4.26) is elliptic and bounded in V_0 (see [43, Lemma 2]).

Corollary 4.9. *Let* $k \in \{1, \ldots, n\}$, $\delta > 0$ *be fixed and* t_k *denote the kth time step. Let the mapping* $\mathrm{D}\, J_k : V_D \to {V_0}^*$ *be defined* $\mathrm{D}\, J_k(v) := \mathrm{D}\, J_k(v\,;\circ)$ *as in (4.22), and* $(\mathrm{D}\, J_k)^o : V_D \to \mathcal{L}(V_0, {V_0}^*)$ *be defined* $(\mathrm{D}\, J_k)^o\,(v) := (\mathrm{D}\, J_k)^o\,(v\,;\diamond,\circ)$ *as in* (4.26). *Then, the Slant Newton iteration*

$$v^{j+1} = v^j - \left[(\mathrm{D}\, J_k)^o\,(v^j)\right]^{-1} \mathrm{D}\, J_k(v^j)$$

converges super-linearly to the solution u_k *of Problem 4.5, provided that* $\|v^0 - u_k\|_V$ *is sufficiently small, and that* $\phi_{k-1}(\tilde{T}_k(\varepsilon(v)))$ *as in* (4.23) *is in* $L^{2+\delta}(\Omega)$ *for all* $v \in V_D$.

Proof. We check the assumptions of Theorem 4.8 for the choice $F = \mathrm{D}\, J_k$. Let $v \in V_D$ be arbitrarily fixed. The mapping $(\mathrm{D}\, J_k)^o\,(v) : V_0 \to {V_0}^*$ serves as a slanting function for $\mathrm{D}\, J_k$ at v, since $\phi_{k-1}(\tilde{T}_k(\varepsilon(v)))$ is in $L^{2+\delta}(\Omega)$. Moreover, $(\mathrm{D}\, J_k)^o\,(v) : V_0 \to {V_0}^*$ is bijective if and only if there exists a unique element w in V_0 such, that for arbitrary but fixed $f \in {V_0}^*$ there holds

$$(\mathrm{D}\, J_k)^o\,(v\,;w,\bar{w}) = f(\bar{w}) \quad \forall \bar{w} \in V_0\,. \tag{4.28}$$

Since the bilinear form $(\mathrm{D}\, J_k)^o\,(v)$ is elliptic and bounded (see [43, Lemma 4.9]), we apply the Lax-Milgram Theorem to ensure the existence of a unique solution w to (4.28). Finally, with κ_1 denoting the v-independent ellipticity constant for $(\mathrm{D}\, J_k)^o\,(v;\diamond,\circ)$, the uniform boundedness of $\left[(\mathrm{D}\, J_k)^o\,(\cdot)\right]^{-1} : V_D \to \mathcal{L}({V_0}^*, V_0)$ follows from the estimate

$$\begin{aligned}
\|[(\mathrm{D}\, J_k)^o\,(v)]^{-1}\| &= \sup_{w^* \in {V_0}^*} \frac{\|[(\mathrm{D}\, J_k)^o\,(v)]^{-1}\; w^*\|}{\|w^*\|_{{V_0}^*}} = \sup_{w \in V_0} \frac{\|w\|_V}{\|(\mathrm{D}\, J_k)^o\,(v\,;w,\cdot)\|_{{V_0}^*}} \\
&= \sup_{w \in V_0} \inf_{\bar{w} \in V_0} \frac{\|w\|_V\,\|\bar{w}\|_V}{|(\mathrm{D}\, J_k)^o\,(v\,;w,\bar{w})|} \leq \sup_{w \in V_0} \frac{\|w\|_V^2}{|(\mathrm{D}\, J_k)^o\,(v\,;w,w)|} \leq \frac{1}{\kappa_1}\,.
\end{aligned}$$

□

Remark 4.10. Notice the required assumption on the integrability of $\phi_{k-1}(\tilde{T}_k(\varepsilon(v)))$. It is still an open question, under which extra conditions this property can be satisfied for all $v \in V_D$, or, at least for all Newton iterates v^j. The local super-linear convergence in the spatially discrete case (after FE-discretization) can be shown without any additional assumption, see [43, Theorem 4.14].

4.3 Numerical Examples

Finite Element Method with nodal linear shape functions was used in the test examples below. The interested reader is referred to [44, 45, 46] for more convergence tables and numerical examples. The super-linear convergence was observed in both 2D and 3D computations.

4.3.1 2D-Example

We simulate the deformation of a screw-wrench under pressure, the problem geometry is shown in Figure 2. A screw-wrench sticks on a screw (homogeneous Dirichlet boundary condition) and a surface load g is applied to a part of the wrench's handhold in interior normal direction. The material parameters are set

$$\lambda = 1.15e8\,\tfrac{N}{m}\,, \qquad \mu = 7.7e7\,\tfrac{N}{m}\,, \qquad T_y = 2e6\,\tfrac{N}{m}\,, \qquad H = 0.001\,,$$

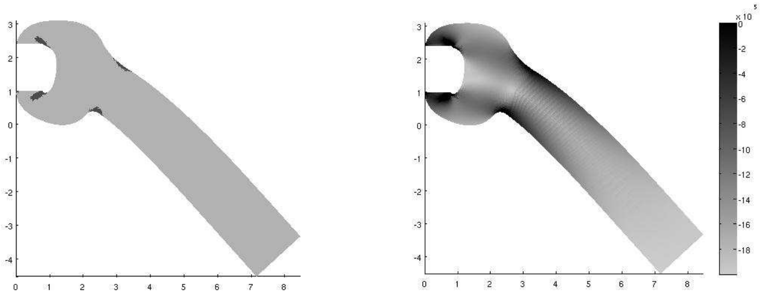

Figure 3: Elastoplastic zones (left) and yield function (right) of the deformed wrench geometry. The displacement is magnified by a factor 10 for visualization reasons.

and the traction intensity amounts $|g| = 6e4\,\frac{N}{m}$. Figure 3 shows the yield function (right) and the elastoplastic zones (left), where purely elastic zones are light, and plastic zones are dark. Table 1 reports on the super-linear convergence of the Newton-like method for graded uniform meshes. The implementation was done in Matlab.

DOF:	202	...	10590	41662	165246	658174	2627070
j=1:	1.000e+00	...	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
j=2:	6.510e-04	...	3.394e-01	4.344e-01	4.682e-01	5.038e-01	5.417e-01
j=3:	4.238e-09	...	4.018e-02	5.786e-02	8.919e-02	1.892e-01	2.552e-01
j=4:	1.266e-12	...	1.009e-03	3.076e-03	1.642e-02	2.253e-02	3.049e-02
j=5:		...	2.679e-07	4.550e-05	1.473e-03	7.595e-04	1.294e-03
j=6:			3.817e-13	2.244e-09	1.014e-04	6.519e-05	1.264e-04
j=7:				6.000e-13	2.628e-08	7.342e-09	8.528e-06
j=8:					1.047e-12	1.892e-12	4.153e-08
j=9:							3.638e-12

Table 1: The relative error in displacements $|v^j - v^{j-1}|_\varepsilon / \left(|v^j|_\varepsilon + |v^{j-1}|_\varepsilon\right)$ is displayed for increasing degrees of freedom (DOF), where $|v|_\varepsilon := \left(\int_\Omega \langle \varepsilon(v)\,,\, \varepsilon(v)\rangle_F \,\mathrm{d}x\right)^{1/2}$.

4.3.2 3D-Example

This three dimensional test example is similar to a two dimensional example in [100]. Figure 4 shows the quarter of a thin plate $(-10, 10) \times (-10, 10) \times (0, 2)$ with a circular hole of the radius $r = 1$ in the middle. One elastoplastic time step is performed, where a surface load g with the intensity $|g| = 450\,\frac{N}{m^2}$ is applied to the plate's upper and lower edge in outer normal direction. Due to the symmetry of the domain, the solution is calculated on one quarter of the domain only. Thus, homogeneous Dirichlet boundary conditions in the normal direction (gliding conditions) are considered for both symmetry axes. The material parameters are set

$$\lambda = 110744\,\frac{N}{m^2}\,, \quad \mu = 80193.8\,\frac{N}{m^2}\,, \quad \sigma_Y = 450\,\sqrt{2/3}\,\frac{N}{m^2}\,, \quad H = \frac{1}{2}\,.$$

Differently to the original problem in [100], the modulus of hardening H is nonzero, i.e., hardening effects are considered. Figure 5 shows the norm of the plastic strain field p (right) and the

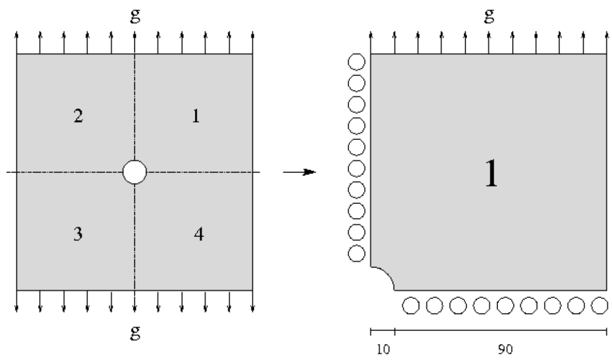

Figure 4: Here, the geometry of the example domain is outlined. Due to reasons of symmetry, only one of the quarters is solved.

coarsest refinement of the geometry (left). Table 2 reports on the convergence of the Slant Newton Method. The implementation was done in C++ using the NETGEN/NGSolve software package developed by J. Schöberl [90].

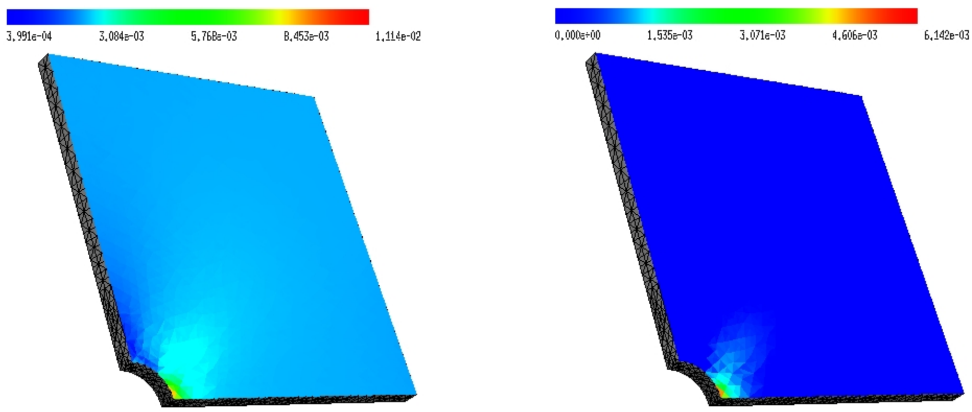

Figure 5: The Frobenius norms of the total strain ε (left) and of the plastic strain p (right).

