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Homogenization in gradient plasticity

Hauke Hanke[1]

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[1] Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: hanke@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper yields a two-scale homogenization result for a rate-independent elastoplastic system. The presented model is a generalization of the classical model of linearized elastoplacticity with hardening, which is extended by a gradient term of the plastic variables. The associated stored elastic energy density has periodically oscillating coefficients, where the period is scaled by $\varepsilon > 0$. The additional gradient term of the plastic variables z is contained in the elastic energy with a prefactor ε^γ ($\gamma \geq 0$). We derive different limiting models for $\varepsilon \to 0$ in dependence of γ. For $\gamma > 1$ the limiting model is the two-scale model derived in [MT07], where no gradient term was present. For $\gamma = 1$ the gradient term of the plastic variable survives on the microscopic cell poblem, while for $\gamma \in [0, 1)$ the limit model is defined in terms of a plastic variable without microscopic fluctuation. The latter model can be simplified to a purely macroscopic elastoplasticity model by homogenisation of the elastic part.

1 Introduction

Our aim is to provide homogenization of an elastoplastic model with an additional gradient term of the vector of internal variables. The modeling is done in the framework of the energetic formulation for rate-independent problems. This framework allows us to apply the theory of Γ-convegence for rate-independent systems developed in [MRS08].

The energetic formulation is based on the energy-storage functional $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}$ and the dissipation potential $\mathcal{R} : \mathcal{Q} \to [0, \infty]$, which is convex, continuous and positively homogeneous of degree 1, i.e. $\mathcal{R}(0) = 0$ and $\mathcal{R}(\beta q) = \beta \mathcal{R}(q)$ for all $\beta > 0$ and every $q \in \mathcal{Q}$. Here $\mathcal{Q}$ is a Hilbert space with dual $\mathcal{Q}^*$ and dual pairing $\langle \cdot, \cdot \rangle_{\mathcal{Q}} : \mathcal{Q}^* \times \mathcal{Q} \to \mathbb{R}$ and the fact that $\mathcal{R} : \mathcal{Q} \to [0, \infty]$ is positively homogeneous of degree 1 ensures us the rate-independence. In our case the energy functional is defined via $\mathcal{E}(t, q) := \frac{1}{2}\langle \mathcal{A}q, q\rangle_{\mathcal{Q}} - \langle \ell(t), q\rangle_{\mathcal{Q}}$, where $\ell \in \mathrm{C}^1([0, T]; \mathcal{Q}^*)$ is a loading and $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}^*$ is a continuous, linear, positive definite, symmetric operator.

The evolutionary problem is given by the stability condition (S) and the energy balance (E), which read as follows:

$$\text{(S)} \quad \mathcal{E}\big(t, q(t)\big) \leq \mathcal{E}\big(t, q'\big) + \mathcal{R}\big(q' - q(t)\big) \quad \text{for all } q' \in \mathcal{Q},$$

$$\text{(E)} \quad \mathcal{E}\big(t, q(t)\big) + \mathrm{Diss}_{\mathcal{R}}\big(q; [0, t]\big) = \mathcal{E}\big(0, q(0)\big) + \int_0^t \partial_s \mathcal{E}\big(s, q(s)\big) \mathrm{d}s,$$

where the total dissipation $\mathrm{Diss}_{\mathcal{R}}(q; [r, s])$ is defined via $\mathrm{Diss}(q; [r, s]) := \int_r^s \mathcal{R}(\dot{q}(t))\, \mathrm{d}t$. A function $q : [0, T] \to \mathcal{Q}$ which solves the energetic formulation (S)&(E) for every $t \in [0, T]$ is called energetic solution of the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$.

In homogenization we consider problems depending on a small length-scale parameter $\varepsilon > 0$, that denotes the size of a micro cell inside the domain $\Omega \subset \mathbb{R}^d$. For fixed $\varepsilon > 0$

the elastoplastic evolution of the body described by $\Omega \subset \mathbb{R}^d$ is given by the energetic formulation of the rate-independent system $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where $\mathcal{Q} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{H}^1(\Omega)^m$. The goal is to derive homogenized limit models as $\varepsilon \to 0$ and to study the link between the energetic solutions q_ε of the systems $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ and the energetic solutions of the limit models. In the applications we have in mind, the energy functional $\mathcal{E}_\varepsilon : [0,T] \times \mathcal{Q} \to \mathbb{R}$ with periodic coefficients and the dissipation potential $\mathcal{R}_\varepsilon \colon \mathcal{Q} \to [0,\infty]$ are defined as

$$
\begin{aligned}
\mathcal{E}_\varepsilon(t,u,z) &= \frac{1}{2} \int\limits_\Omega \left\langle\!\!\left\langle \mathbb{A}\left(\frac{x}{\varepsilon}\right) \begin{pmatrix} \mathbf{e}(u)(x) \\ z(x) \\ \varepsilon^\gamma \nabla z(x) \end{pmatrix}, \begin{pmatrix} \mathbf{e}(u)(x) \\ z(x) \\ \varepsilon^\gamma \nabla z(x) \end{pmatrix} \right\rangle\!\!\right\rangle \mathrm{d}x - \langle \ell(t), u \rangle, \quad \gamma \geq 0, \\
\mathcal{R}_\varepsilon(z) &= \int\limits_\Omega \rho\left(\frac{x}{\varepsilon}, z(x)\right) \mathrm{d}x,
\end{aligned}
$$

where $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is the displacement and $\mathbf{e}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the linearized strain tensor. The vector of the internal variables $z \in \mathrm{H}^1(\Omega)^m$ describes the inelastic effects caused by plastic hardening and plastic strains. Here the tensor valued mapping $\mathbb{A} \colon \mathcal{Y} \to \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m \times \mathbb{R}^{m\times d})$ and the function $\rho(\cdot, z) : \mathcal{Y} \to [0,\infty]$ are Λ-periodic in $y \in \mathcal{Y}$ with respect to the d-dimensional periodic lattice Λ and $\mathcal{Y} = \mathbb{R}^d/_\Lambda$, see section 2. The model which we consider here is a generalization of a problem treated in [MT07].

In [MT07] a similar energetic formulation was considered, but without the gradient term of $z \in \mathrm{H}^1(\Omega)^m$ in the energy functional; i.e. there $\varepsilon^\gamma \equiv 0$ was considered. This additional term has a couple of good properties. Firstly, this term can be seen as a regularization which, depending on $\gamma \geq 0$, leads to more regularity of the solution of the homogenized problem. Secondly, this term enables us to deduce a one-scale homogenized model in the case of $\gamma \in [0,1)$. In general the homogenized model will be a two-scale one; see [MT07]. Finally, the inelastic effects of a single point influences its neighborhood due to this gradient term.

The task is to find a function space $\mathbf{Q}_\gamma$ and limit functionals $\mathbf{E}_\gamma : [0,T] \times \mathbf{Q}_\gamma \to \mathbb{R}$ and $\mathbf{R}_\gamma \colon \mathbf{Q}_\gamma \to [0,\infty]$, so that the energetic solutions $q_\varepsilon \colon [0,T] \to \mathcal{Q}$ of the family of energetic formulations $(\mathrm{S}^\varepsilon)\&(\mathrm{E}^\varepsilon)$ in some sense converge to the energetic solution $\mathbf{q} \colon [0,T] \to \mathbf{Q}_\gamma$ of formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$, if ε tends to zero. The index γ denotes the γ-dependence of the two-scale homogenized rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$.

This kind of convergence will be a variant of the strong and weak two-scale convergence introduced in [Vis07a, Vis07b] and used for elastoplasticity in [MT07], which is closely linked with the classical two-scale convergence introduced by G. Nguetseng in [Ngu89] and further developed by G. Allaire in [All92] for some more details. As common in two-scale convergence, it will not be possible to find a homogenized model with functionals defined only on Ω again in general. That means $\mathbf{E}_\gamma : [0,T] \times \mathbf{Q}_\gamma \to \mathbb{R}$ and $\mathbf{R}_\gamma : \mathbf{Q}_\gamma \to [0,\infty]$ are two-scale functionals, i.e. they are defined as integrals over $\Omega \times \mathcal{Y}$. But as mentioned before, an equivalent one-scale plasticity model can be given in the case of $\gamma \in [0,1)$, by using special minimization properties of the energetic solution of the two-scale rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$. It turns out, that in this case the effective elasticity tensor is the same as in the linear elastic case. Moreover, we obtain the same two-scale homogenized model as in [MT07] in the case of $\gamma > 1$. The limit passage $\varepsilon \to 0$ is a special case of the general theory of Γ-convergence for rat-independent systems as developed in [MRS08]. In addition to the Γ-convergence of $\mathcal{E}_\varepsilon$ to $\mathbf{E}_\gamma$ and $\mathcal{R}_\varepsilon$ to $\mathbf{R}_\gamma$ we will need a suitable "joint recovery condition", see Proposition 4.3.

The current paper is structured as follows: In the following section we introduce the main tools, which are necessary to obtain the homogenization results of section 4. Thereby, we mainly follow [MT07]. We start with introducing the two-scale convergence with the help of an embedding of the one-scale space $\mathrm{L}^p(\Omega)$ into the two-scale space $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$. The embedding, called periodic unfolding operator, is given by $(\mathcal{T}_\varepsilon u)(x, y) := u^{\mathrm{ex}}(\mathcal{N}_\varepsilon(x) + \varepsilon y)$, where u^{ex} is the extension of the one-scale function $u : \Omega \to \mathbb{R}$ by 0 to all of $\mathbb{R}^d$ and $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ maps every point $x \in \mathbb{R}^d$ to its nearest latice point $\mathcal{N}_\varepsilon(x) \in \varepsilon\Lambda$, cf. [CDG02, CDD04]. In subsection 2.3 we introduce our notion of strong and weak two-scale convergence:

$$u_\varepsilon \xrightarrow{s} U \text{ in } \mathrm{L}^p(\Omega \times \mathcal{Y}) \quad \Leftrightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \to U \text{ in } \mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y}),$$

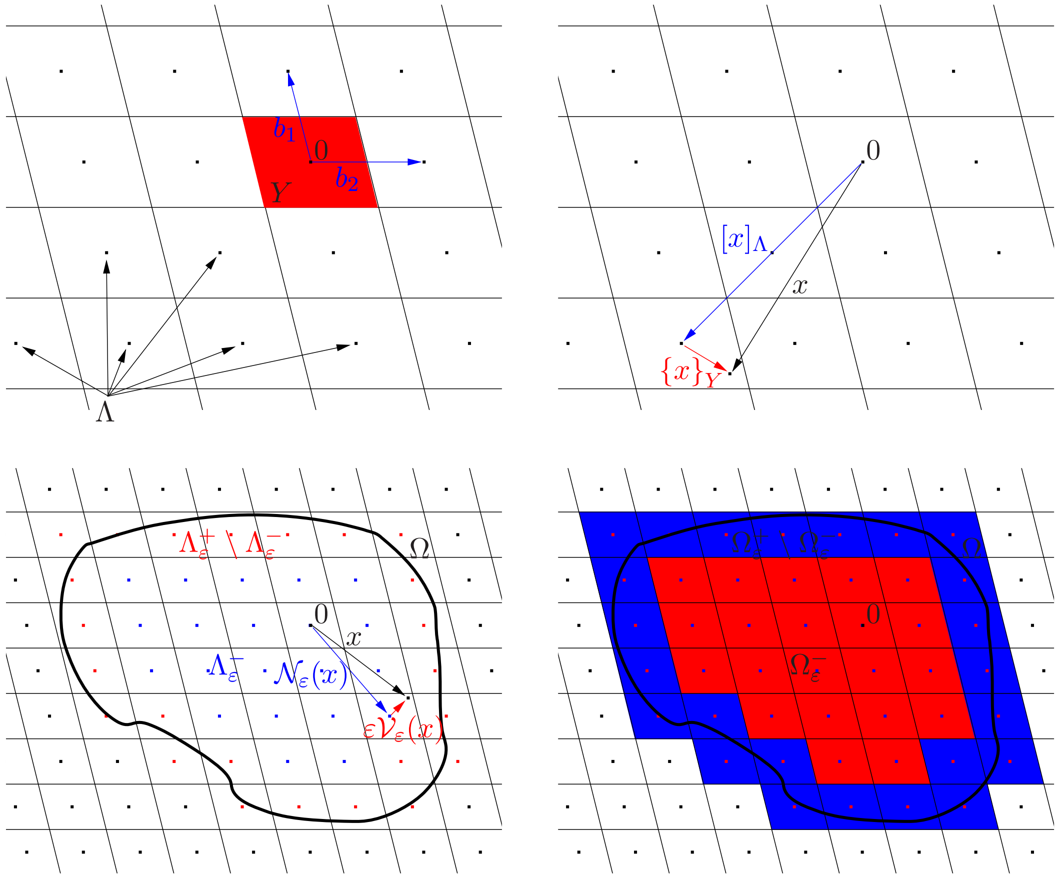
$$u_\varepsilon \overset{w}{\rightharpoonup} U \text{ in } \mathrm{L}^p(\Omega \times \mathcal{Y}) \quad \Leftrightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U \text{ in } \mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y}).$$

This definition is an adaptation of the definitions in [Vis04] to the case that Ω has a boundary. The main results of this section are proposition 2.9, which is a two-scale convergence statement for sequences $(v_\varepsilon)_{\varepsilon>0}$ of $\mathrm{H}^1(\Omega)$ with $\sup_{\varepsilon>0}(\|v_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \varepsilon^\gamma \|\nabla v_\varepsilon\|_{\mathrm{L}^2(\Omega)^d}) \leq C$, and proposition 2.11, which gives a construction of a folding operator $\mathcal{G}_\varepsilon$ from the two-scale domain $\Omega \times \mathcal{Y}$ to the one-scale domain Ω. The operators will be crucial to construct the "joint recovery sequences" needed for the Γ-convergence theory in [MRS08].

Section 3 contains all elastoplastic models, including the limit models, which are considered and discussed in the sections 4 and 5. We start with introducing the energetic formulation in the way it will be used in the following subsections. For more details and a more general setting of this formulation we refer to [Mie03, MR07, MRS08]. The ε-dependent elastoplastic model, which will be homogenized, is stated in subsection 3.2. It turns out, that the z-component of the unique energetic solution of this system is bounded in the sense mentioned above. Furthermore, the two-scale and one-scale homogenized model are described in subsection 3.3 and 3.4, respectively, and existence results are proved.

In section 4 we prove the convergence of the ε-dependent elastoplastic model with Λ-periodic coefficients to the two-scale model defined in subsection 3.3, assuming strong convergence of the initial dates. The proof is an adapted variant of the techniques of [MT07] and is done for all $\gamma \geq 0$. That means, the functionals $\mathbf{E}_\gamma$ and $\mathbf{R}_\gamma$ are the Γ-limits of $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ and $(\mathcal{R}_\varepsilon)_{\varepsilon>0}$, respectively, and that the energetic solutions of the rate-independent systems $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converge strongly to the energetic solution of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$. Thereby, the convergence has to be understood as a kind of two-scale convergence, defined in section 2.

Finally, in section 5 we focus on the case $\gamma \in [0, 1)$ and show the equivalence of the two-scale system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ and the one-scale model defined in subsection 3.4. This is actually a consequence of the homogenization result for nonlinear and quasiconvex integrals of D. Cioranescu, A. Damlamian and R. De Arcangelis in [CDD04] and [CDD06], respectively. For these parameters γ the two-scale dissipation potential $\mathbf{R}_\gamma$ actually depends on a one-scale function z_0 only. This property is the reason why it is possible to construct a homogenized model in the classical sense, i.e. the functionals $\mathcal{E}_\varepsilon : [0, T] \times \mathcal{Q} \to \mathbb{R}$ and $\mathcal{R}_\varepsilon : \mathcal{Q} \to [0, \infty]$ as well as their Γ-limits $\mathcal{E}_\gamma : [0, T] \times \mathcal{Q}_\gamma \to \mathbb{R}$ and $\mathcal{R}_\gamma : \mathcal{Q}_\gamma \to [0, \infty]$ are defined as integrals over the domain $\Omega \subset \mathbb{R}^d$, only. Since $\mathcal{Q}_0 = \mathcal{Q}$ if $\gamma = 0$, we especially obtain that $\mathcal{E}_0 : [0, T] \times \mathcal{Q} \to \mathbb{R}$ and $\mathcal{R}_0 : \mathcal{Q} \to [0, \infty]$ are the classical Γ-limits of $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ and $(\mathcal{R}_\varepsilon)_{\varepsilon>0}$, respectively, by combining the results from section 4 and 5.



2 Two-scale convergence

2.1 Definitions

Let $d \in \mathbb{N}$ be the space dimension. To describe periodicity let Λ be a d-dimensional, periodic lattice given by

$$\Lambda := \left\{ \lambda = \sum_{j=1}^{d} k_j b_j \,|\, k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d \right\},$$

where $\{b_1, b_2, \ldots, b_d\}$ is an arbitrary basis in $\mathbb{R}^d$. An almost everywhere on $\mathbb{R}^d$ defined function f is called Λ-periodic, if $f(x) = f(x+\lambda)$ for all $\lambda \in \Lambda$ and almost every $x \in \mathbb{R}^d$. Furthermore, let $Y := \{x = \sum_{j=1}^{d} \beta_j b_j \,|\, \beta_j \in [-\frac{1}{2}, \frac{1}{2})\} \subset \mathbb{R}^d$ be the to Λ associated unit cell, so that $\mathbb{R}^d$ is the disjoint union of all translated cells $\lambda + Y$, e.g. λ ranges all of Λ. We also want to distinguish the unit cell Y from the periodicity cell $\mathcal{Y} := \mathbb{R}^d/_\Lambda$, as it is done in [Vis04]. In the following we may be inconsistent and use y to denote elements of Y and $\mathcal{Y}$ simultaneously.

Now we introduce the mappings $[\cdot]_\Lambda$ and $\{\cdot\}_Y$ on $\mathbb{R}^d$, so that

$$[\cdot]_\Lambda \colon \mathbb{R}^d \to \Lambda, \quad \{\cdot\}_Y \colon \mathbb{R}^d \to Y, \quad \text{and} \quad x = [x]_\Lambda + \{x\}_Y \quad \text{for all } x \in \mathbb{R}^d.$$

Let $x \in \mathbb{R}^d$ be in the cell $\lambda + Y$, then is $[x]_\Lambda$ the center λ of the cell and $\{x\}_Y$ is determinable as $\{x\}_Y = x - [x]_\Lambda$. For $\varepsilon > 0$ we have the following decomposion for $x \in \mathbb{R}^d$:

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon \mathcal{V}_\varepsilon(x), \quad \text{with } \mathcal{N}_\varepsilon(x) := \varepsilon \left[\frac{x}{\varepsilon}\right]_\Lambda \text{ and } \mathcal{V}_\varepsilon(x) := \left\{\frac{x}{\varepsilon}\right\}_Y,$$

where $\mathcal{N}_\varepsilon(x)$ denotes the macroscopic center of the cell $\mathcal{N}_\varepsilon(x) + \varepsilon Y$ that contains x and $\mathcal{V}_\varepsilon(x)$ is the microscopic part of x in Y. Thereby, an on $\mathbb{R}^d$ defined function f is Λ-periodic, if $f(x) = f(\{x\}_Y)$ for almost every $x \in \mathbb{R}^d$. Moreover, $\mathrm{L}^p(Y)$ and $\mathrm{L}^p(\mathcal{Y})$ may be identified, but $\mathrm{C}^k(Y)$ and $\mathrm{C}^k(\mathcal{Y}) = \mathrm{C}^k_{\mathrm{per}}(\overline{Y})$ have to be distinguished. Also $\mathrm{H}^1(\mathcal{Y}) = \mathrm{H}^1_{\mathrm{per}}(\overline{Y})$ differs from $\mathrm{H}^1(Y)$.

