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Variational approach in weighted Sobolev spaces to scattering by unbounded rough surfaces

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Abstract. We consider the problem of scattering of time harmonic acoustic waves by an unbounded sound soft surface which is assumed to lie within a finite distance of some plane. The paper is concerned with the study of an equivalent variational formulation of this problem set in a scale of weighted Sobolev spaces. We prove well-posedness of this variational formulation in an energy space with weights which extends previous results in the unweighted setting (Chandler-Wilde & Monk, SIAM J Math Anal 37 (2005), 598-618) to more general inhomogeneous terms in the Helmholtz equation. In particular, in the two-dimensional case, our approach covers the problem of plane wave incidence, whereas in the 3D case incident spherical and cylindrical waves can be treated. As a further application of our results we analyse a finite section type approximation, whereby the variational problem posed on an infinite layer is approximated by a variational problem on a bounded region.

1. Introduction

This paper is concerned with the analysis of problems of scattering by unbounded surfaces, in particular with what are termed *rough surface scattering problems* in the engineering literature. By the phrase *rough surface*, we will denote throughout a surface which is a (usually non-local) perturbation of an infinite plane surface such that the surface lies within a finite distance of the original plane. Rough surface scattering problems in this sense arise frequently in applications, for example in modeling acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces, and have been studied extensively in the physics and engineering literature from the points of view of developing effective numerical algorithms or asymptotic or statistical approximation methods (see e.g. Ogilvy [30], Voronovich [39], Saillard & Sentenac [32], Warnick & Chew [40], DeSanto [18], and Elfouhaily and Guerin [19]).

Despite this extensive practical interest, relatively little mathematical analysis of these problems has been carried out. In particular, only in the last four years have the first results been obtained establishing well-posedness for three-dimensional rough surface scattering problems, using integral equation methods (see Chandler-Wilde, Heinemeyer & Potthast [13, 14], Thomas [36]) or variational formulations (see Chandler-Wilde, Monk & Thomas [11, 15], Thomas [36]). The variational approach proposed in [11] for the sound soft acoustic problem leads to explicit bounds on the solution in terms of the data and applies to a rather general class of non-smooth unbounded surfaces. The approach in [11] is extended to more general acoustic scattering problems in [36], including problems of scattering by impedance surfaces and by inhomogeneous layers (and see [15]).

In contrast to the general case of a non-locally perturbed plane surface, there is already a vast literature on the variational approach applied to periodic diffractive structures (diffraction gratings) or to locally perturbed plane scatterers; see, e.g., Kirsch [25], Bonnet-Bendhia & Starling [6], Elschner & Schmidt [20], Bao & Dobson [5], Elschner, Hinder, Penzel & Schmidt [21], Ammari, Bao & Wood [1], and Elschner & Yamamoto [22]. The assumption made in all of these papers leads to a variational problem over a bounded region, so that compact imbedding arguments can be applied and the sesquilinear form that arises satisfies a Gårding inequality which simplifies the mathematical arguments considerably compared to the cases studied in [11], [15] and [36].

In this paper we will rigorously analyze time harmonic acoustic scattering, seeking to solve the Helmholtz equation with wave number $k > 0$,

$$\Delta u + k^2 u = g\,,$$

in the perturbed half-plane or half-space $D \subset \mathbb{R}^n$, $n = 2, 3$. The scattering surface $\Gamma := \partial D$ is assumed to lie within a finite distance of some plane; for example it may be the graph of an arbitrary bounded continuous function. While the methods we use and results we derive can be adapted to other boundary conditions, to keep things specific and to make use of earlier results [11, 15], we will restrict our attention to the simplest case when a homogeneous Dirichlet boundary condition $u = 0$ holds on Γ. The problem formulation is completed by a suitable radiation condition, expressing that the wave scattered by the surface must radiate away from the surface.

This paper is closest in its results to Chandler-Wilde & Monk [11], who studied the same Dirichlet scattering problem. Following [11], we introduce an equivalent variational formulation of this problem set in an infinite layer S_0 of finite thickness between the surface Γ and some plane Γ_0 lying above that surface on which the solution is required to satisfy a non-local boundary condition involving the exact Dirichlet to Neumann map T. This condition is often used in a formal manner in the rough surface scattering literature (e.g. [18]), that, above the rough surface Γ and the support of g, the solution can be represented in integral form as a superposition of upward traveling and evanescent plane waves. This radiation condition is equivalent to the upward propagating radiation condition proposed for two-dimensional rough surface scattering problems in [10], and has recently been analyzed carefully in the 2D case by Arens and Hohage [4]. Arens and Hohage also propose a further equivalent radiation condition (a 'pole condition').

In Sections 2 and 3 we formulate the boundary value problem and its variational formulation precisely, and give the details about our assumptions on D and about the radiation condition we impose. Section 3 is also devoted to new continuity properties of the DtN map T in weighted Sobolev spaces on Γ_0.

In Section 4 we study the well-posedness of the variational formulation in an energy space with weights which decay or increase polynomially as a function of radial distance within the layer S_0. Our main result, Theorem 4.1, is to show, for a range of increasing and decreasing weights, that the problem is well-posed in the weighted space setting if and only if it is well-posed in the unweighted space setting. This result depends on technical estimates of the commutator of the DtN map T and the operation of multiplication by the weight function; see Theorem 3.1. Combining this result with previous results on well-posedness in the unweighted setting for sound soft scattering [11], we are able to show well-posedness in a weighted space setting.

In Section 5, to illustrate the importance of these results, we make two applications. First, in the two-dimensional case, we prove existence of solution to the problem of plane wave scattering by an unbounded sound soft surface, extending previous results derived for the case when the boundary is the graph of a sufficiently smooth function [17] to much more general surface profiles. We note that, even in the well-studied case when the boundary is periodic (is a diffraction grating) the uniqueness result we obtain is a significant extension of the results known to date [22]. We briefly discuss why our methods do not extend to the case of plane wave incidence in the three-dimensional case (indeed, why this problem may not be well-posed), and apply our results to prove existence of solution to the three-dimensional problems of scattering of incident spherical and cylindrical waves. In the second application of our results we analyse approximation of the variational problem in the infinite layer S_0 by a variational problem in a bounded region (to which finite element methods can then be applied), this bounded region coinciding with the original layer S_0 inside a ball of radius R. We prove stability and convergence of this approximation procedure and use our weighted space results to prove error estimates as a function of R.

The final Section 6 is concerned with the proof of our crucial commutator estimates stated in Theorem 3.1. Note that the DtN map T is a pseudodifferental operator on $\mathbb{R}^{n-1}$ with a non-smooth symbol, so that the standard calculus of pseudifferential operators acting in weighted Sobolev spaces (see, e.g., [31]) is not sufficient to obtain the result.

2. The Boundary Value Problem And Radiation Conditions

Let $\mathrm{x} = (x, x_n) \in \mathbb{R}^n$ $(n = 2, 3)$ with $x \in \mathbb{R}^{n-1}$, and let $D \subset \mathbb{R}^n$ be an unbounded domain such that, for some $b < 0$,

$$U_0 \subset D \subset U_b\,, \quad U_b := \{\mathrm{x} : x_n > b\}\,. \tag{2.1}$$

For $h \in \mathbb{R}$, let $\Gamma_h := \{\mathrm{x} : x_n = h\}$ and $S_h := D\backslash\overline{U}_h$. The variational problem will be posed on the open set S_0 which lies between the rough surface $\Gamma = \partial D$ and the plane (or line) Γ_0. In addition to (2.1) we will assume that D satisfies the condition that

$$\mathrm{x} \in D \Rightarrow \mathrm{x} + se_n \in D\,, \quad \text{for all } s > 0\,, \tag{2.2}$$

where e_n denotes the unit vector in direction x_n. Condition (2.2) is satisfied if Γ is the graph of a continuous function, but also allows more general domains.

We now introduce weighted L^2 and Sobolev spaces. For $\varrho \in \mathbb{R}$, $l \in \mathbb{N}$ and a domain $G \subset \mathbb{R}^n$, define the Hilbert spaces

$$L^2_\varrho(G) := (1 + x^2)^{-\varrho/2} L^2(G)\,, \quad H^l_\varrho(G) := (1 + x^2)^{-\varrho/2} H^l(G)\,,$$

equipped with the corresponding canonical norm and scalar product. The space $V_{h,\varrho}$ is then defined, for $h \geq 0$, as the closure of $\{u|_{S_h} : u \in C_0^\infty(D)\}$ in the norm

$$\|u\|_{V_{h,\varrho}} = \|u\|_{H^1_\varrho(S_h)} = \left(\int_{S_h} \left(\left|(1 + x^2)^{\varrho/2} u\right|^2 + \left|\nabla((1 + x^2)^{\varrho/2} u)\right|^2\right) d\mathrm{x}\right)^{1/2}. \tag{2.3}$$

We set $V_{0,\varrho} = V_\varrho$ in the following, which is the energy space for our variational problem. Moreover, we introduce

$$H^s_\varrho(\Gamma_h) := (1 + x^2)^{-\varrho/2} H^s(\Gamma_h)\,, \quad s, \varrho \in \mathbb{R}\,,$$

where $H^s(\Gamma_h)$ is identified with the Sobolev space $H^s(\mathbb{R}^{n-1})$ with norm

$$\|v\|_{H^s(\mathbb{R}^{n-1})} = \left(\int_{\mathbb{R}^{n-1}} (1 + \xi^2)^s |Fv|^2 d\xi\right)^{1/2}.$$

Here Fv denotes the Fourier transform of v defined by

$$Fv(\xi) = (2\pi)^{-(n-1)/2} \int_{R^{n-1}} \exp(-ix \cdot \xi) v(x)\, dx\,,\ \xi \in \mathbb{R}^{n-1}\,,$$

with the inverse transform given by

$$F^{-1}w(x) = (2\pi)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} \exp(ix \cdot \xi) w(\xi)\, d\xi\,,\ x \in \mathbb{R}^{n-1}\,.$$

Note that F is an isometry of $L^2(\mathbb{R}^{n-1})$ onto itself and also an isometry of $L^2_\varrho(\mathbb{R}^{n-1})$ onto $H^\varrho(\mathbb{R}^{n-1})$ since

$$\|Fu\|^2_{H^\varrho} = \int_{\mathbb{R}^{n-1}} (1 + x^2)^\varrho |F^2 u|^2 dx = \int_{\mathbb{R}^{n-1}} (1 + x^2)^\varrho |u|^2 dx\,,\ u \in C_0^\infty(\mathbb{R}^{n-1})\,.$$

More generally, F is an isomorphism of $H^s_\varrho(\mathbb{R}^{n-1})$ onto $H^\varrho_s(\mathbb{R}^{n-1})$ for all $s, \varrho \in \mathbb{R}$; see [31] and [38]. From time to time we will make use of the following lemma.

Lemma 2.1. *Suppose $h \geq 0$ and $\varrho \in \mathbb{R}$. For $u \in V_{h,\varrho}$ let*

$$\|u\|' := \left(\int_{S_h} (1+x^2)^\varrho \left(|u|^2 + |\nabla u|^2 \right) d\mathrm{x} \right)^{1/2} .$$

Then $\|\cdot\|'$ is a norm on $V_{h,\varrho}$ that is equivalent to the norm $\|\cdot\|_{V_{h,\varrho}}$; precisely, for some constants C_1 and C_2 depending only on ϱ, $\|u\|_{V_{h,\varrho}} \leq C_1 \|u\|' \leq C_2 \|u\|_{V_{h,\varrho}}$, for all $u \in V_{h,\varrho}$, and all $h \geq 0$.

We now state our boundary value problem, formulated in a weighted space setting. As indicated in the introduction, it is the study of this problem in weighted spaces, and application of the new results this gives rise to, which is the object of this paper. Indeed a main result of this paper will be to show that solvability of this boundary value problem in weighted spaces, for the range of ϱ indicated, is a consequence of its solvability without weights, i.e. for the case $\varrho = 0$. (And this is useful since solvability for the simpler case without weights is already established in [11].) After stating the boundary value problem we will comment in the remainder of this section on how the radiation condition is to be understood, in particular when $\varrho < 0$. We will also comment on the restriction on the range of ϱ ($|\varrho| < 1$) in the statement of the boundary value problem, explaining why this range is natural and sharp. Precisely, we will point out that the radiation condition (2.4) does not make sense for all $u \in H^{1/2}_\varrho$ when $\varrho \leq -1$, and will show that the boundary value problem is not, in general, solvable for $\varrho \geq 1$.