5 Rate-independent evolutionary processes – Temporal regularity of solutions

This section is devoted to the subclass of quasistatic, rate-independent evolutionary processes. The time-evolution of a system can be considered as rate-independent if the time scales imposed to the system from the exterior are much larger than the intrinsic ones, i.e. if the external loadings evolve much slower than the internal variables. Throughout this section we will apply the energetic formulation of a rate-independent process. This approach does not use the classical formulation (2.1)–(2.5) but considers the energy functional $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty =: \mathbb{R} \cup \{\infty\}$ and the dissipation distance $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$ related to the evolution equation (2.3) in an appropriate state space $\mathcal{Q}$, which is assumed to be a Banach space with dual $\mathcal{Q}^*$. An energetic solution of the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ is defined as follows

DOF:	717	5736	45888	367104
j=1:	1.000e+00	1.000e+00	1.000e+00	1.000e+00
j=2:	1.013e-01	1.254e-01	1.367e-01	1.419e-01
j=3:	7.024e-03	6.919e-03	7.159e-03	6.993e-03
j=4:	1.076e-04	9.359e-05	1.263e-04	1.176e-04
j=5:	2.451e-08	6.768e-07	1.744e-06	1.849e-06
j=6:	7.149e-15	6.887e-12	4.874e-09	1.001e-08
j=7:			4.298e-13	2.368e-14

Table 2: This table outlines the convergence of the Slant Newton Method in 3D. We observe super-linear convergence and (almost) a constant number of iterations at each refinement.

Definition 5.1. The process $q=(u,z):[0,T]\to\mathcal{Q}$ is an energetic solution of the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if $t\mapsto\partial_t\mathcal{E}(t,q(t))\in L^1((0,T))$, if for all $t\in[0,T]$ we have $\mathcal{E}(t,q(t))<\infty$ and if the global stability inequality (S) and the global energy balance (E) are satisfied:

$$\textsf{Stability}: \quad \text{for all } \tilde{q}\in\mathcal{Q}: \ \mathcal{E}(t,q(t))\le\mathcal{E}(t,\tilde{q})+\mathcal{D}(q(t),\tilde{q}), \tag{S}$$

$$\textsf{Energy balance}: \quad \mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(q,[0,t])=\mathcal{E}(0,q(0))+\int_0^t\partial_\xi\mathcal{E}(\xi,q(\xi))\,\mathrm{d}\xi \tag{E}$$

with $\mathrm{Diss}_{\mathcal{D}}(q,[0,t]):=\sup\sum_{j=1}^N\mathcal{D}(q(\xi_{j-1}),q(\xi_j))$, where the supremum is taken over all partitions of $[0,t]$.

In Section 5.1.1 we will clarify the relations between the classical and the energetic formulation. Since the conditions (S) & (E) do not require that $\dot{q}$ exists, an energetic solution may in general have jumps with respect to time. In particular, (S) provides the uniform boundedness of $\mathcal{E}(t,q(t))$ and hence (E) yields that $q:[0,T]\to\mathcal{Q}$ is only of bounded variation in time with respect to the dissipation distance providing an L^1-norm in space. This means that in general the time derivative $\dot{q}$ is only given as a Radon-measure. Therefore, Section 5.2 pays special attention to the temporal regularity of energetic solutions. It is investigated how their temporal regularity can be improved due to additional convexity assumptions on the energy $\mathcal{E}$. In Section 5.2.1 it is explained that strict convexity of $\mathcal{E}$ on $\mathcal{Q}$ yields continuity of the solutions with respect to time. Section 5.2.2 deals with the Hölder- and Lipschitz-continuity of energetic solutions, which can be obtained by claiming a kind of uniform convexity on $\mathcal{E}$. In Section 5.3 the theory introduced in Section 5.2 is applied to evolutionary processes modeling plasticity, damage or phase transformations in shape memory alloys and we give examples on stored elastic energy densities that lead to such improved temporal regularity.

5.1 The energetic formulation of rate-independent processes

The outline of this section is to clarify the energetic formulation of rate-indepedent processes. Thereto Section 5.1.1 indicates the relation of energetic solutions to the concept of solution used in the Sections 2, 3. Moreover Section 5.1.2 gives a short introduction to the existence theory of energetic solutions. At this point we want to start our discussion with the mathematical characterization of rate-independence.

The energetic formulation of a rate-independent process is solely based on an energy functional $\mathcal{E}:[s,T]\times\mathcal{Q}\to\mathbb{R}_\infty$, which depends on time t and the state q, and a dissipation potential

$\mathcal{R} : \mathcal{Q} \to [0, \infty]$ depending on the velocity $\dot{q}$. It is assumed that the potential $\mathcal{R}$ is convex and positively 1-homogeneous, i.e. $\mathcal{R}(0) = 0$ and $\mathcal{R}(\lambda v) = \lambda \mathcal{R}(v)$ for all $\lambda > 0$ and all $v \in \mathcal{Q}$. Due to these two properties $\mathcal{R}$ satisfies a triangle inequality, i.e. for all $q_1, q_2, q_3 \in \mathcal{Q}$ it holds

$$\mathcal{R}(q_1 - q_2) = 2\mathcal{R}\big(\tfrac{1}{2}(q_1 - q_3) + \tfrac{1}{2}(q_3 - q_2)\big) \le 2\big(\tfrac{1}{2}\mathcal{R}(q_1 - q_3) + \tfrac{1}{2}\mathcal{R}(q_3 - q_2)\big) = \mathcal{R}(q_1 - q_3) + \mathcal{R}(q_3 - q_2)\,.$$

Hence the dissipation potential generates a dissipation distance

$$\mathcal{D}(q, \tilde{q}) = \mathcal{R}(\tilde{q} - q)\,, \tag{5.1}$$

which is an extended pseudo-distance on the state space $\mathcal{Q}$. This means that $\mathcal{D}$ satisfies the axioms of a metric (positivity, triangle inequality), except symmetry and it may attain the value ∞, as we will see in the examples of Section 5.3.

Rate-independence of a process $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ with the initial condition $q(s) = q_0 \in \mathcal{Q}$, the given external loadings $b \in C^1([s,T], \mathcal{Q}^*)$ and a solution $q : [s,T] \to \mathcal{Q}$ can be defined using an input-output operator

$$\mathcal{H}_{[s,T]} : \mathcal{Q} \times C^1([s,T], \mathcal{Q}^*) \to L^\infty([s,T], \mathcal{Q}) \cap BV_{\mathcal{D}}([s,T], \mathcal{Q}),\ (q_0, b) \mapsto q\,, \tag{5.2}$$

where $BV_{\mathcal{D}}([s,T], \mathcal{Q}) := \{q : [s,T] \to \mathcal{Q} \,|\, \mathrm{Diss}_{\mathcal{D}}(q, [s,T]) < \infty\}$. Thus, the input-output operator maps the given data (q_0, b) onto a solution of the problem. Therewith the rate-independence of the system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ can be characterized as follows

Definition 5.2. An evolutionary process $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$, which can be expressed by (5.2), is called rate-independent if for all $s_\star < T_\star$ and all $\alpha \in C^1([s_\star, T_\star])$ with $\dot{\alpha} > 0$ and $\alpha(s_\star) = s$, $\alpha(T_\star) = T$ the following holds:

$$\mathcal{H}_{[s_\star, T_\star]}(q_0, b \circ \alpha) = \mathcal{H}_{[s,T]}(q_0, b) \circ \alpha\,. \tag{5.3}$$

We verify now that the positive 1-homogeneity of $\mathcal{R}$ implies (5.3). We prove this implication for input-output operators $\mathcal{H}_{[s,t]} : \mathcal{Q} \times C^1([s,t], \mathcal{Q}^*) \to W^{1,1}([s,t], \mathcal{Q})$. Thereby, $\mathcal{Q}$ is in general a Lebesgue or Sobolev space defined with respect to a domain $\Omega \subset \mathbb{R}^d$. By mollification, see also [7], one can therefore show that for any $q \in BV_{\mathcal{D}}([0,T], \mathcal{Q})$ there is a sequence $(q_n)_{n\in\mathbb{N}} \subset C^\infty([0,T], \mathcal{Q})$ satisfying $q_n \to q$ in $L^1([0,T] \times \Omega)$, $\mathrm{Diss}_{\mathcal{D}}(q_n, [0,t]) < C$ and $\mathrm{Diss}_{\mathcal{D}}(q_n, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(q, [0,t])$ for all $t \in [0,T]$. Thus, the above mentioned implication also holds true for the input-output operators from (5.2).

Proposition 5.3. *Let* $\mathcal{H}_{[s,T]} : \mathcal{Q} \times C^1([s,T], \mathcal{Q}^*) \to W^{1,1}([s,t], \mathcal{Q})$, $(q_0, b) \mapsto q$, *be the input-output-operator for the rate-independent system* $(\mathcal{Q}, \mathcal{E}_b, \mathcal{R})$, *where* $\mathcal{E}_b$ *depends continuously on the external loading* b *and where* $\mathcal{R}$ *is convex and positively* 1*-homogeneous. Then* (5.3) *holds true.*

Proof. Let $s_\star < T_\star$ and $\alpha \in C^1([s_\star, T_\star])$ with $\dot{\alpha} > 0$ and $\alpha(s_\star) = s$, $\alpha(T_\star) = T$. In particular it holds $s_\star = \alpha^{-1}(s)$, $T_\star = \alpha^{-1}(T)$ and $(\alpha^{-1})' > 0$. Assume that $q : [s,T] \to \mathcal{Q}$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}_b, \mathcal{R},)$ satisfying $q(s) = q_0$. Hence (S)&(E) are satisfied for all $t \in [s,T]$. Now the time interval is rescaled, i.e. $t = \alpha(t_\star)$ for all $t \in [s,T]$. Then (S) implies that $\mathcal{E}_{b\circ\alpha}(t_\star, q \circ \alpha(t_\star)) \le \mathcal{E}_{b\circ\alpha}(t_\star, \tilde{q}) + \mathcal{D}(q \circ \alpha(t_\star), \tilde{q})$ for all $\tilde{q} \in \mathcal{Q}$, i.e. (S) holds true for all $t_\star \in [s_\star, T_\star]$ for $q \circ \alpha : [s_\star, T_\star] \to \mathcal{Q}$ and the system $(\mathcal{Q}, \mathcal{E}_{b\circ\alpha}, \mathcal{R})$.