Following [Vis04], it is now possible to construct a decomposing map $\mathcal{D}_\varepsilon$ and a composing map $\mathcal{S}_\varepsilon$ as follows:

$$\mathcal{D}_\varepsilon\colon \left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R}^d \times \mathcal{Y}, \\ x & \mapsto & (\mathcal{N}_\varepsilon(x), \mathcal{V}_\varepsilon(x)), \end{array} \right. \qquad \mathcal{S}_\varepsilon\colon \left\{ \begin{array}{ccc} \mathbb{R}^d \times \mathcal{Y} & \to & \mathbb{R}^d, \\ (x, y) & \mapsto & \mathcal{N}_\varepsilon(x) + \varepsilon y, \end{array} \right.$$

where in the last sum $y \in \mathcal{Y}$ is identified with $y \in Y \subset \mathbb{R}^d$. For the periodic unfolding operator and a variant of the folding operator constructed in [CDG02], the following properties are essential:

$$\mathcal{D}_\varepsilon(\mathcal{S}_\varepsilon(x, y)) = (\mathcal{N}_\varepsilon(x), y) \quad \text{and} \quad \mathcal{S}_\varepsilon(\mathcal{D}_\varepsilon(x)) = x \quad \text{for all } (x, y) \in \mathbb{R}^d \times \mathcal{Y}.$$

Since $\Omega \subset \mathbb{R}^d$ should be a body, we want to consider a domain Ω, which does not coincide with $\mathbb{R}^d$. Due to this there are some technical problems, because of the fact that in this case the images of $\mathcal{D}_\varepsilon$ and $\mathcal{S}_\varepsilon$ are not contained in $\Omega \times \mathcal{Y}$ and Ω, respectively. To handle this and to make sure that our periodic unfolding operator is well-defined we consider now the following subsets of Λ :

$$\Lambda_\varepsilon^- := \left\{\lambda \in \Lambda \,|\, \varepsilon(\lambda + Y) \subset \overline{\Omega}\right\} \quad \text{and} \quad \Lambda_\varepsilon^+ := \left\{\lambda \in \Lambda \,|\, \varepsilon(\lambda + Y) \cap \Omega \neq \emptyset\right\}.$$

With this we define the sets Ω_ε^- and Ω_ε^+ as:

$$\Omega_\varepsilon^\pm := \mathrm{int}\big(\bigcup_{\lambda \in \Lambda_\varepsilon^\pm} \varepsilon(\lambda + Y)\big).$$

Let $\mathrm{diam}(M)$ be the diameter and $N_{\varepsilon_0}(M)$ the ε_0-neighborhood of the set M for some $\varepsilon_0 > 0$. Then Ω_ε^- and Ω_ε^+ have the following properties:

$$\Omega_\varepsilon^- \subset \Omega \subset \Omega_\varepsilon^+, \quad \left[\Omega_\varepsilon^\pm\right]_\varepsilon^\pm = \Omega_\varepsilon^\pm, \quad \Omega \subset N_{\varepsilon \mathrm{diam}(Y)}(\Omega_\varepsilon^-) \quad \text{and} \quad \Omega_\varepsilon^+ \subset N_{\varepsilon \mathrm{diam}(Y)}(\Omega). \tag{2.1}$$

Furthermore, we set $[\Omega \times \mathcal{Y}]_\varepsilon := \mathcal{S}_\varepsilon^{-1}(\Omega) = \{(x, y) \,|\, \mathcal{S}_\varepsilon(x, y) \in \Omega\}$. For this set we have the relation

$$\Omega_\varepsilon^- \times \mathcal{Y} \subset [\Omega \times \mathcal{Y}]_\varepsilon \subset \overline{\Omega_\varepsilon^+} \times \mathcal{Y}, \tag{2.2}$$

which will be needed below.

Let Ω be a bounded open set in $\mathbb{R}^d$ satisfying $\mathcal{L}_d(\partial\Omega) = 0$, then we get for $\varepsilon \to 0$

$$\mathcal{L}_d(\Omega \backslash \Omega_\varepsilon^-) + \mathcal{L}_d(\Omega_\varepsilon^+ \backslash \Omega) \to 0. \tag{2.3}$$

This can be seen by considering the characteristic function $\mathbb{1}_\varepsilon$ of the set $N_{\varepsilon \mathrm{diam}(Y)}(\partial\Omega)$. We now have on the one hand $\Omega \backslash \Omega_\varepsilon^- \cup \Omega_\varepsilon^+ \backslash \Omega \subset N_{\varepsilon \mathrm{diam}(Y)}(\partial\Omega)$ and on the other hand $\mathbb{1}_\varepsilon(x) \to 0$ for all $x \notin \partial\Omega$ and $\varepsilon \to 0$. Thereby, it follows $\mathcal{L}_d(\Omega \backslash \Omega_\varepsilon^-) + \mathcal{L}_d(\Omega_\varepsilon^+ \backslash \Omega) \leq \mathcal{L}_d(N_{\varepsilon \mathrm{diam} Y}(\partial\Omega)) = \int_{\mathbb{R}^d} \mathbb{1}_\varepsilon(x) \, \mathrm{d}x \to 0$ for $\varepsilon \to 0$.

2.2 Folding and periodic unfolding operators

Henceforth let Ω be a bounded open set in $\mathbb{R}^d$ satisfying $\mathcal{L}_d(\partial\Omega) = 0$ and we may also assume

$$\mathcal{L}_d(Y) = 1.$$

The two-scale convergence is linked to a "suitable" two-scale embedding of $\mathrm{L}^p(\Omega)$ in the two-scale space $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$. Such an embedding is called periodic unfolding operator. It is also necessary to find a function u_ε defined on Ω which has a corresponding microscopic behavior like a two-scale function U defined on $\Omega \times \mathcal{Y}$. A mapping from $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$ to $\mathrm{L}^p(\Omega)$ is called folding operator.

The following definition of a periodic unfolding operator was given in [CDG02].

Definition 2.1. *[CDG02] Let $\Omega \subset \mathbb{R}^d$ be open, $\varepsilon > 0$ and $p \in [1, \infty)$. Then the natural candidate of a periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined via:*

$$\mathcal{T}_\varepsilon \colon \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y});\ u \mapsto u^{\mathrm{ex}} \circ \mathcal{S}_\varepsilon,$$

where $u^{\mathrm{ex}} \in \mathrm{L}^p(\mathbb{R}^d)$ is the extension with 0 to $\mathbb{R}^d$ of function u.

With this definition the following product rule is valid:

$$\tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{r}, u \in \mathrm{L}^p(\Omega), v \in \mathrm{L}^q(\Omega) \quad \Rightarrow \quad \mathcal{T}_\varepsilon(uv) = (\mathcal{T}_\varepsilon u)(\mathcal{T}_\varepsilon v) \in \mathrm{L}^r(\mathbb{R}^d \times \mathcal{Y}).$$

Note that in general $\overline{[\Omega \times \mathcal{Y}]_\varepsilon}$ is the support of $\mathcal{T}_\varepsilon v$, and from subsection 2.1 we know that this is not contained in $\Omega \times \mathcal{Y}$. In the next subsection we will use this periodic unfolding operator to introduce the kind of two-scale convergence which is used here.

To indicate a well-defined folding operator $\mathcal{F}_\varepsilon \colon \mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y}) \to \mathrm{L}^p(\Omega)$, we first have to give the definition of the classical projector to piecewise constant functions on every $\varepsilon(\lambda + Y)$.

Definition 2.2. *[MT07] Let $\varepsilon > 0$ and $p' \in (1, \infty)$. On $\mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})$ the classical projector to piecewise constant functions in the x-component is*

$$(\mathcal{P}_\varepsilon U)(x, y) := \fint_{\mathcal{N}_\varepsilon(x) + \varepsilon Y} U(\xi, y) \, \mathrm{d}\xi,$$

where $\fint_A g(a) \, \mathrm{d}a := \frac{1}{|A|} \int_A g(a) \, \mathrm{d}a$ is the average of the function g over A.

This definition yields $(\mathcal{P}_\varepsilon)^2 = \mathcal{P}_\varepsilon, \|\mathcal{P}_\varepsilon U\|_{\mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})} \leq \|U\|_{\mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})}$ and $\mathcal{P}_\varepsilon U \to U$ in $\mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})$ for all $U \in \mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})$.

Now the folding operator $\mathcal{F}_\varepsilon$, which is a variant of the "averaging operator" defined in section 5 of [CDG02], is given as follows.

Definition 2.3. *[MT07] Let $\varepsilon > 0$ and $p' \in (1, \infty)$. Then the folding operator $\mathcal{F}_\varepsilon$ is defined as follows:*

$$\mathcal{F}_\varepsilon \colon \mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y}) \to \mathrm{L}^{p'}(\Omega);\ U \mapsto (\mathcal{P}_\varepsilon(\mathbb{1}_\varepsilon U) \circ \mathcal{D}_\varepsilon)|_\Omega \quad \textit{with} \quad \mathbb{1}_\varepsilon := \mathbb{1}_{[\Omega \times \mathcal{Y}]_\varepsilon} = \mathcal{T}_\varepsilon \mathbb{1}_\Omega.$$

Note that this folding operator is defined on the space of functions with supports contained in $\mathbb{R}^d \times \mathcal{Y}$ and that it takes values in the space of functions, which are defined on Ω. This is guaranteed by the construction with the characteristic function $\mathbb{1}_\varepsilon$, because of $\operatorname{supp}(\mathbb{1}_\varepsilon \circ \mathcal{D}_\varepsilon) = \overline{\Omega}$ and $\mathbb{1}_\varepsilon = \mathcal{P}_\varepsilon \mathbb{1}_\varepsilon$, which follow directly from the definition of $[\Omega \times \mathcal{Y}]_\varepsilon$.

Basic properties of the periodic unfolding operator $\mathcal{T}_\varepsilon$ and the folding operator $\mathcal{F}_\varepsilon$ are listed in the following proposition.

Proposition 2.4. *[MT07] Let $p \in (1,\infty)$, $p' := \frac{p}{p-1}$ and $\varepsilon > 0$. Then the periodic unfolding operators $\mathcal{T}_\varepsilon \colon \mathrm{L}^p(\Omega) \to L^p(\mathbb{R}^d \times \mathcal{Y}), \widetilde{\mathcal{T}}_\varepsilon \colon \mathrm{L}^{p'}(\Omega) \to \mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})$ and the folding operator $\mathcal{F}_\varepsilon \colon \mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y}) \to \mathrm{L}^{p'}(\Omega)$ have the following properties:*

(a) $\|\mathcal{T}_\varepsilon u\|_{\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})} = \|u\|_{\mathrm{L}^p(\Omega)}$ *and* $\operatorname{supp}(\mathcal{T}_\varepsilon u) \subset \overline{[\Omega \times \mathcal{Y}]_\varepsilon}$ *for all* $u \in \mathrm{L}^p(\Omega)$.
(b) $\|\mathcal{F}_\varepsilon U\|_{\mathrm{L}^{p'}(\Omega)} \leq \|U\|_{\mathrm{L}^{p'}(\mathbb{R}^d\times\mathcal{Y})}$ *for all* $U \in \mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})$.
(c) $\mathcal{F}_\varepsilon$ *is the adjoint of* $\mathcal{T}_\varepsilon$*, i.e.* $\mathcal{F}_\varepsilon = (\mathcal{T}_\varepsilon)'$.
(d) $\mathcal{F}_\varepsilon \circ \widetilde{\mathcal{T}}_\varepsilon = \mathrm{id}_{\mathrm{L}^{p'}(\Omega)}$ *and* $\big(\widetilde{\mathcal{T}}_\varepsilon \circ \mathcal{F}_\varepsilon\big)^2 = \widetilde{\mathcal{T}}_\varepsilon \circ \mathcal{F}_\varepsilon = \mathbb{1}_\varepsilon \mathcal{P}_\varepsilon$.

2.3 Strong and weak two-scale-convergence

In this section we give the definition of the two-scale convergence used in this paper following the lines in [MT07]; the strong and weak two-scale convergence, respectively. But we want to start by defining the classical two-scale convergence introduced by G. Nguetseng in 1989 ([Ngu89]), because of the equivalence of classical and weak two-scale convergence in the bounded case.

Definition 2.5. *[Ngu89] A sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ in $\mathrm{L}^p(\Omega)$ is called two scale convergent to a function $U \in \mathrm{L}^p(\Omega \times \mathcal{Y})$, shortly $u_\varepsilon \overset{2}{\rightharpoonup} U$, if for all testfunctions $\psi : \Omega \times \mathcal{Y} \to \mathbb{R}$ from a set of testfunctions Ψ we have:*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi\Big(x, \Big\{\frac{x}{\varepsilon}\Big\}_Y\Big)\,\mathrm{d}x = \int_{\Omega\times\mathcal{Y}} U(x,y)\psi(x,y)\,\mathrm{d}y\mathrm{d}x.$$

Here the choice of the set of testfunctions is very important. Choosing $\Psi = \mathrm{C}_c^\infty(\Omega \times \mathcal{Y})$, then the classical two-scale convergence corresponds to a kind of distributional convergence. Let $p' := \frac{p}{p-1}$ be the dual of $p \in (1,\infty)$, then $\Psi = \mathrm{L}^{p'}(\Omega, \mathrm{C}(\mathcal{Y}))$ guarantees the weak convergence of $(u_\varepsilon)_{\varepsilon>0}$ to $\int_\mathcal{Y} U(\cdot,y)\,\mathrm{d}y$ in $\mathrm{L}^p(\Omega)$. The following definition of strong and weak two-scale convergence was given in [MT07] and will be used here mainly.

Definition 2.6. *[MT07] Let $p \in (1,\infty)$ and let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathrm{L}^p(\Omega)$. Then*

(a) *u_ε converges strongly two-scale to $U \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, $u_\varepsilon \overset{s}{\to} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, if $\mathcal{T}_\varepsilon u_\varepsilon \to U^{\mathrm{ex}}$ in $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$.*
(b) *u_ε converges weakly two-scale to $U \in \mathrm{L}^p(\Omega \times \mathcal{Y})$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, $u_\varepsilon \overset{w}{\rightharpoonup} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, if $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U^{\mathrm{ex}}$ in $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$.*

Because of the fact that for all $\varepsilon > 0$ the support of the function $\mathcal{T}_\varepsilon u_\varepsilon$ is contained in $\overline{[\Omega \times \mathcal{Y}]_\varepsilon} \subset \overline{\Omega}_\varepsilon^+ \times \mathcal{Y}$, we conclude with (2.2) and (2.3), that the support of a possible accumulation point U of the sequence $(\mathcal{T}_\varepsilon u_\varepsilon)_{\varepsilon>0}$ has to be in $\overline{\Omega} \times \mathcal{Y}$. Moreover, we have $\mathrm{L}^p(\Omega\times\mathcal{Y}) = \mathrm{L}^p(\overline{\Omega}\times\mathcal{Y})$ because of $\mathcal{L}_d(\partial\Omega) = 0$ and so every accumulation point of $(\mathcal{T}_\varepsilon u_\varepsilon)_{\varepsilon>0}$

can be uniquely identified with an element of $\mathrm{L}^p(\Omega \times \mathcal{Y})$. But notice that it is important to determine the convergence in $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$ and not in $\mathrm{L}^p(\Omega \times \mathcal{Y})$. We refer to [MT07] where it is shown in example 2.3 that convergence in $\mathrm{L}^p(\Omega \times \mathcal{Y})$ is not sufficient.

Using property (c) of proposition 2.4, an equivalent definition of the weak two-scale convergence is given as follows:

$$u_\varepsilon \overset{w}{\rightharpoonup} U \text{ in } \mathrm{L}^p(\Omega \times \mathcal{Y}) \;\Leftrightarrow\; \int\limits_{\Omega} u_\varepsilon \mathcal{F}_\varepsilon V^{\mathrm{ex}} \mathrm{d}x \overset{\varepsilon \to 0}{\longrightarrow} \int\limits_{\Omega \times \mathcal{Y}} UV \mathrm{d}y \mathrm{d}x \quad \forall V \in \mathrm{L}^{p'}(N_{\varepsilon_0}(\Omega) \times \mathcal{Y}),$$

where $\varepsilon_0 > 0$ is arbitrary but fixed. Because of the characteristic function $\mathbb{1}_\varepsilon$ in the definition 2.4 of the folding operator $\mathcal{F}_\varepsilon$ and property (2.1) and (2.2), it is sufficient to consider all functions V of $\mathrm{L}^{p'}(N_{\varepsilon_0}(\Omega) \times \mathcal{Y})$ instead of $\mathrm{L}^{p'}(\mathbb{R}^d \times \mathcal{Y})$.

For convenience we list further convergence properties in proposition 2.7 and refer to [MT07] for the proofs.

Proposition 2.7. *[MT07] Let $p \in (1, \infty)$, $p' := \frac{p}{p-1}$ and $\varepsilon > 0$. Then*

(a) $u_\varepsilon \overset{w}{\rightharpoonup} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y}) \quad \Rightarrow \quad \|u_\varepsilon\|_{\mathrm{L}^p(\Omega)}$ *is bounded for all* $\varepsilon > 0$.
(b) $u_\varepsilon \overset{w}{\rightharpoonup} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y}) \quad \Rightarrow \quad u_\varepsilon \overset{2}{\rightharpoonup} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$.
(c) $u_\varepsilon \overset{w}{\rightharpoonup} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$ *and* $\|u_\varepsilon\|_{\mathrm{L}^p(\Omega)} \to \|U\|_{\mathrm{L}^p(\Omega \times \mathcal{Y})} \;\Leftrightarrow\; u_\varepsilon \overset{s}{\to} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$.
(d) $u_\varepsilon \overset{w}{\rightharpoonup} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$ *and* $v_\varepsilon \overset{s}{\to} V$ *in* $\mathrm{L}^{p'}(\Omega \times \mathcal{Y}) \Rightarrow \langle u_\varepsilon, v_\varepsilon \rangle_{\mathrm{L}^2(\Omega)} \to \langle U, V \rangle_{\mathrm{L}^2(\Omega \times \mathcal{Y})}$.
(e) *For all* $U \in \mathrm{L}^p(\Omega \times \mathcal{Y})$ *there exists a sequence* $(u_\varepsilon)_{\varepsilon > 0}$ *so that* $u_\varepsilon \overset{s}{\to} U$ *in* $\mathrm{L}^{p'}(\Omega \times \mathcal{Y})$ *(for example* $u_\varepsilon = \mathcal{F}_\varepsilon U^{\mathrm{ex}}$*).*
(f) *For all* $w \in \mathrm{L}^p(\Omega)$ *we have* $w \overset{s}{\to} Ew$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$*, where* $E : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega \times \mathcal{Y})$ *is defined via* $Ev(x, y) := v(x)$.
(g) $w_\varepsilon \to w$ *in* $\mathrm{L}^p(\Omega) \;\Rightarrow\; w_\varepsilon \overset{s}{\to} Ew$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$.
(h) *For* $p \in (1, \infty), q \in (1, \infty]$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$ *let* $u_\varepsilon \overset{w}{\rightharpoonup} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$ *and* $v_\varepsilon \overset{s}{\to} V$ *in* $\mathrm{L}^q(\Omega \times \mathcal{Y})$*. Then* $u_\varepsilon v_\varepsilon \overset{w}{\rightharpoonup} UV$ *in* $\mathrm{L}^r(\Omega \times \mathcal{Y})$*. If additionally* $u_\varepsilon \overset{s}{\to} U$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})$*, then* $u_\varepsilon v_\varepsilon \overset{s}{\to} UV$ *in* $\mathrm{L}^r(\Omega \times \mathcal{Y})$.

The following proposition is an extension of property (h) of proposition 2.7.

Proposition 2.8. *[MT07] Let $p \in [1, \infty)$, $\varepsilon > 0$ and $(u_\varepsilon)_\varepsilon$ be a sequence in $\mathrm{L}^p(\Omega)$ with $u_\varepsilon \overset{s}{\to} U$ in $\mathrm{L}^p(\Omega \times \mathcal{Y})$. Furthermore, let $(m_\varepsilon)_\varepsilon$ be in $\mathrm{L}^\infty(\Omega)$ so that $\mathcal{T}_\varepsilon m_\varepsilon(x, y) \to M(x, y)$ for almost every $(x, y) \in \Omega \times \mathcal{Y}$. Then $m_\varepsilon u_\varepsilon \overset{s}{\to} MU$ in $\mathrm{L}^p(\Omega \times \mathcal{Y})$.*

2.4 Two-scale convergence of Sobolev-functions

In this subsection we will consider sequences of $\mathrm{H}^1(\Omega)$ which are bounded in the sense of relation (2.4) below. Especially we will need the function space

$$\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}) := \left\{ v \in \mathrm{H}^1(\mathcal{Y}) | \int_{\mathcal{Y}} v(y) \,\mathrm{d}y = 0 \right\}.$$

To describe the weak two-scale convergence of gradients multiplied by a scaling parameter ε^γ $(\varepsilon > 0, \gamma \geq 0)$, we introduce the function space $\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$, which is the space of functions $V \in \mathrm{L}^2(\Omega \times \mathcal{Y}) = \mathrm{L}^2(\Omega; \mathrm{L}^2(\mathcal{Y}))$, with $\int_{\mathcal{Y}} V(x, y) \,\mathrm{d}y = 0$ for almost every $x \in \Omega$ and $\nabla_y V \in \mathrm{L}^2(\Omega \times \mathcal{Y})^d$ in the sense of distribution.

Proposition 2.9. *Let $\gamma \geq 0$ be given and $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathrm{H}^1(\Omega)$ so that*

$$\|v_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \varepsilon^\gamma \|\nabla v_\varepsilon\|_{\mathrm{L}^2(\Omega)^d} \leq C \tag{2.4}$$

for all $\varepsilon > 0$ and a $C \geq 0$. Then there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'>0}$ of $(v_\varepsilon)_{\varepsilon>0}$ and

$\gamma = 0:$ *functions $v_0 \in \mathrm{H}^1(\Omega)$ and $V_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ so that*

$$\begin{aligned} v_{\varepsilon'} &\xrightarrow{s} Ev_0 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y}), \\ \nabla v_{\varepsilon'} &\xrightharpoonup{w} \nabla_x Ev_0 + \nabla_y V_1 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^d. \end{aligned}$$

$\gamma \in (0,1):$ *functions $v_0 \in \mathrm{L}^2(\Omega)$ and $V_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ so that*

$$\begin{aligned} v_{\varepsilon'} &\xrightharpoonup{w} Ev_0 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y}), \\ \varepsilon^\gamma \nabla v_{\varepsilon'} &\xrightharpoonup{w} \nabla_y V_1 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^d. \end{aligned}$$

$\gamma = 1:$ *a function $V_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ so that*

$$\begin{aligned} v_{\varepsilon'} &\xrightharpoonup{w} V_1 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y}), \\ \varepsilon \nabla v_{\varepsilon'} &\xrightharpoonup{w} \nabla_y V_1 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^d. \end{aligned}$$

$\gamma > 1:$ *a function $V \in \mathrm{L}^2(\Omega \times \mathcal{Y})$ so that*

$$\begin{aligned} v_{\varepsilon'} &\xrightharpoonup{w} V && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y}), \\ \varepsilon^\gamma \nabla v_{\varepsilon'} &\xrightharpoonup{w} 0 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^d. \end{aligned}$$

Proof 2.9. Theorem 3.1.4 in [Pe07] yields the result. □

To simplify notation, we introduce the function space $\mathbf{X}_\gamma$ of the limit functions of proposition 2.9 and a map $\mathcal{L}^\gamma \colon \mathbf{X}_\gamma \to \mathrm{L}^2(\Omega \times \mathcal{Y})^{d+1}$.