The boundary value problem (BVP). Given $g \in L^2_\varrho(D)$, with $|\varrho| < 1$ and supp $g \subset \bar{S}_0$, find $u \in H^1_{loc}(D)$ such that $u|_{S_h} \in V_{h,\varrho}$ for every $h > 0$,

$$(\Delta + k^2)u = g \quad \text{in} \quad D \,,$$

in a distributional sense, and the following radiation condition is satisfied:

$$\begin{aligned} u(\mathrm{x}) &= F^{-1} \exp\left(-x_n \sqrt{\xi^2 - k^2} \right) F u_0(\xi) \\ &= (2\pi)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} \exp\left(-x_n \sqrt{\xi^2 - k^2} + i x \cdot \xi \right) F u_0(\xi) \, d\xi \,, \ \mathrm{x} \in U_0 \,, \end{aligned} \tag{2.4}$$

where $u_0 = u|_{\Gamma_0} \in H^{1/2}_\varrho(\Gamma_0)$ (from the trace theorem) and $\sqrt{\xi^2 - k^2} = -i\sqrt{k^2 - \xi^2}$ when $|\xi| < k$.

We explain in this paragraph and the next in what sense (2.4) is to be understood and why in the above formulation we restrict ϱ to the range $\varrho > -1$ (the restriction to $\varrho < 1$ is explained at the end of this section). For $\varrho \geq 0$ the integral (2.4) exists in the ordinary Lebesgue sense since $F u_0 \in L^2(\mathbb{R}^{n-1})$. Further, for $u_0 \in L^2(\Gamma_0)$, the radiation condition (2.4) can be written in the alternative form

$$u(\mathrm{x}) = 2 \int_{\Gamma_0} \frac{\partial \Phi(\mathrm{x}, \mathrm{y})}{\partial y_n} u(\mathrm{y}) \, ds(\mathrm{y}) = 2 \int_{\mathbb{R}^{n-1}} \frac{\partial \Phi(\mathrm{x}, \mathrm{y})}{\partial y_n} u_0(y) \, dy \,, \ \mathrm{x} \in U_0 \,, \tag{2.5}$$

where the fundamental solution Φ of the Helmholtz equation is given by

$$\Phi(\mathrm{x}, \mathrm{y}) := \begin{cases} \dfrac{1}{4\pi} \dfrac{\exp(ik|\mathrm{x} - \mathrm{y}|)}{|\mathrm{x} - \mathrm{y}|} & \text{if } n = 3 \\ \dfrac{i}{4} H_0^{(1)}(k|\mathrm{x} - \mathrm{y}|) & \text{if } n = 2 \,, \end{cases}$$

for $\mathrm{x} = (x, x_n), \mathrm{y} = (y, y_n) \in \mathbb{R}^n$, $\mathrm{x} \neq \mathrm{y}$. Here $H_0^{(1)}$ is the Hankel function of the first kind of order zero. In the case $n = 2$, (2.5) is just the upward propagating radiation condition (UPRC) proposed in [10], and we refer to [7, Chap. 5.1.1] for $n = 3$. For further discussion of the rationale for the radiation condition (2.4) and its relationship to other proposed radiation conditions for rough surface scattering we refer the reader to [11].

For $\varrho < 0$ we understand (2.4) by extending the mapping $u_0 \mapsto u(\mathrm{x})$, given by (2.4), to a bounded linear functional on $H^{1/2}_\varrho(\Gamma_0)$. The restriction $\varrho > -1$ arises precisely because this extension is possible only for the range $\varrho > -1$. To see this we observe that, since F is an isomorphism from $H^{1/2}_\varrho(\mathbb{R}^{n-1})$ to $H^{\varrho}_{1/2}(\mathbb{R}^{n-1})$, the mapping $u_0 \mapsto u(\mathrm{x})$ given by (2.4) extends to a bounded linear functional on $H^{1/2}_\varrho(\Gamma_0)$ for $\varrho < 0$ if and only if $f_\mathrm{x} \in H^{-\varrho}_{-1/2}(\mathbb{R}^{n-1})$, where $f_\mathrm{x}(\xi) := \exp\left(-x_n\sqrt{\xi^2 - k^2} + i x \cdot \xi\right)$. But this holds precisely for $\varrho > -1$; in fact (this can be deduced from (2.5) and Parseval's theorem) the Fourier transform of f_x is given by

$$F f_\mathrm{x}(y) = 2(2\pi)^{(n-1)/2} \frac{\partial \Phi(\mathrm{x}, \mathrm{y})}{\partial y_n}\Big|_{y_n=0},$$

and straightforward explicit calculations (see [10] for the case $n = 2$) yield that

$$|F f_\mathrm{x}(y)| \sim c_n x_n |y|^{-(1+n)/2} \tag{2.6}$$

as $|y| \to \infty$, where the constant c_n depends only on k and on the dimension n. From this one deduces that $F f_\mathrm{x} \in H^{-1/2}_{-\varrho}(\mathbb{R}^{n-1})$, so that $f_\mathrm{x} \in H^{-\varrho}_{-1/2}(\mathbb{R}^{n-1})$, if and only if $\varrho > -1$. For $\varrho > -1$ the extension of the mapping $u_0 \mapsto u(\mathrm{x})$, given by (2.4), to a bounded linear functional on $H^{1/2}_\varrho(\Gamma_0)$ is given explicitly by (2.5), the asymptotics (2.6) guaranteeing the existence of the integral (2.5). Thus (2.5) makes explicit the meaning of (2.4) in the case $-1 < \varrho < 0$.

Remark 2.2. We note (and this is important in our later applications) that there is a degree of arbitrariness in our radiation conditions (2.4) and (2.5). By this we mean that one could replace x_n in (2.4) by $x_n - c$, for any $c > 0$ (in fact for any $c \in \mathbb{R}$ such that supp $g \subset \bar{S}_c$ and $U_c \subset D$); the corresponding change to (2.5) would be to replace Γ_0 by Γ_c. We will show in Theorem 4.1 below that the boundary value problem is uniquely solvable. Clearly (by a simple shift in the vertical direction of the axes), one can deduce from this that the above boundary value problem with x_n replaced by $x_n - c$ in (2.4) is also uniquely solvable. We reassure the reader that these unique solutions are the same! This is demonstrated for the case $\varrho = 0$ in [11] and this result, together with the density of $L^2(S_0)$ in $L^2_\varrho(S_0)$ for $\varrho < 0$ and the stability results proved in Theorem 4.1, implies that the solutions are the same also for $|\varrho| < 1$.

We have explained above the restriction to $\varrho > -1$ in the boundary value problem formulation. We make the restriction $\varrho < 1$ because we cannot, in general, expect the boundary value problem to be solvable for $\varrho \geq 1$ (with the solution satisfying that $u|_{S_h} \in V_{h,\varrho}$ for every $h > 0$). To see this we consider the instructive case where D is a half-plane or half-space, i.e. $D = U_c$, for some $c \in (b, 0)$. Moreover, let us consider the case when g is compactly supported in a ball centred on some point $\mathrm{z} \in S_0$ of radius $\epsilon > 0$ sufficiently small so that the ball lies in $\bar{S}_0$, and further let us assume that $g(\mathrm{x}) \equiv 1$ inside the ball.

We have remarked already that it is shown in [11] that the boundary value problem is uniquely solvable when $\varrho = 0$. For the specific case we are considering the unique solution to this boundary value problem can be written down very explicitly. Let $G_c(\mathrm{x}, \mathrm{y})$ denote the Dirichlet Green's function for U_c, given by

$$G_c(\mathrm{x}, \mathrm{y}) := \Phi(\mathrm{x}, \mathrm{y}) - \Phi(\mathrm{x}, \mathrm{y}'_c),$$

where y'_c denotes the image of the point y in the plane $y_n = c$. Then the solution to the boundary value problem is

$$u(\mathrm{x}) = \begin{cases} k^{-2}(1 + Af(k|\mathrm{x}-\mathrm{z}|)), & |\mathrm{x}-\mathrm{z}| \le \epsilon, \\ Ck^{-2}G_c(\mathrm{x},\mathrm{z}), & |\mathrm{x}-\mathrm{z}| > \epsilon, \end{cases} \tag{2.7}$$

where the function f is defined by

$$f(r) := \begin{cases} J_0(r), & n = 2, \\ \dfrac{\sin r}{r}, & n = 3, \end{cases}$$

J_0 is the Bessel function of the first kind of order zero, and the constants A and C are chosen to ensure C^1 continuity of u across the boundary of the ball; for example, in 3D,

$$A = Q^{-1}(i\kappa - 1) \text{ and } C = 4\pi Q^{-1}\exp(-i\kappa)(\kappa\cos(\kappa) - 1),$$

where $Q = \kappa\cos(\kappa) - 1 + \sin(\kappa)(1 - i\kappa)$ and $\kappa = k\epsilon$. It is easy to check that this expression does satisfy $\Delta u + k^2 u = g$ in $D = U_c$, and that $u|_{S_h} \in V_{h,\varrho}$, for all $h > 0$, if $\rho = 0$ (indeed for all $\varrho < 1$) follows from the asymptotics (2.8) below. To see that u, given by (2.7), does satisfy the boundary value problem it remains to check that u satisfies the radiation condition; to do this one can show that u satisfies the form (2.5) of the radiation condition by applications of Green's theorem to $G_0(\mathrm{x},\cdot)$ and u in U_0.

A first observation is that this example demonstrates that, if g is chosen carefully enough, then the above boundary value problem is solvable for all $\varrho \in \mathbb{R}$. For certainly it is true in this example that g, being compactly supported, satisfies $g \in L^2_\varrho(D)$ for all $\varrho \in \mathbb{R}$ and, if ϵ is chosen so that $C = 0$, then u is also compactly supported and so $u|_{S_h} \in V_{h,\varrho}$, for all $h > 0$ and $\varrho \in \mathbb{R}$. But the example, slightly more subtly, also illustrates that, in general, even if g is compactly supported and so satisfies $g \in L^2_\varrho(D)$ for all $\varrho > 0$, one cannot expect that $u|_{S_h} \in V_{h,\varrho}$ for any $h > 0$ and $\varrho \ge 1$, since $u|_{S_0} \notin V_{0,\varrho}$ for $\varrho = 1$ if $C \neq 0$. To see that this is true it is enough to examine the asymptotics of $G(\mathrm{x},\mathrm{z})$ as $\mathrm{x} \to \infty$ in S_0. From [12, equation (4.2)] in the 2D case, and by simple direct calculations in the 3D case, we see that (cf. (2.6))

$$|G(\mathrm{x},\mathrm{z})| \sim c'_n k^{4-n}(x_n - c)(z_n - c)(k|\mathrm{x}|)^{-(1+n)/2}, \text{ as } |\mathrm{x}| \to \infty, \tag{2.8}$$

uniformly in $\mathrm{x} \in S_0$, for some constant $c'_n > 0$ depending only on n. From these asymptotics it is an easy calculation to see that $u|_{S_0} \in L^2_\varrho(S_0)$ for $\varrho < 1$ but not for $\varrho = 1$, so that $u|_{S_0} \notin V_{0,1}$. This example explains why the boundary value problem is not, in general, solvable in the case $\varrho \ge 1$.