For a function $q \in W^{1,1}([s,T], \mathcal{Q})$ it holds that $\mathrm{Diss}_{\mathcal{D}}(q, [s,t]) = \int_s^t \mathcal{R}(\dot{q}(\xi))\,\mathrm{d}\xi$, which can be verified by applying the positive 1-homogeneity of $\mathcal{R}$ and the mean value theorem of differentiability to the definition of $\mathrm{Diss}_{\mathcal{D}}(q, [s,t])$. Then, for $s = \alpha(s_\star)$ and $t = \alpha(t_\star)$ the application of the chain rule on $q(\alpha(t_\star))$ together with the positive 1-homogeneity of $\mathcal{R}$ imply that

$\int_s^t \mathcal{R}(\dot q(\xi))\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \mathcal{R}(\partial_\alpha q(\alpha(\xi)))\dot\alpha(\xi)\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \mathcal{R}(\partial_\alpha q(\alpha(\xi))\dot\alpha(\xi))\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \mathcal{R}(\partial_\xi q\circ\alpha(\xi))\,\mathrm{d}\xi$, which proves that $\mathrm{Diss}_{\mathcal{D}}(q,[s,t]) = \mathrm{Diss}_{\mathcal{D}}(q\circ\alpha,[s_\star,t_\star])$. Again by the chain rule we calculate that $\int_s^t \partial_\xi\mathcal{E}_b(\xi,q(\xi))\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \partial_\alpha\mathcal{E}_{b\circ\alpha}(\xi,q(\xi))\dot\alpha(\xi)\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \partial_\xi\mathcal{E}_{b\circ\alpha}(\xi,q(\xi))\,\mathrm{d}\xi$ and hence (E) is verified for all $t_\star\in[s_\star,T_\star]$ for $q\circ\alpha$ and $(\mathcal{Q},\mathcal{E}_{b\circ\alpha},\mathcal{R})$. Moreover the initial condition is satisfied since $q_0 = q(s) = q\circ\alpha(s_\star)$.

With the same arguments we can verify for an energetic solution $q_\star:[s_\star,T_\star]\to\mathcal{Q}$ of $(\mathcal{Q},\mathcal{E}_{b\circ\alpha},\mathcal{R})$ with $q_\star(s_\star)=q_0$ that $q_\star\circ\alpha^{-1}$ satisfies (S)&(E) with $(\mathcal{Q},\mathcal{E}_b,\mathcal{R})$ for all $t\in[s,T]$ and with $q_0 = q_\star(s_\star) = q_\star\circ\alpha^{-1}(s)$. Thus, (5.3) is proved. □

5.1.1 Different concepts of solutions and their relations

In this section we clarify the relation of energetic solutions with other types of solutions. To do so, we only treat the simplest case here, namely when $\mathcal{E}:[0,T]\times\mathcal{Q}\to\mathbb{R}_\infty$ is quadratic, i.e.

$$\mathcal{E}(t,q) := \tfrac{1}{2}\langle\mathcal{A}\,q,q\rangle - \langle b(t),q\rangle \tag{5.4}$$

for the given linear, symmetric, positive definite operator $\mathcal{A}:\mathcal{Q}\to\mathcal{Q}^*$ and the given external loading $b\in C^1([0,T],\mathcal{Q}^*)$. Thereby $\mathcal{Q}$ is a Banach space and $q_n\to q$ in $\mathcal{Q}$ indicates the convergence of a sequence $(q_n)\subset\mathcal{Q}$ in the weak topology of $\mathcal{Q}$. As it can be easily verified in this setting, $\mathcal{E}$ satisfies

1. **Continuity:** If $\|q_n-q\|_{\mathcal{Q}}\to 0$, then $|\mathcal{E}(t,q_n)-\mathcal{E}(t,q)|\to 0$ for all $t\in[0,T]$.
2. **Coercivity:** There is a constant $c>0$ such that $\mathcal{E}(t,q)\geq c\|q\|^2_{\mathcal{Q}}$ for all $q\in\mathcal{Q}$ and all $t\in[0,T]$. (Cf. **R2** in Section 3.1.1)
3. **Uniform convexity:** There is a constant $c_{\mathcal{A}}>0$ such that for all $t\in[0,T]$, all $q_0,q_1\in\mathcal{Q}$ and all $\theta\in[0,1]$ it holds
$$\mathcal{E}(t,\theta q_1+(1-\theta)q_0) \leq \theta\mathcal{E}(t,q_1)+(1-\theta)\mathcal{E}(t,q_0) - c_{\mathcal{A}}\theta(1-\theta)\|q_1-q_0\|^2_{\mathcal{Q}}. \tag{5.5}$$
4. **Uniform control of the powers:** For all $q\in\mathcal{Q}$ with $\mathcal{E}(t_\star,q)<\infty$ for some $t_\star\in[0,T]$ we have $\partial_t\mathcal{E}(\cdot,q)\in L^1([0,T])$ with $\partial\mathcal{E}(t,q) = -\langle\dot b(t),q\rangle$ and there are constants $c_1>0$, $c_2\geq 0$ such that $|\partial_t\mathcal{E}(t,q)|\leq c_1(\mathcal{E}(t,q)+c_2)$.
5. **Uniform continuity of the powers:** For all $(t,q_n)\to(t,q)$ in $\mathcal{Q}$ it holds $\partial_t\mathcal{E}(t,q_n)\to\partial_t\mathcal{E}(t,q)$.
6. **Closedness of stable sets:** If (t_n,q_n) satisfy (S) for all $n\in\mathbb{N}$ and $(t_n,q_n)\to(t,q)$ in $[0,T]\times\mathcal{Q}$, then also (t,q) satisfies (S).
7. **Differentiability:** For all $t\in[0,T]$ and all $q\in\mathcal{Q}$ the energy functional $\mathcal{E}(t,\cdot)$ is Gâteaux-differentiable with $\mathrm{D}_q\mathcal{E}(t,q) = \mathcal{A}q - b(t)$.

Thereby Items 1-5 and 7 can be easily verified using the properties of $\mathcal{A}$ and b. Item 6 can be obtained by choosing $\tilde q_n = q_n+v-q$ with $v\in\mathcal{Q}$ for all $n\in\mathbb{N}$, which yields $\mathcal{D}(q_n,\tilde q_n) = \mathcal{R}(\tilde q_n - q_n) = \mathcal{R}(v-q)$ for all $n\in\mathbb{N}$. Since b is continuous in time we have $\langle b(t_n),\tilde q_n\rangle\to\langle b(t),v\rangle$ and since $\mathcal{A}\in\mathrm{Lin}(\mathcal{Q},\mathcal{Q}^*)$ it holds $\langle\mathcal{A}(v-q),q_n\rangle\to\langle\mathcal{A}(v-q),q\rangle$. Using these observations in (S) for all $n\in\mathbb{N}$ one recovers (S) for the limit (t,q).

In Section 5.1.2 it is explained that the properties 1–6 together with the properties of the extended pseudo-distance $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}_\infty$ allow to prove the existence of an energetic solution. Furthermore in Section 5.2.2 it is discussed that property 3 yields Lipschitz-continuity of the energetic solution $q : [0, T] \to \mathcal{Q}$ with respect to time, i.e. there is a constant $C_L > 0$ such that $\|q(s)-q(t)\|_\mathcal{Q} \leq C_L|s-t|$. Hence $q \in W^{1,\infty}([0, T], \mathcal{Q})$, which means that $\dot{q}$ exists a.e. in $[0, T]$.

Since the dissipation potential $\mathcal{R} : \mathcal{Q} \to [0, \infty]$ is convex and positively 1-homogeneous but not necessarily differentiable we introduce its subdifferential

$$\partial_v \mathcal{R}(v) := \{q^* \in \mathcal{Q}^* \,|\, \mathcal{R}(w) \geq \mathcal{R}(v) + \langle q^*, w-v\rangle \text{ for all } w \in \mathcal{Q}\}\,. \tag{5.6}$$

Due to the validity of 1–7 and (5.6) we may consider the subdifferential formulation (SDF) and the formulation as a variational inequality (VI), which directly use $\dot{q}$, as alternative formulations to the energetic one. The subdifferential formulation of the evolutionary process reads as follows

Definition 5.4 (Subdifferential formulation)**.** For a given initial condition $q_0 \in \mathcal{Q}$ find $q : [0, T] \to \mathcal{Q}$ such that for a.e. $t \in [0, T]$ it holds

$$0 \in \partial\mathcal{R}(\dot{q}(t)) + \mathrm{D}_q\mathcal{E}(t, q(t)) \subset \mathcal{Q}^* \quad \text{and } q(0) = q_0 \in \mathcal{Q}\,. \tag{SDF}$$

Moreover (SDF) is equivalent to $-\mathrm{D}_q\mathcal{E}(t, q) \in \partial\mathcal{R}(\dot{q})$ and due to the definition of the subdifferential we may equivalently formulate the rate-independent process as a variational inequality

Definition 5.5 (Variational inequality)**.** For a given inital condition $q_0 \in \mathcal{Q}$ find $q : [0, T] \to \mathcal{Q}$ such that for a.e. $t \in [0, T]$ and for all $v \in \mathcal{Q}$ it holds

$$\langle \mathrm{D}_q\mathcal{E}(t, q), v - \dot{q}\rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}) \geq 0 \quad \text{and } q(0) = q_0 \in \mathcal{Q}\,. \tag{VI}$$

Between the three different formulations (S) & (E), (SDF) and (VI) the following relation holds

Lemma 5.6. *If $\mathcal{E} : \mathcal{Q} \to \mathbb{R}_\infty$ satisfies the properties 1–7, if $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$ is an extended pseudo-distance and lower semicontinuous on the Banach space $\mathcal{Q}$ and if q_0 satisfies* (S) *at $t = 0$, every energetic solution of the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ also is a solution in the sense of* (SDF) *as well as* (VI) *and vice versa, i.e.* (S) & (E) $\Leftrightarrow$ (SDF) $\Leftrightarrow$ (VI).

Proof. Let $q : [0, T] \to \mathcal{Q}$ solve (S) & (E). By Theorem 5.13 we have $q \in W^{1,\infty}([0, T], \mathcal{Q})$, so that $\mathrm{Diss}_\mathcal{D}(q, [0, t]) = \int_0^t \mathcal{R}(\dot{q}(\xi))\,\mathrm{d}\xi$ for all $t \in [0, T]$. Hence (E) reads $\mathcal{E}(t, q(t)) + \int_0^t \mathcal{R}(\dot{q}(\xi))\,\mathrm{d}\xi = \mathcal{E}(0, q(0)) + \int_0^t \partial_t\mathcal{E}(\xi, q(\xi))\,\mathrm{d}\xi$. Applying $\frac{\mathrm{d}}{\mathrm{d}t}$ leads to $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, q(t)) + \mathcal{R}(\dot{q}(t)) = \partial_t\mathcal{E}(t, q(t))$ for almost all $t \in [0, T]$. Using the chain rule on $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, q(t))$ yields

$$\langle \mathrm{D}_q\mathcal{E}(t, q(t)), \dot{q}(t)\rangle + \mathcal{R}(\dot{q}(t)) = 0\,. \tag{E$_\mathrm{loc}$}$$

Furthermore, inserting $q(t) + hv$ for $v \in \mathcal{Q}$ in (S) together with Item 7 results in

$$\langle \mathrm{D}_q\mathcal{E}(t, q(t)), v\rangle + \mathcal{R}(v) \geq 0 \quad \text{for all } v \in \mathcal{Q} \tag{S$_\mathrm{loc}$}$$

and subtracting (E_loc) from (S_loc) finally yields (VI), which is equivalent to (SDF).