$\gamma \in$	$\mathbf{X}_\gamma :=$	$\mathcal{L}^\gamma(\mathcal{V}_\gamma) \mapsto$
$\{0\}$	$\mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$	$(Ev_0, \nabla_x Ev_0 + \nabla_y V_1)$
$(0,1)$	$\mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$	$(Ev_0, \nabla_y V_1)$
$\{1\}$	$\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$	$(V_1, \nabla_y V_1)$
$(1,\infty)$	$\mathrm{L}^2(\Omega \times \mathcal{Y})$	$(V_1, 0)$

(2.5)

where $\mathcal{V}_\gamma$ denotes the Elements of $\mathbf{X}_\gamma$ and is defined via

$$\mathcal{V}_\gamma := \begin{cases} (v_0, V_1) \in \mathbf{X}_\gamma & \text{if } \gamma \in [0,1), \\ V_1 \in \mathbf{X}_\gamma & \text{if } \gamma \geq 1. \end{cases} \tag{2.6}$$

Furthermore, we equip $\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ with the norm $\|V_1\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))} := \|\nabla_y V_1\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})^d}$ and set $\|\cdot\|_{\mathbf{X}_\gamma} \colon \mathbf{X}_\gamma \to [0,\infty)$ as the norm of the product space ($\gamma \in [0,1)$).

Definition 2.10. *Let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathrm{H}^1(\Omega)$. Then strong and weak two-scale-γ-convergence in $\mathbf{X}_\gamma$ are defined as follows:*

$$v_\varepsilon \xrightarrow{s\gamma} \mathcal{V}_\gamma \text{ in } \mathbf{X}_\gamma \quad :\Longleftrightarrow \quad (v_\varepsilon, \varepsilon^\gamma \nabla v_\varepsilon) \xrightarrow{s} \mathcal{L}^\gamma(\mathcal{V}_\gamma) \text{ in } \mathrm{L}^2(\Omega \times \mathcal{Y})^{d+1}.$$

$$v_\varepsilon \xrightharpoonup{w\gamma} \mathcal{V}_\gamma \text{ in } \mathbf{X}_\gamma \quad :\Longleftrightarrow \quad (v_\varepsilon, \varepsilon^\gamma \nabla v_\varepsilon) \xrightharpoonup{w} \mathcal{L}^\gamma(\mathcal{V}_\gamma) \text{ in } \mathrm{L}^2(\Omega \times \mathcal{Y})^{d+1}.$$

Note, if $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1(\Omega)$ satisfies estimate (2.4), then there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'>0}$ of $(v_\varepsilon)_{\varepsilon>0}$ and a function $\mathcal{V}_\gamma$ in $\mathbf{X}_\gamma$ so that $v_{\varepsilon'} \overset{w\gamma}{\rightharpoonup} \mathcal{V}_\gamma$ in $\mathbf{X}_\gamma$ (see propostion 2.9).

Now we want to prove a kind of density result, namely, that for every function $\mathcal{V}_\gamma \in \mathbf{X}_\gamma$ there exists a sequence $(v_\varepsilon)_{\varepsilon>0}$ in $\mathrm{H}^1(\Omega)$, so that this sequence converges strongly to $\mathcal{V}_\gamma$ in the sense of definition 2.10, i.e. $v_\varepsilon \overset{s\gamma}{\to} \mathcal{V}_\gamma$ in $\mathbf{X}_\gamma$. This sequence is called "recovery sequence" and will be used in section 4 to prove stability of the limit function of a stable sequence.

Thereto, we extend an idea from [MT07] from the case $\gamma = 0$ to $\gamma \geq 0$. We start with defining $\mathbf{L} := \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$ and for $\varepsilon > 0$ and $\gamma \geq 0$ the two norm-preserving and linear operators $\mathbb{T}^\gamma_\varepsilon$ and $\mathbb{F}^\gamma_\varepsilon$ as follows:

$$\mathbb{T}^\gamma_\varepsilon \colon \begin{cases} \mathrm{H}^1(\Omega) & \to \quad \mathbf{L}^{d+1}, \\ v_0 & \mapsto \quad (\mathcal{T}_\varepsilon v_0, \mathcal{T}_\varepsilon(\varepsilon^\gamma \nabla v_0)), \end{cases} \qquad \mathbb{F}^\gamma_\varepsilon \colon \begin{cases} \mathbf{X}_\gamma & \to \quad \mathbf{L}^{d+1}, \\ \mathcal{V}_\gamma & \mapsto \quad (\mathcal{L}^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}. \end{cases}$$

In the case of $\gamma = 0$ we obtain

$$\|(\nabla_x E v_0 + \nabla_y V_1)^{\mathrm{ex}}\|^2_{\mathbf{L}^d} = \|(\nabla_x E v_0)^{\mathrm{ex}}\|^2_{\mathbf{L}^d} + \|(\nabla_y V_1)^{\mathrm{ex}}\|^2_{\mathbf{L}^d} \tag{2.7}$$

for $v_0 \in \mathrm{H}^1(\Omega)$ and $V_1 \in \mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ by using $\|\cdot\|^2_{\mathbf{L}^d} = \langle \cdot, \cdot \rangle_{\mathbf{L}^d}$, integration by parts and the facts that Ev_0 is constant and V_1 is periodic in the y-component. That indicates the norm-preservation of $\mathbb{F}^0_\varepsilon$ (in the case of $\gamma > 0$ it is obvious).

By these definitions the images $\mathcal{X}^\varepsilon_{\mathbb{T}^\gamma} := \mathbb{T}^\gamma_\varepsilon \mathrm{H}^1(\Omega)$ and $\mathcal{X}^\varepsilon_{\mathbb{F}^\gamma} := \mathbb{F}^\gamma_\varepsilon \mathbf{X}_\gamma$ are closed subsets of $\mathbf{L}^{d+1}$, so that we are able to consider the orthogonal projections $\mathbb{Q}^\varepsilon_{\mathbb{T}^\gamma}$ and $\mathbb{Q}^\varepsilon_{\mathbb{F}^\gamma}$ of $\mathbf{L}^{d+1}$ on $\mathcal{X}^\varepsilon_{\mathbb{T}^\gamma}$ and $\mathcal{X}^\varepsilon_{\mathbb{F}^\gamma}$, respectively. Now for $\varepsilon > 0$ and $\gamma \geq 0$ we introduce the folding operator $\mathcal{T}^\gamma_\varepsilon$ and the unfolding operator $\mathcal{G}^\gamma_\varepsilon$.

$$\mathcal{T}^\gamma_\varepsilon \colon \begin{cases} \mathrm{H}^1(\Omega) & \to \quad \mathbf{X}_\gamma, \\ v & \mapsto \quad (\mathbb{F}^\gamma_\varepsilon)^{-1}(\mathbb{Q}^\varepsilon_{\mathbb{F}^\gamma}(\mathbb{T}^\gamma_\varepsilon(v))), \end{cases} \qquad \mathcal{G}^\gamma_\varepsilon \colon \begin{cases} \mathbf{X}_\gamma & \to \quad \mathrm{H}^1(\Omega), \\ \mathcal{V}_\gamma & \mapsto \quad (\mathbb{T}^\gamma_\varepsilon)^{-1}(\mathbb{Q}^\varepsilon_{\mathbb{T}^\gamma}(\mathbb{F}^\gamma_\varepsilon(\mathcal{V}_\gamma))). \end{cases}$$

Because of the fact that the operators $\mathcal{T}^\gamma_\varepsilon$ and $\mathcal{G}^\gamma_\varepsilon$ are compositions of norm-preserving operators and orthogonal projections, they have a norm less or equal 1. The following proposition will show that the operator $\mathcal{G}^\gamma_\varepsilon$ is related to an elliptic problem and that it will help us to construct a "recovery sequence" which converges in the sense of definition 2.10.

Proposition 2.11. *Let $\varepsilon > 0, \gamma \geq 0$ and $\mathcal{V}_\gamma \in \mathbf{X}_\gamma$ be given. Then the function $\mathcal{G}^\gamma_\varepsilon(\mathcal{V}_\gamma)$ is uniquely characterized as the solution $v \in \mathrm{H}^1(\Omega)$ of the following elliptic problem:*

$$\int_\Omega \Big(\big(v - \mathcal{F}_\varepsilon(\mathcal{L}^\gamma_1(\mathcal{V}_\gamma))^{\mathrm{ex}}\big) w + \big\langle \varepsilon^\gamma \nabla v - \mathcal{F}_\varepsilon(\mathcal{L}^\gamma_2(\mathcal{V}_\gamma))^{\mathrm{ex}}, \varepsilon^\gamma \nabla w \big\rangle_d \Big) \mathrm{d}x = 0 \quad \text{for all } w \in \mathrm{H}^1(\Omega),$$

where $\mathcal{L}^\gamma_1 \colon \mathbf{X}_\gamma \to \mathrm{L}^2(\Omega \times \mathcal{Y})$ and $\mathcal{L}^\gamma_2 \colon \mathbf{X}_\gamma \to \mathrm{L}^2(\Omega \times \mathcal{Y})^d$ are the first and second component of the mapping $\mathcal{L}^\gamma \colon \mathbf{X}_\gamma \to \mathrm{L}^2(\Omega \times \mathcal{Y})^{d+1}$ defined in (2.5). Furthermore, for $\varepsilon \to 0$:

$$\mathcal{G}^\gamma_\varepsilon(\mathcal{V}_\gamma) \overset{s\gamma}{\to} \mathcal{V}_\gamma \text{ in } \mathbf{X}_\gamma.$$

Proof 2.11. 1. Let first $\varepsilon > 0$ be fix and set $v := \mathcal{G}^\gamma_\varepsilon(\mathcal{V}_\gamma)$. Since $\mathbb{T}^\gamma_\varepsilon v$ is the orthogonal projection of $\mathbb{F}^\gamma_\varepsilon(\mathcal{V}_\gamma)$ on $\mathcal{X}^\varepsilon_{\mathbb{T}^\gamma} := \mathbb{T}^\gamma_\varepsilon \mathrm{H}^1(\Omega)^d$ the following is fulfilled for all $w \in \mathrm{H}^1(\Omega)^d$:

$$0 \;=\; \langle \mathbb{T}^\gamma_\varepsilon v - \mathbb{F}^\gamma_\varepsilon(\mathcal{V}_\gamma), \mathbb{T}^\gamma_\varepsilon w \rangle_{\mathbf{L}^{d+1}}$$

$$= \int_{\mathbb{R}^d\times\mathcal{Y}} \Big(\big(\mathcal{T}_\varepsilon v - (\mathcal{L}_1^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}\big)\mathcal{T}_\varepsilon w + \big\langle \mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla v) - (\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}, \mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla w)\big\rangle_d\Big)\mathrm{d}y\mathrm{d}x$$

$$= \int_\Omega \Big(vw - \mathcal{F}_\varepsilon(\mathcal{L}_1^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}w + \langle\varepsilon^\gamma\nabla v, \varepsilon^\gamma\nabla w\rangle_d - \langle\mathcal{F}_\varepsilon(\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}, \varepsilon^\gamma\nabla w\rangle_d\Big)\mathrm{d}x.$$

Here, we used the definitions of $\mathbb{T}_\varepsilon^\gamma$ and $\mathbb{F}_\varepsilon^\gamma$ plus property (c) and (d) of proposition 2.4.

2. Because of the definition of the operator $\mathcal{F}_\varepsilon$ and proposition 2.4(c), it is sufficient to show the desired convergence on the dense subset

$$\mathcal{C}_\gamma := \begin{cases} \mathrm{C}_\mathrm{c}^\infty(\Omega)\times\mathrm{C}_\mathrm{c}^\infty(\Omega\times\mathcal{Y}) & \text{if } \gamma\in[0,1) \\ \mathrm{C}_\mathrm{c}^\infty(\Omega\times\mathcal{Y}) & \text{if } \gamma\geq 1 \end{cases}$$

of $\mathbf{X}_\gamma$. For $\mathcal{V}_\gamma\in\mathcal{C}_\gamma$ we set $v_\varepsilon^{(\gamma)} := \mathcal{G}_\varepsilon^\gamma(\mathcal{V}_\gamma)$ and split v_ε as follows:

$$v_\varepsilon^{(\gamma)}(x) = \vartheta_\varepsilon^{(\gamma)}(x) + g_\varepsilon^{(\gamma)}(x) \quad \text{with } \vartheta_\varepsilon^{(\gamma)}(x) = \begin{cases} v_0(x) + \varepsilon^{1-\gamma}V_1\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right) & \text{if } \gamma\in[0,1) \\ V_1\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right) & \text{if } \gamma\geq 1 \end{cases}.$$

Thereby, $g_\varepsilon^{(\gamma)}$ is the solution of the following elliptic problem:

$$\int_\Omega \Big(g_\varepsilon^{(\gamma)}w + \big\langle\varepsilon^\gamma\nabla g_\varepsilon^{(\gamma)}, \varepsilon^\gamma\nabla w\big\rangle_d\Big)\mathrm{d}x = l_\varepsilon(w) \quad \text{for all } w\in\mathrm{H}^1(\Omega),$$

with $$l_\varepsilon(w) = \int_\Omega \Big(\big(\mathcal{F}_\varepsilon(\mathcal{L}_1^\gamma(\mathcal{Z}_\gamma))^{\mathrm{ex}} - \vartheta_\varepsilon^{(\gamma)}\big)w + \big\langle\mathcal{F}_\varepsilon(\mathcal{L}_2^\gamma(\mathcal{Z}_\gamma))^{\mathrm{ex}} - \varepsilon^\gamma\nabla\vartheta_\varepsilon^{(\gamma)}, \varepsilon^\gamma\nabla w\big\rangle_d\Big)\mathrm{d}x.$$

3. Immediately we obtain $\vartheta_\varepsilon^{(\gamma)} \xrightarrow{s} \mathcal{L}_1^\gamma(\mathcal{Z}_\gamma)$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$ by using the definition of the periodic unfolding operator $\mathcal{T}_\varepsilon$ and the continuity of the function $\vartheta_\varepsilon^{(\gamma)}$.

4. Furthermore, we have

$$\begin{aligned} &\|\mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla\vartheta_\varepsilon^{(\gamma)}) - (\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}\|_{\mathbf{L}^d} \\ &= \begin{cases} \left\|\mathcal{T}_\varepsilon\big(\varepsilon^\gamma\nabla v_0 + \varepsilon\nabla_x V_1\left(\cdot, \left\{\frac{\cdot}{\varepsilon}\right\}_Y\right) + \nabla_y V_1\left(\cdot, \left\{\frac{\cdot}{\varepsilon}\right\}_Y\right)\big) - (\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}\right\|_{\mathbf{L}^d} & \text{if } \gamma\in[0,1) \\ \left\|\mathcal{T}_\varepsilon\big(\varepsilon^\gamma\nabla_x V_1\left(\cdot, \left\{\frac{\cdot}{\varepsilon}\right\}_Y\right) + \varepsilon^{\gamma-1}\nabla_y V_1\left(\cdot, \left\{\frac{\cdot}{\varepsilon}\right\}_Y\right)\big) - (\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}\right\|_{\mathbf{L}^d} & \text{if } \gamma\geq 1 \end{cases} \end{aligned}$$

where the terms multiplied by an ε-factor can be split off by using the Minkowski inequality. Thereby, these terms converge to zero by using the norm-preservation of $\mathcal{T}_\varepsilon$ and the boundedness of the functions. The remaining term of $\mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla\vartheta_\varepsilon^{(\gamma)})$ converges pointwise to $(\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}$ in $\mathbb{R}^d\times\mathcal{Y}$ by using the definition of the periodic unfolding operator $\mathcal{T}_\varepsilon$. Hence, we obtain $\lim_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla\vartheta_\varepsilon^{(\gamma)}) - (\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}}\|_{\mathbf{L}^d} = 0$, i.e. $\varepsilon^\gamma\nabla\vartheta_\varepsilon^{(\gamma)} \xrightarrow{s} \mathcal{L}_2^\gamma(\mathcal{V}_\gamma)$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})^d$.

5. Now it is sufficient to show $\|g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)} \to 0$ and $\|\varepsilon^\gamma\nabla g_\varepsilon^{(\gamma)}\|^2_{\mathrm{L}(\Omega)^d} \to 0$ for $\varepsilon\to 0$. With the elliptic problem of step 2 and proposition 2.4(a), (c) and (d) we have

$$\begin{aligned} &\tfrac{1}{2}\Big[\|g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon^\gamma\nabla g_\varepsilon^{(\gamma)}\|^2_{\mathrm{L}(\Omega)^d}\Big]^2 \leq \|g_\varepsilon^{(\gamma)}\|^2_{\mathrm{L}^2(\Omega)} + \|\varepsilon^\gamma\nabla g_\varepsilon^{(\gamma)}\|^2_{\mathrm{L}^2(\Omega)^d} = l_\varepsilon(g_\varepsilon^{(\gamma)}) \\ &= \int_\Omega \Big(\big(\mathcal{F}_\varepsilon(\mathcal{L}_1^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \vartheta_\varepsilon^{(\gamma)}\big)g_\varepsilon^{(\gamma)} + \big\langle\mathcal{F}_\varepsilon(\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \varepsilon^\gamma\nabla\vartheta_\varepsilon^{(\gamma)}, \varepsilon^\gamma\nabla g_\varepsilon^{(\gamma)}\big\rangle_d\Big)\mathrm{d}x \\ &= \int_{\mathbb{R}^d\times\mathcal{Y}} \Big(\big((\mathcal{L}_1^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \mathcal{T}_\varepsilon\vartheta_\varepsilon^{(\gamma)}\big)\mathcal{T}_\varepsilon g_\varepsilon^{(\gamma)} + \big\langle(\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla\vartheta_\varepsilon^{(\gamma)}), \mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla g_\varepsilon^{(\gamma)})\big\rangle_d\Big)\mathrm{d}y\mathrm{d}x \end{aligned}$$

$$\leq \|(\mathcal{L}_1^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \mathcal{T}_\varepsilon \vartheta_\varepsilon^{(\gamma)}\|_{\mathbf{L}} \|g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)} + \|(\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \mathcal{T}_\varepsilon(\varepsilon^\gamma \nabla \vartheta_\varepsilon^{(\gamma)})\|_{\mathbf{L}^d} \|\varepsilon^\gamma \nabla g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)^d}$$

$$\leq \Big[\|(\mathcal{L}_1^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \mathcal{T}_\varepsilon \vartheta_\varepsilon^{(\gamma)}\|_{\mathbf{L}} + \|(\mathcal{L}_2^\gamma(\mathcal{V}_\gamma))^{\mathrm{ex}} - \mathcal{T}_\varepsilon(\varepsilon^\gamma \nabla \vartheta_\varepsilon^{(\gamma)})\|_{\mathbf{L}^d}\Big]\Big[\|g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon^\gamma \nabla g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)^d}\Big].$$

Dividing this by $\|g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon^\gamma \nabla g_\varepsilon^{(\gamma)}\|_{\mathrm{L}^2(\Omega)^d}$ and going to the limit $\varepsilon \to 0$ we obtain the result. □

In the following section we introduce an elastoplastic model described by the displacement $u : \Omega \to \mathbb{R}^d$ and a vector of internal variables $z : \Omega \to \mathbb{R}^m$. Furthermore, we expect a special Dirichlet boundary condition for the displacement. That is why we have to make some modifications in the construction of the "recovery sequence", to make sure that the "recovery sequence" used in section 4 has the same boundary condition as its limit. Thereto, we define

$$\mathcal{G}_\varepsilon : \begin{cases} \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y})) & \to & \mathrm{H}_0^1(\Omega), \\ (v, V) & \mapsto & \mathbb{T}_\varepsilon^{-1}(\mathbb{Q}_{\mathbb{T}}^\varepsilon(\mathbb{F}_\varepsilon(v, V))), \end{cases}$$

where the mappings $\mathbb{T}_\varepsilon : \mathrm{H}_0^1(\Omega) \to \mathbf{L}^{d+1}$ and $\mathbb{F}_\varepsilon : \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y})) \to \mathbf{L}^{d+1}$ are given by $\mathbb{T}_\varepsilon = \mathbb{T}_\varepsilon^0|_{\mathrm{H}_0^1(\Omega)}$ and $\mathbb{F}_\varepsilon = \mathbb{F}_\varepsilon^0|_{\mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}))}$, respectively. Furthermore, is $\mathbb{Q}_{\mathbb{T}}^\varepsilon$ the orthogonal projection of $\mathbf{L}^{d+1}$ on $\mathbb{T}_\varepsilon \mathrm{H}_0^1(\Omega)$.

Corollary 2.12. *[MT07] Let $\varepsilon > 0$ and $(u_0, U_1) \in \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}))$ be given. Then the function $\mathcal{G}_\varepsilon(u_0, U_1) \in \mathrm{H}_0^1(\Omega)$ is uniquely characterized as the solution $v \in \mathrm{H}_0^1(\Omega)$ of the following elliptic problem:*

$$\int_\Omega \Big(\big(v - \mathcal{F}_\varepsilon(Eu_0)^{\mathrm{ex}}\big) w + \big\langle \nabla v - \mathcal{F}_\varepsilon(\nabla_x E u_0 + \nabla_y U_1)^{\mathrm{ex}}, \nabla w \big\rangle_d \Big) \mathrm{d}x = 0 \quad \textit{for all } w \in \mathrm{H}_0^1(\Omega).$$

Furthermore, for $\varepsilon \to 0$:

$$\mathcal{G}_\varepsilon(u_0, U_1) \overset{s0}{\to} (u_0, U_1) \ \textit{in } \mathbf{X}_0.$$

2.5 Two-scale Γ-convergence

In this subsection we will discuss the question, in which sense functionals behave under two-scale convergence.