3. The Dirichlet to Neumann Map and Variational Formulation

We now consider a variational formulation in weighted Sobolev spaces of the above boundary value problem, which involves the Dirichlet-to-Neumann operator on the artifical boundary Γ_0. As in [11] for $\varrho = 0$, there exist continuous trace operators

$$\gamma_- : V_\varrho \to H^{1/2}_\varrho(\Gamma_0)\,,\ \gamma_+ : H^1_\varrho(U_0 \backslash \bar{U}_h) \to H^{1/2}_\varrho(\Gamma_0)\,,\ h > 0\,.$$

Moreover, if $u_0 \in C_0^\infty(\Gamma_0)$ and u is given by (2.4), then

$$\left.\frac{\partial u}{\partial x_n}\right|_{\Gamma_0} = -T\gamma_+ u\,,$$

where the Dirichlet-to-Neumann map T is given by the pseudodifferential operator

$$Tv(x) := F^{-1}t(\xi)Fv(\xi)\,,\ t(\xi) := \sqrt{\xi^2 - k^2}\,. \tag{3.1}$$

Note that the symbol t of T is not smooth which makes the study of (3.1) in weighted Sobolev spaces more complicated than in the case $\varrho = 0$ (which was treated in [11]). The following commutator estimate is crucial for our analysis and its proof is carried out in Section 6.

Theorem 3.1. *Consider the commutator*

$$C := T - (a^2 + x^2)^{\varrho/2} T (a^2 + x^2)^{-\varrho/2} \cdot \tag{3.2}$$

with parameter $a > 0$. Then, for $ka \geq 1$ and $|\varrho| < 1$, the norm of C on $L^2(\mathbb{R}^{n-1})$ is bounded by $c(\varrho)\sqrt{k/a}$.

Here and in the following $c(\varrho)$ denotes a positive constant which only depends on ϱ. We remark (cf. the comments at the end of section 2) that the range $|\varrho| < 1$ in this theorem is optimal, i.e. this result does not hold for $\varrho = \pm 1$. This follows in part from the duality exhibited between positive and negative values of ϱ in the proof of Theorem 4.1 in section 6 below, which shows that the statement in this theorem holds for $\varrho = -1$ iff it holds for $\varrho = 1$. Further, if the above theorem were to hold for $\rho = 1$ then the proof of Theorem 4.1 below would extend to the case $\rho = 1$, which would contradict the example of a solution of the boundary value problem with $g \in L^2_1(D)$ but $u_0 \notin V_{0,1}$ at the end of section 2.

Sometimes the following weaker version of Theorem 3.1 is sufficient, the proof of which is analogous but simpler.

Lemma 3.2. *For fixed $k > 0$ and $a = 1$, the norm of* (3.2) *on $L^2(\mathbb{R}^{n-1})$ is bounded by some constant $c(\varrho)$ for $|\varrho| < 1$.*

The following lemma describes the continuity properties of T.

Lemma 3.3. *(i) For any $s \in \mathbb{R}$, $\quad T : H^s(\mathbb{R}^{n-1}) \to H^{s-1}(\mathbb{R}^{n-1})$ is bounded.*

(ii) For $|\varrho| < 1$, $0 \leq s \leq 1$, $\quad T : H^s_\varrho(\mathbb{R}^{n-1}) \to H^{s-1}_\varrho(\mathbb{R}^{n-1})$ is bounded.

Proof. (i) follows by taking Fourier transform and using the estimate $|t(\xi)| \leq c(1+\xi^2)^{1/2}$ on $\mathbb{R}^{n-1}$; see also [11]. To prove (ii) for $\varrho \neq 0$, we apply the commutator estimate of Lemma 3.2 to get the continuity

$$\tilde{C} := (1 + x^2)^{-\varrho/2} T - T(1 + x^2)^{-\varrho/2} \cdot : L^2(\mathbb{R}^{n-1}) \to L^2_\varrho(\mathbb{R}^{n-1}),$$

and by the continuous imbeddings $H^s_\varrho \subset L^2_\varrho \subset H^{s-1}_\varrho$, the operator $\tilde{C} : H^s(\mathbb{R}^{n-1}) \to H^{s-1}_\varrho(\mathbb{R}^{n-1})$ is bounded. Moreover, by (i), $(1 + x^2)^{-\varrho/2} T : H^s(\mathbb{R}^{n-1}) \to H^{s-1}_\varrho(\mathbb{R}^{n-1})$ is bounded, so that $T(1 + x^2)^{-\varrho/2} \cdot$ is bounded there, too. This implies the result. $\square$

To state the variational formulation of (BVP), we use the notation

$$(u, v) := \int_{S_0} u\bar{v} \, d\mathrm{x},$$

and define the continuous sesquilinear form $B : V_\varrho \times V_{-\varrho} \to \mathbb{C}$ by

$$B(u, v) := (\nabla u, \nabla v) - k^2(u, v) + \int_{\Gamma_0} \gamma_- \bar{v} \; T \gamma_- u \, ds(\mathrm{x}). \tag{3.3}$$

Note that this sesquilinear form is well-defined and continuous on $V_\varrho \times V_{-\varrho}$ for $|\varrho| < 1$ as a consequence of Lemma 3.3 with $s = 1/2$.

The variational formulation (V). Given $g \in L^2_\varrho(S_0)$, $|\varrho| < 1$, find $u \in V_\varrho$ such that

$$B(u, v) = -(g, v), \quad \forall v \in V_{-\varrho}. \tag{3.4}$$

As in [11], the equivalence of (BVP) and (V) follows from the following weighted version of Lemma 3.2 in that paper.

Lemma 3.4. *Let* $|\varrho| < 1$.

(i) If (2.4) *holds with* $u_0 \in H^{1/2}_\varrho(\Gamma_0)$*, then* $u \in H^1_\varrho(U_0 \backslash \bar{U}_h)$ *for every* $h > 0$.

(ii) Furthermore, we have $(\Delta + k^2)u = 0$ *in* U_0*,* $\gamma_+ u = u_0$*, and*

$$\int_{\Gamma_0} \bar{v}\, T\gamma_+ u\, dx + k^2 \int_{U_0} u\bar{v}\, \mathrm{dx} - \int_{U_0} \nabla u \cdot \nabla \bar{v}\, \mathrm{dx} = 0 \quad \forall v \in C_0^\infty(D)\,.$$

As in [11] (for $\varrho = 0$), assertion (ii) is a consequence of (i). We will prove Lemma 3.4 (i) in Section 6 applying our commutator estimates. Following [11], Lemma 3.4 then implies

Lemma 3.5. *If* u *is a solution of* (BVP)*, then* $u|_{S_0}$ *satisfies the variational problem* (V)*. Conversely, let* w *be a solution of* (V)*. If we set* $u = w$ *in* S_0 *and define* u *in* U_0 *to be the right-hand side of* (2.4) *with* $u_0 = \gamma_- w$*, and extend the definition of* g *to* D *by setting* $g = 0$ *in* U_0*, then* u *is a solution of* (BVP).

Remark 3.6. We note that the equivalence of (BVP) and (V) stated in Lemma 3.5 holds whenever (2.1) holds. In particular, the proof is not dependent on (2.2). Further, we note that there is no requirement that $\Gamma_0 \subset \overline{S_0}$; it may be the case, for example, that S_0 is a bounded open set, which need not necessarily be connected. In the case that $\Gamma_0 \not\subset \overline{S_0}$, the action of the trace operator γ_- on V_ϱ in (3.3) should be understood by extending the definition of the functions in V_ϱ by zero to the whole of the strip $U_b \setminus \bar{U}_0$, from S_0 which is their initial domain of definition. This implies that, for $u \in V_\varrho$, $\gamma_- u = 0$ on $\Gamma_0 \setminus \overline{S_0}$.

4. Existence and uniqueness results in weighted spaces

We shall establish that the boundary value problem (BVP) and the equivalent variational problem (V) are uniquely solvable by using the result of [11, Thm. 4.1] in the non-weighted case and a perturbation argument based on the commutator estimates. By Lemma 3.3 (ii) the form (3.3) generates a continuous linear operator $\mathcal{B}_\varrho : V_\varrho \to V^*_{-\varrho}$, where $V^*_{-\varrho}$ is the dual of $V_{-\varrho}$ (the space of continous anti-linear functionals on $V_{-\varrho}$) with respect to the scalar product $(\cdot,\cdot)$ in $L^2(S_0)$.

Theorem 4.1. *Under the assumptions* (2.1)*,* (2.2) *and* $|\varrho| < 1$*, the operator* $\mathcal{B}_\varrho$ *is invertible. In particular,* (V) *and hence* (BVP) *have a unique solution for all* $g \in L^2_\varrho(S_0)$.

For $\varrho = 0$, this was proved in [11, Thm. 4.1], using a Rellich identity and the generalized Lax-Milgram theorem. Moreover, an explicit bound for the norm of $\mathcal{B}_0^{-1}$ in terms of k and $|b|$ was given there (using wave number dependent Sobolev norms on S_0 and Γ_0; see Remark 4.2 below). A significant idea in the proof of this theorem for the case $\varrho \neq 0$ is the use of a perturbation argument, reducing the proof of invertibility for $\varrho \neq 0$ to that for $\varrho = 0$. This idea has been used previously to study rough surface scattering in 2D ($n = 2$) in weighted spaces of continuous functions via integral equation methods in [2, 3]. A commutator result for boundary integral operators (cf. Theorem 4.1) plays in [2] an important role, but the idea there is to prove that the commutator is compact, or at least preserves Fredholmness, rather than to show the stronger and more constructive result that the commutator is sufficiently small in norm. (And it should be noted that the proof of properties of the commutator in [2] is very much more straightforward than the proof of Theorem 3.1, not least because the kernels of the boundary integral operators in [2] are absolutely integrable.) A key ingredient in our commutator estimate, Theorem 3.1, is the parameter a in the weight facto r. We note that the idea of introducing such a parameter into commutator estimates goes back at least to Shubin [33, Theorem 5.3], though we seem to be the first to use this idea in an estimate of the commutator of a convolution operator with multiplication by a weight function in the case when the convolution operator has a non-smooth symbol.

Proof for $\varrho \neq 0$. Introduce equivalent norms $\|u\|_{L^2_\varrho} = \|(a^2+x^2)^{\varrho/2}u\|_{L^2}$ with parameter $a > 0$ and modify the norm (2.3) in V_ϱ correspondingly. We will choose $a > 0$ sufficiently large, and set, for $u \in V_\varrho$, $\varphi \in V_{-\varrho}$,

$$v = (a^2 + x^2)^{\varrho/2}u \in V_0 \,, \quad \psi = (a^2 + x^2)^{-\varrho/2}\varphi \in V_0 \,.$$

Then we obtain from (3.3)

$$B(u, \varphi) = B(v, \psi) + K(v, \psi) \,, \tag{4.1}$$

where $K = K_1 + K_2$ with

$$\begin{aligned} K_1(v, \psi) &= (\nabla(a^2 + x^2)^{-\varrho/2}v, \nabla(a^2 + x^2)^{\varrho/2}\psi) - (\nabla v, \nabla\psi) \\ &= (v\nabla(a^2 + x^2)^{-\varrho/2}, \psi\nabla(a^2 + x^2)^{\varrho/2} + (\nabla v, \psi(a^2 + x^2)^{-\varrho/2}\nabla(a^2 + x^2)^{\varrho/2}) \\ &+ (v(a^2 + x^2)^{\varrho/2}\nabla(a^2 + x^2)^{-\varrho/2}, \nabla\psi) \end{aligned} \tag{4.2}$$

and

$$K_2(v, \psi) = \int_{\Gamma_0} \left\{(a^2 + x^2)^{\varrho/2}\bar{\psi}\, T(a^2 + x^2)^{-\varrho/2}v - \bar{\psi}\, Tv\right\}\, dx = -\int_{\Gamma_0} \bar{\psi}\, Cv\, dx \tag{4.3}$$

with C defined in (3.2). For (4.2) we obtain the estimate

$$\begin{aligned} |K_1(v, \psi)| &\le \left(\frac{|\varrho|}{2a}\right)^2 \|v\|_{L^2(S_0)}\|\psi\|_{L^2(S_0)} + \left(\frac{|\varrho|}{2a}\right) \big(\|\nabla v\|_{L^2(S_0)}\|\psi\|_{L^2(S_0)} \\ &+ \|v\|_{L^2(S_0)}\|\nabla\psi\|_{L^2(S_0)}\big) \le \frac{|\varrho|}{2a} \max\left(1,\, \frac{|\varrho|}{2a}\right) \|v\|_{V_0}\|\psi\|_{V_0} \,. \end{aligned} \tag{4.4}$$

Note that

$$\sup_{S_0} \left|\nabla(a^2 + x^2)^{|\varrho|/2}\right|(a^2 + x^2)^{-|\varrho|/2} \le |\varrho|/2a$$

and compare [15, Sec. 4].