Assume now that q solves (VI) and (SDF) for a.e. $t \in [0, T]$. Multiply (VI) by $h > 0$ and put $v = \frac{\tilde{q}}{h}$. For $h \to 0$ one obtains (S_loc). Due to the convexity and the Gâteaux-differentiablility of $\mathcal{E}(t, \cdot)$ for all $q \in \mathcal{Q}$ we find from (S_loc) with $v = \tilde{q} - q(t)$ that $0 \leq \langle \mathrm{D}_q\mathcal{E}(t, q(t)), \tilde{q} - q(t)\rangle + \mathcal{R}(\tilde{q} - q(t)) \leq \mathcal{E}(t, \tilde{q}) - \mathcal{E}(t, q(t)) + \mathcal{R}(\tilde{q} - q(t))$ for a.e. $t \in [0, T]$. But since $q : [0, T] \to \mathcal{Q}$ is

Lipschitz-continuous in time and since $\mathcal{E}(\cdot,\tilde{q})$ is continuous for all $\tilde{q} \in \mathcal{Q}$ we observe that (S) holds for all $t \in [0,T]$. Now (E) has to be proven. Choosing thereto $v = \dot{q}(t)$ in ($\mathrm{S_{loc}}$) gives $\langle \mathrm{D}_q\mathcal{E}(t,q(t)),\dot{q}(t)\rangle + \mathcal{R}(\dot{q}(t)) \geq 0$ and $v = 0$ in (VI) yields $\langle \mathrm{D}_q\mathcal{E}(t,q(t)),-\dot{q}(t)\rangle - \mathcal{R}(\dot{q}(t)) \geq 0$, which proves ($\mathrm{E_{loc}}$). By integrating ($\mathrm{E_{loc}}$) over $[0,t]$ we verify that (E) holds for all $t \in [0,T]$. $\square$

The equivalence established in Lemma 5.6 is in general only true for energies satisfying the uniform convexity inequality in property 3. For convex energies it can be verified if energetic solutions are supplied with sufficient temporal regularity. In the case of nonconvex energies, or energies which are convex but not jointly convex in $q = (u,z)$, energetic solutions are of bounded variation with respect to time. Hence they may have jumps in time and the time-derivative is only a Radon-measure. Relations between the three different formulations with $\dot{q}$ as a Radon-measure are discussed in [70]. Furthermore it comments on their relations in the case of doubly nonlinear problems, which were introduced in [27] and where $\mathcal{E}$ is only subdifferentiable but not Gâteaux-differentiable.

In many applications the dissipation potential only depends on the internal variable z, not on the full state $q = (u,z)$, i.e. $\mathcal{R}(\dot{q}) = \widetilde{\mathcal{R}}(\dot{z})$, so that $\partial\mathcal{R}(\dot{q}) = \partial_{\dot{u}}\widetilde{\mathcal{R}}(\dot{z}) \times \partial_{\dot{z}}\widetilde{\mathcal{R}}(\dot{z}) = \{0\} \times \partial\widetilde{\mathcal{R}}(\dot{z})$. This is also the case in the setting of plasticity studied in Sections 2, 3. Using the duality theory of functionals one can establish a relation between the flow rule given by (2.3) and (2.9) and the dissipation potential $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ under the assumption that $\mathcal{Z}$ is a reflexive Banach space. In view of the definition of the subdifferential $\partial\mathcal{R}(z) = \{z^* \in \mathcal{Z}^* \,|\, \mathcal{R}(\tilde{z})-\mathcal{R}(z) \geq \langle z^*,\tilde{z}-z\rangle \text{ for all } \tilde{z} \in \mathcal{Z}\}$ the direct calculation of the Legendre-Fenchel transform of the positively 1-homogeneous dissipation potential $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ yields that its dual functional is given as the indicator function of $\partial\mathcal{R}(0)$, i.e. $\mathcal{R}^*(z^*) = \sup_{z\in\mathcal{Z}}\big(\langle z^*,z\rangle - \mathcal{R}(z)\big) = I_{\partial\mathcal{R}(0)}(z^*)$ for all $z^* \in \mathcal{Z}^*$, where $I_{\partial\mathcal{R}(0)}(z^*) = 0$ if $z \in \partial\mathcal{R}(0)$ and $I_{\partial\mathcal{R}(0)}(z^*) = \infty$ otherwise.

Since $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ is assumed to be convex and lower semicontinuous on the reflexive Banach space $\mathcal{Z}$ the theorem of Fenchel-Moreau implies that $\mathcal{R} = (\mathcal{R}^*)^*$, see [51]. Assume now that the dissipation potential is an integral functional, i.e. for all $z \in \mathcal{Z}$ it is $\mathcal{R}(z) = \int_\Omega R(z(x))\,\mathrm{d}x$, where R is a positively 1-homogeneous, convex density and $\Omega \subset \mathbb{R}^d$ is a d-dimensional domain. Then [51, p. 296, Th. 1] states that $\mathcal{R}^*(\cdot) = \Big(\int_\Omega R(\cdot)\,\mathrm{d}x\Big)^* = \int_\Omega R^*(\cdot)\,\mathrm{d}x$, i.e. for the density $R : V \to [0,\infty]$, where $V \in \{\mathbb{R},\mathbb{R}^d,\mathbb{R}^{d\times d}\}$, holds the analogous relation to its Legendre-Fenchel transformed: $R(z) = R^{**}(z)$ for all $z \in V$. Thus, between the subdifferential formulation (SDF) of Definition 5.4 and the flow rule given by (2.3) and (2.9) we have established the relation $\dot{z} \in g(-\nabla_z\psi(e,z)) = \partial R^*(-\nabla_z\psi(e,z))$, where R^* is the Legendre-Fenchel transformed of the density R of the positively 1-homogeneous dissipation potential $\mathcal{R}$.

Throughout this chapter we will in general consider dissipation potentials $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ of the form

$$\mathcal{R}(z) = \int_\Omega R(z)\,\mathrm{d}x \quad \text{with } R : V \to [0,\infty],\; R(z) = \begin{cases} \varrho|z| & \text{if } z \in A \subset V\,, \\ \infty & \text{otherwise}\,, \end{cases} \tag{5.7}$$

where $0 < \varrho_0 \leq \varrho \in L^\infty(\Omega)$.

Example 5.7. For $K_{vM}{=}\{\tau \in \mathbb{R}^{d\times d}_{\mathrm{sym,dev}} \,|\, |\tau| \leq c_0\}$ from Example 3.5 it is $R_{vM}(\varepsilon_p) = c_0|\varepsilon_p|$ for all $\varepsilon_p \in \mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$.

5.1.2 Existence of energetic solutions

The quasistatic evolution of mechanical processes in solids such as elasto-plastic deformations, damage, crack propagation or contact angle hystheresis of droplets have been analyzed in various contributions, amongst these e.g. [28, 63, 40, 29, 16, 30]. All these processes can be described in terms of an energy functional $\mathcal{E}$ and a dissipation distance $\mathcal{D}$, so that the energetic formulation from Definition 5.1 applies. Within the works [65, 71, 36, 69] an abstract existence theory for energetic solutions of rate-independent processes has been developed. It is based on the assumption that $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ satisfies

$$\textit{Quasi-distance:} \quad \forall\, z_1, z_2, z_3 \in \mathcal{Z} : \quad \begin{array}{l} \mathcal{D}(z_1, z_2) = 0 \;\Leftrightarrow z_1 = z_2 \text{ and} \\ \mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3); \end{array} \tag{D1}$$

$$\textit{Lower semi-continuity:} \quad \mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty] \text{ is weakly seq. lower semi-continuous.} \tag{D2}$$

and it uses the following assumptions on the energy $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$

$$\begin{array}{l} \textit{Compactness of energy sublevels:} \quad \forall\, t\in[0, T]\ \forall\, E\in\mathbb{R} : \\ \qquad L_E(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}(t, q) \le E\} \text{ is weakly seq. compact.} \end{array} \tag{E1}$$

$$\begin{array}{l} \textit{Uniform control of the power:} \ \exists\, c_0\in\mathbb{R}\ \exists\, c_1{>}0\ \forall\, (t_q, q)\in[0, T]{\times}\mathcal{Q} \text{ with } \mathcal{E}(t_q, q) < \infty : \\ \qquad \mathcal{E}(\cdot, q) \in C^1([0, T]) \text{ and } |\partial_t\mathcal{E}(t, q)| \le c_1(c_0{+}\mathcal{E}(t, q)) \text{ for all } t\in[0, T]. \end{array} \tag{E2}$$

These properties ensure the following existence result for energetic solutions of rate-independent processes.

Theorem 5.8 ([69])**.** *Let* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *satisfy conditions* (E1)*,* (E2) *and* (D1)*,* (D2)*. Moreover, let the following compatibility conditions hold: For every sequence* $(t_k, q_k)_{k\in\mathbb{N}}$ *with* $(t_k, q_k) \rightharpoonup (t, q)$ *in* $[0, T] \times \mathcal{Q}$ *and* (t_k, q_k) *satisfying* (S) *for all* $k \in \mathbb{N}$ *we have*

$$\partial_t\mathcal{E}(t, q_k) \to \partial_t\mathcal{E}(t, q)\,, \tag{C1}$$

$$(t, q) \textit{ satisfies } \text{(S)}\,. \tag{C2}$$

Then, for each initial condition $(t = 0, q_0)$ *satisfying* (S) *there exists an energetic solution* $q : [0, T] \to \mathcal{Q}$ *for* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *with* $q(0) = q_0$*.*

The proof of Theorem 5.8 is based on a time-discretization, where conditions (E1), (D2) ensure the existence of a minimizer for the time-incremental minimization problem at each time-step. Thereto the direct method of the calculus of variations is applied. In particular conditions (E1) and (D2) can be verified if $\mathcal{E}$ and $\mathcal{D}$ are convex and coercive. Hence, for a given partition $\Pi := \{0 = t_0 < t_1 < \ldots < t_M = T\}$, for every $k = 1, \ldots, M$ one has to

$$\text{find } q_k \in \operatorname{argmin}\{\mathcal{E}(t_k, \tilde{q}) + \mathcal{D}(z_{k-1}, \tilde{z}) \,|\, \tilde{q} = (\tilde{u}, \tilde{z}) \in \mathcal{Q}\}\,. \tag{IP}$$

One then defines a piecewise constant interpolant q^Π with $q^\Pi(t) := q_{k-1}$ for $t \in [t_{k-1}, t_k)$ and $q^\Pi(T) = q_M$. Choosing a sequence $(\Pi_m)_{m\in\mathbb{N}}$ of partitions, where the fineness of Π_m tends to 0 as $m \to \infty$, it is possible to apply a version of Helly's selection principle to the sequence $(q^{\Pi_m})_{m\in\mathbb{N}}$, see thereto [65]. Using (E2) and the compatibility conditions (C1), (C2) it can be shown that the limit function fulfills the properties (S) and (E) of an energetic solution. See e.g. [69] for a detailed proof.