Definition 2.13. *[MT07] A function $\mathcal{W} : \mathcal{Y} \times \mathbb{R}^n \to \mathbb{R}_\infty$ $n \in \mathbb{N}$, is called a convex normal integrand, if for all $v \in \mathbb{R}^n$ the function $y \mapsto \mathcal{W}(y, v)$ is measurable and for almost every $y \in \mathcal{Y}$ the function $v \mapsto \mathcal{W}(y, v)$ is lower semicontinuous and convex.*

Now the two-scale Γ-convergence for the following functionals is given in definition 2.14 below.

$$W_\varepsilon : \begin{cases} \mathrm{L}^p(\Omega)^n & \to & \mathbb{R}_\infty, \\ v & \mapsto & \int_\Omega \mathcal{W}\big(\{\tfrac{x}{\varepsilon}\}_Y, v(x)\big)\,\mathrm{d}x \end{cases}$$

and (2.8)

$$\mathbf{W} : \begin{cases} \mathrm{L}^p(\Omega \times \mathcal{Y})^n & \to & \mathbb{R}_\infty, \\ V & \mapsto & \int_{\Omega \times \mathcal{Y}} \mathcal{W}(y, V(x, y))\,\mathrm{d}y\mathrm{d}x. \end{cases}$$

Definition 2.14. *The functional $\mathbf{W} : \mathrm{L}^p(\Omega \times \mathcal{Y})^n \to \mathbb{R}_\infty$ is called the two-scale-Γ-limit of the sequence $(W_\varepsilon)_{\varepsilon > 0}$, if the following two conditions are satisfied for all $V \in \mathrm{L}^p(\Omega \times \mathcal{Y})^n$:*

(i) (lim inf-*inequality*) *For every sequence* $(v_\varepsilon)_{\varepsilon>0} \in \mathrm{L}^p(\Omega)^n$, *which satisfies* $v_\varepsilon \stackrel{w}{\rightharpoonup} V$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})^n$, *it follows:*
$$\mathbf{W}(V) \leq \liminf_{\varepsilon>0} W_\varepsilon(v_\varepsilon).$$

(ii) (lim sup-*inequality*) *There exists a sequence* $(\widetilde{v}_\varepsilon)_{\varepsilon>0} \in \mathrm{L}^p(\Omega)^n$, *which satisfies* $\widetilde{v}_\varepsilon \stackrel{s}{\rightarrow} V$ *in* $\mathrm{L}^p(\Omega \times \mathcal{Y})^n$ *and*
$$\mathbf{W}(V) \geq \limsup_{\varepsilon>0} W_\varepsilon(\widetilde{v}_\varepsilon).$$

The following two lemmata show under which assumptions the functionals W_ε are two-scale-Γ-convergent to $\mathbf{W}$. The lemmata are proved in [MT07].

Lemma 2.15. *[MT07] Let* $p \in (1, \infty), \varepsilon > 0$ *and* $\mathcal{W}\colon \mathcal{Y} \times \mathbb{R}^n \to \mathbb{R}_\infty$ *be a convex normal integrand. Furthermor let* $\mathcal{W}(y, \vec{v}) \geq 0$ *for almost every* $(y, \vec{v}) \in \mathcal{Y} \times \mathbb{R}^n$. *Then:*

$$v_\varepsilon \stackrel{w}{\rightharpoonup} V \text{ in } \mathrm{L}^p(\Omega \times \mathcal{Y})^n \quad \Rightarrow \quad \mathbf{W}(V) \leq \liminf_{\varepsilon\to 0} W_\varepsilon(v_\varepsilon).$$

Lemma 2.16. *[MT07] Let* $p \in (1, \infty)$ *and* $\varepsilon > 0$.

(a) *Let* $\mathcal{W} : \mathcal{Y} \times \mathbb{R}^n \to \mathbb{R}_\infty$ *be a Carathéodory-function. Furthermore, there exists a function* $h \in \mathrm{L}^1(\mathcal{Y})$, *so that* $|\mathcal{W}(y, \vec{v})| \leq h(y) + C(1 + \|\vec{v}\|_n)^p$ *for all* $\vec{v} \in \mathbb{R}^n$ *and almost every* $y \in \mathcal{Y}$ *for a constant* $C > 0$. *Then:*

$$v_\varepsilon \stackrel{s}{\rightarrow} V \text{ in } \mathrm{L}^p(\Omega \times \mathcal{Y})^n \quad \Rightarrow \quad \mathbf{W}(V) = \lim_{\varepsilon\to 0} W_\varepsilon(v_\varepsilon).$$

This implies especially $W_\varepsilon(\mathcal{F}_\varepsilon V^{\mathrm{ex}}) \to \mathbf{W}(V)$.

(b) *Let* $\mathcal{W}\colon \mathcal{Y} \times \mathbb{R}^n \to \mathbb{R}_\infty$ *be a convex normal integrand. Furthermore, there exists a function* $h \in \mathrm{L}^1(\mathcal{Y})$, *so that* $|\mathcal{W}(y, 0)| \leq h(y)$ *for almost every* $y \in \mathcal{Y}$. *Then:*

$$\mathbf{W}(V) = \lim_{\varepsilon\to 0} W_\varepsilon(\mathcal{F}_\varepsilon V^{\mathrm{ex}}) \quad \text{for all } V \in \mathrm{L}^p(\Omega \times \mathcal{Y})^n.$$

Altogether we obtain the following corollary with lemma 2.15 and 2.16.

Corollary 2.17. *[MT07] Let* $p \in (1, \infty)$ *and let* $\mathcal{W}\colon \mathcal{Y} \times \mathbb{R}^n \to \mathbb{R}_\infty$ *be a convex normal integrand. Moreover, there exists a function* $h \in \mathrm{L}^1(\mathcal{Y})$, *so that* $\mathcal{W}(y, \vec{v}) \geq -h(y)$ *for all* $\vec{v} \in \mathbb{R}^n$ *and almost every* $y \in \mathcal{Y}$ *and* $\mathcal{W}(y, 0) \leq h(y)$ *for almost every* $y \in \mathcal{Y}$. *Then* $\mathbf{W}$ *is the two-scale-Γ-limit of* $(W_\varepsilon)_{\varepsilon>0}$, *where* $\mathbf{W}$ *and* W_ε *are defined as in* (2.8).

3 Existence and uniqueness of solutions

In the following subsection we shortly introduce the energetic formulation and cite an existence and uniqueness result from [MTh04] and [Mie05]. In subsection 3.2 we consider an ε-dependent periodic model and clarify the existence and uniqueness of a solution. Since the task is to pass to the limit $\varepsilon \to 0$ in this model, we also state a two-scale and a one-scale model, which turn out to be the limits of the ε-dependent periodic one. Finally, we show existence and uniqueness of a solution for these two models, too. Whereas the analysis of the convergence of the ε-dependent periodic model to these two models is done in section 4 and 5.

3.1 Energetic formulation

We start with a Hilbert space $\mathcal{Q}$ with a dual $\mathcal{Q}^*$, a dual pairing $\langle\cdot,\cdot\rangle_{\mathcal{Q}} : \mathcal{Q}^* \times \mathcal{Q} \to \mathbb{R}$ and a positive definite, continuous and symmetric operator $\mathcal{A} \in \mathrm{Lin}(\mathcal{Q},\mathcal{Q}^*)$, i.e. $\mathcal{A} = \mathcal{A}^*$ and $\langle \mathcal{A}q, q\rangle_{\mathcal{Q}} \geq \alpha\|q\|^2_{\mathcal{Q}}$. For a function $\ell \in \mathrm{C}^1([0,T];\mathcal{Q}^*)$ we define the energy functional $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}$ via $\mathcal{E}(t,q) := \frac{1}{2}\langle \mathcal{A}q,q\rangle_{\mathcal{Q}} - \langle \ell(t), q\rangle_{\mathcal{Q}}$. Furthermore, let a dissipation potential $\mathcal{R} : \mathcal{Q} \to [0,\infty]$ be given which is convex, lower semicontinuous and positively homogeneous of degree 1, i.e $\mathcal{R}(0) = 0$ and $\mathcal{R}(\beta q) = \beta\mathcal{R}(q)$ for all $\beta > 0$ and every $q \in \mathcal{Q}$. Then the energetic formulation (S)&(E) of the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{R})$ is given by:

$$\text{(S)}\qquad \mathcal{E}\big(t,q(t)\big) \leq \mathcal{E}\big(t,q'\big) + \mathcal{R}_\varepsilon\big(q'-q(t)\big) \quad \text{for all } q' \in \mathcal{Q}.$$

$$\text{(E)}\qquad \mathcal{E}\big(t,q(t)\big) + \mathrm{Diss}_{\mathcal{R}}\big(q;[0,T]\big) \;=\; \mathcal{E}\big(0,q(0)\big) + \int_0^t \partial_s\mathcal{E}\big(s,q(s)\big)\,\mathrm{d}s,$$

where $\partial_s\mathcal{E}(s,q(s)) = -\langle\dot\ell(s),q(s)\rangle$ and $\mathrm{Diss}_{\mathcal{R}}(q;[r,s]) = \int_s^r \mathcal{R}(\dot q(t))\,\mathrm{d}t$ is called total dissipation. We call $q\colon [0,T] \to \mathcal{Q}$ satisfying (S)&(E) for all $t \in [0,T]$ an energetic solution associated with $(\mathcal{Q},\mathcal{E},\mathcal{R})$. An equivalent definition of the global stability condition (S) can be given by defining the set of stable states $\mathcal{S}(t)$ for $t \in [0,T]$ via

$$\mathcal{S}(t) := \big\{ q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) \leq \mathcal{E}(t,q') - \mathcal{R}(q'-q) \quad \text{for all } q' \in \mathcal{Q}\big\}\,.$$

Then (S) just means $q(t) \in \mathcal{S}(t)$.

Theorem 3.1. *Let $\mathcal{Q}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{Q}}\colon \mathcal{Q} \to [0,\infty)$. Furthermore, let $\mathcal{A} \in \mathrm{Lin}(\mathcal{Q},\mathcal{Q}^*)$ be a positive definite, continuous and symmetric operator, i.e. we have:*

$$\exists\,\alpha > 0 : \forall\, q \in \mathcal{Q} : \quad \langle \mathcal{A}q,q\rangle_{\mathcal{Q}} \geq \alpha\|q\|^2_{\mathcal{Q}}, \tag{3.1}$$

where $\alpha > 0$ is the ellipticity constant. Moreover, let $\mathcal{R} : \mathrm{H}^1(\Omega)^m \to [0,\infty]$ be convex, lower semicontinuous and positively homogeneous of degree 1 and $\ell \in \mathrm{C}^1([0,T];\mathcal{Q}^*)$. Then, for a given $q^0 \in \mathcal{S}(0)$, there exists a unique solution $q \in \mathrm{C}^{0,1}([0,T];\mathcal{Q})$ of the energetic formulation (S)&(E) of the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{R})$ with:

$$q(0) = q^0, \qquad \sup_{t\in[0,T]} \|q(t)\|_{\mathcal{Q}} \leq \|q^0\|_{\mathcal{Q}} + \tfrac{T\Theta}{\alpha}, \qquad \operatorname*{ess\,sup}_{t\in[0,T]} \|\dot q(t)\|_{\mathcal{Q}} \leq \tfrac{\Theta}{\alpha},$$

where $\Theta = \|\dot\ell\|_{\mathrm{L}^\infty([0,T];\mathcal{Q}^*)}$.

Proof 3.1. The proof of this theorem can be found in [MTh04] and [Mie05], respectively, in a more general setting. □

3.2 Elastoplasticity with periodic coefficients

Let the elastoplastic body be given by a bounded open domain $\Omega \subset \mathbb{R}^d$ with Lipschitz-boundary $\partial\Omega$. Moreover, let $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ be a part of the boundary of the set Ω with a positive $(d-1)$-dimensional measure. Now we consider the Hilbert space $\mathcal{Q} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{H}^1(\Omega)^m$ where $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega) := \big\{ u \in \mathrm{H}^1(\Omega) | u|_{\Gamma_{\mathrm{Dir}}} = 0\big\}$, i.e. the displacement vanishes on Γ_{Dir}. We denote the elements of $\mathcal{Q}$ by $q = (u,z)$, where $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ denotes the displacement and

$z \in \mathrm{H}^1(\Omega)^m$ denotes a vector of internal variables which describe the inelastic effects like plastic hardening. For $\varepsilon > 0$ and $\gamma \geq 0$ we equip $\mathcal{Q}$ with the following ε-dependent norm:

$$\|q\|^2_{\mathcal{Q},\varepsilon} := \|u\|^2_{\mathrm{H}^1(\Omega)^d} + \|z\|^2_{\mathrm{L}^2(\Omega)^m} + \|\varepsilon^\gamma \nabla z\|^2_{\mathrm{L}^2(\Omega)^{m\times d}}. \tag{3.2}$$

To describe the problem by the energetic formulation, we denote by $\langle\!\langle \cdot,\cdot \rangle\!\rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the scalarproduct of $\mathbb{R}^n$ and define the function $\mathcal{W}\colon \mathcal{Y} \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\mathcal{W}(y,\xi,\zeta,\eta) := \left\langle\!\!\left\langle \mathbb{A}(y) \begin{pmatrix} \xi \\ \zeta \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \\ \eta \end{pmatrix} \right\rangle\!\!\right\rangle,$$

where $n := d^2 + m + md, \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \zeta \in \mathbb{R}^m, \eta \in \mathbb{R}^{m\times d}$ and $y \in \mathcal{Y}$. Thereby, the mapping $\mathbb{A}\colon \mathcal{Y} \to \mathbb{R}^n$ has the block structure

$$\mathbb{A}(y) = \begin{pmatrix} \mathbb{C}(y) & -\mathbb{C}(y)\mathbb{B}(y) & 0 \\ -\mathbb{B}^T(y)\mathbb{C}(y) & \mathbb{H}(y) + \mathbb{B}^T(y)\mathbb{C}(y)\mathbb{B}(y) & 0 \\ 0 & 0 & \mathbb{F}(y) \end{pmatrix}$$

and the elasticity tensor $\mathbb{C} \in \mathrm{L}^\infty(\mathcal{Y}, \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ (tensor of fourth order), the hardening tensor $\mathbb{H} \in \mathrm{L}^\infty(\mathcal{Y}, \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^m, \mathbb{R}^m))$ (tensor of second order) and the tensor $\mathbb{F} \in \mathrm{L}^\infty(\mathcal{Y}, \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{m\times d}, \mathbb{R}^{m\times d}))$ (tensor of fourth order) have the following properties:

$$\text{The tensors are uniformly positive definite and symmetric.} \tag{3.3}$$

Thereby, the fourth order tensors are symmetric in the following way: $\mathbb{F}_{i\kappa j\mu}(y) = \mathbb{F}_{j\mu i\kappa}(y)$ and $\mathbb{C}_{ijkl}(y) = \mathbb{C}_{klij}(y) = \mathbb{C}_{jikl}(y) = \mathbb{C}_{ijlk}(y)$ for almost every $y \in \mathcal{Y}$ and $i,j,k,l = 1,\ldots,d, \kappa,\mu = 1,\ldots,m$. Furthermore, let $\mathbb{B} \in \mathrm{L}^\infty(\mathcal{Y}, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^{d\times d}_{\mathrm{sym}}))$.

Let $\langle\cdot,\cdot\rangle\colon (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^* \times \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \to \mathbb{R}$ be the dual pairing and $\mathbf{e}(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$ for $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ the linearized strain tensor. Then we define the energy functional $\mathcal{E}_\varepsilon \colon [0,T] \times \mathcal{Q} \to \mathbb{R}$ as

$$\mathcal{E}_\varepsilon(t,u,z) = \frac{1}{2}\int_\Omega \mathcal{W}\big(\left\{\tfrac{x}{\varepsilon}\right\}_Y, \mathbf{e}(u)(x), z(x), \varepsilon^\gamma \nabla z(x)\big)\mathrm{d}x - \big\langle \ell(t), u\big\rangle, \tag{3.4}$$

where

$$\big\langle \ell(t), u\big\rangle := \int_\Omega \big\langle \widetilde{u}(x), f_{ap}(t,x)\big\rangle_d \mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_{\mathrm{Dir}}} \big\langle \widetilde{u}(\xi), g_{ap}(t,\xi)\big\rangle_d \mathrm{d}\xi,$$

for $f_{ap} \in \mathrm{C}^1([0,T], \mathrm{L}^2(\Omega)^d), g_{ap} \in \mathrm{C}^1([0,T], \mathrm{L}^2(\partial\Omega \setminus \Gamma_{\mathrm{Dir}})^d)$ and $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, so that $\ell \in \mathrm{C}^1([0,T], (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$.

Let $\mathcal{Q}^*$ be the dual of $\mathcal{Q}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Q}} \colon \mathcal{Q}^* \times \mathcal{Q} \to \mathbb{R}$ the dual pairing. Then, with respect to the quadratic structure of the function $\mathcal{W}$, the energy functional $\mathcal{E}_\varepsilon$ can be equivalently represented as

$$\mathcal{E}_\varepsilon(t,u,z) = \tfrac{1}{2}\big\langle \mathcal{A}_\varepsilon q, q\big\rangle_{\mathcal{Q}} - \big\langle \ell(t), u\big\rangle,$$

where $\mathcal{A}_\varepsilon \in \mathrm{Lin}(\mathcal{Q}, \mathcal{Q}^*)$ is the continuous, linear, positive definite and symmetric operator given by $\langle \mathcal{A}_\varepsilon q, q\rangle_{\mathcal{Q}} := \int_\Omega \mathcal{W}\big(\left\{\tfrac{x}{\varepsilon}\right\}_Y, \mathbf{e}(u)(x), z(x), \varepsilon^\gamma \nabla z(x)\big)\mathrm{d}x$.

Finally, let a convex, lower semicontinuous and positively homogeneous of degree 1 dissipation potential $\mathcal{R}_\varepsilon\colon \mathrm{H}^1(\Omega)^m \to [0,\infty]$ be given by

$$\mathcal{R}_\varepsilon(z) := \int_\Omega \rho\left(\left\{\frac{x}{\varepsilon}\right\}_Y, z(x)\right) \mathrm{d}x \tag{3.5}$$

where $\rho\colon \mathcal{Y}\times\mathbb{R}^m \to [0,\infty]$ satisfies the following properties:

$\rho\colon \mathcal{Y}\times\mathbb{R}^m \to [0,\infty]$ is a Cartheodory-function and there exists a constant $C>0$ and a function $h\in \mathrm{L}^1(\mathcal{Y})$, so that $|\rho(y,\vec{z})| \le h(y) + C(1+|\vec{z}|_m)^2$ for all $\vec{z}\in\mathbb{R}^m$ and almost every $y\in\mathcal{Y}$. Furthermore, $\rho(y,\cdot)\colon \mathbb{R}^m\to[0,\infty]$ is convex and positive homogeneous of degree 1 for almost every $y\in\mathcal{Y}$. (3.6)

Now the energetic formulation (S^ε)&(E^ε) for the elastoplastic model with periodic coefficients is indicated. The ε-dependent energy balance (E^ε) and the global stability condition (S^ε) read for $\varepsilon>0$ as follows:

$$(\mathrm{S}^\varepsilon)\qquad \mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big) \le \mathcal{E}_\varepsilon\big(t,u',z'\big) + \mathcal{R}_\varepsilon\big(z'-z_\varepsilon(t)\big) \quad \text{for all } q'=(u',z')\in\mathcal{Q}.$$

$$(\mathrm{E}^\varepsilon)\qquad \mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big) + \mathrm{Diss}_{\mathcal{R}_\varepsilon}\big(z_\varepsilon;[0,t]\big) \;=\; \mathcal{E}_\varepsilon\big(0,u_\varepsilon(0),z_\varepsilon(0)\big) - \int_0^t \big\langle \dot\ell(s), u_\varepsilon(s)\big\rangle \mathrm{d}s,$$

where $\mathrm{Diss}_{\mathcal{R}_\varepsilon}(z;[0,t]) := \int_0^t \mathcal{R}_\varepsilon(\dot z(s))\,\mathrm{d}s$. The set of stable states $\mathcal{S}_\varepsilon(t)$ for $t\in[0,T]$ is defined via:

$$\mathcal{S}_\varepsilon(t) := \big\{ q=(u,z)\in\mathcal{Q} \,|\, \mathcal{E}_\varepsilon(t,u,z) \le \mathcal{E}_\varepsilon(t,u',z') - \mathcal{R}_\varepsilon(z'-z) \quad \text{for all } q'=(u',z')\in\mathcal{Q}\big\}.$$

Under all these assumptions we obtain the following existence and uniqueness result.