Applying Theorem 3.1 to (4.3), we get

$$|K_2(v, \psi)| \le c(\varrho)\sqrt{k/a}\, \|\gamma_- v\|_{L^2(\Gamma_0)}\, \|\gamma_- \psi\|_{L^2(\Gamma_0)} \,, \tag{4.5}$$

and since $\|\gamma_- v\|_{L^2(\Gamma_0)} \le \|\gamma_- v\|_{H^{1/2}(\Gamma_0)} \le c\|v\|_{V_0}$, (4.4) and (4.5) then imply that the norm of the operator $\mathcal{K}_0 : V_0 \to V_0^*$ generated by the form K tends to zero as $a \to \infty$. Finally, from (4.1) we have

$$\mathcal{B}_\varrho = (a^2 + x^2)^{-\varrho/2}(\mathcal{B}_0 + \mathcal{K}_0)(a^2 + x^2)^{\varrho/2} \cdot \,. \tag{4.6}$$

Since $\mathcal{B}_0$ is invertible, this operator is invertible provided that a is sufficiently large. $\square$

Remark 4.2. Introducing norms dependent on the wave number k and/or the parameter $a > 0$, defined by

$$
\begin{aligned}
|v|_{H^{1/2}(\Gamma_0)} &= \left(\int_{\mathbb{R}^{n-1}} (k^2 + \xi^2) |Fv|^2 \, d\xi \right)^{1/2}, \\
|u|_{V_\varrho} &= \left(\int_{S_0} \left(k^2 |(a^2 + x^2)^{\varrho/2} u|^2 + |\nabla \left((a^2 + x^2)^{\varrho/2} u \right)|^2 \right) dx \right)^{1/2}, \\
|g|_{L^2_\varrho(S_h)} &= \left(\int_{S_h} \left(|(a^2 + x^2)^{\varrho/2} g|^2 \, dx \right)^{1/2}
\end{aligned}
$$

on $H^{1/2}(\Gamma_0)$, V_ϱ, and $L^2_\varrho(S_h)$, respectively, we can obtain a bound of the norm $\|\mathcal{B}_\varrho^{-1}\|$ of $\mathcal{B}_\varrho^{-1} : V^*_{-\varrho} \to V_\varrho$ in terms of $\kappa = k|b|$, provided that the parameter $a > 0$ is chosen sufficiently large. From (4.4) we see that

$$
|K_1(v, \psi)| \le \frac{|\varrho|}{2ka} \left(1 + \frac{|\varrho|}{2ka} \right) |v|_{V_0} |\psi|_{V_0},
$$

and since (cf. [11])

$$
\|\gamma_- v\|_{L^2(\Gamma_0)} \le k^{-1/2} |\gamma_- v|_{H^{1/2}(\Gamma_0)} \le k^{-1/2} |v|_{V_0},
$$

(4.5) implies that

$$
|K_2(v, \psi)| \le \frac{c(\varrho)}{\sqrt{ka}} |v|_{V_0} |\psi|_{V_0}.
$$

Thus we have, for $ka \ge 1$ and $|\varrho| < 1$,

$$
|K_0(v, \psi)| \le \left(\frac{|\varrho|}{2ka} \left(1 + \frac{|\varrho|}{2ka} \right) + \frac{c(\varrho)}{\sqrt{ka}} \right) |v|_{V_0} |\psi|_{V_0} \le \frac{|\varrho| + c(\varrho)}{\sqrt{ka}} |v|_{V_0} |\psi|_{V_0},
$$

so that $\|\mathcal{K}_0\| \le (|\varrho| + c(\varrho))/\sqrt{ka}$. Taking the bound

$$
\|\mathcal{B}_0^{-1}\| \le \gamma := 1 + \sqrt{2}\kappa(\kappa + 1)^2
$$

from [11, Thm. 4.1] and using (4.6), we obtain the norm estimate

$$
\|\mathcal{B}_\varrho^{-1}\| \le 2\gamma, \tag{4.7}
$$

provided that

$$
\|\mathcal{K}_0\| \le (|\varrho| + c(\varrho))/\sqrt{ka} \le \frac{1}{2\gamma} \le \frac{1}{2} \|\mathcal{B}_0^{-1}\|,
$$

which holds for $a \ge 4\gamma^2(|\varrho| + c(\varrho))^2/k$. Since (V) written in operator form is the equation $\mathcal{B}_\varrho u = \tilde{g}$, where $\tilde{g} \in V^*_{-\varrho}$ is defined by $\tilde{g}(v) = (g, v)$, $v \in V_{-\varrho}$, this implies that the solution u of (V) satisfies

$$
|u|_{V_\varrho} \le 2\gamma |\tilde{g}|_{V^*_{-\varrho}} \le 2\gamma k^{-1} |g|_{L^2_\varrho(S_0)}, \tag{4.8}
$$

provided $ka \ge \max(1, 4\gamma^2(|\varrho| + c(\varrho))^2)$.

5. Applications

5.1. **Plane Wave Incidence, Diffraction Gratings, and Other Scattering Problems.** As an application of Theorem 4.1, the problem of plane wave incidence in the 2D case ($n = 2$) can be treated. That is, it can be shown, in appropriate function spaces, that the scattering problem for plane wave incidence has exactly one solution in 2D (for a brief discussion of what goes wrong in the 3D case, see Remark 5.5 below, and see Remark 5.6 for details of 3D scattering problems which can be tackled by Theorem 4.1). The incident plane wave has the form

$$v^{in}(\mathrm{x}) = \exp(ik[\sin\theta\, x_1 - \cos\theta\, x_2])\,,$$

where θ is the angle of incidence, with $|\theta| < \pi/2$. In this problem we look for the total field $v = v^{sc} + v^{in}$, v^{sc} being the unknown scattered field, such that

$$(\Delta + k^2)v = 0 \ \text{ in } \ D\,, \quad v = 0 \ \text{ on } \ \Gamma\,, \tag{5.1}$$

and v^{sc} satisfies an appropriate radiation condition.

This 2D rough surface scattering problem with plane wave incidence has been treated before, by integral equation methods, in [17] where it is shown that there exists exactly one solution $v \in C^2(D) \cap C(\bar{D})$ such that v is bounded in S_h, for every $h > 0$, and v^{sc} satisfies the radiation condition in the form (2.5) (termed the upwards propagating radiation condition (UPRC) in [17]). However, the proof in [17] is only for the case where ∂D is the graph of a sufficiently smooth ($C^{1,1}$) function (this, or at least a restriction to Lyapunov surfaces, is an essential restriction due to the compactness arguments in the existence proofs in [17]). In this section we will establish unique existence of solution for much more general surfaces, with only the constraints (2.1) and (2.2) on ∂D that we impose throughout the paper.

To use the results of the previous section which are formulated in a Sobolev space setting, including the results of Remark 4.2 which are formulated in terms of wave number dependent norms, we will replace the assumption that v is bounded in S_h by an assumption that $v|_{S_h} \in V_h^\infty$, for all $h > 0$, where

$$V_h^\infty := \{w \in V_{h,-1} : \|w\|_{V_h^\infty} < \infty\}.$$

In this definition the (wave number dependent) norm $\|w\|_{V_h^\infty}$ on V_h^∞ is defined by

$$\|w\|_{V_h^\infty} := \sup_{m\in\mathbb{Z}} \|w|_{S_h^m}\|_{H^1(S_h^m)}\,, \quad S_h^m := \{\mathrm{x} = (x_1, x_2) \in S_h : Am < x_1 < A(m+1)\}\,,$$

$A > 0$ is a parameter at our disposal, and $\|\cdot\|_{H^1(S_h^m)}$ (cf. Remark 4.2) is a wave number dependent norm on $H^1(S_h^m)$, equivalent to the usual norm, defined by

$$\|w\|_{H^1(S_h^m)} := \left(\int_{S_h^m} \left(k^2|w|^2 + |\nabla w|^2\right) d\mathrm{x}\right)^{1/2}.$$

It is easy to see that V_h^∞ is a Banach space and that the different choices of $A > 0$ in the definition of $\|\cdot\|_{V_h^\infty}$ provide a family of equivalent norms on V_h^∞.

In terms of V_h^∞ our formulation of the plane wave scattering problem is as follows:

Plane Wave Scattering Problem (PW). Given $k > 0$ and $\theta \in (-\pi/2, \pi/2)$, find $v \in H^1_{\mathrm{loc}}(D) \cap C^2(D)$ such that $v|_{S_h} \in V_h^\infty$, for every $h > 0$,

$$\Delta v + k^2 v = 0 \quad \text{in } D,$$

and such that $v^{sc} := v - v^{in}$ satisfies the UPRC (2.5).

Our main result in this subsection is the following, which is an immediate consequence of Theorem 5.3 below.

Theorem 5.1. *The plane wave scattering problem has exactly one solution. Moreover, for every $h > 0$ there exists a constant $C_p > 0$, depending only on $\kappa = k|b|$, kh, and kA, such that $\|v|_{S_h}\|_{V_h^\infty} \leq C_p$.*

An interesting application of this result is to the much-studied diffraction grating case where D is periodic in the horizontal direction with some period $A > 0$, i.e.

$$\mathrm{x} = (x_1, x_2) \in D \Leftrightarrow (x_1 + A, x_2) \in D. \tag{5.2}$$

The standard formulation of the problem of plane wave scattering in the diffraction grating case is the following (e.g. [20, 22]). In this formulation the function space $V_h^{qp,\theta}$ denotes the set of functions in $H^1_{\mathrm{loc}}(S_h)$ that vanish on ∂D and are *quasi-periodic* in the x_1-direction with the same period and phase shift as the incident wave; more precisely

$$V_h^{qp,\theta} := \{w \in V_h^\infty : w((x_1 + A, x_2)) = w(\mathrm{x}) \exp(ikA \sin\theta),\ \mathrm{x} \in S_h\}.$$

Note that, for $w \in V_h^{qp,\theta}$, the norm $\|w\|_{V_h^\infty}$ simplifies to

$$\|w\|_{V_h^\infty} = \|w|_{S_h^0}\|_{H^1(S_h^0)}\,.$$

Diffraction Grating Plane Wave Scattering Problem (DGPW) (e.g. [22]). Given $k > 0$ and $\theta \in (-\pi/2, \pi/2)$, find $v \in H^1_{\mathrm{loc}}(D) \cap C^2(D)$ such that $v|_{S_h} \in V_h^{qp,\theta}$, for every $h > 0$,

$$\Delta v + k^2 v = 0 \quad \text{in } D,$$

and such that $v^{sc} := v - v^{in}$ satisfies the *Rayleigh expansion radiation condition*, that

$$v^{sc}(\mathrm{x}) = \sum_{m \in \mathbb{Z}} u_m \exp(ik[\alpha_m x_1 + \beta_m x_2]), \qquad \mathrm{x} \in U_0, \tag{5.3}$$

where the u_m are complex constants, $\alpha_m := \sin\theta + 2\pi m/(kA)$, and

$$\beta_m := \begin{cases} \sqrt{1 - \alpha_m^2}, & |\alpha_m| \leq 1, \\ i\sqrt{\alpha_m^2 - 1}, & |\alpha_m| > 1. \end{cases}$$

It is shown in [22] that (DGPW) has exactly one solution in the case that ∂D is the graph of an (A-periodic) Lipschitz function, by extending well-known arguments (see e.g. [25]), which apply in the case when ∂D is the graph of a smooth function, to the non-smooth Lipschitz case. The following corollary of Theorem 5.1 extends that result further to the much more general case where ∂D is only required to satisfy (2.1), (2.2), and (5.2).