In various works this abstract theory has been applied to prove the existence of energetic solutions to rate-independent processes in the field of plasticity, damage, delamination, crack-propagation, hystheresis or shape memory alloys, amongst these [66, 68, 106, 87, 60, 70, 73, 72]. The way to verify the abstract conditions depends on the properties of the process under consideration. In particular, unidirectional processes such as damage or delamination processes require additional techiques to obtain compatibility condition (C2). In such a setting the dissipation distance takes the form (5.7) with $A \neq V$, where the value ∞ models the unidirectionality, i.e. it prohibits healing. This leads to the fact that the dissipation distance is neither continuous nor weakly continuous on $\mathcal{Z}$, so that (C2) cannot be directly obtained from the stability of the approximating sequence $(t_k, q_k) \to (t, q)$ in $[0,T] \times \mathcal{Q}$. Such unidirectional processes and alternative techniques to prove (C2) are studied in [68, 106, 87].

Finally it is worth mentioning that the quadratic energy defined in (5.4), which satisfies Items 1–7 fits into the abstract setting of (E1), (E2) and Theorem 5.8.

5.2 The temporal regularity of energetic solutions

The two properties (S) & (E) provide a very weak result on the temporal regularity of an energetic solution only. (S) implies that $\mathcal{E}(t, q(t))$ is uniformly bounded for all $t \in [0,T]$ and under the assumption of coercivity we find $q \in L^\infty([0,T], \mathcal{Q})$. Furthermore one obtains from (E) that $\mathrm{Diss}_{\mathcal{D}}(z, [0,T])$ is finite and hence $z \in BV([0,T], L^1(\Omega))$. Thus neither the component u nor z of an energetic solution has to be continuous – not to mention continuously differentiable in time. In other words, it cannot be excluded that an energetic solution has jumps with respect to time. The aim of this section is to discuss settings which lead to a better temporal regularity of an energetic solution. In particular we want to obtain continuity in time, so that jumps are forbidden.

5.2.1 Continuity with respect to time

In this section we discuss the temporal continuity of energetic solutions, which can be obtained in settings that guarantee unique minimizers of the functional $\mathcal{J}_{z_*} : \mathcal{Q} \to \mathbb{R}_\infty$, $\mathcal{J}_{z_*}(\tilde{q}) = \mathcal{E}(t, \tilde{q}) + \mathcal{D}(z_*, \tilde{z})$ for any $z_* \in \mathcal{Z}$. In the following the results are sketched. The details are developed in [106, Th. 4.2, 4.3].

The uniqueness of the minimizer, which is guaranteed by the strict convexity of $\mathcal{J}_{z_-(t)}$, enables to state the following jump relations

Lemma 5.9 (Jump relations)**.** *Assume that $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfies (E1)—(C2). Moreover,*

$$\forall\, t \in [0,T]\ \forall\, q = (u,z) \in \mathcal{S}(t): \quad \{u\} = \operatorname*{Argmin}_{\tilde{u} \in \mathcal{U}} \mathcal{E}(t, \tilde{u}, z). \tag{5.8}$$

Then, for all $t \in [0,T]$ the weak limits $q_-(t) = \text{w-lim}_{\tau \to t^-} q(\tau)$ and $q_+(t) = \text{w-lim}_{\tau \to t^+} q(\tau)$ (where $q_-(0) := q(0)$ and $q_+(T) = q(T)$) exist and satisfy

$$\begin{aligned} \mathcal{E}(t, q_-(t)) &= \mathcal{E}(t, q(t)) + \mathcal{D}(q_-(t), q(t)), \\ \mathcal{E}(t, q(t)) &= \mathcal{E}(t, q_+(t)) + \mathcal{D}(q(t), q_+(t)), \\ \mathcal{D}(q_-(t), q_+(t)) &= \mathcal{D}(q_-(t), q(t)) + \mathcal{D}(q(t), q_+(t)). \end{aligned} \tag{5.9}$$

The existence of the limits $z_-(t) = \text{w-lim}_{\tau\to t^-} z(\tau)$ and $z_+(t) = \text{w-lim}_{\tau\to t^+} z(\tau)$ is due to $\text{Diss}_{\mathcal{D}}(z,[0,T]) < \infty$ for an energetic solution, see [65]. From (E1) one finds $u(t_k^\pm) \rightharpoonup v_\pm$ for $t_k^\pm \to t$ and (C2) yields that $(t, v_\pm, z_\pm)$ satisfy (S). Due to assumption (5.8) the limits $v^\pm$ are uniquely determined and thus they are the desired left and right limits to $u_\pm(t_h)$ in the weak sense. To verify the jump relations (5.9) the energy balance for the energetic solution $q(t)$ is used

$$\mathcal{E}(s,q(s)) + \text{Diss}_{\mathcal{D}}(z,[r,s]) = \mathcal{E}(r,z(r)) + \int_r^s \partial_\tau \mathcal{E}(\tau,q(\tau))\,\mathrm{d}\tau \quad \text{for all } 0 \le r < s \le T\,.$$

The first and the second identity in (5.9) are based on the fact that both $q_-(t)$ and $q_+(t)$ as well as $q(t)$ satisfy (S). Hence they can be obtained by considering $s = t$ together with $r \to t^-$ and $r = t$ together with $s \to t^+$. The third identity is due to (D1) and the first two identities.

The next theorem provides the temporal continuity of the energetic solution $q = (u,z) : [0,T] \to \mathcal{Q} = \mathcal{U}\times\mathcal{Z}$ in the case that the energy $\mathcal{E}(t,\cdot)$ is strictly convex on $\mathcal{Q}$. This requirement is satisfied for an energy, which is defined via a stored elastic energy density $W : \mathbb{R}^m \to \mathbb{R}_\infty$ being strictly convex on $\mathbb{R}^m$, i.e. for a bounded domain $\Omega \subset \mathbb{R}^d$ it is $\mathcal{E}(t,u,z) := \int_\Omega W\big(F(\tilde{u}+u_D(t),\tilde{z})\big)\,\mathrm{d}x - \langle b(t), \tilde{u}+u_D(t)\rangle$. Thereby $F(u,z)$ stands for all occuring components of the pair (u,z) and all occuring derivatives, e.g. $F(u,z) = (e(u),z)$ for kinematic hardening, whereas $F(u,z) = (e(u),z,\nabla z)$ for damage. In particular, $F(u,z)$ has to be of such a form that it induces a norm for (u,z) on $\mathcal{Q}$.

Theorem 5.10. *Let the stored elastic energy density $W : \mathbb{R}^m \to \mathbb{R}_\infty$ be continuous and strictly convex on $\mathbb{R}^m$. Let the the given data satisfy $u_D \in C^1([0,T],\mathcal{U})$, $b \in C^1([0,T],\mathcal{U}^*)$. Then for all $t \in [0,T]$, $z_* \in \mathcal{Z}$ the functional $\mathcal{J}_{z_*}(t,\tilde{q}) = \int_\Omega W(F(\tilde{u}+u_D(t),\tilde{z}))\,\mathrm{d}x - \langle b(t),\tilde{u}+u_D(t)\rangle + \mathcal{D}(z_*,\tilde{z})$ is strictly convex in $\tilde{q}$. Assume that $q = (u,z) : [0,T] \to \mathcal{Q}$ is an energetic solution to $(\mathcal{Q},\mathcal{E},\mathcal{D})$. Then q is (norm-) continuous with respect to time, i.e. $q \in C^0([0,T],\mathcal{Q})$.*

The strict convexity allows us to show that energetic solutions $q = (u,z) : [0,T] \to \mathcal{Q}$ have weak left and right limits $q_-(t)$ and $q_+(t)$ for all $t \in [0,T]$. Exploiting the jump relations one obtains that $q_-(t)$, $q(t)$ and $q_+(t)$ all provide the same value $\mathcal{J}_{z_-(t)}(t,q_-(t))$, which has to be the global minimum by stability of $q_-(t)$. Since the strict convexity of $\mathcal{J}_{z_-(t)}$ guarantees a unique minimizer, all three states must coincide and weak continuity follows. Strong continuity is deduced from a result of Visintin [109, § 2 & Th. 8], which converts weak convergence and energy convergence into strong convergence by exploiting the strict convexity once again.

5.2.2 Hölder- and Lipschitz-continuity in time

The temporal Hölder- or Lipschitz-continuity is based on the uniform convexity of the functional $\mathcal{J}_{z_*}(t,q) = \mathcal{E}(t,q) + \mathcal{D}(z_*,z)$ on a subset of a suitable Banach space $\mathcal{V}$. As we will see in the examples of Section 5.3, the Banach space $\mathcal{V}$ may differ significantly from the state space $\mathcal{Q}$ that is used to prove existence. This is due to fact that the choice of $\mathcal{V}$ influences the temporal regularity obtained, so that the use of a bigger space may lead to a better temporal regularity result. The uniform convexity is defined as follows

Definition 5.11. The functional $\mathcal{J} : \mathcal{V} \to \mathbb{R}_\infty$ is uniformly convex on the convex set $\mathcal{M} \subset \mathcal{V}$, if there exist constants $c_\star > 0$, $2 \le \alpha < \infty$, such that for all convex combinations $q_\theta := \theta q_1 + (1-\theta) q_0$ with $\theta \in (0,1)$ and $q_0, q_1 \in \mathcal{M}$ the following holds

$$\mathcal{J}(t,q_\theta) \le \theta\mathcal{J}(t,q_1) + (1-\theta)\mathcal{J}(t,q_0) - \theta(1-\theta)c_\star\|q_1-q_0\|_{\mathcal{V}}^\alpha\,. \tag{5.10}$$

For a better understanding of this notion of convexity we first investigate the definition for real valued, scalar functions. A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly convex if there are constants $2 \le \alpha < \infty$, $c_\star > 0$ such that for all convex combinations $q_\theta = (1-\theta)q_0 + \theta q_1$ with $\theta \in (0,1)$, q_0, $q_1 \in \mathbb{R}$ the following holds

$$f(q_\theta) \le \theta f(q_1) + (1-\theta) f(q_0) - \theta(1-\theta) c_\star |q_1 - q_0|^\alpha \,. \tag{5.11}$$

In other words, if $f : \mathbb{R} \to \mathbb{R}$ is uniformly convex, then for any two points $f(q_0)$, $f(q_1)$ of its graph there fits some polynomial that is quadratic in θ, between the function and the chord, see Fig. 6. Hence uniform convexity implies strict convexity.

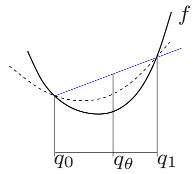

Figure 6: Uniformly convex function.

The meaning of the exponent α can be understood from the following example.