Proposition 3.2. *Let $\varepsilon>0, \gamma\ge 0$ and let $\mathcal{Q} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{H}^1(\Omega)^m$ be equipped with the norm given by* (3.2). *Moreover, let $\mathcal{E}_\varepsilon\colon [0,T]\times\mathcal{Q}\to\mathbb{R}$ and $\mathcal{R}_\varepsilon\colon \mathrm{H}^1(\Omega)^m\to[0,\infty]$ be defined by* (3.4) *and* (3.5), *respectively, and let the conditions* (3.3) *and* (3.6) *be satisfied. Then, for a given $q^0_\varepsilon\in\mathcal{S}_\varepsilon(0)$, there exists a unique solution $q_\varepsilon\in \mathrm{C}^{0,1}([0,T];\mathcal{Q})$ of the energetic formulation* (S^ε)&(E^ε) *of the rate-independent system $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with:*

$$q_\varepsilon(0)=q^0_\varepsilon,\qquad \sup_{t\in[0,T]} \|q_\varepsilon(t)\|_{\mathcal{Q},\varepsilon} \le \|q^0_\varepsilon\|_{\mathcal{Q},\varepsilon} + \tfrac{T\Theta}{\alpha},\qquad \operatorname*{ess\,sup}_{t\in[0,T]} \|\dot q_\varepsilon(t)\|_{\mathcal{Q},\varepsilon} \le \tfrac{\Theta}{\alpha},$$

where α is given in (3.1) *and $\Theta = \|\dot\ell\|_{\mathrm{L}^\infty([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)}$.*

Proof 3.2. To apply theorem 3.1, it only remains to show the coercivity condition (3.1), which is a straightforward consequence of Young and Korn inequality. □

3.3 The two-scale homogenized problem

Since we want to consider the two-scale-Γ-limits of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, the limit function space $\mathbf{Q}_\gamma$ is basically given by $\mathbf{X}_\gamma$ defined in (2.5). The only thing we have to keep in mind is that we assumed a Dirichlet boundary condition in the first component, so that $\mathbf{Q}_\gamma$ has the following form:

$$\mathbf{Q}_\gamma := \mathbf{H}^d\times\mathbf{X}^m_\gamma \text{ where } \mathbf{H} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)\times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})) \text{ and } \mathbf{X}_\gamma \text{ defined in (2.5).} \tag{3.7}$$

Thereby, we equip $\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ with the norm $\|V\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))} := \|\nabla_y V\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})^d}$ and set $\|\cdot\|_{\mathbf{Q}_\gamma} : \mathbf{Q}_\gamma \to [0,\infty)$ as the norm of the product space. Analog to (2.6), the elements of $\mathbf{X}^m_\gamma$ are denoted by $\mathcal{Z}_\gamma$, so that $\mathbf{q}_\gamma = (u_0, U_1, \mathcal{Z}_\gamma)$ denotes the elements of $\mathbf{Q}_\gamma$. With the help of the mapping $\mathcal{L}^\gamma$ given by (2.5), we specify the two-scale-functional as follows:

$$\mathbf{E}_\gamma(t,\mathbf{q}_\gamma) = \frac{1}{2}\int\limits_{\Omega\times\mathcal{Y}} \mathcal{W}\big(y, \mathbf{e}_x(u_0)(x,y) + \mathbf{e}_y(U_1)(x,y), \mathcal{L}^\gamma(\mathcal{Z}_\gamma)(x,y)\big)\mathrm{d}y\mathrm{d}x - \big\langle \ell(t), u_0\big\rangle. \quad (3.8)$$

The two-scale dissipation potential $\mathbf{R}_\gamma\colon \mathbf{X}^m_\gamma \to [0,\infty]$ has the form:

$$\mathbf{R}_\gamma(\mathcal{Z}_\gamma) := \int\limits_{\Omega\times\mathcal{Y}} \rho\big(y, \mathcal{L}^\gamma_1(\mathcal{Z}_\gamma)(x,y)\big)\mathrm{d}y\mathrm{d}x, \quad (3.9)$$

where $\mathcal{L}^\gamma_1$ is the first component of the function $\mathcal{L}^\gamma\colon \mathbf{X}^m_\gamma \to \mathrm{L}^2(\Omega\times\mathcal{Y})^m \times \mathrm{L}^2(\Omega\times\mathcal{Y})^{m\times d}$. Note , if $\gamma\in[0,1)$, then the two-scale dissipation potential $\mathbf{R}_\gamma$ actually depends on a one-scale function z_0 only. This property is the reason why it is possible to construct an equivalent one-scale model, if $\gamma\in[0,1)$, see section 5.

The energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ is now given by:

$$(\mathbf{S}^\gamma) \qquad \mathbf{E}_\gamma\big(t,\mathbf{q}_\gamma(t)\big) \le \mathbf{E}_\gamma\big(t,\mathbf{q}'_\gamma\big) + \mathbf{R}_\gamma\big(\mathcal{Z}'_\gamma - \mathcal{Z}_\gamma(t)\big) \quad \text{for all } \mathbf{q}'_\gamma = (u'_0, U'_1, \mathcal{Z}'_\gamma) \in \mathbf{Q}_\gamma,$$

$$(\mathbf{E}^\gamma) \qquad \mathbf{E}_\gamma\big(t,\mathbf{q}_\gamma(t)\big) + \mathrm{Diss}_{\mathbf{R}_\gamma}\big(\mathcal{Z}_\gamma;[0,t]\big) \;=\; \mathbf{E}_\gamma\big(0,\mathbf{q}_\gamma(0)\big) - \int_0^t \big\langle \dot\ell(s), u_0(s)\big\rangle \mathrm{d}s,$$

where $\mathrm{Diss}_{\mathbf{R}_\gamma}(\mathcal{Z}_\gamma;[0,t]) := \int_0^t \mathbf{R}_\gamma(\dot{\mathcal{Z}}_\gamma(s))\,\mathrm{d}s$. Note, that in the case of $\gamma\in(0,1)$ the energetic solution $\mathbf{q}_\gamma = (u_0, U_1, z_0, Z_1) : [0,T] \to \mathbf{Q}_\gamma$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ satisfies $Z_1 \equiv 0$. This follows by testing the stability condition $(\mathbf{S}^\gamma)$ with special testfunctions and will be specified in section 5.

Finally, the set of stable states $\widetilde{\mathcal{S}}_\gamma(t)$ for $t\in[0,T]$ is defined via

$$\widetilde{\mathcal{S}}_\gamma(t) := \big\{\mathbf{q}\in\mathbf{Q}_\gamma \,|\, \mathbf{E}_\gamma(t,\mathbf{q}) \le \mathbf{E}_\gamma(t,\mathbf{q}') - \mathbf{R}_\gamma(\mathcal{Z}'_\gamma - \mathcal{Z}_\gamma) \quad \text{for all } \mathbf{q}' = (u'_0, U'_1, \mathcal{Z}'_\gamma) \in \mathbf{Q}_\gamma\big\}\,.$$

The abstract existence and uniqueness theorem 3.1 now yields the following result.

Proposition 3.3. *Let $\mathbf{Q}_\gamma$ be given by* (3.7) *and equipped with the norm described above. Furthermore, let $\mathbf{E}_\gamma : [0,T]\times\mathbf{Q}_\gamma \to \mathbb{R}$ and $\mathbf{R}_\gamma : \mathrm{H}^1(\Omega)^m \to [0,\infty]$ be defined by* (3.8) *and* (3.9), *respectively, and let the conditions* (3.3) *and* (3.6) *be satisfied. Then, for a given $\mathbf{q}^0_\gamma \in \widetilde{\mathcal{S}}_\gamma(0)$, there exists a unique solution $\mathbf{q}_\gamma \in \mathrm{C}^{0,1}([0,T],\mathbf{Q}_\gamma)$ of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ with*

$$\mathbf{q}_\gamma(0) = \mathbf{q}^0_\gamma, \quad \sup_{t\in[0,T]} \|\mathbf{q}_\gamma(t)\|_{\mathbf{Q}_\gamma} \le \|\mathbf{q}^0_\gamma\|_{\mathbf{Q}_\gamma} + \tfrac{T\Theta}{\alpha}, \quad \operatorname*{ess\,sup}_{t\in[0,T]} \|\dot{\mathbf{q}}_\gamma(t)\|_{\mathbf{Q}_\gamma} \le \tfrac{\Theta}{\alpha},$$

where α is defined in (3.1) *and $\Theta = \|\dot\ell\|_{\mathrm{L}^\infty([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)}$.*

Proof 3.3. It only remains to show the coercivity condition (3.1). With condition (2.7) this follows analogously to proof 3.2. □

3.4 Equivalent one-scale model in the case of $\gamma \in [0,1)$

In this subsection we introduce a one-scale model which turns out to be equivalent (section 5) to the two-scale model of the previous subsection if $\gamma \in [0,1)$ and if $\mathbb{B}$ is constant, i.e. $\mathbb{B} \in \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^{d\times d}_{\mathrm{sym}})$. To prove this (section 5), we will use a special property of the solution of the two-scale energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$, namely, that the solution minimizes the energy functional $\mathbf{E}_\gamma$. Furthermore, we will use the fact that the two-scale dissipation potential $\mathbf{R}_\gamma$ in this case ($\gamma \in [0,1)$) depends on a one-scale function z_0 only.

Thereto, we introduce the space $\mathcal{Q}_\gamma := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{X}^m_\gamma$ where

$$\mathrm{X}_\gamma := \begin{cases} \mathrm{H}^1(\Omega); & \|\cdot\|_{\mathcal{Q}_0} := \|\cdot\|_{\mathcal{Q}} \quad \text{if } \gamma = 0, \\ \mathrm{L}^2(\Omega); & \|(u,z)\|^2_{\mathcal{Q}_\gamma} := \|u\|^2_{\mathrm{H}^1(\Omega)^d} + \|z\|^2_{\mathrm{L}^2(\Omega)^m} \quad \text{if } \gamma \in (0,1). \end{cases} \tag{3.10}$$

Now we consider two minimizing problems which correspond to the minimizing property of the two-scale solution of the energetic formulation $(\mathbf{S})_\gamma\&(\mathbf{E})_\gamma$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ as it is shown in section 5. Let $C_{\mathrm{eff}} : \mathbb{R}^{d\times d} \to \mathbb{R}$, $H_{\mathrm{eff}} : \mathbb{R}^m \to \mathbb{R}$ and $F_{\mathrm{eff}} : \mathbb{R}^{m\times d} \to \mathbb{R}$ for $\xi \in \mathbb{R}^{d\times d}, \zeta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^{m\times d}$ be given by

$$C_{\mathrm{eff}}(\xi) = \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d} I_1(\xi,v) := \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d} \int_{\mathcal{Y}} \langle \mathbb{C}(y)(\xi + \mathbf{e}(v)(y)), \xi + \mathbf{e}(v)(y)\rangle_{d\times d}\, \mathrm{d}y, \tag{3.11}$$

$$H_{\mathrm{eff}}(\zeta) = \langle \mathbb{H}_{\mathrm{eff}}\zeta, \zeta\rangle_m := \int_{\mathcal{Y}} \langle \mathbb{H}(y)\zeta, \zeta\rangle_m\, \mathrm{d}y,$$

$$F_{\mathrm{eff}}(\eta) = \min_{w\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m} I_2(\eta,w) := \min_{w\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m} \int_{\mathcal{Y}} \langle \mathbb{F}(y)(\eta + \nabla w(y)), \eta + \nabla w(y)\rangle_{m\times d}\, \mathrm{d}y. \tag{3.12}$$

Before describing the one-scale energetic formulation with the help of these mappings, we want to show the existence and uniqueness of solutions of (3.11) and (3.12).

Lemma 3.4. *Let $\mathbb{C} \in \mathrm{L}^\infty(\mathcal{Y}; \mathrm{Lin}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d}))$ and $\mathbb{F} \in \mathrm{L}^\infty(\mathcal{Y}; \mathrm{Lin}(\mathbb{R}^{m\times d}, \mathbb{R}^{m\times d}))$ be uniformly positive definite and symmetric. Then there exist unique solutions of minimizing problems* (3.11) *and* (3.12) *and symmetric, linear mappings $\mathbb{C}_{\mathrm{eff}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\mathbb{F}_{\mathrm{eff}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{m\times d}, \mathbb{R}^{m\times d})$, so that $C_{\mathrm{eff}}(\cdot) = \langle \mathbb{C}_{\mathrm{eff}}\cdot, \cdot\rangle_{d\times d}$ and $F_{\mathrm{eff}}(\cdot) = \langle \mathbb{F}_{\mathrm{eff}}\cdot, \cdot\rangle_{m\times d}$.*

Proof 3.4. 1. Let $\xi \in \mathbb{R}^{d\times d}$ and $\eta \in \mathbb{R}^{m\times d}$ be given and fixed. Since the functionals $I_1(\xi,\cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d \to \mathbb{R}$ and $I_2(\eta,\cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m \to \mathbb{R}$ are strictly convex and continuous, the existence and uniqueness of minimizers $v^* \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ and $w^* \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m$ of (3.11) and (3.12) follows. Furthermore, by applying the Lax-Milgram theorem to the Euler-Lagrange equations of $I_1(\xi,\cdot)\colon \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d \to \mathbb{R}$ and $I_2(\eta,\cdot)\colon \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m \to \mathbb{R}$ there exist linear mappings $\mathcal{L}_{\mathbb{C}}\colon \mathbb{R}^{d\times d} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ and $\mathcal{L}_{\mathbb{F}}\colon \mathbb{R}^{m\times d} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m$, which satisfy $\mathcal{L}_{\mathbb{C}}(\xi) = v^*$ and $\mathcal{L}_{\mathbb{F}}(\eta) = w^*$. Therefore, we used crucially the uniqueness of the minimizeres of (3.11) and (3.12) as well as the uniqueness of the solutions of the Euler-Lagrange equations.

2. It remains to show that $C_{\mathrm{eff}} : \mathbb{R}^{d\times d} \to \mathbb{R}$ and $F_{\mathrm{eff}} : \mathbb{R}^{m\times d} \to \mathbb{R}$ actually have quadratic structure. We show this only for the second one. According to step 1 this mapping can be written in the following form:

$$F_{\mathrm{eff}}(\eta) = \int_{\mathcal{Y}} \langle \mathbb{F}(y)(\eta + \nabla\mathcal{L}_{\mathbb{F}}(\eta)(y)), \eta + \nabla\mathcal{L}_{\mathbb{F}}(\eta)(y)\rangle_{m\times d}\, \mathrm{d}y.$$

Testing this function with $\eta = e_{ij} \in \mathbb{R}^{m\times d}$, where $(e_{ij})_{kl} := \delta_{ij,kl}$ and $\delta_{ij,kl}$ for $i,k \in \{1,\dots,m\}$ and $j,l \in \{1,\dots d\}$ is the Kronecker delta, enables us to define the effective tensor $\mathbb{F}_{\text{eff}}$ of forth order as follows:

$$\mathbb{F}_{\text{eff}_{ijkl}} := \int\limits_{\mathcal{Y}} \langle \mathbb{F}(y)(e_{ij} + \nabla\mathcal{L}_{\mathbb{F}}(e_{ij})(y)), e_{kl} + \nabla\mathcal{L}_{\mathbb{F}}(e_{kl})(y)\rangle_{m\times d}\,\mathrm{d}y. \tag{3.13}$$

By this definition, the symmetry of $\mathbb{F}_{\text{eff}}$ is obvious and the quadratic structure can be seen by writing $\eta \in \mathbb{R}^{m\times d}, \eta := (\eta_{ij})_{i\in\{1,\dots m\}}^{j\in\{1,\dots,d\}}$ as $\eta = \sum_{i=1}^m \sum_{j=1}^d \eta_{ij} e_{ij}$. $\square$

Note that the effective tensor $\mathbb{C}_{\text{eff}} \in \text{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$, defined analogously to (3.13), is exactly the same as in the linear elastic case, see [Ngu89] subsection 6.2. There, equation (6.10) is in fact the Euler-Lagrange equation of $I_1(\xi,\cdot) : \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d \to \mathbb{R}$ with $\xi = -e_{jl}$, if one extends the result of [Ngu89] to the d-dimensional case, i.e. $u \in \mathrm{H}^1_0(\Omega)^d$ instead of $u \in \mathrm{H}^1_0(\Omega)$. Furthermore, in our case $u \in \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d$ instead of $u \in \mathrm{H}^1_0(\Omega)^d$, but this is only an easy generalization. Then the definition of the effective tensor of [Ngu89] given in condition (6.14) is the same as ours in definition (3.13).

The one-scale energy functional $\mathcal{E}_\gamma \colon [0,T]\times\mathcal{Q}_\gamma \to \mathbb{R}$ is given by

$$\mathcal{E}_\gamma(t,u,z) = \frac{1}{2}\int\limits_{\Omega} \left\langle\!\!\left\langle \mathbb{A}_\gamma \begin{pmatrix} \mathbf{e}(u)(x) \\ z(x) \\ \widetilde{\nabla} z(x) \end{pmatrix}, \begin{pmatrix} \mathbf{e}(u)(x) \\ z(x) \\ \widetilde{\nabla} z(x) \end{pmatrix} \right\rangle\!\!\right\rangle \mathrm{d}x - \langle \ell(t), u\rangle, \tag{3.14}$$

where

$$\mathbb{A}_\gamma = \begin{pmatrix} \mathbb{C}_{\text{eff}} & -\mathbb{C}_{\text{eff}}\mathbb{B} & 0 \\ -\mathbb{B}^T\mathbb{C}_{\text{eff}} & \mathbb{H}_{\text{eff}} + \mathbb{B}^T\mathbb{C}_{\text{eff}}\mathbb{B} & 0 \\ 0 & 0 & \mathbb{F}_\gamma \end{pmatrix} \quad \text{with} \quad \mathbb{F}_\gamma := \begin{cases} \mathbb{F}_{\text{eff}} & \text{if } \gamma = 0, \\ 0 & \text{if } \gamma \in (0,1) \end{cases}$$

and

$$\widetilde{\nabla} : \begin{cases} \mathrm{X}^m_\gamma & \to \quad \mathrm{L}^2(\Omega)^{m\times d} \\ z & \mapsto \quad \begin{cases} \nabla z & \text{if } \gamma = 0, \\ 0 & \text{if } \gamma \in (0,1). \end{cases} \end{cases} \tag{3.15}$$

The one-scale dissipation potential $\mathcal{R}_\gamma \colon \mathrm{X}^m_\gamma \to [0,\infty]$ reads as follows

$$\mathcal{R}_\gamma(z) := \int\limits_{\Omega} \rho_{\text{eff}}(z(x))\,\mathrm{d}x, \tag{3.16}$$

where $\rho_{\text{eff}} := \int_{\mathcal{Y}} \rho(y,\cdot)\,\mathrm{d}y \colon \mathbb{R}^m \to [0,\infty]$. With this definition we obtain $\mathcal{R}_\gamma(z_0) = \widetilde{\mathbf{R}}_\gamma(z_0)$ for every $z_0 \in \mathrm{H}^1(\Omega)^m$, where $\mathbf{R}_\gamma(z_0, Z_1) \equiv \widetilde{\mathbf{R}}_\gamma(z_0) := \int_{\Omega\times\mathcal{Y}} \rho(y, z_0(x))\,\mathrm{d}y\mathrm{d}x$ for all $(z_0, Z_1) \in \mathbf{X}^m_\gamma$.

Now the energetic formulation (S^γ)&(E^γ) of the rate-independent system $(\mathcal{Q}_\gamma, \mathcal{E}_\gamma, \mathcal{R}_\gamma)$ reads as follows:

$$(\mathrm{S}^\gamma) \qquad \mathcal{E}_\gamma\big(t,u(t),z(t)\big) \le \mathcal{E}_\gamma\big(t,u',z'\big) + \mathcal{R}_\gamma\big(z' - z(t)\big) \quad \text{for all } (u',z') \in \mathcal{Q}_\gamma,$$

$$(\mathrm{E}^\gamma) \qquad \mathcal{E}_\gamma\big(t,u(t),z(t)\big) + \text{Diss}_{\mathcal{R}_\gamma}\big(\dot{z};[0,t]\big) = \mathcal{E}_\gamma\big(0,u(0),z(0)\big) - \int_0^t \big\langle \dot{\ell}(s), u(s)\big\rangle \mathrm{d}s,$$

where $\mathrm{Diss}_{\mathcal{R}_\gamma}(\dot{z};[0,t]) := \int_0^t \mathcal{R}_\gamma(\dot{z}(s))\,\mathrm{d}s$. Furthermore, we define the set of stable states $\mathcal{S}_\gamma(t)$ for $t \in [0,T]$ via

$$\mathcal{S}_\gamma(t) := \left\{ q \in \mathcal{Q}_\gamma \,|\, \mathcal{E}_\gamma(t,q) \le \mathcal{E}_\gamma(t,q') - \mathcal{R}_\gamma(z'-z) \quad \text{for all } q' \in \mathcal{Q}_\gamma \right\}.$$

Proposition 3.5. *Let $\mathcal{Q}_\gamma$ be given as above and be equipped with the norm introduced by* (3.10). *Furthermore, let $\mathcal{E}_\gamma\colon [0,T]\times\mathcal{Q}_\gamma \to \mathbb{R}$ and $\mathcal{R}_\gamma\colon \mathrm{X}^m_\gamma \to [0,\infty]$ be defined by* (3.14) *and* (3.16), *respectively, and let the conditions* (3.3) *and* (3.6) *be satisfied. Then, for a given $q^0 \in \mathcal{S}_\gamma(0)$, there exists a unique solution $q \in \mathrm{C}^{0,1}([0,T],\mathcal{Q}_\gamma)$ of the energetic formulation* $(\mathrm{S}^\gamma)\&(\mathrm{E}^\gamma)$ *of the rate-independent system $(\mathcal{Q}_\gamma,\mathcal{E}_\gamma,\mathcal{R}_\gamma)$, with*

$$q(0)=q^0, \qquad \sup_{t\in[0,T]} \|q\|_{\mathcal{Q}_\gamma} \le \|q^0\|_{\mathcal{Q}_\gamma} + \tfrac{T\Theta}{\alpha}, \qquad \operatorname*{ess\,sup}_{t\in[0,T]} \|\dot{q}\|_{\mathcal{Q}_\gamma} \le \tfrac{\Theta}{\alpha},$$

where α is defined in (3.1) *and* $\Theta = \|\dot{\ell}\|_{\mathrm{L}^\infty([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)}$.