Corollary 5.2. *Suppose that* (5.2) *holds. Then (DGPW) has exactly one solution, and this is the unique solution of (PW).*

Proof. Suppose that v satisfies (DGPW). Then it is clear that v satisfies (PW), provided we can show that v satisfying the Rayleigh expansion radiation condition implies that v satisfies the UPRC (2.5). But this is shown in [8]. Conversely, suppose that v satisfies (PW). Then

$$v((x_1 + A, x_2)) = v(\mathrm{x}) \exp(ikA \sin\theta), \quad \mathrm{x} \in D, \tag{5.4}$$

for otherwise w, defined by $w(\mathrm{x}) = v((x_1 + A, x_2))$, is another, distinct solution of (PW), which contradicts Theorem 5.1. Thus v satisfies (DGPW) provided that (5.3) holds. But, in the case that (5.4) is satisfied, it is shown in [8] that (2.5) can be written in the form (5.3). Thus v satisfies (PW) iff v satisfies (DGPW), and the corollary follows from Theorem 5.1. $\square$

We will prove Theorem 5.1 by applications of Theorem 4.1 and the observations in Remark 4.2. To apply these results we introduce the function $u \in H^1_{\mathrm{loc}}(D) \cap C^1(D)$, related to the solution v of (PW) by the formula

$$u(\mathrm{x}) := v^{in}(\mathrm{x})\chi(x_2) + v^{sc}(\mathrm{x}) = v(\mathrm{x}) + (\chi(x_2) - 1)v^{in}(\mathrm{x}), \quad \mathrm{x} \in D\,, \tag{5.5}$$

where $\chi \in C^1(\mathbb{R})$ is defined by

$$\chi(t) := \begin{cases} 1, & t < |b|, \\ \cos^2\left(\dfrac{\pi(t-|b|)}{2|b|}\right), & |b| \le t \le 2|b|, \\ 0, & t > |b|. \end{cases}$$

Moreover, define $g_P \in L^\infty(D)$ by

$$g_P(\mathrm{x}) = (\chi''(x_2) - 2ik\cos\theta\,\chi'(x_2))\,v^{in}(\mathrm{x}), \quad \mathrm{x} \in D, \tag{5.6}$$

so that $(\Delta + k^2)u = g_P$ in a distributional sense in D, supp $g_P \subset S_{-b}$, and $g_P \in L^2_\varrho(D)$ for $\varrho < -1/2$. Then Theorem 5.1 is an immediate consequence of the following result whose proof is largely applications of Theorem 4.1.

Theorem 5.3. *Define $g_P \in L^\infty(D)$ by* (5.6)*. Then v satisfies (PW) if and only if v and u are related by* (5.5) *and u satisfies the following boundary value problem: given $\varrho \in (-1, -1/2)$, find $u \in H^1_{loc}(D)$ such that $u|_{S_h} \in V_{h,\varrho}$, for every $h > 0$,*

$$(\Delta + k^2)u = g_P \quad in \quad D\,,$$

in a distributional sense, and u satisfies the radiation condition (2.4) *with $x_n (= x_2)$ replaced by $x_2 - b$. Moreover, for every $h > 0$ there exists a constant $C'_p > 0$, depending only on $\kappa = k|b|$, kh, and kA, such that $|u|_{S_h}|_{V_h^\infty} \le C'_p$.*

Proof. It is almost immediate from the observations immediately above the theorem that if v satisfies (PWSC) then u, defined by (5.5), satisfies the above boundary value problem. The only difficulty is to show the radiation condition. To see this we note that v^{sc} satisfies the radiation condition (2.5), from which it follows (see [9] and cf. Remark 2.2) that v^{sc} satisfies (2.5) with Γ_0 replaced with Γ_c, for all $c > 0$, in particular with $c = -b$. Since $u = v^{sc}$ in U_c it is immediate that v satisfies (2.5) with Γ_0 replaced by Γ_{-b}, which is equivalent (see Remark 2.2) to (2.4) with x_2 replaced by $x_2 + b$.

We next observe that it follows from Theorem 4.1 that the boundary value problem for u has exactly one solution (u satisfies exactly a boundary value problem of the form of Section 2 after vertical translation of the axes by a distance $|b|$). The theorem is thus proved if we can show that this solution satisfies that $u|_{S_h} \in V_h^\infty$, for every $h > 0$, and the bound $|u|_{S_h}|_{V_h^\infty} \le C'_p$.

To see this we make the following construction. Given $h > 0$ set $\tilde{h} := \max(|b|, h)$ and, for $j \in \mathbb{Z}$, define D_j, $g_j \in L^\infty(D_n)$, and $u_j \in H^1_{\mathrm{loc}}(D_j)$, by

$$D_j := \{(x_1 + jA, x_2 - \tilde{h}) : \mathrm{x} = (x_1, x_2) \in D\},$$

$$g_j(\mathrm{x}) := g_P((x_1 - jA, x_2 + \tilde{h}))\,, \quad u_j(\mathrm{x}) := u((x_1 - jA, x_2 + \tilde{h}))\,, \quad \mathrm{x} \in D_j\,.$$

Then u_j satisfies (BVP), with D replaced by D_j and g replaced by g_j in (BVP). (Since u satisfies (2.4) with x_2 replaced by $x_2 - b$, it follows in the first instance that u_j satisfies (2.4) with x_2 replaced with $x_2 - b - \tilde{h}$, but this implies that (2.4) holds as written, by Remark 2.2.) Thus Theorem 4.1 and Remark 4.2 apply for $\varrho \in (-1, -1/2)$. In particular, choosing ka sufficiently large (by Remark 4.2, how large is sufficient depends only the values of κ and ϱ), it follows from (4.8) that, for some constant $c_\kappa > 0$ depending only on $\tilde{\kappa} := k(|b| + \tilde{h})$

and ϱ, $|u_j|_{V_\varrho^j} \leq c_\kappa k^{-1} |g_j|_{L^2_\varrho(S_{0,j})}$, for $j \in \mathbb{Z}$, where $S_{0,j}$ and V_ϱ^j denote S_0 and V_ϱ, respectively, in the case $D = D_j$. Since, for some constant $c > 0$ depending only on ϱ,

$$|g_j|_{L^2_\varrho(S_{0,j})} \leq \left(\int_{S_{0,j}} (a^2 + x_1^2)^\varrho \, d\mathrm{x} \right)^{1/2} \|g_j\|_{L^\infty(S_{0,j})} \leq c|b|^{1/2} a^{\varrho+1/2} \|g_P\|_{L^\infty(S_{-b})} ,$$

we see that

$$|u_j|_{V_\varrho^j} \leq c'_\kappa |b|^{-1/2} a^{\varrho+1/2}, \quad j \in \mathbb{Z},$$

for some constant c'_κ depending only on $\tilde{\kappa}$ and ϱ. Careful calculations yield that, again for some constant $c > 0$ depending only on ϱ, and where $S^0_{0,j}$ denotes S^0_0 in the case $D = D_j$,

$$|u_j|_{S^0_{0,j}}|_{H^1(S^0_{0,j})} \leq c(a^2 + A^2)^{-\varrho/2} |u_j|_{V_\varrho^j}, \quad j \in \mathbb{Z},$$

Thus

$$|u|_{S_h}|_{V_h^\infty} \leq |u|_{S_{\tilde{h}}}|_{V_{\tilde{h}}^\infty} = \sup_{j \in \mathbb{Z}} |u_j|_{S^0_{0,j}}|_{H^1(S^0_{0,j})} \leq c''_\kappa \left(1 + A^2/a^2\right)^{|\varrho|/2} (ka)^{1/2},$$

where $c''_\kappa > 0$ depends only on $\tilde{\kappa}$ and ϱ. Now this bound holds for all ka sufficiently large, but how large is sufficient depends only on $\tilde{\kappa}$ and ϱ. Thus, choosing such a sufficiently large ka and choosing (say) $\varrho = -3/4$, we see that the theorem is proved. □

Remark 5.4. Theorem 5.3 explains how, in the 2D case, the plane wave scattering problem (PW) is equivalent to an instance of (BVP). This enables us to prove solvability of (PW) via the reformulation of (BVP) in variational form as (V). This is convenient for the purpose of proving Theorem 5.1 above, but, as a starting point for numerical computation (cf. Section 5.2 below), one would choose rather to reformulate (PW) directly in variational form. Arguing analogously to the proof of Lemma 3.5, in particular using Lemma 3.4 which applies to v^{sc}, one can show that v satisfies (PW) if and only if, for some $\varrho \in (-1, -1/2)$, $v|_{S_0}$ satisfies the variational problem: find $v \in V_\varrho$ such that

$$B(u, w) = G(w) , \quad \forall w \in V_{-\varrho} . \tag{5.7}$$

where $G \in V^*_{-\varrho}$ is defined by

$$G(w) = \int_{\Gamma_0} \gamma_- \bar{w} \left(\frac{\partial v^{in}}{\partial x_2} + T\gamma_- v^{in} \right) ds(\mathrm{x}) , \quad w \in V_{-\varrho} . \tag{5.8}$$

The restriction to the range $\varrho < -1/2$ arises since $v^{in} \in V_\varrho$ for $\varrho < -1/2$ but not for $\varrho = 1/2$. Having solved this variational problem to determine $v|_{S_0}$, v is determined throughout D through (2.5) satisfied by v^{sc}. Of course this variational formulation is well-posed, by Theorem 4.1.

Remark 5.5. The above results show that the problem of plane wave incidence is well posed in the 2D case. In the 3D case it seems to us likely that a solution to the problem of plane wave incidence does not exist for every choice of domain D satisfying (2.1) and (2.2). Certainly, the methods of argument above do not extend to the 3D case, for, in the 3D case, g_P in Theorem 5.3 is in $L^2_\varrho(D)$ only for $\varrho < -1$, and G given by (5.8) is in $V^*_{-\varrho}$ only for $\varrho < -1$, so that Theorem 4.1 does not apply. Further, even the formulation of the 3D plane wave problem appears problematic in 3D. Precisely, just as the radiation condition (2.4) does not extend to a bounded linear functional on $H^{1/2}_\varrho(\Gamma_0)$ for $\varrho < -1$, it does not extend to a bounded linear functional on $L^\infty(\Gamma_0)$ (which would require that the integral in (2.5) be absolutely convergent for every $u_0 \in L^\infty(\Gamma_0)$, which is true in 2D but not in 3D, as a consequence of the asymptotics (2.8)). Thus it is difficult to envisage that the radiation condition (2.4) or (2.5) can hold in general in the case of 3D plane wave incidence.

Remark 5.6. In the 3D case, the above approach does apply to prove unique existence of solution in weighted spaces in the cases of incoming spherical or cylindrical waves. That is, it applies to the cases, respectively, where

$$v^{in}(\mathrm{x}) := \Phi(\mathrm{x}, \mathrm{z}) = \frac{1}{4\pi} \frac{\exp(ik|\mathrm{x}-\mathrm{z}|)}{|\mathrm{x}-\mathrm{z}|}, \quad \mathrm{x} \in \mathbb{R}^3 \setminus \{\mathrm{z}\}, \tag{5.9}$$

for some $\mathrm{z} \in D$, and

$$v^{in}(\mathrm{x}) = H_0^{(1)}\left(k\sqrt{x_1^2 + H^2}\right), \quad \mathrm{x} \in \mathbb{R}^3 \setminus \Sigma_L, \tag{5.10}$$

for some $H \in \mathbb{R}$ such that $\Sigma_L := \{(0, x_2, H) : x_2 \in \mathbb{R}\} \subset D$. This second case is the case of an incident cylindrical wave generated by a line source occupying the line Σ_L which, without loss of generality we choose to lie in the plane $x_1 = 0$. The problem in each case is to find the unknown scattered field v^{sc} and the total field $v = v^{sc} + v^{in}$, such that

$$(\Delta + k^2)v^{sc} = 0 \text{ in } D, \quad v = 0 \text{ on } \Gamma,$$

and v^{sc} satisfies an appropriate radiation condition.