Example 5.12. First, consider the function $f(q) = q^2$. We immediately see that f is strictly convex, since $f''(q) = 2 > 0$ for all $q \in \mathbb{R}$ and by simple calculation we verify $f(q_\theta) = \theta f(q_1) + (1-\theta) f(q_0) - \theta(1-\theta)(q_1 - q_0)^2$. But there are also functions being strictly convex although $f''(q) = 0$ for some $q \in \mathbb{R}$. Such a candidate is e.g. $f(q) = q^4$ with $f''(0) = 0$. Since f is continuously differentiable, the uniform convexity inequality (5.11) is equivalent to $f(q_1) - f(q_0) \ge f'(q_0)(q_1 - q_0) + c_\star |q_1 - q_0|^\alpha$ and hence equivalent to $(f'(q_1) - f'(q_0))(q_1 - q_0) - 2c_\star |q_1 - q_0|^\alpha \ge 0$. Therewith we verify for $c_\star = 1/4$ and $\alpha = 4$ that $(f'(q_1) - f'(q_0))(q_1 - q_0) - 2c_\star |q_1 - q_0|^\alpha = \frac{1}{2}(q_1 - q_0)^4 + \frac{6}{2}(q_1^2 - q_0^2)^2 \ge 0$ and thus we conclude that (5.11) holds for $f(q) = q^4$ with $c_\star = 1/4$ and $\alpha = 4$.

This notion of convexity is now transfered to the context of energy functionals. The theorem below generalizes the ideas developed in [70, 74], where Lipschitz-continuity with respect to time was derived. The generalization has two aspects. First it is emphasized that the convexity properties can be formulated with respect to a norm $\|\cdot\|_\mathcal{V}$ that may differ significantly from that underlying the state space $\mathcal{Q}$. In particular, if $\mathcal{Q}$ is chosen as small as possible under preservation of the coercivity of $\mathcal{E}$ (see (E1)), it may be an advantage to investigate the temporal regularity of energetic solutions with respect to the norm of a larger Banach space $\mathcal{V} \supset \mathcal{Q}$, since temporal regularity may improve. Second, as can be seen from (5.10) the notion of uniform convexity is not restricted to the exponent $\alpha = 2$, so that a weaker lower bound is admissible due to $\alpha \ge 2$. Previous work [70, 74] asked $\alpha = 2$ and $\beta = 1$ and enforced the uniform convexity condition on whole $\mathcal{Q}$, while the theorem below only requires it on sublevels. In fact, the formulation of the conditions on sublevels is sufficient, since an energetic solution $q : [0,T] \to \mathcal{Q}$ satisfies $q(t) \in L_{E_\star}(s)$ for some fixed $E_\star > 0$ and all $s, t \in [0,T]$. This is due to stability (S) and the temporal Lipschitz-estimate $|\mathcal{E}(s,q) - \mathcal{E}(t,q)| \le c_E |s - t|$ for a constant $c_E > 0$ and for all fixed states $q \in \mathcal{Q}$ with $\mathcal{E}(r,q) < E$ for some $r \in [0,T]$, which is a direct consequence of (E2) and Gronwall's inequality.

Theorem 5.13 (Temporal Hölder-continuity)**.** *Let $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ be a rate-independent system, where $\mathcal{Q}$ is a closed, convex subset of a Banach space $\mathcal{X}$. Let $L_E(t) = \{q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) \le E\}$. Assume* *that there is a Banach space $\mathcal{V}$ and that there are constants $\alpha \geq 2$, $\beta \leq 1$ such that for all E_* there exist constants C_*, $c_* > 0$ so that for all $t \in [0,T]$, $q_0, q_1 \in L_{E_*}(t)$ and all $\theta \in [0,1]$ the following holds:*

$$\mathcal{E}(t,q_\theta) + \mathcal{D}(z_0,z_\theta) + c_*\theta(1-\theta)\|q_1-q_0\|_{\mathcal{V}}^\alpha \leq (1-\theta)\big(\mathcal{E}(t,q_0)+\mathcal{D}(z_0,z_0)\big) + \theta\big(\mathcal{E}(t,q_1)+\mathcal{D}(z_0,z_1)\big) \tag{5.12a}$$

$$|\partial_t\mathcal{E}(t,q_1) - \partial_t\mathcal{E}(t,q_0)| \leq C_*\|q_1 - q_0\|_{\mathcal{V}}^\beta, \tag{5.12b}$$

where $(u_\theta, z_\theta) = q_\theta = (1-\theta)q_0 + \theta q_1$.

Then, any energetic solution $q : [0,T] \to \mathcal{Q}$ of $(\mathcal{Q},\mathcal{E},\mathcal{D})$ is Hölder-continuous from $[0,T]$ to $\mathcal{V}$ with the exponent $1/(\alpha-\beta)$, i.e. there is a constant $C_{\mathrm{H}} > 0$ such that

$$\|q(s)-q(t)\|_{\mathcal{V}} \leq C_{\mathrm{H}}|t-s|^{1/(\alpha-\beta)} \quad \textit{for all } s,t \in [0,T]\,. \tag{5.13}$$

The main idea of the proof is to use uniform convexity inequality (5.12a) to derive an improved stability estimate, which contains the additional term $c_*\theta(1-\theta)\|q_1-q_0\|_{\mathcal{V}}^\alpha$. Using assumption (5.12b) one obtains an upper estimate for $\|q_1-q_0\|_{\mathcal{V}}^\alpha$ from the energy balance. Finally the Hölder estimate (5.13) can be proved with the aid of a differential inequality and Gronwall's lemma. The details are carried out in [106].

5.3 Applications

In this section we discuss examples for uniformly convex stored elastic energy densities arising from various types of rate-independent processes, such as plasticity, phase transformations in shape memory alloys and damage. All these applications can be treated as rate-independent processes in terms of the energetic formulation. As the unknowns their models involve the the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u+\nabla u^T)$ in terms of the displacement field $u : \Omega \to \mathbb{R}^d$ and an internal variable z which may be scalar-, vector- or tensor valued depending on the problem. The way, how u and z are linked in the model differs and here we distinguish between energies, which compose the different variables additively, such as in the Example 3.5 for kinematic hardening, and energies which use a multiplicative composition of the variables, such as in the case of damage, see Examples 5.16–5.18.

5.3.1 Additive energies: Plasticity, phase transformations in shape memory alloys

In the following we treat two applications with quadratic energies. We will obtain that $\mathcal{V} = \mathcal{Q}$ in these settings, that $\alpha = 2$ and $\beta = 1$, so that energetic solutions are Lipschitz-continuous with respect to time. This regularity is in good accordance with the results proven in [70] and with classical existence results for elastoplasticity.

Example 5.14. As a first example for Theorem 5.13 we consider the particular situation where $\mathcal{E}(t,\cdot)$ is quadratic. Let $\mathcal{Q}$ be a reflexive Banach space and assume that $\mathcal{A} \in \mathrm{Lin}(\mathcal{Q},\mathcal{Q}^*)$ is a linear, bounded operator with $\langle \mathcal{A}q, q\rangle \geq c\|q\|_{\mathcal{Q}}^2$ for all $q \in \mathcal{Q}$ and for some constant $c > 0$. Given $q_D \in C^1([0,T],\mathcal{Q})$ and $b \in C^1([0,T],\mathcal{Q}^*)$ the energy $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathcal{R}$ is defined by

$$\mathcal{E}(t,q) = \tfrac{1}{2}\langle \mathcal{A}(q+q_D(t)), (q+q_D(t))\rangle - \langle b(t), q+q_D(t)\rangle\,.$$

Moreover assume that the dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$ is defined as $\mathcal{D}(z_1, z_2) = \mathcal{R}(z_2 - z_1)$ with $\mathcal{R} : \mathcal{Z} \to [0,\infty)$ being positively 1-homogeneous, convex, weakly sequentially lower semicontinuous and satisfying $\mathcal{R}(z) \leq c_{\mathcal{R}}\|z\|_{\mathcal{Z}}$ for all $z \in \mathcal{Z}$ and for a constant $c_{\mathcal{R}} > 0$. Then, for all $q_i \in \mathcal{Q}$, the system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfies the assumptions (5.12) with $\mathcal{V} = \mathcal{Q}$, $\alpha = 2$ and $\beta = 1$. Thus, from (5.13) we obtain that energetic solutions $q : [0,T] \to \mathcal{Q}$ are Lipschitz-continuous with $\|q(s) - q(t)\|_{\mathcal{Q}} \leq C_H|s - t|^{\frac{1}{2-1}}$.

Thereby the uniform convexity inequality (5.12a) is a direct consequence of (5.5) and the convexity of $\mathcal{D}$. Estimate (5.12b) can be vierified by straight forward calculations.

Observe that the models of elastoplasticity with linear kinematic hardening and of elastoplasticity with Cosserat micropolar effects from Examples 3.5 and 3.6 fit into this framework. Let us finally note that the result on the temporal Lipschitz-continuity due to Theorem 5.13 is in accordance with known results for equations of this type, see e.g. [17, 47].

Example 5.15 (The Souza-Auricchio model for thermally driven phase transformations in shape memory alloys[73])**.** In the context of phase transformations in shape memory alloys the internal variable $z : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$ is the mesoscopic transformation strain reflecting the phase distribution. The dissipation distance, which measures the energy dissipated due to phase transformation, is assumed to take the form $\mathcal{D}(z, \tilde{z}) = \varrho\|z - \tilde{z}\|_{L^1(\Omega)}$ with $\varrho > 0$.

The phase transformations are considered to be thermally induced. For a body that is small in at least one direction, it is reasonable to assume that the temperature $\vartheta \in C^1([0,T], H^1(\Omega))$, with $C_\vartheta := \|\vartheta\|_{C^1([0,T],H^1(\Omega))}$, is a priori given, since it influences the transformation process like an applied load, see [10]. Thus the energy density takes the form

$$W(F(u,z),\vartheta) = \tfrac{1}{2}\big(e(u) - z\big) : \mathbb{B}(\vartheta) : \big(e(u) - z\big) + h(z,\vartheta) + \tfrac{\sigma}{2}|\nabla z|^2$$

with the constant $\sigma > 0$ and the elasticity tensor $\mathbb{B} \in C^1([\vartheta_{\min}, \vartheta_{\max}], \mathbb{R}^{(d\times d)\times(d\times d)})$ being symmetric and positive definite for all ϑ, i.e. there are constants $c_1^{\mathbb{B}}, c_2^{\mathbb{B}} > 0$ so that $c_1^{\mathbb{B}}|A|^2 \leq A : \mathbb{B} : A \leq c_2^{\mathbb{B}}|A|^2$ for all $A \in \mathbb{R}^{d\times d}$. Moreover, let $c_\vartheta^{\mathbb{B}} := \|\mathbb{B}\|_{C^1([\vartheta_{\min},\vartheta_{\max}],\mathbb{R}^{(d\times d)\times(d\times d)})}$. The function $h : \mathbb{R}^{d\times d}_{\mathrm{sym,dev}} \times \mathbb{R} \to \mathbb{R}$ is given by

$$h(z,\vartheta) := c_1(\vartheta)\sqrt{\delta^2 + |z|^2} + c_2(\vartheta)|z|^2 + \tfrac{1}{\delta}\big(|z| - c_3(\vartheta)\big)^3_+ \,,$$