Proof 3.5. It only remains to show the coercivity condition (3.1). Analog to condition (2.7), we obtain for all $v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$

$$\int_{\mathcal{Y}} \langle \mathbb{C}(y)(\xi+\mathbf{e}(v)(y)), \xi+\mathbf{e}(v)(y)\rangle_{d\times d}\mathrm{d}y \ge \int_{\mathcal{Y}} c|\xi+\mathbf{e}(v)(y)|^2_{d\times d}\,\mathrm{d}y \overset{(2.7)}{=} c\Big[|\xi|^2_{d\times d}+\|\mathbf{e}(v)\|^2_{\mathrm{L}^2(\mathcal{Y})^{d\times d}}\Big],$$

where we used that $\mathbb{C} \in \mathrm{L}^\infty(\mathcal{Y};\mathrm{Lin}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d}))$ is uniformly positive definite. Since this is true for every $\xi \in \mathbb{R}^{d\times d}$, this means that $\mathbb{C}_{\mathrm{eff}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d})$ is positive definite. Analog we obtain that $\mathbb{F}_{\mathrm{eff}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{m\times d},\mathbb{R}^{m\times d})$ $(\gamma=0)$ is positive definite and thereby follows coercivity condition (3.1) analog to Proof 3.2. ☐

4 Convergence results

The following theorem yields one of the main results of this paper: Assuming strong two-scale convergence (in sense of definition 2.10) of the initial data we obtain strong two-scale convergence (in sense of definition 2.10) of the solutions $(u_\varepsilon, z_\varepsilon)$ of the energetic formulation $(\mathrm{S}^\varepsilon)\&(\mathrm{E}^\varepsilon)$ of the rate-independent systems $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to the solution $(u_0,U_1,\mathcal{Z}_\gamma)$ of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma,\mathbf{E}_\gamma,\mathbf{R}_\gamma)$.

Theorem 4.1. *Let $\gamma \ge 0, \varepsilon > 0$ and let $q^0_\varepsilon = (u^0_\varepsilon, z^0_\varepsilon) \in \mathcal{S}_\varepsilon(0)$ be given. Furthermore, let $q_\varepsilon = (u_\varepsilon,z_\varepsilon) \in \mathrm{C}^{0,1}([0,T],\mathcal{Q})$ be the unique solution of the energetic formulation* $(\mathrm{S}^\varepsilon)\&(\mathrm{E}^\varepsilon)$ *of the rate-independent system $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with $q_\varepsilon(0)=q^0_\varepsilon$. Moreover, let $u_\varepsilon(0) \overset{s0}{\to} (u^0_0, U^0_1)$ in $\mathbf{H}^d$ and let $z_\varepsilon(0) \overset{s\gamma}{\to} \mathcal{Z}^0_\gamma$ in $\mathbf{X}^m_\gamma$. Then $\mathbf{q}^0_\gamma := (u^0_0,U^0_1,\mathcal{Z}^0_\gamma) \in \widetilde{\mathcal{S}}_\gamma(0)$ and for all $t \in [0,T]$ it follows: $u_\varepsilon(t) \overset{s0}{\to} (u_0(t),U_1(t))$ in $\mathbf{H}^d$ and $z_\varepsilon(t) \overset{s\gamma}{\to} \mathcal{Z}_\gamma(t)$ in $\mathbf{X}^m_\gamma$ where $\mathbf{q}_\gamma := (u_0,U_1,\mathcal{Z}_\gamma) \in \mathrm{C}^{0,1}([0,T],\mathbf{Q}_\gamma)$ is the unique solution of the energetic formulation* $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ *of the rate-independent system $(\mathbf{Q}_\gamma,\mathbf{E}_\gamma,\mathbf{R}_\gamma)$ with $\mathbf{q}_\gamma(0)=\mathbf{q}^0_\gamma$.*

One of the main difficulties by proving this result is to show that the weak limit (in sense of definition 2.10) of a stable sequence is stable again. That is why we first start with two propositions before proving theorem 4.1. In [MRS08] was introduced a sufficient condition to get this result, namely, the existence of a “joint recovery sequence“. In the context of this paper, “joint recovery sequences“ are defined as follows:

Definition 4.2. *The functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ fulfill the "joint recovery condition", if for every stable sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}$ (i.e. $(u_\varepsilon, z_\varepsilon) \in \mathcal{S}_\varepsilon(t)$ for every $\varepsilon > 0$) with $u_\varepsilon \overset{w0}{\rightharpoonup} (u_0, U_1)$ in $\mathbf{H}^d$ and $z_\varepsilon \overset{w\gamma}{\rightharpoonup} \mathcal{Z}_\gamma$ in $\mathbf{X}_\gamma^m$ and every $\widetilde{\mathbf{q}}_\gamma = (\widetilde{u}_0, \widetilde{U}_1, \widetilde{\mathcal{Z}}_\gamma) \in \mathbf{Q}_\gamma$, there exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}$ with $\widetilde{u}_\varepsilon \overset{w0}{\rightharpoonup} (\widetilde{u}_0, \widetilde{U}_1)$ in $\mathbf{H}^d$ and $\widetilde{z}_\varepsilon \overset{w\gamma}{\rightharpoonup} \widetilde{\mathcal{Z}}_\gamma$ in $\mathbf{X}_\gamma^m$ so that:*

$$\limsup_{\varepsilon\to 0} \big(\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) + \mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon - z_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon)\big) \le \mathbf{E}_\gamma(t, \widetilde{\mathbf{q}}_\gamma) + \mathbf{R}_\gamma(\widetilde{\mathcal{Z}}_\gamma - \mathcal{Z}_\gamma) - \mathbf{E}_\gamma(t, \mathbf{q}_\gamma),$$

where $\mathbf{q}_\gamma = (u_0, U_1, \mathcal{Z}_\gamma) \in \mathbf{Q}_\gamma$. The sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}$ is called "joint recovery sequence".

The following proposition shows the existence of such "joint recovery sequences" and that this is sufficient for the stability of a limit of stable states.

Proposition 4.3. *Let $\gamma \ge 0$ and $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathcal{Q}$ with $(u_\varepsilon, z_\varepsilon) \in \mathcal{S}_\varepsilon(t)$ for $t \in [0, T]$. Furthermore, let $u_\varepsilon \overset{w0}{\rightharpoonup} (u_0, U_1)$ in $\mathbf{H}^d$ and $z_\varepsilon \overset{w\gamma}{\rightharpoonup} \mathcal{Z}_\gamma$ in $\mathbf{X}_\gamma^m$. Then:*

(a) *For every $\widetilde{\mathbf{q}}_\gamma = (\widetilde{u}_0, \widetilde{U}_1, \widetilde{\mathcal{Z}}_\gamma) \in \mathbf{Q}_\gamma$ exists a "joint recovery sequence".*
(b) *$\mathbf{q}_\gamma = (u_0, U_1, \mathcal{Z}_\gamma) \in \widetilde{\mathcal{S}}_\gamma(t)$.*

Proof 4.3. 1. With the help of the in subsection 2.4 introduced folding operators $\mathcal{G}_\varepsilon$ and $\mathcal{G}_\varepsilon^\gamma$, the joint recovery sequence can be constructed explicitly:

$$\begin{pmatrix} \widetilde{u}_\varepsilon \\ \widetilde{z}_\varepsilon \end{pmatrix} := \begin{pmatrix} u_\varepsilon \\ z_\varepsilon \end{pmatrix} + \begin{pmatrix} \widetilde{u}_0 - u_0 + \mathcal{G}_\varepsilon(0, \widetilde{U}_1 - U_1) \\ \mathcal{G}_\varepsilon^\gamma(\widetilde{\mathcal{Z}}_\gamma - \mathcal{Z}_\gamma) \end{pmatrix}.$$

Note, that the first component of this sequence has the correct boundary value because of $u_\varepsilon + \widetilde{u}_0 - u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $\mathcal{G}_\varepsilon(0, \widetilde{U}_1 - U_1) \in \mathrm{H}^1_0(\Omega)^d$. Now proposition 2.7 (f) and corollary 2.12 yield

$$\widetilde{u}_\varepsilon - u_\varepsilon = \widetilde{u}_0 - u_0 + \mathcal{G}_\varepsilon(0, \widetilde{U}_1 - U_1) \overset{s0}{\to} (\widetilde{u}_0 - u_0, \widetilde{U}_1 - U_1) \text{ in } \mathbf{H}^d \tag{4.1}$$

and proposition 2.11 yields

$$\widetilde{z}_\varepsilon - z_\varepsilon = \mathcal{G}_\varepsilon^\gamma(\widetilde{\mathcal{Z}}_\gamma - \mathcal{Z}_\gamma) \overset{s\gamma}{\to} (\widetilde{\mathcal{Z}}_\gamma - \mathcal{Z}_\gamma) \text{ in } \mathbf{X}_\gamma^m. \tag{4.2}$$

Condition (4.1) and (4.2) imply $\widetilde{u}_\varepsilon \overset{w0}{\rightharpoonup} (\widetilde{u}_0, \widetilde{U}_1)$ in $\mathbf{H}^d$ and $\widetilde{z}_\varepsilon \overset{w\gamma}{\rightharpoonup} \widetilde{\mathcal{Z}}_\gamma$ in $\mathbf{X}_\gamma^m$ because of the assumed convergence $u_\varepsilon \overset{w0}{\rightharpoonup} (u_0, U_1)$ in $\mathbf{H}^d$ and $z_\varepsilon \overset{w\gamma}{\rightharpoonup} \mathcal{Z}_\gamma$ in $\mathbf{X}_\gamma^m$, respectively.

2. Since $\rho : \mathcal{Y} \times \mathbb{R}^m \to [0, \infty]$ has the properties (3.6) and $\widetilde{z}_\varepsilon - z_\varepsilon \overset{s}{\to} \mathcal{L}_1^\gamma(\widetilde{\mathcal{Z}}_\gamma - \mathcal{Z}_\gamma)$ in $\mathrm{L}^2(\Omega \times \mathcal{Y})^m$, according to definition 2.10 and condition (4.2), we obtain with lemma 2.16(a):

$$\lim_{\varepsilon\to 0} \mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon - z_\varepsilon) = \lim_{\varepsilon\to 0} \int_\Omega \rho\left(\left\{\frac{x}{\varepsilon}\right\}_Y, \widetilde{z}_\varepsilon - z_\varepsilon\right) \mathrm{d}x = \int_{\Omega\times\mathcal{Y}} \rho\big(y, \mathcal{L}_1^\gamma(\widetilde{\mathcal{Z}}_\gamma - \mathcal{Z}_\gamma)\big) \mathrm{d}y\mathrm{d}x = \mathbf{R}_\gamma(\widetilde{\mathcal{Z}}_\gamma - \mathcal{Z}_\gamma).$$

3. Because of the quadratic structure of the energy functionals $\mathcal{E}_\varepsilon$, we obtain

$$\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) =$$

$$\frac{1}{2}\int\limits_{\Omega}\left\langle\!\!\left\langle \mathbb{A}\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big)\begin{pmatrix}\mathbf{e}(\widetilde{u}_\varepsilon-u_\varepsilon)(x)\\ \widetilde{z}_\varepsilon(x)-z_\varepsilon(x)\\ \varepsilon^\gamma\nabla(\widetilde{z}_\varepsilon(x)-z_\varepsilon(x))\end{pmatrix},\begin{pmatrix}\mathbf{e}(\widetilde{u}_\varepsilon+u_\varepsilon)(x)\\ \widetilde{z}_\varepsilon(x)+z_\varepsilon(x)\\ \varepsilon^\gamma\nabla(\widetilde{z}_\varepsilon(x)+z_\varepsilon(x))\end{pmatrix}\right\rangle\!\!\right\rangle \mathrm{d}x-\langle\ell(t),\widetilde{u}_\varepsilon-u_\varepsilon\rangle.$$

Using definition 2.10 and the results of step 2, we have firstly

$$\begin{pmatrix}\mathbf{e}(\widetilde{u}_\varepsilon+u_\varepsilon)\\ \widetilde{z}_\varepsilon+z_\varepsilon\\ \varepsilon^\gamma\nabla(\widetilde{z}_\varepsilon+z_\varepsilon)\end{pmatrix}\stackrel{w}{\rightharpoonup}\begin{pmatrix}\mathbf{e}_x(E(\widetilde{u}_0+u_0))+\mathbf{e}_y(\widetilde{U}_1+U_1)\\ \mathcal{L}_1^\gamma(\widetilde{\mathcal{Z}}_\gamma+\mathcal{Z}_\gamma)\\ \mathcal{L}_2^\gamma(\widetilde{\mathcal{Z}}_\gamma+\mathcal{Z}_\gamma)\end{pmatrix}\text{ in }\mathrm{L}^2(\Omega\times\mathcal{Y})^n,$$

and secondly with condition (4.1) and (4.2)

$$\begin{pmatrix}\mathbf{e}(\widetilde{u}_\varepsilon-u_\varepsilon)\\ \widetilde{z}_\varepsilon-z_\varepsilon\\ \varepsilon^\gamma\nabla(\widetilde{z}_\varepsilon-z_\varepsilon)\end{pmatrix}\stackrel{s}{\to}\begin{pmatrix}\mathbf{e}_x(E(\widetilde{u}_0-u_0))+\mathbf{e}_y(\widetilde{U}_1-U_1)\\ \mathcal{L}_1^\gamma(\widetilde{\mathcal{Z}}_\gamma-\mathcal{Z}_\gamma)\\ \mathcal{L}_2^\gamma(\widetilde{\mathcal{Z}}_\gamma-\mathcal{Z}_\gamma)\end{pmatrix}\text{ in }\mathrm{L}^2(\Omega\times\mathcal{Y})^n,$$

where $n:=d^2+m+md$.

The last result allows us now to apply proposition 2.8 to $m_\varepsilon:=\mathbb{A}\big(\{\frac{\cdot}{\varepsilon}\}_Y\big)$, i.e.

$$\mathbb{A}\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big)\begin{pmatrix}\mathbf{e}(\widetilde{u}_\varepsilon-u_\varepsilon)\\ \widetilde{z}_\varepsilon-z_\varepsilon\\ \varepsilon^\gamma\nabla(\widetilde{z}_\varepsilon-z_\varepsilon)\end{pmatrix}\stackrel{s}{\to}\mathbb{A}\begin{pmatrix}\mathbf{e}_x(E(\widetilde{u}_0-u_0))+\mathbf{e}_y(\widetilde{U}_1-U_1)\\ \mathcal{L}_1^\gamma(\widetilde{\mathcal{Z}}_\gamma-\mathcal{Z}_\gamma)\\ \mathcal{L}_2^\gamma(\widetilde{\mathcal{Z}}_\gamma-\mathcal{Z}_\gamma)\end{pmatrix}\text{ in }\mathrm{L}^2(\Omega\times\mathcal{Y})^n,$$

where we used $\mathcal{T}_\varepsilon m_\varepsilon(x,y)=\mathbb{A}(y)$.

4. Now, with step 3, all assumptions of proposition 2.7(d) are fulfilled and we conclude

$$\int\limits_{\Omega}\left\langle\!\!\left\langle \mathbb{A}\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big)\begin{pmatrix}\mathbf{e}(\widetilde{u}_\varepsilon-u_\varepsilon)(x)\\ \widetilde{z}_\varepsilon(x)-z_\varepsilon(x)\\ \varepsilon^\gamma\nabla(\widetilde{z}_\varepsilon(x)-z_\varepsilon(x))\end{pmatrix},\begin{pmatrix}\mathbf{e}(\widetilde{u}_\varepsilon+u_\varepsilon)(x)\\ \widetilde{z}_\varepsilon(x)+z_\varepsilon(x)\\ \varepsilon^\gamma\nabla(\widetilde{z}_\varepsilon(x)+z_\varepsilon(x))\end{pmatrix}\right\rangle\!\!\right\rangle \mathrm{d}x\stackrel{\varepsilon\to0}{\to}$$

$$\int\limits_{\Omega\times\mathcal{Y}}\left\langle\!\!\left\langle \mathbb{A}(y)\begin{pmatrix}\widehat{\mathbf{e}}((\widetilde{u}_0,\widetilde{U}_1)-(u_0,U_1))(x,y)\\ \mathcal{L}_1^\gamma(\widetilde{\mathcal{Z}}_\gamma-\mathcal{Z}_\gamma)(x,y)\\ \mathcal{L}_2^\gamma(\widetilde{\mathcal{Z}}_\gamma-\mathcal{Z}_\gamma)(x,y)\end{pmatrix},\begin{pmatrix}\widehat{\mathbf{e}}((\widetilde{u}_0,\widetilde{U}_1)+(u_0,U_1))(x,y)\\ \mathcal{L}_1^\gamma(\widetilde{\mathcal{Z}}_\gamma+\mathcal{Z}_\gamma)(x,y)\\ \mathcal{L}_2^\gamma(\widetilde{\mathcal{Z}}_\gamma+\mathcal{Z}_\gamma)(x,y)\end{pmatrix}\right\rangle\!\!\right\rangle \mathrm{d}y\mathrm{d}x,$$

where $\widehat{\mathbf{e}}(\widehat{u}_0,\widehat{U}_1)=\mathbf{e}_x(E\widehat{u}_0)+\mathbf{e}_y(\widehat{U}_1)$, which gives:

$$\lim_{\varepsilon\to0}\big(\mathcal{E}_\varepsilon(t,\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)-\mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon)\big)=\mathbf{E}_\gamma(t,\widetilde{u}_0,\widetilde{U}_1,\widetilde{\mathcal{Z}}_\gamma)-\mathbf{E}_\gamma(t,u_0,U_1,\mathcal{Z}_\gamma),$$

where we used $\widetilde{u}_\varepsilon-u_\varepsilon\rightharpoonup\widetilde{u}_0-u_0$ in $\mathrm{H}^1(\Omega)^d$. Thereby, we proved part (a).

5. Part (b) is a direct consequence of part (a). Let $(u_0,U_1,\mathcal{Z}_\gamma)\in\mathbf{Q}_\gamma$ be the weak two-scale-γ-limit of the stable states $(u_\varepsilon,z_\varepsilon)\in\mathcal{Q}$ and let $(\widetilde{u}_0,\widetilde{U}_1,\widetilde{\mathcal{Z}}_\gamma)\in\mathbf{Q}_\gamma$ be an arbitrary teststate. Replacing $(\widetilde{u},\widetilde{z})\in\mathcal{Q}$ in the stability condition (S^ε) by the constructed “joint recovery sequence“ $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)\in\mathcal{Q}$ from step (a), leads to

$$0\le\mathcal{E}_\varepsilon(t,\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)+\mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon-z_\varepsilon)-\mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon).\tag{4.3}$$

According to part (a), the right-hand side of (4.3) converges for $\varepsilon\to0$ and we obtain $0\le\mathbf{E}_\gamma(t,\widetilde{u}_0,\widetilde{U}_1,\widetilde{\mathcal{Z}}_\gamma)+\mathbf{R}_\gamma(\widetilde{\mathcal{Z}}_\gamma-\mathcal{Z}_\gamma)-\mathbf{E}_\gamma(t,u_0,U_1,\mathcal{Z}_\gamma)$. Thereby, stability is proven since $(\widetilde{u}_0,\widetilde{U}_1,\widetilde{\mathcal{Z}}_\gamma)\in\mathbf{Q}_\gamma$ was arbitrary. $\square$

Finally, before proving theorem 4.1, we show that $\mathbf{E}_\gamma$ and $\mathbf{R}_\gamma$ are the two-scale Γ-limits (in sense of definition 2.14) of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, respectively.

Proposition 4.4. *Let $\gamma \geq 0$ and let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathcal{Q}$. Then for $t \in [0,T]$:*

(a) *If $u_\varepsilon \overset{w0}{\rightharpoonup} (u_0, U_1)$ in $\mathbf{H}^d$ and $z_\varepsilon \overset{w\gamma}{\rightharpoonup} \mathcal{Z}_\gamma$ in $\mathbf{X}^m_\gamma$ then*

$$\begin{cases} \mathbf{E}_\gamma(t, u_0, U_1, \mathcal{Z}_\gamma) & \leq \liminf\limits_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon), \\ \mathbf{R}_\gamma(\mathcal{Z}_\gamma) & \leq \liminf\limits_{\varepsilon\to 0} \mathcal{R}_\varepsilon(z_\varepsilon). \end{cases}$$

(b) *For all $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{Z}_\gamma) \in \mathbf{Q}_\gamma$ there exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ in $\mathcal{Q}$ so that $\widetilde{u}_\varepsilon \overset{s0}{\to} (\widetilde{u}_0, \widetilde{U}_1)$ in $\mathbf{H}^d$, $\widetilde{z}_\varepsilon \overset{s\gamma}{\to} \mathcal{Z}_\gamma$ in $\mathbf{X}^m_\gamma$ and*

$$\begin{cases} \mathbf{E}_\gamma(t, \widetilde{u}_0, \widetilde{U}_1, \widetilde{\mathcal{Z}}_\gamma) & = \lim\limits_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon), \\ \mathbf{R}_\gamma(\widetilde{\mathcal{Z}}_\gamma) & = \lim\limits_{\varepsilon\to 0} \mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon). \end{cases}$$

Proof 4.4. (a) This is an easy consequence of lemma 2.15. We only have to show that $\mathcal{W} : \mathcal{Y} \times \mathbb{R}^n \to [0, \infty]$ and $\rho : \mathcal{Y} \times \mathbb{R}^m \to [0, \infty]$ satisfy its assumptions. Thereby, let $n := d^2 + m + md$. Since $\mathbb{A} \in \mathrm{L}^\infty(\mathcal{Y}; \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^n))$ is uniformly positive definite, we have $\mathcal{W}(y, (A, b, C)) \geq 0$ for all $(A, b, C) \in \mathbb{R}^n$ and almost every $y \in \mathcal{Y}$. Furthermore, $(A, b, C) \mapsto \mathcal{W}(y, (A, b, C))$ is continuous and because of its quadratic structure it is also convex for almost every $y \in \mathcal{Y}$.