One way to make use of Theorem 4.1 to study these scattering problems is to formulate each scattering problem as an instance of (BVP) in section 2. To do this one can adapt the construction already used around equation (2.7). Precisely, in the spherical wave case one chooses $\epsilon > 0$ such that $\mathrm{dist}(\mathrm{z}, \partial D) > \epsilon$ and replaces v^{in} by $\tilde{v}^{in}$ which coincides with v^{in} except within distance ϵ of the source z, given by

$$\tilde{v}^{in}(\mathrm{x}) := \begin{cases} v^{in}(\mathrm{x}), & |\mathrm{x}-\mathrm{z}| > \epsilon, \\ A + B\dfrac{\sin kr}{r}, & |\mathrm{x}-\mathrm{z}| \le \epsilon, \end{cases}$$

where the constants A and B are chosen to ensure that $\tilde{v}^{in} \in C^1(\mathbb{R}^3)$ (which is possible provided that ϵ is chosen sufficiently small, e.g. if $k\epsilon < 1$). Then $\tilde{v}^{in} \in H^2_{loc}(\mathbb{R}^3)$ with $(\Delta + k^2)v^{in} = g_S$, where $g_S(\mathrm{x}) := Ak^2$, $|\mathrm{x}-\mathrm{z}| < \epsilon$, $g_S(\mathrm{x}) := 0$, otherwise. Similarly, in the cylindrical wave case one chooses $\epsilon > 0$ such that $\mathrm{dist}(\Sigma_L, \partial D) > \epsilon$ and replaces v^{in} by $\tilde{v}^{in}$ which coincides with v^{in} except within distance ϵ of the source Σ_L, given by

$$\tilde{v}^{in}(\mathrm{x}) := \begin{cases} v^{in}(\mathrm{x}), & \sqrt{x_1^2 + (x_3 - H)^2} > \epsilon, \\ A + BJ_0\left(k\sqrt{x_1^2 + (x_3 - H)^2}\right), & \sqrt{x_1^2 + (x_3 - H)^2} \le \epsilon, \end{cases}$$

where the constants A and B are chosen to ensure that $\tilde{v}^{in} \in C^1(\mathbb{R}^3)$ (again this is possible provided ϵ is chosen sufficiently small). Then $\tilde{v}^{in} \in H^2_{loc}(\mathbb{R}^3)$ with $(\Delta + k^2)v^{in} = g_C$, where $g_C(\mathrm{x}) := Ak^2$, $\sqrt{x_1^2 + (x_3 - H)^2} < \epsilon$, $g_C(\mathrm{x}) := 0$, otherwise. We observe that g_S is compactly supported so that $g_S \in L^2_\varrho(D)$ for every $\varrho \in \mathbb{R}$. Further, it is an easy calculation to see that $g_C \in L^2_\varrho(D)$ for $\varrho < -1/2$, but not for $\varrho = -1/2$.

Since $v^{in} = \tilde{v}^{in}$, except in neighbourhoods of z or Σ_L which do not intersect ∂D, so that, in particular, $v^{in} = \tilde{v}^{in}$ in a neighbourhood of ∂D, the substitution of v^{in} by $\tilde{v}^{in}$ does not change the scattered field v^{sc}. Further, since $\tilde{v}^{in} \in H^2_{loc}(\mathbb{R}^3)$, the scattering problem with this modified incident field can be formulated as an instance of (BVP). Precisely, in the spherical wave case, we can formulate the scattering problem as seeking the total field $\tilde{v} = \tilde{v}^{in} + v^{sc}$ which satisfies (BVP) for some $\varrho \in (-1, 1)$, with $g := g_S \in L^2_\varrho(D)$ (and if supp $g \not\subset S_0$ one needs to replace x_n by $x_n - a$ in (2.4), for some $a > 0$ such that supp $g \subset S_a$). Theorem 4.1 tells us that there is exactly one solution to this boundary value problem, and that this solution $\tilde{v}$ satisfies $\tilde{v}|_{S_h} \in V_\varrho^h$, for every $h \ge 0$ and $\varrho \in (-1, 1)$. In the cylindrical wave case, we seek the total field $\tilde{v} = \tilde{v}^{in} + v^{sc}$ which satisfies (BVP) for some $\varrho \in (-1, -1/2)$, with

$g := g_C \in L^2_\varrho(D)$ (again, if supp $g \not\subset S_0$ one needs to replace x_n by $x_n - a$ in (2.4), for some $a > 0$ such that supp $g \subset S_a$). Theorem 4.1 again tells us that there is exactly one solution, and that this solution $\tilde{v}$ satisfies $\tilde{v}|_{S_h} \in V^h_\varrho$, for every $h \geq 0$ and $\varrho \in (-1, -1/2)$.

5.2. **Analysis of a finite section method for the variational formulation.** An obvious approach to computing the solution to (BVP) numerically is to solve the variational formulation (V) by a finite element method. This is a standard approach for the numerical treatment of the diffraction grating problem (DGPW) [20, 22], but in that case the corresponding variational formulation, thanks to the periodicity, reduces to one on S_0^0, a single period of S_0. In the case of (V) the region of integration is the whole infinite region S_0. Thus a necessary first step towards solving (V) numerically is to approximate (V) by a variational formulation on a domain of finite size, to which standard FEMs can then be applied.

We are not aware of any analysis of such an approximation for the variational formulation (V), or for any similar variational formulations for rough surface scattering problems. However, the analogous approximation when boundary integral equation methods are applied to (BVP), namely truncation of the region of integration, which is the infinite boundary ∂D, to a finite part of that boundary (a so-called *finite section* approximation) has been analysed in both the 2D case [27] (and see [28, 23]) and, very recently, in the much more difficult 3D case [24]. In [27, 24] convergence and stability of modifications of the finite section method are proved. In the theses [28, 23] (cf. [16]), for the easier 2D case, convergence rates are also established, via results on stability and convergence of the finite section method in weighted spaces of continuous functions.

In this section we prove stability and convergence of an approximation method in the same spirit for (V). This approximation method consists simply in replacing S_0 by a finite region $S^{(R)} \subset S_0$ which coincides with S_0 in the region $|x| < R+1$, and making the same approximation for D, so that D is replaced by $D^{(R)} \subset D$, with $S^{(R)} = D^{(R)} \setminus \bar{U}_0$. The only constraint on the choice of $S^{(R)}$ is that (2.1) and (2.2) should apply to $D^{(R)}$; this is the case, for example, for the simple explicit choice $S^{(R)} := \{\mathrm{x} = (x, x_n) \in S_0 : |x| < R+1\}$. In addition to proving stability and convergence, we also establish rates of convergence for the error measured in weighted spaces. In the case when g, the source of the acoustic waves, is compactly supported, these results imply that, locally in the energy norm, the error converges at a rate $O(R^{\epsilon-2})$, for every $\epsilon > 0$. This convergence rate is consiste nt with those obtained previously by methods specific to the 2D case for boundary integral equation formulations [28, 16, 23]. For example, the results in [28], in the case when the boundary Γ is the graph of a function which is sufficiently smooth, imply a convergence rate R^{-2} locally in the uniform norm for the solution of a boundary integral formulation when the source of the acoustic waves is compactly supported and the finite section that is taken coincides with Γ inside a ball of radius R.

Given $g \in L^2_\varrho(S_0)$, with $|\varrho| < 1$, let $u \in V_\varrho$ be the unique solution of the variational problem (V), so that

$$B(u, v) = -(g, v)\,, \quad \forall v \in V_{-\varrho}\,. \tag{5.11}$$

For $R > 0$, we approximate problem (5.11) by a corresponding variational equation on the bounded domain $S^{(R)}$. For $\varrho \in \mathbb{R}$ and $R > 0$ let $V^{(R)}_\varrho$ denote the Hilbert space V_ϱ in the case that we replace D by $D^{(R)}$; explicitly $V^{(R)}_\varrho$ denotes the completion of $\{u|_{S_0^{(R)}} : u \in C_0^\infty(D^{(R)})\}$

in the norm

$$(5.12) \qquad \|u\|_{V_\varrho^{(R)}} = \left(\int_{S_0^{(R)}} \left(\left|(1+x^2)^{\varrho/2} u\right|^2 + \left|\nabla((1+x^2)^{\varrho/2} u)\right|^2 \right) d\mathrm{x} \right)^{1/2} .$$

We remark, as is easily seen from Lemma 2.1, that the norms $\|\cdot\|_{V_\varrho^{(R)}}$, $\varrho \in \mathbb{R}$, are equivalent since $S_0^{(R)}$ is bounded, so that, as linear spaces, for $\varrho \in \mathbb{R}$, $V_\varrho^{(R)} = V^{(R)} := V_0^{(R)}$. The approximating variational problem is the following: find $u^{(R)} \in V^{(R)}$ such that

$$(5.13) \qquad B^{(R)}(u^{(R)}, v) = -(g, v) , \quad \forall v \in V^{(R)} .$$

Here $B^{(R)}$ is the continuous sesquilinear form on $V^{(R)} \times V^{(R)}$ defined by (3.3) with D replaced by $D^{(R)}$, i.e. defined by

$$(5.14) \qquad B^{(R)}(u, v) := \int_{S_0^{(R)}} (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v})\, d\mathrm{x} + \int_{\Gamma_0^{(R)}} \gamma_- \bar{v}\; T\gamma_- u\, ds(\mathrm{x}) ,$$

where $\Gamma_0^{(R)} := \overline{S_0^{(R)}} \cap \Gamma_0$ (see Remark 3.6 for the interpretation of γ_- in this case).

Making the observation that we can view $V_\varrho^{(R)}$ as a closed subspace of V_ϱ (the elements of $V_\varrho^{(R)}$ become elements of V_ϱ if we extend them by zero from $S_0^{(R)}$ to S_0), the analysis of the error in approximating u by $u^{(R)}$ follows the usual pattern for analysing the Galerkin method for variational problems via a generalized Céa's lemma. Precisely, if $\tilde{u} \in V^{(R)} \subset V_\varrho$, then, for $v \in V^{(R)}$, applying (5.11),

$$B^{(R)}(\tilde{u}, v) = B(\tilde{u}, v) = B(\tilde{u} - u, v) - (g, v) .$$

Subtracting this equation from (5.13) we see that

$$(5.15) \qquad B^{(R)}(\tilde{u} - u^{(R)}, v) = B(\tilde{u} - u, v) , \quad \forall v \in V^{(R)} .$$

Now recall from Section 4 that $\mathcal{B}_\varrho : V_\varrho \to V_{-\varrho}^*$ is our notation for the bounded linear operator induced by the continuous sesquilinear form B. Similarly, let $\mathcal{B}_\varrho^{(R)} : V_\varrho^{(R)} \to V_{-\varrho}^{(R)^*}$ denote the operator induced by the sesquilinear form $B^{(R)}$; in other words $\mathcal{B}_\varrho^{(R)}$ is just $\mathcal{B}_\varrho$ in the case that D is replaced by $D^{(R)}$. From Theorem 4.1 it is clear that $\mathcal{B}_\varrho^{(R)}$ is invertible for every $R > 0$ and $\varrho \in (-1, 1)$. From Remark 4.2 it is clear, moreover, that $\|(\mathcal{B}_\varrho^{(R)})^{-1}\|$ is bounded uniformly for $R > 0$, with a bound which depends only on $|b|$, k, and ϱ. Thus from (5.15) it follows that, for $\varrho_1 \in (-1, 1)$,

$$\|\tilde{u} - u^{(R)}\|_{V_{\varrho_1}^{(R)}} \leq c \|\tilde{u} - u\|_{V_{\varrho_1}} ,$$

where the constant $c > 0$, which depends only on $|b|$, k, and ϱ_1, is an upper bound for $\|\mathcal{B}_{\varrho_1}\| \sup_{R>0} \|(\mathcal{B}_{\varrho_1}^{(R)})^{-1}\|$. Thus

$$(5.16) \qquad \|u - u^{(R)}\|_{V_{\varrho_1}} \leq (1 + c) \inf_{\tilde{u} \in V_{\varrho_1}^{(R)}} \|\tilde{u} - u\|_{V_{\varrho_1}} .$$