where $\delta > 0$ is constant and $c_i \in C^1([\vartheta_{\min}, \vartheta_{\max}])$ with $0 < c_i^1 \leq c_i(\vartheta)$ for all $\vartheta \in [\vartheta_{\min}, \vartheta_{\max}]$ and $c_i^\vartheta := \|c_i\|_{C^1([\vartheta_{\min},\vartheta_{\max}])}$, $i = 1,2,3$. Thereby $c_1(\vartheta)$ is an activation threshold for the initiation of martensitic phase transformations, $c_2(\vartheta)$ measures the occurence of an hardening phenomenon with respect to the internal variable z and $c_3(\vartheta)$ represents the maximum modulus of transformation strain that can be obtained by alignment of martensitic variants. Furthermore $(f)_+ := \max\{0, f\}$. For given data $b \in C^1([0,T], H^{-1}(\Omega, \mathbb{R}^d))$ and $u_D \in C^1([0,T], H^1(\Omega, \mathbb{R}^d))$ the energy functional is defined by $\mathcal{E}(t,q) = \int_\Omega W(F(u + u_D(t), z), \vartheta)\,\mathrm{d}x - \langle b(t), u + u_D(t)\rangle$. Hence we have

$$\begin{aligned}
&\partial_t\mathcal{E}(t,q) = \int_\Omega \big(\partial_u W(F(u+u_D, z),\vartheta) : e(\dot{u}_D) + \dot{\vartheta}\,\partial_\vartheta W(F(u+u_D,z),\vartheta)\big)\,\mathrm{d}x - \langle \dot{b}, u+u_D\rangle - \langle b, \dot{u}_D\rangle,\\
&\partial_u W(F(u+u_D,z),\vartheta) : \dot{u}_D = (e(u+u_D) - z) : \mathbb{B}(\vartheta) : e(\dot{u}_D)\,,\\
&\dot{\vartheta}\,\partial_\vartheta W(F(u+u_D,z),\vartheta) = \dot{\vartheta}\,\big((e(u+u_D) - z) : \partial_\vartheta\mathbb{B}(\vartheta) : (e(u+u_D) - z) + \partial_\vartheta h(\vartheta, z)\big)\,.
\end{aligned}$$

To gain a Lipschitz-estimate for $\partial_t\mathcal{E}(t,\cdot)$ for the present model it is important that Theorem 5.13 is formulated for energy-sublevels $L_{E_\star}(t) = \{q \in \mathcal{Q}\,|\,\mathcal{E}(t,q) \leq E_\star\}$, since this provides the bound

$\|u_i\|_{H^1}+\|z_i\|_{H^1} \leq C_{E_\star}$. Thus for all $(u_0,z_0),(u_1,z_1) \in L_{E_\star}(t)$ it holds

$$\int_\Omega |\dot\vartheta\big(e(u_1-u_0)-(z_1-z_0)\big):\partial_\vartheta\mathbb{B}(\vartheta):\big(e(u_1-u_0)-(z_1-z_0)\big)|\,\mathrm{d}x \leq C_\vartheta c_\vartheta^{\mathbb{B}}\big(\|e(u_1-u_0)\|_{L^2}+\|z_1-z_0\|_{L^2}\big)^2$$
$$\leq C_\vartheta c_\vartheta^{\mathbb{B}}\big(\sum_{i=0}^{1}\|e(u_i)\|_{L^2}+\|z_i\|_{L^2}\big)\big(\|e(u_1-u_0)\|_{L^2}+\|z_1-z_0\|_{L^2}\big)$$
$$\leq 2C_{E_\star}C_\vartheta c_\vartheta^{\mathbb{B}}\big(\|u_1-u_0\|_{H^1}+\|z_1-z_0\|_{L^2}\big)\,.$$

Furthermore the application of the main theorem on differentiable functions yields

$$|\sqrt{\delta^2+|z_1|^2}-\sqrt{\delta^2+|z_0|^2}| \leq |z_1-z_0|,$$
$$||z_1|^2-|z_0|^2| \leq 2(|z_1|+|z_0|)|z_1-z_0|,$$
$$|(|z_1|-c_3(\vartheta))_+^3-(|z_0|-c_3(\vartheta))_+^3| \leq 2(|z_1|+|z_0|)^2|z_1-z_0|,$$

so that

$$\int_\Omega |\partial_\vartheta h(\vartheta,z_1)-\partial_\vartheta h(\vartheta,z_0)| \leq \|z_1-z_0\|_{L^1}\big(c_1^\vartheta+2(\mathcal{L}^d(\Omega)C_{E_\star})^{\frac12}c_2^\vartheta+\tfrac{6}{\delta}c_3^\vartheta C_{E_\star}\big) \leq \tilde C_\star\|z_1-z_0\|_{L^2}$$

with $\tilde C_\star := \mathcal{L}^d(\Omega)^{\frac12}\big(c_1^\vartheta+2(\mathcal{L}^d(\Omega)C_{E_\star})^{\frac12}c_2^\vartheta+\frac{6}{\delta}c_3^\vartheta C_{E_\star}\big)$, where $\mathcal{L}^d(\Omega)$ denotes the d-dimensional Lebesgue-measure of Ω. Therefore Lipschitz-estimate (5.12b) holds true with $\beta=1$ and with $C_\star = (\tilde C_\star + 2C_{E_\star}C_\vartheta c_\vartheta^{\mathbb{B}} + c_\vartheta^{\mathbb{B}}c_D + c_l)$.

Now it has to be verified that the density W is uniformly convex with respect to $F(u,z)$. Thereto we first calculate that $w_\theta{:}\mathbb{B}(\vartheta){:}w_\theta \leq \theta w_1{:}\mathbb{B}(\vartheta){:}w_1+(1-\theta)w_0{:}\mathbb{B}(\vartheta){:}w_0-\theta(1-\theta)c_1^{\mathbb{B}}|w_1-w_0|^2$ for $w_i{=}e_i-z_i$ with $(e_i,z_i)\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$, $i=0,1$, $w_\theta=\theta w_1+(1-\theta)w_0$ with $\theta\in(0,1)$. Thereby a binomic formula and the positive definiteness of $\mathbb{B}(\vartheta)$ for all ϑ were applied. The uniform convexity of $|\nabla z|^2=\nabla z:\nabla z$ can be obtained similarly. We now show that h is uniformly convex. We immediately see that $\tilde h_1(z) := (\delta^2+|z|^2)^{\frac12}$ is convex in z. Furthermore, since $\tilde h_3(z) := (|z|-c_3(\vartheta))_+^3$ is the composition of the monotone function x^3 and the convex function $(\cdot)_+$, we conclude that also $\tilde h_3(z)$ is convex in z. Additionally we obtain with similar calculations as applied for the other quadratic terms that $\tilde h_2(z) := |z|^2$ is uniformly convex. Since $c_i(\vartheta)\geq c_i^1>0$ for all $\vartheta\in[\vartheta_{\min},\vartheta_{\max}]$ and $i=1,2,3$ we have proven that h is uniformly convex in z with $h(z_\theta,\vartheta)\leq\theta h(z_1,\vartheta)+(1-\theta)h(z_0,\vartheta)-\theta(1-\theta)c_2^1|z_1-z_0|^2$. Summing up all terms and taking into account all prefactors yields a uniform convexity estimate for W, which leads to

$$\mathcal{E}(t,q_\theta)\leq\theta\mathcal{E}(t,q_1)+(1-\theta)\mathcal{E}(t,q_0)-\theta(1-\theta)\left(\tfrac{c_1^{\mathbb{B}}}{2}\|w_1-w_0\|_{L^2}^2+\tfrac{\sigma}{2}\|\nabla(z_1-z_0)\|_{L^2}^2+c_2^1\|z_1-z_0\|_{L^2}^2\right).$$

Thereby we have used that the term describing the work of the external loadings is linear in u. Moreover we find with Korn's inequality that $\|w_1-w_0\|_{L^2}^2\geq\frac12\|e(u_1)-e(u_0)\|_{L^2}^2-\|z_1-z_0\|_{L^2}^2\geq\frac{1}{2C_K^2}\|u_1-u_0\|_{H^1}^2-\|z_1-z_0\|_{L^2}^2$. Under the assumption that $(c_2^1-(c_1^{\mathbb{B}}/2))>0$ we conclude that (5.12a) holds for $\alpha=2$, $c_* := \min\big\{c_1^{\mathbb{B}}/(4C_K^2),\sigma/2,(c_2^1-(c_1^{\mathbb{B}}/2))\big\}$ and the space

$$\mathcal{V}=\mathcal{Q}=\{\tilde u\in H^1(\Omega,\mathbb{R}^d)\,|\,\tilde u=0\text{ on }\Gamma_D\}\times\{\tilde z\in H^1(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym,dev}})\}.$$

Hence any energetic solution $q:[0,T]\to\mathcal{Q}$ is temporally Lipschitz-continuous: $q\in C^{0,1}([0,T],\mathcal{Q})$.

5.3.2 Multiplicative energies: Damage

In the following we apply the temporal regularity results stated in Theorems 5.10 and 5.13 to energies used in the modeling of partial, isotropic damage processes. Thereby, damage means the creation and growth of cracks and voids on the micro-level of a solid material. To describe the influence of damage on the elastic behavior of the material one defines an internal variable, the damage variable $z(t,x) \in [z^\star, 1]$, as the volume fraction of undamaged material in a neighbourhood of material dependent size around $x \in \Omega$ at time $t \in [0,T]$. Thus $z(t,x)$=1 means that the material around x is perfectly undamaged, whereas $z(t,x)$=$z^\star \geq 0$ stands for maximal damage of the neighbourhood. The condition $z^\star > 0$ models partial damage and the fact that z is scalar valued reflects the isotropy of the damage process, which means that the cracks and voids are presumed to have a uniform orientation distribution in the material. Furthermore it is assumed that damage is a unidirectional process, so that healing is forbidden and $\dot{z}(t,x) \leq 0$. This condition is preserved by the dissipation distance, i.e. for $\varrho > 0$ it is

$$\mathcal{D}(z_0, z_1) := \begin{cases} \int_\Omega \varrho(z_0 - z_1)\,\mathrm{d}x & \text{if } z_1 \leq z_0, \\ \infty & \text{else,} \end{cases} \tag{5.14}$$

which punishes a decrease of damage with the value ∞. The energy in the framework of damage is given by

$$\mathcal{E}(t,u,z) := \int_\Omega \widetilde{W}(e(u+u_D(t)), z)\,\mathrm{d}x + \int_\Omega \frac{\kappa}{r}|\nabla z|^r\,\mathrm{d}x - \int_\Omega l(t)(u+u_D(t))\,\mathrm{d}x\,. \tag{5.15}$$

The first term in (5.15) is the stored elastic energy, the second describes the influence of damage with $1 < r < \infty$ and $\kappa > 0$ and the third term accounts for the work of the external loadings.