Let now $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}$ with $u_\varepsilon \overset{w0}{\rightharpoonup} (u_0, U_1)$ in $\mathbf{H}^d$ and $z_\varepsilon \overset{w\gamma}{\rightharpoonup} \mathcal{Z}_\gamma$ in $\mathbf{X}^m_\gamma$ be given. Following definition 2.10, this means $v_\varepsilon \overset{w}{\rightharpoonup} V_\gamma$ in $\mathrm{L}^2(\Omega \times \mathcal{Y})^n$, where $v_\varepsilon := (\mathbf{e}(u_\varepsilon), z_\varepsilon, \varepsilon^\gamma \nabla z_\varepsilon) \in \mathrm{L}^2(\Omega)^n$ and $V_\gamma := (\mathbf{e}_x(u_0) + \mathbf{e}_y(U_1), \mathcal{L}^\gamma(\mathcal{Z}_\gamma)) \in \mathrm{L}^2(\Omega \times \mathcal{Y})^n$. Thereby, the assumptions of lemma 2.15 are fulfilled and we have

$$\mathbf{W}(V_\gamma) = \int\limits_{\Omega\times\mathcal{Y}} \mathcal{W}\big(y, V_\gamma(x,y)\big) \mathrm{d}y\mathrm{d}x \leq \liminf_{\varepsilon\to 0} W_\varepsilon(v_\varepsilon) = \liminf_{\varepsilon\to 0} \int\limits_\Omega \mathcal{W}\left(\left\{\frac{x}{\varepsilon}\right\}_Y, v_\varepsilon(x)\right) \mathrm{d}x.$$

Because of the weak convergence of u_ε to u_0 in $\mathrm{H}^1(\Omega)^d$, it follows : $\langle \ell(t), u_0 \rangle = \lim\limits_{\varepsilon\to 0} \langle \ell(t), u_\varepsilon \rangle$. Finally, we obtain:

$$\mathbf{E}_\gamma(t, u_0, U_1, \mathcal{Z}_\gamma) = \mathbf{W}(V_\gamma) - \langle \ell(t), u_0 \rangle \leq \liminf_{\varepsilon\to 0} W_\varepsilon(v_\varepsilon) - \lim_{\varepsilon\to 0} \langle \ell(t), u_\varepsilon \rangle = \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon).$$

Since $\rho \colon \mathcal{Y} \times \mathbb{R}^m \to [0, \infty]$ has the properties (3.6) and $z_\varepsilon \overset{s}{\to} \mathcal{L}^\gamma_1(\mathcal{Z}_\gamma)$ in $\mathrm{L}^2(\Omega \times \mathcal{Y})^m$, we also have

$$\mathbf{R}_\gamma(\mathcal{Z}_\gamma) = \int\limits_{\Omega\times\mathcal{Y}} \rho\big(y, \mathcal{L}^\gamma_1(\mathcal{Z}_\gamma)(x,y)\big) \mathrm{d}y\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int\limits_\Omega \rho\left(\left\{\frac{x}{\varepsilon}\right\}_Y, z_\varepsilon(x)\right) \mathrm{d}x = \liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon(z_\varepsilon)$$

according to lemma 2.15. Lemma 2.16(a) states that this is even an equality.

(b) With the help of the in subsection 2.4 given folding operators $\mathcal{G}_\varepsilon$ and $\mathcal{G}^\gamma_\varepsilon$, the sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}$ is explicitly given in the form

$$\begin{pmatrix} \widetilde{u}_\varepsilon \\ \widetilde{z}_\varepsilon \end{pmatrix} := \begin{pmatrix} \widetilde{u}_0 + \mathcal{G}_\varepsilon(0, \widetilde{U}_1) \\ \mathcal{G}^\gamma_\varepsilon(\widetilde{\mathcal{Z}}_\gamma) \end{pmatrix}.$$

Proposition 2.11 immediately yields the wanted convergence and analog to step 2 of proof 4.3 we obtain $\lim_{\varepsilon\to 0}\mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon) = \mathbf{R}_\gamma(\widetilde{\mathcal{Z}}_\gamma)$ with lemma 2.16(a). Finally, we conclude analog to step 3 and 4 of proof 4.3 that: $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t,\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon) = \mathbf{E}_\gamma(t,\widetilde{u}_0,\widetilde{U}_1,\widetilde{\mathcal{Z}}_\gamma)$. □

Now we are in the position to prove theorem 4.1.

Proof of theorem 4.1. Let $q_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathrm{C}^{0,1}([0,T],\mathcal{Q})$ be the unique solution of the energetic formulation $(\mathrm{S}^\varepsilon)\&(\mathrm{E}^\varepsilon)$ of the rate-independent system $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with $q_\varepsilon(0) = q_\varepsilon^0 \in \mathcal{S}_\varepsilon(0)$.

1. Let $\mathbb{Q} := \mathrm{H}^1(\Omega)^d \times \mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^n$, where $n := d^2+m+md$. Now we consider the subset M of $\mathrm{C}^{0,1}([0,T],\mathbb{Q})$ defined as follows:

$$M := \{\widetilde{q}_\varepsilon := (u_\varepsilon, \mathcal{T}_\varepsilon(\nabla u_\varepsilon), \mathcal{T}_\varepsilon z_\varepsilon, \mathcal{T}_\varepsilon(\nabla z_\varepsilon)) \,|\, (u_\varepsilon,z_\varepsilon) : [0,T]\to\mathcal{Q} \text{ solution of } (\mathrm{S}^\varepsilon)\&(\mathrm{E}^\varepsilon)\}\,.$$

According to proposition 3.2 we obtain

$$\begin{aligned}
\sup_{t\in[0,T]} \left(\|u_\varepsilon(t)\|^2_{\mathrm{H}^1(\Omega)^d} + \|z_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega)^m} + \|\varepsilon^\gamma\nabla z_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega)^{m\times d}}\right)^{\frac12} &\le \|q_\varepsilon^0\|_{\mathcal{Q},\varepsilon} + \frac{T\Theta}{\alpha}, \quad (4.4)\\
\operatorname*{ess\,sup}_{t\in[0,T]} \left(\|\dot u_\varepsilon(t)\|^2_{\mathrm{H}^1(\Omega)^d} + \|\dot z_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega)^m} + \|\varepsilon^\gamma\nabla \dot z_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega)^{m\times d}}\right)^{\frac12} &\le \frac{\Theta}{\alpha}.
\end{aligned}$$

Because of the assumed two-scale convergence (in sense of definition 2.10) of the initial data, there exists a constant $C_1>0$ so that $\|q_\varepsilon^0\|_{\mathcal{Q},\varepsilon}\le C_1$ for every $\varepsilon>0$. Thereby, we obtain that the subset M is bounded in $\mathrm{C}^{0,1}([0,T],\mathbb{Q})$, since $\mathcal{T}_\varepsilon$ is norm-preserving. Now, according to the Arzela-Ascoli theorem, there exists a subsequence $(\widetilde{q}_{\varepsilon'})_{\varepsilon'>0}$ of $(\widetilde{q}_\varepsilon)_{\varepsilon>0}$ and a function $(u_0,\widetilde{U}_1,\widetilde{Z}_\gamma,\widetilde{W}_\gamma)\in\mathrm{C}^{0,1}([0,T];\mathbb{Q})$, so that the following holds true for every $t\in[0,T]$:

$$\begin{pmatrix} u_{\varepsilon'}(t), \mathcal{T}_\varepsilon(\nabla u_{\varepsilon'}(t)) \\ \mathcal{T}_\varepsilon z_{\varepsilon'}(t), \mathcal{T}_\varepsilon((\varepsilon')^\gamma\nabla z_{\varepsilon'}(t)) \end{pmatrix} \rightharpoonup \begin{pmatrix} u_0(t), \widetilde{U}_1(t) \\ \widetilde{Z}_\gamma(t), \widetilde{W}_\gamma(t)\end{pmatrix} \text{ weakly in } \mathbb{Q}. \tag{4.5}$$

Since the supports of the functions $\mathcal{T}_\varepsilon(\nabla u_\varepsilon(t)), \mathcal{T}_\varepsilon(z_\varepsilon(t))$ and $\mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla z_\varepsilon(t))$ are contained in $[\Omega\times\mathcal{Y}]_\varepsilon$ for every $t\in[0,T]$, the supports of the functions $\widetilde{U}_1(t),\widetilde{Z}_\gamma(t)$ and $\widetilde{W}_\gamma(t)$ are contained in $\overline{\Omega}\times\mathcal{Y}$.

2. According to proposition 2.9 (note condition (4.4)), we obtain for every $t\in[0,T]$, that there exists a subsequence $(u_{\varepsilon''}(t))_{\varepsilon''>0}$ of $(u_{\varepsilon'}(t))_{\varepsilon'>0}$ and functions $u_0(t)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $U_1(t)\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$, so that $\nabla u_{\varepsilon''}(t)\overset{w}{\rightharpoonup}\nabla_x E u_0(t)+\nabla_y U_1(t)$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$. Furthermore, $\nabla u_{\varepsilon''}(t)\overset{w}{\rightharpoonup}\widetilde{U}_1(t)$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$ for every $t\in[0,T]$ according to condition (4.5), i.e. we obtain

$$\nabla_y U_1 = \widetilde{U}_1 - \nabla_x E u_0 \tag{4.6}$$

as a possible representation of the function $\nabla_y U_1 : [0,T]\to\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$. Following step 1, the right-hand side of (4.6) is Lipschitz-continuous, i.e. $U_1\in\mathrm{C}^{0,1}([0,T];\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)$.

Since the same argument works for the z-component, we showed: There exists a subsequence $(u_{\varepsilon'},z_{\varepsilon'})_{\varepsilon'>0}$ of $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ and a function $(u_0,U_1,\mathcal{Z}_\gamma)\in\mathrm{C}^{0,1}([0,T];\mathbf{Q}_\gamma)$, so that

$$u_{\varepsilon'}(t)\overset{w0}{\rightharpoonup}(u_0(t),U_1(t)) \text{ in } \mathbf{H}^d \text{ and } z_{\varepsilon'}(t)\overset{w\gamma}{\rightharpoonup}\mathcal{Z}_\gamma(t) \text{ in } \mathbf{X}^m_\gamma \tag{4.7}$$

for every $t \in [0,T]$. Since every subsequence converges to the same limit, we obtain the convergence of the whole sequence.

3. It remains to show that $(u_0, U_1, \mathcal{Z}_\gamma) \in \mathrm{C}^{0,1}([0,T]; \mathbf{Q}_\gamma)$ is the unique solution of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ with the initial value $(u_0(0), U_1(0), \mathcal{Z}_\gamma(0)) = (u_0^0, U_1^0, \mathcal{Z}_\gamma^0) \in \widetilde{\mathcal{S}}_\gamma(0)$. According to prop. 4.3(b), $(u_0(t), U_1(t), \mathcal{Z}_\gamma(t))$ is stable for every $t \in [0,T]$. To show the energy balance $(\mathbf{E}^\gamma)$, we pass in (E^ε) to the limit $\varepsilon \to 0$.

4. Proposition 3.2 yields for almost every $t \in [0,T]$

$$\big\langle \dot{\ell}(t), u_\varepsilon(t) \big\rangle \leq \|\dot{\ell}\|_{\mathrm{L}^\infty([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)} \|u_\varepsilon\|_{\mathrm{C}^0([0,T];\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)} \leq \|\dot{\ell}\|_{\mathrm{L}^\infty([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)} C_2,$$

where $C_2 := C_1 + \alpha^{-1} T \Theta$. Thereby we obtain

$$\lim_{\varepsilon \to 0} \int_0^t \big\langle \dot{\ell}(s), u_\varepsilon(s) \big\rangle \,\mathrm{d}s = \int_0^t \lim_{\varepsilon \to 0} \big\langle \dot{\ell}(s), u_\varepsilon(s) \big\rangle \,\mathrm{d}s = \int_0^t \big\langle \dot{\ell}(s), u_0(s) \big\rangle \mathrm{d}s,$$

by applying Lebesgue's theorem of dominated convergence and making use of $u_\varepsilon(t) \rightharpoonup u_0(t)$ in $\mathrm{H}^1(\Omega)$ for every $t \in [0,T]$.

5. Because of the assumed strong two-scale convergence of the initial condition, we have $\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(0, u_\varepsilon(0), z_\varepsilon(0)) = \mathbf{E}_\gamma(0, U(0), Z(0))$ analog to step 3 and 4 of proof 4.3.

6. According to step 4 and 5, we now obtain for every $t \in [0,T]$ from (E^ε)

$$\Big[\mathcal{E}_\varepsilon\big(t, u_\varepsilon(t), z_\varepsilon(t)\big) + \int_0^t \mathcal{R}_\varepsilon\big(\dot{z}_\varepsilon(s)\big)\mathrm{d}s\Big] \overset{\varepsilon \to 0}{\to} \mathbf{E}_\gamma\big(0, u_0(0), U_1(0), \mathcal{Z}_\gamma(0)\big) + \int_0^t \big\langle \dot{\ell}(s), u_0(s) \big\rangle \mathrm{d}s, \tag{4.8}$$

if we pass in (E^ε) to the limit $\varepsilon \to 0$. Furthermore, because of convergence (4.7) proposition 4.4 can be applied, so that for every $t \in [0,T]$ we have

$$\mathbf{E}_\gamma\big(t, u_0(t), U_1(t), \mathcal{Z}_\gamma(t)\big) \leq \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon\big(t, u_\varepsilon(t), z_\varepsilon(t)\big) \tag{4.9}$$

and for all $t_k \in [0,T]$, where $k = 0, \ldots, N$ and $N \in \mathbb{N}$, we have

$$\sum_{j=1}^N \mathbf{R}_\gamma\big(\mathcal{Z}_\gamma(t_j) - \mathcal{Z}_\gamma(t_{j-1})\big) \leq \liminf_{\varepsilon \to 0} \sum_{j=1}^N \mathcal{R}_\varepsilon\big(z_\varepsilon(t_j) - z_\varepsilon(t_{j-1})\big) \leq \liminf_{\varepsilon \to 0} \int_0^t \mathcal{R}_\varepsilon\big(\dot{z}_\varepsilon(s)\big)\mathrm{d}s.$$

In the last estimation we used $\mathrm{Diss}_{\mathcal{R}_\varepsilon}(z_\varepsilon; [0,T]) := \sup\big\{ \sum_{j=1}^N \mathcal{R}_\varepsilon(z_\varepsilon(t_j) - z_\varepsilon(t_{j-1}))\big\}$ as an equivalent definition of the total dissipation $\mathrm{Diss}_{\mathcal{R}_\varepsilon}(z_\varepsilon; [0,T]) := \int_0^t \mathcal{R}_\varepsilon\big(\dot{z}_\varepsilon(s)\big)\mathrm{d}s$, where $N \in \mathbb{N}$ and the supremum is taken over all finite partitions of $[0,T]$. That means, after taking the supremum over all finite partitions of $[0,T]$, we obtain with the same equivalent definition of $\mathrm{Diss}_{\mathbf{R}_\gamma}(\mathcal{Z}_\gamma; [0,T])$

$$\int_0^t \mathbf{R}_\gamma\big(\dot{\mathcal{Z}}_\gamma(s)\big)\mathrm{d}s \leq \liminf_{\varepsilon \to 0} \int_0^t \mathcal{R}_\varepsilon\big(\dot{z}_\varepsilon(s)\big)\mathrm{d}s. \tag{4.10}$$

Thereby, we finally showed together with condition (4.8)

$$\begin{aligned}
&\mathbf{E}_\gamma\big(t,u_0(t),U_1(t),\mathcal{Z}_\gamma(t)\big)+\int_0^t \mathbf{R}_\gamma\big(\dot{\mathcal{Z}}_\gamma(s)\big)\mathrm{d}s\\
&\qquad\le \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big)+\liminf_{\varepsilon\to 0}\int_0^t \mathcal{R}_\varepsilon\big(\dot z_\varepsilon(s)\big)\mathrm{d}s\\
&\qquad\le \lim_{\varepsilon\to 0}\Big[\mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big)+\int_0^t \mathcal{R}_\varepsilon\big(\dot z_\varepsilon(s)\big)\mathrm{d}s\Big]\\
&\qquad\qquad\qquad= \mathbf{E}_\gamma\big(0,u_0(0),U_1(0),\mathcal{Z}_\gamma(0)\big)+\int_0^t\big\langle\dot\ell(s),u_0(s)\big\rangle\mathrm{d}s \qquad (4.11)
\end{aligned}$$

for every $t\in[0,T]$. Because of the stability of $(u_0(t),U_1(t),\mathcal{Z}_\gamma(t))$ for every $t\in[0,T]$ showed in step 3, we obtain immediately the opposite inequality to (4.11) and altogether we have the energy balance $(\mathbf{E}^\gamma)$ for every $t\in[0,T]$. That means

$$\mathbf{E}_\gamma\big(t,u_0(t),U_1(t),\mathcal{Z}_\gamma(t)\big)+\int_0^t \mathbf{R}_\gamma\big(\dot{\mathcal{Z}}_\gamma(s)\big)\mathrm{d}s=\mathbf{E}_\gamma\big(0,u_0(0),U_1(0),\mathcal{Z}_\gamma(0)\big)+\int_0^t\big\langle\dot\ell(s),u_0(s)\big\rangle\mathrm{d}s.$$

Hence, it is shown that $(u_0,U_1,\mathcal{Z}_\gamma)\in \mathrm{C}^{0,1}([0,T];\mathbf{Q}_\gamma)$ is the unique solution of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma,\mathbf{E}_\gamma,\mathbf{R}_\gamma)$ with the initial condition $(u_0(0),U_1(0),\mathcal{Z}_\gamma(0))=(u_0^0,U_1^0,\mathcal{Z}_\gamma^0)\in\widetilde{\mathcal{S}}_\gamma(0)$.

7. According to step 6, $(u_0,U_1,\mathcal{Z}_\gamma)\in \mathrm{C}^{0,1}([0,T];\mathbf{Q}_\gamma)$ has to satisfy the energy balance $(\mathbf{E}^\gamma)$, so that

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big)+\liminf_{\varepsilon\to 0}\int_0^t \mathcal{R}_\varepsilon\big(\dot z_\varepsilon(s)\big)\mathrm{d}s=\lim_{\varepsilon\to 0}\Big[\mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big)+\int_0^t \mathcal{R}_\varepsilon\big(\dot z_\varepsilon(s)\big)\mathrm{d}s\Big]$$

following condition (4.11). But this means that the limits $\lim_{\varepsilon\to0}\mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big)$ and $\lim_{\varepsilon\to0}\int_0^t\mathcal{R}_\varepsilon\big(\dot z_\varepsilon(s)\big)\mathrm{d}s$ have to exist. Furthermore, we have the following equation from the energy balance $(\mathbf{E}^\gamma)$ and (4.11)

$$\underbrace{\mathbf{E}_\gamma\big(t,u_0(t),U_1(t),\mathcal{Z}_\gamma(t)\big)-\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big)}_{\le 0 \text{ see (4.9)}}+\underbrace{\int_0^t \mathbf{R}_\gamma\big(\dot{\mathcal{Z}}_\gamma(s)\big)\mathrm{d}s-\lim_{\varepsilon\to 0}\int_0^t\mathcal{R}_\varepsilon\big(\dot z_\varepsilon(s)\big)\mathrm{d}s}_{\le 0 \text{ see (4.10)}}=0,$$

i.e. altogether we have

$$\mathbf{E}_\gamma\big(t,u_0(t),U_1(t),\mathcal{Z}_\gamma(t)\big)=\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon\big(t,u_\varepsilon(t),z_\varepsilon(t)\big)\ \&\ \int_0^t \mathbf{R}_\gamma\big(\dot{\mathcal{Z}}_\gamma(s)\big)\mathrm{d}s=\lim_{\varepsilon\to 0}\int_0^t\mathcal{R}_\varepsilon\big(\dot z_\varepsilon(s)\big)\mathrm{d}s.$$

8. In this step all arguments are valid for every $t\in[0,T]$, even if it is not explicitly mentioned. By using the in (4.7) showed weak convergence, we now show strong convergence. Thereto, let $q_\varepsilon(t):=(u_\varepsilon(t),z_\varepsilon(t))$ and $\mathbf{q}_\gamma(t):=(u_0(t),U_1(t),\mathcal{Z}_\gamma(t))$. Furthermore, we consider the sequence $(\widetilde q_\varepsilon(t))_{\varepsilon\in(0,\varepsilon_0)}$, which is defined via $\widetilde q_\varepsilon(t):=(\widetilde u_\varepsilon(t),\widetilde z_\varepsilon(t))=(u_0(t)+\mathcal{G}_\varepsilon(0,U_1(t)),\mathcal{G}_\varepsilon^\gamma(\mathcal{Z}_\gamma(t)))$. Applying proposition 2.11, we have $\widetilde u_\varepsilon(t)\overset{s0}{\to}(u_0(t),U_1(t))$ in $\mathbf{H}^d$ and $\widetilde z_\varepsilon(t)\overset{s\gamma}{\to}\mathcal{Z}_\gamma(t)$ in $\mathbf{X}_\gamma^m$. Moreover, because of the ellipticity of the operator $\mathcal{A}_\varepsilon\in\mathrm{Lin}(\mathcal{Q},\mathcal{Q}^*)$ (see (3.1)), we obtain

$$\begin{aligned}
&\tfrac{\alpha}{2}\|\widetilde q_\varepsilon(t)-q_\varepsilon(t)\|_{\mathcal{Q}}^2\\
&\qquad\le \tfrac12\langle\mathcal{A}_\varepsilon(\widetilde q_\varepsilon(t)-q_\varepsilon(t)),\widetilde q_\varepsilon(t)-q_\varepsilon(t)\rangle_{\mathcal{Q}}\\
&\qquad=\mathcal{E}_\varepsilon(t,q_\varepsilon(t))-\mathcal{E}_\varepsilon(t,\widetilde q_\varepsilon(t))+\langle\mathcal{A}_\varepsilon\widetilde q_\varepsilon(t),\widetilde q_\varepsilon(t)-q_\varepsilon(t)\rangle_{\mathcal{Q}}-\langle\ell(t),\widetilde u_\varepsilon(t)-u_\varepsilon(t)\rangle.
\end{aligned}$$

According to step 7 we have $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t, q_\varepsilon(t)) = \mathbf{E}_\gamma(t, \mathbf{q}_\gamma(t))$ and moreover we obtain firstly $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t, \widetilde{q}_\varepsilon(t)) = \mathbf{E}_\gamma(t, \mathbf{q}_\gamma(t))$ and secondly $\lim_{\varepsilon\to 0}\langle \mathcal{A}_\varepsilon \widetilde{q}_\varepsilon(t), \widetilde{q}_\varepsilon(t) - q_\varepsilon(t)\rangle_{\mathcal{Q}} = 0$ analog to step 3 and 4 of proof 4.3. Finally, we have $\lim_{\varepsilon\to 0}\langle \ell(t), \widetilde{u}_\varepsilon(t) - u_\varepsilon(t)\rangle = 0$ because of $\widetilde{u}_\varepsilon(t) - u_\varepsilon(t) \rightharpoonup 0$ in $\mathrm{H}^1(\Omega)^d$.