To obtain a more concrete error estimate, choose a cut-off function $\chi_R \in C_0^\infty(\mathbb{R}^{n-1})$ such that, for all $R > 0$,

$$\chi_R(x) := 1 \text{ for } |x| < R \,, \chi_R(x) := 0 \text{ for } |x| > R + 1 , \quad \sup_{\mathbb{R}^{n-1}} \{|\chi_R| + |\nabla \chi_R|\} \leq c_1 ,$$

for some constant $c_1 > 0$ independent of R. Defining $\tilde{u} \in V_{\varrho_1}^{(R)}$ by $\tilde{u}(\mathrm{x}) := u(\mathrm{x})\chi_R(x)$, $\mathrm{x} \in S_0$, we see using Lemma 2.1 that, for $-1 < \varrho_1 \leq \varrho$, where $\tilde{S}_0^R := \{\mathrm{x} \in S_0 : |x| > R\}$ and c_2, c_3, and c_4 denote further constants dependent only on ϱ and $|b|$,

$$\begin{aligned}\|\tilde{u}-u\|_{V_{\varrho_1}} &= \|(1-\chi_R)u\|_{V_{\varrho_1}} \\ &\leq c_2\left(\int_{\tilde{S}_0^R}(1+x^2)^{\varrho_1}\left(|u|^2+|\nabla u|^2\right)d\mathrm{x}\right)^{1/2} \\ &\leq c_3\,R^{\varrho_1-\varrho}\left(\int_{\tilde{S}_0^R}(1+x^2)^{\varrho}\left(|u|^2+|\nabla u|^2\right)d\mathrm{x}\right)^{1/2} \\ &\leq c_4\,R^{\varrho_1-\varrho}\,\|u\|_{V_\varrho}\,.\end{aligned}$$

We see that we have proved the following result:

Theorem 5.7. *Suppose $g \in L^2_\varrho(S_0)$, with $|\varrho| < 1$, and let $u \in V_\varrho$ be the unique solution of the variational problem (V). Choose, for $R > 0$, approximating domains $D^{(R)} \subset D$ which satisfy* (2.1) *and* (2.2)*, and are such that $S^{(R)} \subset S_0$ is bounded and $S^{(R)} \supset \Omega_R := \{\mathrm{x} \in S_0 : |x| < R+1\}$. Then the approximating variational problem* (5.11) *on the finite region $S^{(R)}$ has exactly one solution $u^{(R)}$ for every $R > 0$. Further, for some constant $c > 0$ dependent only on k, ϱ, ϱ_1, and $|b|$, it holds for $-1 < \varrho_1 < \varrho$ that*

$$\|u-u^{(R)}\|_{V_{\varrho_1}^{(R)}} \leq c\,R^{\varrho_1-\varrho}\,\|u\|_{V_\varrho}.$$

As a consequence, for every $R_1 > 0$, it holds that $\|u-u^{(R)}\|_{H^1(\Omega_{R_1})} = O(R^{\varrho_1-\varrho})$ as $R \to \infty$. In particular, if $g \in L^2_\varrho(S_0)$ for every $\varrho < 1$, which holds for example if g is compactly supported, then, for every $\epsilon > 0$ and $R_1 > 0$,

$$\|u-u^{(R)}\|_{H^1(\Omega_{R_1})} = O(R^{\epsilon-2}) \quad as \quad R \to \infty\,.$$

6. Commutator estimates

This section is devoted to the proofs of Theorem 3.1 and Lemma 3.4 (i). Let $k > 0$, $a > 0$, and consider the pseudodifferential operator T_a on $\mathbb{R}^m$, $m = n-1 = 1, 2$, with symbol $t_a(\xi)$:

(6.1) $$T_a u(x) = F^{-1}t_a(\xi)Fu(\xi)\,, \quad t_a(\xi) := a^{-1}\sqrt{\xi^2-k^2a^2}\,.$$

Here and in the following the square root is chosen so that its argument lies in $[-\pi/2, 0]$:

(6.2) $$t_a(\xi) = (-i/a)\sqrt{k^2a^2-\xi^2},\ |\xi| \leq ka; \quad t_a(\xi) = (1/a)\sqrt{\xi^2-k^2a^2},\ |\xi| > ka\,.$$

We have $T_1 = T$, $t_1 = t$, where T and t are defined in (3.1).

Using a scaling argument, we reduce the assertion of Theorem 3.1 to a corresponding estimate for the commutator defined by

(6.3) $$C_a := T_a - (1+x^2)^{\varrho/2}T_a(1+x^2)^{-\varrho/2}\cdot\ .$$

With $S_a u(x) := u(ax)$ we obtain $FS_a u = a^{-m}S_{1/a}Fu$, and the same relation holds with F replaced by F^{-1}. Hence

$$a^m S_a F^{-1}tFu = F^{-1}(S_{1/a}t)S_{1/a}Fu = a^m F^{-1}(S_{1/a}t)FS_a u\,,$$

giving $S_a T u = T_a S_a u$, where T_a is the operator (6.1) with the symbol $t(\xi/a) = t_a(\xi)$. From (3.2) and (6.3) we then have

(6.4) $$S_a C = T_a S_a - (1+x^2)^{\varrho/2}T_a(1+x^2)^{-\varrho/2}S_a = C_a S_a\,.$$

Using the relation $\|S_a u\|_{L^2(\mathbb{R}^m)} = a^{-m/2}\|u\|_{L^2(\mathbb{R}^m)}$ and (6.4), we now observe that Theorem 3.1 is equivalent to the following

Theorem 6.1. *For $ka \geq 1$ and $|\varrho| < 1$, the commutator C_a defined in* (6.3) *has norm $\leq c(\varrho)\sqrt{k/a}$ on $L^2(\mathbb{R}^m)$.*

It is enough to consider $\varrho \in (0,1)$ since the case of negative ϱ then follows by duality (with respect to the scalar product on $L^2(\mathbb{R}^m)$). We split the symbol t_a as

$$t_a = t^{(0)} + t^{(1)} =: \chi(|\xi|)\, t_a(\xi) + (1-\chi(|\xi|))\, t_a(\xi)\,, \tag{6.5}$$

where χ is a suitable cut-off funtion (see below), and consider the corresponding decomposition of T_a,

$$T_a = T^{(0)} + T^{(1)}\,, \tag{6.6}$$

where $T^{(j)}$ is defined by (6.1) with $t^{(j)}$ in place of t_a. Then the commutator C_a takes the form

$$\begin{aligned} C_a = C^{(0)} + C^{(1)} =: &\big(T^{(0)} - (1+x^2)^{\varrho/2} T^{(0)} (1+x^2)^{-\varrho/2}\cdot\big) \\ &+ \big(T^{(1)} - (1+x^2)^{\varrho/2} T^{(1)} (1+x^2)^{-\varrho/2}\cdot\big)\,. \end{aligned} \tag{6.7}$$

We will estimate the norm of the operators

$$N := (1+x^2)^{-\varrho/2} C^{(0)} : L^2(\mathbb{R}^m) \to L^2_\varrho(\mathbb{R}^m)\,, \tag{6.8}$$

$$\hat{N}u := FNF^{-1}u, \quad \hat{N} : L^2(\mathbb{R}^m) \to H^\varrho(\mathbb{R}^m)\,, \tag{6.9}$$

$$\hat{T} := t^{(1)}(\xi)\cdot : H^\varrho(\mathbb{R}^m) \to H^\varrho(\mathbb{R}^m)\,. \tag{6.10}$$

In view of (6.7)–(6.10), and recalling that F is an isometry of $L^2_\varrho(\mathbb{R}^m)$ onto $H^\varrho(\mathbb{R}^m)$ for every $\varrho \in \mathbb{R}$, Theorem 6.1 then follows from

Theorem 6.2. *(i) For $\varrho \in (0,1]$, the norm of $\hat{N}$ is bounded by $c(\varrho)\sqrt{k/a}$.*

(ii) For $\varrho \in [0,1)$, the norm of $\hat{T}$ is bounded by $c(\varrho)\sqrt{k/a}$, too.

We now choose the cut-off function $\chi \in C^\infty[0,\infty)$ with $0 \leq \chi \leq 1$ and

$$\chi(r) = 0 \quad \text{on} \quad |r-ka| \leq 1/3\,, \quad \chi(r) = 1 \quad \text{on} \quad |r-ka| \geq 2/3\,, \tag{6.11}$$

and such that, for some $c > 0$ independent of $ka \geq 1$.

$$|\partial_r \chi(r)| \leq c \quad \text{on} \quad \mathbb{R}^+\,. \tag{6.12}$$

Note that (6.11) implies

$$\partial_r \chi(r) = 0 \quad \text{on} \quad \{|r-ka| \leq 1/3\} \cup \{|r-ka| \geq 2/3\}\,. \tag{6.13}$$

To prove Theorem 6.2, we need some auxiliary results.

Lemma 6.3. *For $ka \geq 1$, where $r = |\xi|$, we have $|\partial_r t^{(0)}(\xi)| \leq c\sqrt{k/a}$ on $\mathbb{R}^+$.*

Proof. Setting $h(r) := at_a(\xi) = (r-ka)^{1/2}(r+ka)^{1/2}$ for $r > ka$ and $h(r) := iat_a(\xi) = (ka-r)^{1/2}(r+ka)^{1/2}$ for $r < ka$, we obtain

$$\partial_r h = \begin{cases} r/h, & r > ka, \\ -r/h, & 0 \leq r < ka, \end{cases} \qquad \partial_r^2 h = -k^2a^2/h^3\,, \quad r \neq ka\,,$$

which implies that $\partial_r^j h$ $(j = 1,2)$ do not change sign on $(0,ka)$ and (ka,∞). Therefore, the maximum of h on $1/3 \leq |r-ka| \leq 2/3$ is attained at $r = ka - 2/3$ or $r = ka + 2/3$, while the maximum of $|\partial_r h|$ on $|r-ka| \geq 1/3$ is attained at $r = ka-1/3$ or $r = ka+1/3$, and both maxima are bounded above by $c\sqrt{ka}$. Together with (6.11)–(6.13), this easily implies the result. □

To prove Theorem 6.2 (i), we write (cf. (6.7)–(6.9))

$$\hat{N}Fu(\xi) \;= \textstyle\int_{\mathbb{R}^m} b_\varrho(\xi-\eta)\,(t^{(0)}(\eta)-t^{(0)}(\xi))\,Fu(\eta)d\eta\,, \quad u\in C_0^\infty(\mathbb{R}^m)\,, \tag{6.14}$$

with $b_\varrho := F(1+x^2)^{-\varrho/2}$. Here the integral in (6.14) is well defined since Fu is rapidly decreasing and $b_\varrho \in L^1(\mathbb{R}^m)$ for $\varrho>0$ (see the next lemma), and we have used the relation $F(1+x^2)^{-\varrho/2}v = b_\varrho * Fv$ for a function v of rapid decay, with $*$ denoting convolution.

Lemma 6.4. *For any $\varrho>0$, the functions b_ϱ and $|\xi|\,\nabla_\xi b_\varrho$ are rapidly decreasing as $|\xi|\to\infty$ and belong to $L^1(\mathbb{R}^m)$.*

For the proof of this, we refer to [29, Chap. 8.1]; see also [34, Chap. 5.3].