As in the previous sections we set $W(F(u,z)) = \widetilde{W}(e(u+u_D(t)), z) + \frac{\kappa}{r}|\nabla z|^r$. In engineering, see e.g. [64], a typical ansatz for the stored elastic energy density is the following

$$\widetilde{W}(e,z) := f_1(z)W_1(e) + W_2(e) + f_2(z) \quad \text{and} \quad \partial_z W(e,z) \geq 0\,. \tag{5.16}$$

In Section 5.2.1 we obtained that the joint strict convexity of $\widetilde{W}$ in (z,e) will ensure the temporal continuity of the energetic solution. But the crucial point, which may spoil this regularity in the case of damage is, that not many stored elastic energy densities $\widetilde{W}(e,z) := f_1(z)W_1(e)$, that satisfy $\partial_z \widetilde{W}(e,z) \geq 0$, are also jointly strictly convex, although both f_1, W_1 may be convex. As a negative example we present the wellknown $(1-\mathsf{d})$-model for isotropic damage, see e.g. [64]:

Example 5.16. For the symmetric, positive definite fourth order tensor $\mathbb{B}$ the stored elastic energy density

$$\hat{W}(e,\mathsf{d}) = \frac{(1-\mathsf{d})}{2} e{:}\mathbb{B}{:}e = \frac{z}{2} e{:}\mathbb{B}{:}e = \widetilde{W}(e,z)$$

is not jointly convex in (e,z). This can be seen from calculating the Hessian; evaluating it in (e,z)=$(e,1)$, $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, in the direction $(\tilde{e},\tilde{z})$=$(-\frac{e}{2},1)$ yields $\mathrm{D}^2\widetilde{W}(e,z)[(\tilde{e},\tilde{z}),(\tilde{e},\tilde{z})] = z\tilde{e}{:}\mathbb{B}{:}\tilde{e} + 2\tilde{z}e{:}\mathbb{B}{:}\tilde{e} = -\frac{3}{4}\,e{:}\mathbb{B}{:}e < 0$.

To find a positive example on stored elastic energy densities satisfying (5.16) one may use the ideas of [88].

Example 5.17. For $\mathbb{B}$ as in Example 5.16 the energy density $\widehat{W}(e,z) := \frac{e:\mathbb{B}:e}{2(2-z)}$ is jointly convex in (e,z) and

$$\widetilde{W}(e,z) := \frac{e:\mathbb{B}:e}{2(2-z)} + \frac{z^2}{2}$$

is strictly convex in (e,z). Calculating the Hessian yields

$$\mathrm{D}^2\widehat{W}(e,z)[(\tilde{e},\tilde{z}),(\tilde{e},\tilde{z})] = \frac{\tilde{z}e:\mathbb{B}:\tilde{z}e}{(2-z)^3} - 2\frac{\tilde{z}e:\mathbb{B}:\tilde{e}}{(2-z)^2} + \frac{\tilde{e}:\mathbb{B}:\tilde{e}}{(2-z)} = \frac{1}{2-z}(\hat{e}-\tilde{e}):\mathbb{B}:(\hat{e}-\tilde{e}) \geq 0$$

with $\hat{e} := \tilde{z}e/(2-z)$ for all $(e,\tilde{z}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [z^\star,1]$. Since we have $\mathrm{D}^2\widehat{W}(e,z)[(\tilde{e},\tilde{z}),(\tilde{e},\tilde{z})] = \mathbf{0}$ for all $(\mathbf{0},\tilde{z})$ whenever $e=\mathbf{0}$, we find that $\widehat{W}$ is jointly, but not strictly convex. We conclude that $\widetilde{W}$ is jointly strictly convex due to the term $f(z) = \frac{z^2}{2}$, since $f''(z) = 1$, so that $f''(z)\tilde{z}^2 > 0$ for all $\tilde{z} \neq 0$, which ensures that $\mathrm{D}^2\widetilde{W}(e,z)[(\tilde{e},\tilde{z}),(\tilde{e},\tilde{z})] > 0$ for all $(\tilde{e},\tilde{z}) \neq 0$ and for all $(e,z) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [z^\star,1]$.

Finally we discuss an example which refers to Theorem 5.13 on the Hölder-and Lipschitz-continuity of energetic solutions. With this example we want to point out the importance of the Banach space $\mathcal{V}$. We will see that its choice is not unique and that it may lead to different constants $\alpha > 2$. This is due to the fact that the energy will be chosen non-quadratic in contrast to the Examples 5.14–5.15. We will clarify how the space $\mathcal{V}$ influences the magnitude of the Hölder constant and explain how to achieve better regularity by a clever identification of $\mathcal{V}$.

Example 5.18 (The effective use of $\mathcal{V}$)**.** For $\mathbb{B}$ as above and constants $a, \hat{a}, c > 0$ consider

$$W(e,z,\nabla z) := \frac{e:\mathbb{B}:e}{2\sqrt{2-z}} + G(e) + \tfrac{a}{2}z^2 + \tfrac{\kappa}{2}|\nabla z|^2 \quad \text{with} \quad G(e) := \tfrac{c}{4}(\hat{a}+|\operatorname{dev} e|^2)^2 \tag{5.17}$$

with the deviator $\operatorname{dev} e := e - \frac{\operatorname{tr} e}{d}\mathrm{Id}$ and the energy

$$\mathcal{E}(t,u,z) := \int_\Omega W(e+e_D(t),z,\nabla z)\,\mathrm{d}x - \int_\Omega b(t)(u+u_D(t))\,\mathrm{d}x.$$

We now determine $\mathcal{V}$ suitably. We first treat the case of time-dependent Dirichlet data, as investigated in [106]. Similarly to the ideas applied in Example 5.12 we thereto deduce the following uniform convexity inequality for W

$$W(e_\theta,z_\theta) \leq (1-\theta)W(e_0,z_0) + \theta W(e_1,z_1) - \theta(1-\theta)\tilde{c}\big(|E|^2 + |Z|^2 + |\operatorname{dev} E|^4 + |\nabla Z|^2\big) \tag{5.18}$$

with $E := e_1 - e_0$, $Z := z_1 - z_0$. For q_0, $q_1 \in L_{E_\star}(t)$ we can verify

$$\mathcal{E}(t,q_\theta) \leq (1-\theta)\mathcal{E}(t,q_0) + \theta\mathcal{E}(t,q_1) - \theta(1-\theta)c_\star\big(\|E\|_{L^2} + \|Z\|_{L^2} + \|\operatorname{dev} E\|_{L^4} + \|\nabla Z\|_{L^2}\big)^\alpha \tag{5.19}$$

for $\alpha = 4$, $c_\star = 2^{-3}\tilde{c}\,\min\{(2E_\star)^{2-\alpha},(2E_\star)^{4-\alpha}\}$. This estimate determines the Banach space

$$\mathcal{V}_1 := \{\tilde{u} \in H^1(\Omega,\mathbb{R}^d)\,|\,\operatorname{dev} e(\tilde{u}) \in L^4(\Omega,\mathbb{R}^{d\times d})\} \times \{\tilde{z} \in H^1(\Omega)\}\,.$$

At this point we notice that the right-hand side of (5.19) is increased if we use the $L^{\tilde{p}}(\Omega,\mathbb{R}^{d\times d})$-norm for some $1 < \tilde{p} \leq 4$, which would lead to a smaller $\alpha = \max\{2,\tilde{p}\}$ and hence to a Hölder exponent closer to 1.

In order to find out whether the choice of $\tilde{p} = 2$ is suitable, assumption (5.12b) has to be investigated. Thereto we calculate

$$\partial_t \mathcal{E}(t,u,z) = \int_\Omega \partial_e W(e(u)+e_D(t),z,\nabla z):\dot{e}_D(t)\,\mathrm{d}x - \int_\Omega \dot{b}(t)(u+u_D(t))\,\mathrm{d}x - \int_\Omega b(t)\dot{u}_D(t)\,\mathrm{d}x\,.$$

The term $\mathrm{D}G(\operatorname{dev} e){:}\hat{e} = c(\hat{a}+|\operatorname{dev} e|^2)^{\frac{4-2}{2}}(\operatorname{dev} e){:}\hat{e}$, with G defined in (5.17), plays the decisive role in estimate (5.12b). Using Taylor expansion one can prove that

$$\big|\mathrm{D}G(\operatorname{dev}(\tilde{e}_1)){:}\dot{e}_D(t) - \mathrm{D}G(\operatorname{dev}(\tilde{e}_0)){:}\dot{e}_D(t)\big| \le C\big(1+W_0+W_1\big)^{\frac{p-2}{p}}|\operatorname{dev} E|,$$

where $W_i = W(\tilde{e}_i, z_i, \nabla z_i)$, $\tilde{e}_i = e_i + e_D(t)$, $e_D(t) = e(u_D(t))$ and $\dot{e}_D(t) \in C^0([0,T], W^{1,\infty}(\Omega,\mathbb{R}^{d\times d}))$. Thus integration and Hölder's inequality with $\tilde{p}=2$ and $\tilde{p}'=2$ yield

$$\int_\Omega \big|\mathrm{D}G(\operatorname{dev}(e_1+e_D(t))){:}\dot{e}_D(t) - \mathrm{D}G(\operatorname{dev}(e_0+e_D(t))){:}\dot{e}_D(t)\big|\,\mathrm{d}x \le C_1\|\operatorname{dev} E\|_{L^{\frac{4}{2}}} \le C_2\|u_1-u_0\|_{H^1}$$

with $e_i = e(u_i)$ for $(u_i, z_i) \in L_{E_\star}(t)$. This implies $\beta = 1$ and it is suitable to introduce the Banach space

$$\mathcal{V}_2 := \{\tilde{u} \in H^1(\Omega,\mathbb{R}^d)\} \times \{\tilde{z} \in H^1(\Omega)\}\,.$$

With this choice of $\mathcal{V} = \mathcal{V}_2$ we have $\alpha = 2$, which leads to the Hölder exponent $\frac{1}{\alpha-1} = 1$, so that an energetic solution $q : [0,T] \to \mathcal{Q}$ satisfies $q \in C^{0,1}([0,T],\mathcal{V}_2)$, whereas $\mathcal{V} = \mathcal{V}_1$ yields $q \in C^{0,\frac{1}{3}}([0,T],\mathcal{V}_1)$.

Finally we consider the case of time-independent Dirichlet data u_D, i.e. $\dot{u}_D(t) = 0$ for all $t \in [0,T]$. Then we have $\partial_t\mathcal{E}(t,q) = -\int_\Omega \dot{b}(t)(u+u_D)\,\mathrm{d}x$. Therefore we may drop $\|E\|_{L^p}$ in (5.19) and choose $\mathcal{V} = \mathcal{Q}$. For this choice we find $\alpha = 2$ and the Hölder-exponent $1/(\alpha-1) = 1$, which means that the energetic solution is Lipschitz-continuous in time. This is in accordance to the regularity result obtained [70], where only time-independent Dirichlet data were applied.

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