Thereby, the above yields $\lim_{\varepsilon\to 0}\|\widetilde{q}_\varepsilon(t) - q_\varepsilon(t)\|_{\mathcal{Q}} = 0$ and the following estimation gives us the wanted strong convergence.

$$
\begin{aligned}
&\|\mathcal{T}_\varepsilon u_\varepsilon - (Eu_0)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^d} + \|\mathcal{T}_\varepsilon(\nabla u_\varepsilon) - (\nabla_x Eu_0 + \nabla_y U_1)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{d\times d}} \\
&\qquad \le \|\mathcal{T}_\varepsilon(u_\varepsilon(t) - \widetilde{u}_\varepsilon(t))\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^d} + \|\mathcal{T}_\varepsilon(\nabla u_\varepsilon(t) - \nabla\widetilde{u}_\varepsilon(t))\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{d\times d}} + \delta_\varepsilon^{(1)} \\
&\qquad \le \sqrt{2}\|q_\varepsilon(t) - \widetilde{q}_\varepsilon(t)\|_{\mathcal{Q}} + \delta_\varepsilon^{(1)}, \\
&\|\mathcal{T}_\varepsilon z_\varepsilon - \mathcal{L}_1^\gamma(\mathcal{Z}_\gamma)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{m\times d}} + \|\mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla z_\varepsilon) - \mathcal{L}_2^\gamma(\mathcal{Z}_\gamma)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{m\times d}} \\
&\qquad \le \|\mathcal{T}_\varepsilon(z_\varepsilon(t) - \widetilde{z}_\varepsilon(t))\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^m} + \|\mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla z_\varepsilon(t) - \varepsilon^\gamma\nabla\widetilde{z}_\varepsilon(t))\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{m\times d}} + \delta_\varepsilon^{(2)} \\
&\qquad \le \sqrt{2}\|q_\varepsilon(t) - \widetilde{q}_\varepsilon(t)\|_{\mathcal{Q}} + \delta_\varepsilon^{(2)}
\end{aligned}
$$

where $\delta_\varepsilon^{(1)} := \|\mathcal{T}_\varepsilon\widetilde{u}_\varepsilon - (Eu_0)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^d} + \|\mathcal{T}_\varepsilon(\nabla\widetilde{u}_\varepsilon) - (\nabla_x Eu_0 + \nabla_y U_1)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{d\times d}}$ and $\delta_\varepsilon^{(2)} := \|\mathcal{T}_\varepsilon\widetilde{z}_\varepsilon - \mathcal{L}_1^\gamma(\mathcal{Z}_\gamma)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^m} + \|\mathcal{T}_\varepsilon(\varepsilon^\gamma\nabla\widetilde{z}_\varepsilon) - \mathcal{L}_2^\gamma(\mathcal{Z}_\gamma)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{m\times d}}$. Thereby, we added a nil, then estimated by triangle inequality and finally used the norm-preservation of $\mathcal{T}_\varepsilon$. Now corollary 2.12 and proposition 2.11 yield $\delta_\varepsilon^{(1)} \to 0$ and $\delta_\varepsilon^{(2)} \to 0$, respectively, for $\varepsilon \to 0$, and the proof is done. ☐

5 Equivalence of the two-scale and the one-scale model

Henceforth let $\gamma \in [0, 1)$ and $\mathbb{B}$ be constant in y, i.e. $\mathbb{B} \in \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^{d\times d}_{\mathrm{sym}})$.

Theorem 5.1. *Let $\gamma \in [0, 1)$. Then $q = (u_0, z_0) \in \mathrm{C}^{0,1}([0, T]; \mathcal{Q}_\gamma)$ is the solution of the energetic formulation $(\mathrm{S}^\gamma)\&(\mathrm{E}^\gamma)$ of the rate-independent system $(\mathcal{Q}_\gamma, \mathcal{E}_\gamma, \mathcal{R}_\gamma)$ with $q(0) = q^0$, if and only if $\mathbf{q} = (u_0, U_1 = \mathcal{L}_{\mathbb{C}}(\mathbf{e}(u_0) - \mathbb{B}z_0), z_0, Z_1 = \mathcal{L}_{\mathbb{F}}(\widetilde{\nabla} z_0)) \in \mathrm{C}^{0,1}([0, T]; \mathbf{Q}_\gamma)$ is the solution of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ with $\mathbf{q}(0) = \mathbf{q}^0$, where $\mathcal{L}_{\mathbb{C}}, \mathcal{L}_{\mathbb{F}}$ and $\widetilde{\nabla}$ are defined in subsection 3.4.*

Before proving this, we show that $\mathcal{E}_\gamma$ and $\mathbf{E}_\gamma$ are equal along the energetic solution of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$.

Lemma 5.2. *Let $\gamma \in [0, 1)$ and let $\mathbf{q} = (u_0, U_1, z_0, Z_1) \in \mathrm{C}^{0,1}([0, T]; \mathbf{Q}_\gamma)$ be the unique solution of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ with $\mathbf{q}(0) = \mathbf{q}^0$. Then for every $t \in [0, T]$:*

$$\mathbf{E}_\gamma(t, u_0(t), U_1(t), z_0(t), Z_1(t)) = \mathcal{E}_\gamma(t, u_0(t), z_0(t)).$$

Proof 5.2. 1. Let us start with showing that the solution of the two scale energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ minimizes $\mathbf{E}_\gamma\colon [0, T] \times \mathcal{Q}_\gamma \to \mathbb{R}$. This is the motivation for the in subsection 3.4 defined one-scale energetic formulation $(\mathrm{S}^\gamma)\&(\mathrm{E}^\gamma)$ as well.

Let $\mathbf{q}(t) = (u_0(t), U_1(t), z_0(t), Z_1(t)) \in \mathbf{Q}_\gamma$ be the unique solution of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$ with $\mathbf{q}(0) = \mathbf{q}^0$.

Because of the continuity and the strict convexity of the mapping $\mathbf{E}_\gamma(t,\cdot): [0,T] \to \mathbb{R}$, there exists a unique solution $(U_1^*(t), Z_1^*(t))$ of the following minimization problem:

$$\min\Big\{\mathbf{E}_\gamma(t,u_0(t),U_1,z_0(t),Z_1)\,|\,(U_1,Z_1)\in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d\times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^m\Big\}.$$

By choosing $\widetilde{\mathbf{q}}(t) := (u_0(t), U_1^*(t), z_0(t), Z_1^*(t))$ for $t\in[0,T]$ in the stability condition $(\mathbf{S}^\gamma)$, we obtain $(U_1(t), Z_1(t)) = (U_1^*(t), Z_1^*(t))$ for every $t\in[0,T]$.

2. Considering the functionals $I_3\colon \mathbf{H}^d\times \mathrm{X}^m_\gamma\to\mathbb{R}$, $I_4\colon \mathrm{X}^m_\gamma\to\mathbb{R}$, $I_5\colon \mathbf{X}^m_\gamma\to\mathbb{R}$ given by:

$$\begin{aligned}
I_3(u_0,U_1,z_0) &:= \frac12\int_{\Omega\times\mathcal{Y}}\langle \mathbb{C}(y)(\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)-\mathbb{B}Ez_0),\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)-\mathbb{B}Ez_0\rangle_{d\times d}\,\mathrm{d}y\mathrm{d}x,\\
I_4(z_0) &:= \frac12\int_{\Omega\times\mathcal{Y}}\langle \mathbb{H}(y)Ez_0(x,y),Ez_0(x,y)\rangle_m\mathrm{d}y\mathrm{d}x,\\
I_5(z_0,Z_1) &:= \frac12\int_{\Omega\times\mathcal{Y}}\langle \mathbb{F}(y)\mathcal{L}_2^\gamma(z_0,Z_1)(x,y),\mathcal{L}_2^\gamma(z_0,Z_1)(x,y)\rangle_{m\times d}\,\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

The two-scale energy functional $\mathbf{E}_\gamma\colon[0,T]\times\mathbf{Q}_\gamma\to\mathbb{R}$ now can be represented as

$$\mathbf{E}_\gamma(t,u_0,U_1,z_0,Z_1) = I_3(u_0,U_1,z_0)+I_4(z_0)+I_5(z_0,Z_1)-\langle\ell(t),u_0\rangle.$$

Because of the continuity and strict convexity of $I_3(u_0,\cdot,z_0): \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d\to\mathbb{R}$ and $I_4(z_0,\cdot): \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^m\to\mathbb{R}$ there exist unique minimizers $U_1^*\in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ and $Z_1^*\in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^m$ of these functionals, analog to step 1. Moreover, these minimizers satisfy the Euler-Lagrange equations

$$\mathrm{D}_{\nabla_y U_1}(I_3(u_0,U_1,z_0))[\widetilde{U}_1]=0 \qquad \forall\,\widetilde{U}_1\in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d, \tag{5.1}$$

$$\mathrm{D}_{\nabla_y Z_1}(I_5(z_0,Z_1))[\widetilde{Z}_1]=0 \qquad \forall\,\widetilde{Z}_1\in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^m. \tag{5.2}$$

Furthermore, the one-scale energy functional $\mathcal{E}_\gamma\colon[0,T]\times\mathcal{Q}_\gamma\to\mathbb{R}$ can be written as

$$\begin{aligned}
\mathcal{E}_\gamma(t,u_0,z_0) \;=\; &\int_\Omega I_1\big(\mathbf{e}_x(u_0)(x)-\mathbb{B}z_0(x),\mathcal{L}_\mathbb{C}(\mathbf{e}_x(u_0)(x)-\mathbb{B}z_0(x))\big)\mathrm{d}x\\
&+\; I_4(z_0)+\int_\Omega I_2\big(\widetilde{\nabla}z_0(x),\mathcal{L}_\mathbb{F}(\widetilde{\nabla}z_0(x))\big)\mathrm{d}x-\langle\ell(t),u_0\rangle,
\end{aligned}$$

where $\mathcal{L}_\mathbb{C}: \mathbb{R}^{d\times d}\to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ and $\mathcal{L}_\mathbb{F}: \mathbb{R}^{m\times d}\to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m$ are given in step 1 of proof 3.4, $I_1: \mathbb{R}^{d\times d}\times \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d\to\mathbb{R}$ and $I_2: \mathbb{R}^{m\times d}\times \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m\to\mathbb{R}$ are defined in (3.11) and (3.12), respectively, and $\widetilde{\nabla}: \mathrm{X}^m_\gamma\to \mathrm{L}^2(\Omega)^{m\times d}$ is given by (3.15). In proof 3.4 we have also seen that $\mathcal{L}_\mathbb{C}(\xi)$ and $\mathcal{L}_\mathbb{F}(\eta)$ are the unique solutions of the following Euler-Lagrange equations:

$$\mathrm{D}_{\nabla v}(I_1(\xi,v))[\widetilde{v}]=0 \qquad \forall\,\widetilde{v}\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d, \tag{5.3}$$

$$\mathrm{D}_{\nabla w}(I_2(\eta,w))[\widetilde{w}]=0 \qquad \forall\,\widetilde{w}\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^m. \tag{5.4}$$

3. Now we show that the unique solutions $\mathcal{L}_\mathbb{C}(\mathbf{e}_x(u_0)(x)-\mathbb{B}z_0(x))$ and $\mathcal{L}_\mathbb{F}(\widetilde{\nabla}z_0(x))$ of the Euler-Lagrange equations (5.3) and (5.4) for $\xi=\mathbf{e}_x(u_0)(x)-\mathbb{B}z_0(x)$ and $\eta=\widetilde{\nabla}z_0(x)$

are also the solutions of the Euler-Lagrange equations (5.1) and (5.2), respectively. Note that here $\mathbb{B} \equiv$ *constant* is necessary, because we are only allowed to insert functions which are constant in y. Inserting $\xi = \mathbf{e}_x(u_0)(x) - \mathbb{B}z_0(x)$ and $\eta = \widetilde{\nabla} z_0(x)$ in (5.3) and (5.4), respectively, and integrating over Ω nearly yields the result. The only difference between $\int_\Omega (5.3)\,\mathrm{d}x \,\&\, \int_\Omega (5.4)\,\mathrm{d}x$ and (5.1)&(5.2) is the set of testfunctions.

How we get rid of this difference, we show exemplarily for the first equality (5.3). First we take the special testfunction $\widetilde{v}_i = v_i e_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$, where e_i is the standard basis vector of $\mathbb{R}^d$ and $v_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ for $i \in \{1,\dots,d\}$. Then we multiply (5.3) with an arbitrary function $f_i \in \mathrm{L}^2(\Omega)$ and finally we integrate over Ω. Now, for an arbitrary testfunction $\widetilde{v} = (v_1,\dots,v_d)^T \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$, we repeat this with arbitrary functions $f_1,\dots,f_d \in \mathrm{L}^2(\Omega)$ and the testfunctions $\widetilde{v}_1,\dots,\widetilde{v}_d \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ defined as before. Then we add everything up ($i = 1,\dots,d$), so that we have (5.1) with $\widehat{U}_1 = (f_1v_1,\dots,f_dv_d)^T$. Since the set $\big\{(f_1v_1,\dots,f_mv_m)^T \mid f_i \in \mathrm{L}^2(\Omega), v_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})\big\}$ is dense in $\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ we finally obtain equation (5.1) for all $\widetilde{U}_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$.

Now $U_1^* = \mathcal{L}_{\mathbb{C}}(\mathbf{e}_x(u_0) - \mathbb{B}z_0) \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ is the unique solution of (5.1) for a given $(u_0,z_0) \in \mathcal{Q}_\gamma$ and thereby it is also the minimizer of $I_3(u_0,\cdot,z_0) : \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}$.

Altogether we showed

$$\int_\Omega I_1\big(\mathbf{e}_x(u_0)(x) - \mathbb{B}z_0(x), \mathcal{L}_{\mathbb{C}}(\mathbf{e}_x(u_0)(x) - \mathbb{B}z_0(x))\big)\mathrm{d}x = \min_{U_1\in\mathbb{L}^d} I_3(u_0,U_1,z_0),$$

$$\int_\Omega I_2\big(\widetilde{\nabla} z_0(x), \mathcal{L}_{\mathbb{F}}(\widetilde{\nabla} z_0(x))\big)\mathrm{d}x = \min_{Z_1\in\mathbb{L}^m} I_5(z_0,Z_1)$$

for given a $(u_0,z_0) \in \mathcal{Q}_\gamma$, where $\mathbb{L} := \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$.

4. Let $\mathbf{q}(t) = (u_0(t),U_1(t),z_0(t),Z_1(t)) \in \mathbf{Q}_\gamma$ for all $t \in [0,T]$ be the unique solution of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma,\mathbf{E}_\gamma,\mathbf{R}_\gamma)$ with $\mathbf{q}(0) = \mathbf{q}^0$, then we have

$$\mathbf{E}_\gamma\big(t,\mathbf{q}(t)\big) = \min_{U_1\in\mathbb{L}^d} I_3\big(u_0(t),U_1,z_0(t)\big) + I_4\big(z_0(t)\big) + \min_{Z_1\in\mathbb{L}^m} I_5\big(z_0(t),Z_1\big) - \big\langle \ell(t),u_0(t)\big\rangle,$$

according to step 1. Using step 3 and the representation of $\mathcal{E}_\gamma\colon [0,T]\times\mathcal{Q}_\gamma \to \mathbb{R}$ of step 2, we finally obtain

$$\mathcal{E}_\gamma(t,u_0(t),z_0(t)) = \mathbf{E}_\gamma(t,u_0(t),U_1(t),z_0(t),Z_1(t))$$

for every $t \in [0,T]$. □

Note, if $\gamma \in (0,1)$, we obtain $Z_1 \equiv 0$, because of the minimization property of the energetic solution, so that there is no gradient term of the internal variables in the homogenized model.

Now we are able to prove theorem 5.1.

Proof of theorem 5.1. "⇒" Let $q = (u_0,z_0) \in \mathrm{C}^{0,1}([0,T];\mathcal{Q})$ be the unique solution of the energetic formulation $(\mathrm{S}^\gamma)\&(\mathrm{E}^\gamma)$ of the rate-independent system $(\mathcal{Q}_\gamma,\mathcal{E}_\gamma,\mathcal{R}_\gamma)$ with $q(0) = q^0 = (u_0^0,z_0^0) \in \mathcal{S}_0(0)$. Because of definition (3.16) and (3.9) of the dissipation potential $\mathcal{R}_\gamma$ and $\mathbf{R}_\gamma$, respectively, we have $\mathcal{R}_\gamma(z_0(t)) \equiv \widetilde{\mathbf{R}}_\gamma(z_0(t))$, where $\widetilde{\mathbf{R}}_\gamma\colon \mathrm{X}^m_\gamma \to [0,\infty]$ is defined in subsection 3.4.

Furthermore, we have $\mathcal{E}_\gamma(t, u_0(t), z_0(t)) = \mathbf{E}_\gamma(t, u_0(t), U_1^*(t), z_0(t), Z_1^*(t))$ according to step 4 of proof 5.2, where $U_1^*(t) = \mathcal{L}_\mathbb{C}(\mathbf{e}_x(u_0(t)) - \mathbb{B}z_0(t))$ and $Z_1^*(t) = \mathcal{L}_\mathbb{F}(\nabla_x z_0(t))$. Thereby, $\mathbf{q}^*(t) := (u_0(t), U_1^*(t), z_0(t), Z_1^*(t))$ satisfies the energy balance $(\mathbf{E}^\gamma)$ and the stability condition $(\mathbf{S}^\gamma)$ for every $t \in [0,T]$. Because of the uniqueness of the solution of the energetic formulation $(\mathbf{S}^\gamma)\&(\mathbf{E}^\gamma)$ of the rate-independent system $(\mathbf{Q}_\gamma, \mathbf{E}_\gamma, \mathbf{R}_\gamma)$, it has to be $U_1^*(t) = U_1(t), Z_1^*(t) = Z_1(t)$ and thereby $\mathbf{q}^*(t) = \mathbf{q}(t)$ for every $t \in [0,T]$.

"$\Leftarrow$" Since $\mathbf{R}_\gamma(z_0(t), Z_1(t)) \equiv \mathcal{R}_\gamma(z_0(t))$ this follows immediately from lemma 5.2. $\square$

Remark : Since in the case of $\gamma = 0$ the spaces $\mathcal{Q}_0$ and $\mathcal{Q}$ coincide, $\mathcal{E}_0$ and $\mathcal{R}_0$ are the classical Γ-limits of the sequence $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ and $(\mathcal{R}_\varepsilon)_{\varepsilon>0}$, respectively, according to proposition 4.4 and lemma 5.2.

Furthermore, we obtain that the solution of the energetic formulation $(\mathrm{S}_\varepsilon)\&(\mathrm{E}_\varepsilon)$ converges strongly in $\mathcal{Q}_\gamma$ to the solution of the energetic formulation $(\mathrm{S}^\gamma)\&(\mathrm{E}^\gamma)$ for $\varepsilon \to 0$ according to theorem 4.1, if we assume $q^0_\varepsilon \to q^0$ in $\mathcal{Q}_\gamma$ for $\varepsilon \to 0$.

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