Proof of Theorem 6.2 (i). From (6.14) and Lemma 6.3,

$$\begin{aligned}\|\hat{N}Fu\|_{L^2(\mathbb{R}^m)} &\le \left\|\int_{\mathbb{R}^m} |b_\varrho(\xi-\eta)|\,|\xi-\eta|\,\sup_{\mathbb{R}^+}|\partial_r t^{(0)}|\,|Fu(\eta)|\,d\eta\right\|_{L^2(\mathbb{R}^m)} \\ &\le c\sqrt{k/a}\,\|\,|\xi|b_\varrho\|_{L^1(\mathbb{R}^m)}\,\|Fu\|_{L^2(\mathbb{R}^m)}\end{aligned} \tag{6.15}$$

using the mean value theorem and Young's inequality. Moreover, since

$$\begin{aligned}\nabla_\xi \hat{N}Fu(\xi) = &\int_{\mathbb{R}^m}\left(t^{(0)}(\eta)-t^{(0)}(\xi)\right)\,\nabla_\xi b_\varrho(\xi-\eta)\;Fu(\eta)\,d\eta \\ &+\int_{\mathbb{R}^m} b_\varrho(\xi-\eta)\,(-\nabla_\xi t^{(0)}(\xi))\;Fu(\eta)\,d\eta\,,\end{aligned}$$

we obtain analogously

$$\|\nabla\hat{N}Fu\|_{L^2(\mathbb{R}^m)} \le c\sqrt{k/a}\,\left(\|\,|\xi|\nabla b_\varrho\|_{L^1(\mathbb{R}^m)}+\|b_\varrho\|_{L^1(\mathbb{R}^m)}\right)\,\|Fu\|_{L^2(\mathbb{R}^m)}\,. \tag{6.16}$$

Together with Lemma 6.4, (6.15) and (6.16) imply that for any $\varrho\in(0,1]$ the operators $\hat{N}: L^2(\mathbb{R}^m)\to L^2(\mathbb{R}^m)$ and $\hat{N}: L^2(\mathbb{R}^m)\to H^1(\mathbb{R}^m)$ have norm $\le c(\varrho)\sqrt{k/a}$. By interpolation, we then get the result. $\square$

Proof of Theorem 6.2 (ii). We have to show that the multiplication operator

$$\hat{T}v = (1-\chi(|\xi|))\,a^{-1}\sqrt{\xi^2-k^2a^2}\;v : H^\varrho(\mathbb{R}^m)\to H^\varrho(\mathbb{R}^m)\,,\quad 0\le\varrho<1\,, \tag{6.17}$$

has norm $\le c(\varrho)\sqrt{k/a}$. Note that the support of $1-\chi$ is contained in the set $\mathcal{R} := \{|r-ka|\le 2/3\}$, $r=|\xi|$; see (6.11). By localization, (6.17) can be reduced to an estimate of the form

$$\|qv\|_{H^\varrho(\mathbb{R}^m)} \le c(\varrho)\sqrt{k/a}\,\|v\|_{H^\varrho(\mathbb{R}^m)}\,,\quad 0\le\varrho<1\,, \tag{6.18}$$

where

$$q(\xi) := a^{-1}\sqrt{\xi_m}\;(\xi_m+2ka)^{1/2}\;\psi(\xi_m) \tag{6.19}$$

with $\psi\in C_0^\infty(-2/3,2/3)$ fixed and $\sqrt{\xi_m} = -i|\xi_m|^{1/2}$ for $\xi_m<0$. This reduction is clear for $m=1$, where one has to localize near $\xi_1 = ka$ and $\xi_1=-ka$. For $m=2$, we parametrize the annulus $\mathcal{R}$ by $\xi_2 := r-ka$ and arclength ξ_1 on $|\xi| = r = ka$ and need two local charts again to cover $\mathcal{R}$. Note that the Jacobians of the corresponding coordinate transformations (with respect to the original ξ-coordinates) are uniformly bounded from above and below for $ka\ge 1$. We omit the details since we present an alternative approach in the 3D case below.

To prove (6.18), we first observe that the operator of multiplication by $q_1 := a^{-1}(\xi_m + 2ka)^{1/2}\,\psi$ has norm

$$\sup_{\mathbb{R}}|q_1| \le c\sqrt{k/a} \quad\text{and}\quad \le \sup_{\mathbb{R}}|q_1| + \sup_{\mathbb{R}}|\partial_m q_1| \le c\sqrt{k/a}$$

in $L^2(\mathbb{R}^m)$ and $H^1(\mathbb{R}^m)$, respectively. Note that $ka \ge 1$ implies $1/a \le k$ and

$$a^{-1}(\xi_m + 2ka)^{-1/2}\,|\psi| \le ca^{-1}/\sqrt{ka} \le c\sqrt{k/a}\,.$$

By interpolation, the norm of this multiplication operator in $H^\varrho(\mathbb{R}^m)$ is then bounded by $c(\varrho)\sqrt{k/a}$. It remains to show that

$$\|\sqrt{\xi_m}\,\varphi(\xi_m)\,v\|_{H^\varrho(\mathbb{R}^m)} \le c(\varrho)\,\|v\|_{H^\varrho(\mathbb{R}^m)}\,, \quad 0 \le \varrho < 1\,, \tag{6.20}$$

where φ is a smooth function with somewhat larger support and $\varphi\psi = \psi$.

Let first $m = 1$. Then (6.20) follows for $\varrho \in (1/2, 1)$ since $\sqrt{\xi_1}\,\varphi \in H^\varrho(\mathbb{R})$ for $\varrho < 1$ (but not for $\varrho = 1$) and $H^\varrho(\mathbb{R})$ is a Banach algebra. Since (6.20) is obvious for $\varrho = 0$, we obtain the result by interpolation. Note that the constant $c(\varrho)$ blows up as $\varrho \to 1$.

For $m = 2$, the proof of (6.20) can easily be reduced to the case $m = 1$ by using the relation (cf. [26, Chap. 1])

$$H^\varrho(\mathbb{R}^2) = L^2(\mathbb{R}; H^\varrho(\mathbb{R})) \cap H^\varrho(\mathbb{R}; L^2(\mathbb{R}))$$

and the fact that the function (6.19) is independent of ξ_1. $\square$

Remark 6.5. An alternative proof of Theorem 6.2 (ii) for $m = 2$ can be given by the following more direct reduction to the case $m = 1$. Let (r, θ) be polar coordinates in $\mathbb{R}^2$, and consider a multiplication operator $\mathcal{M} := q\cdot$ on $H^\varrho(\mathbb{R}^2)$, $\varrho \in (0, 1]$, with a continuous function $q = q(r)$ only depending on the radial variable, supp $q \subset [1/3, 2/3]$, and such that $q\cdot$ considered as a multiplication operator on $H^\varrho(\mathbb{R})$ is bounded with norm $\|q\cdot\|_\varrho$. Then the norm of $\mathcal{M}$ on $H^\varrho(\mathbb{R}^2)$ is bounded by a positive constant $c(\varrho)$. Applying this to the operator $\hat{T}$ defined in (6.17) and using Theorem 6.2 (ii) for $m = 1$, we get the result for $m = 2$.

To prove the above norm estimate for $\mathcal{M}$, we first note that $L^2(\mathbb{R}^2)$ is the orthogonal sum of the subspaces

$$\mathcal{H}_j := \{v \in L^2(\mathbb{R}^2) : v = f(r)\,\exp(ij\theta)\,, \quad \int_0^\infty |f(r)|^2\,r\,dr < \infty\}\,, \quad j \in \mathbb{Z}\,,$$

and the Fourier transform leaves each space $\mathcal{H}_j$ invariant; see [35, Chap. 4.1]. Therefore, it is sufficient to verify that

$$\|qf\,\exp(ij\theta)\|_{H^\varrho(\mathbb{R}^2)} \le c(\varrho)\,\|f\,\exp(ij\theta)\|_{H^\varrho(\mathbb{R}^2)} \tag{6.21}$$

for each $j \in \mathbb{Z}$ and $f \in C_0^\infty(0, \infty)$. Furthermore we have, uniformly in j,

$$\begin{aligned}
&\|qf\exp(ij\theta)\|_{L^2(\mathbb{R}^2)} \sim \|r^{1/2}qf\|_{L^2(\mathbb{R})}\,,\\
&\|qf\exp(ij\theta)\|_{H^1(\mathbb{R}^2)} \sim \|qf\exp(ij\theta)\|_{L^2(\mathbb{R}^2)} + \|\partial_r qf\exp(ij\theta)\|_{L^2(\mathbb{R}^2)} + \|qfr^{-1}\partial_\theta\exp(ij\theta)\|_{L^2(\mathbb{R}^2)}\\
&\qquad\qquad \sim \|r^{1/2}qf\|_{H^1(\mathbb{R})} + (1 + |j|)\,\|r^{-1/2}qf\|_{L^2(\mathbb{R})}
\end{aligned}$$

and thus by interpolation,

$$\|qf\,\exp(ij\theta)\|_{H^\varrho(\mathbb{R}^2)} \sim \|r^{1/2}qf\|_{H^\varrho(\mathbb{R})} + (1 + |j|)^\varrho\,\|r^{1/2-\varrho}qf\|_{L^2(\mathbb{R})}\,, \tag{6.22}$$

where $\sim$ means equivalence of norms. Here we used standard interpolation of Sobolev norms and the interpolation theorem for weighted L^2 spaces (see [37, Chap. 1.18.5]). Now (6.22) and the boundedness of $q\cdot$ on $H^\varrho(\mathbb{R})$ imply the estimates

$$\begin{aligned}
\|qf\,\exp(ij\theta)\|_{H^\varrho(\mathbb{R}^2)} &\le c(\varrho)\left\{\|qr^{1/2}f\|_{H^\varrho(\mathbb{R})} + (1 + |j|)^\varrho\,\|qr^{1/2-\varrho}f\|_{L^2(\mathbb{R})}\right\}\\
&\le c(\varrho)\,\max(\|q\cdot\|_\varrho\,, \sup|q|)\left\{\|r^{1/2}f\|_{H^\varrho(\mathbb{R})} + (1 + |j|)^\varrho\,\|r^{1/2-\varrho}f\|_{L^2(\mathbb{R})}\right\}\\
&\le c(\varrho)\,\max(\|q\cdot\|_\varrho\,, \sup|q|)\,\|f\,\exp(ij\theta)\|_{H^\varrho(\mathbb{R}^2)}
\end{aligned}$$

giving (6.21).

Remark 6.6. (i) Repeating the above proofs with a fixed cut-off function χ vanishing in a neighbourhood of $|\xi| = k$, we obtain the norm estimate of Lemma 3.2 for the commutator $T - (1+x^2)^{\varrho/2} T (1+x^2)^{-\varrho/2}\cdot$. Here we need not take care of the dependence of the constants on a and k.

(ii) Note that the symbol $t^{(0)} = \chi t$ of the pseudodifferential operator $T^{(0)} = F^{-1} t^{(0)} F$ is a smooth function satisfying $|t_0| \le c(1+\xi^2)^{1/2}$ and $|\nabla t_0| \le c$ on $\mathbb{R}^m$, and this is enough to obtain the boundedness of the commutators $T^{(0)} - (1+x^2)^{\varrho/2} T^{(0)} (1+x^2)^{-\varrho/2}\cdot$, $|\varrho| \le 1$, on $L^2(\mathbb{R}^m)$; see the estimates (6.15) and (6.16).

Applying this to the operator $\Lambda := F^{-1}(1+\xi^2)^{1/2} F$ which is an isomorphism of $H^1(\mathbb{R}^m)$ onto $L^2(\mathbb{R}^m)$, we observe that Λ is also an isomorphism of $H^1_\varrho(\mathbb{R}^m)$ onto $L^2_\varrho(\mathbb{R}^m)$, and F is an isomorphism of $H^1_\varrho(\mathbb{R}^m)$ onto $H^\varrho_1(\mathbb{R}^m)$, at least for $|\varrho| \le 1$. This is also true for arbitrary $\varrho \in \mathbb{R}$; see [31] and [38].

(iii) Let $\sigma(\xi)$ be a smooth symbol satisfying the estimates $|\sigma| \le c$, $|\nabla_\xi \sigma| \le c(1+\xi^2)^{-1/2}$ on $\mathbb{R}^m$. Then, for $A := F^{-1}\sigma F$ and $|\varrho| \le 1$, the commutator

$$A - (1+x^2)^{\varrho/2} A (1+x^2)^{-\varrho/2}\cdot : L^2(\mathbb{R}^m) \to H^1(\mathbb{R}^m)$$

is bounded. This follows from (ii) applied to the operator $B = \Lambda A$ with the symbol $(1+\xi^2)^{1/2}\sigma(\xi)$ and the relation

$$\begin{aligned}\Lambda^{-1}B - (1-x^2)^{\varrho/2}\Lambda^{-1}B(1+x^2)^{-\varrho/2}\cdot &= \Lambda^{-1}\big(B - (1+x^2)^{\varrho/2}B(1+x^2)^{-\varrho/2}\cdot\big) \\ &\quad + \big(\Lambda^{-1} - (1+x^2)^{\varrho/2}\Lambda^{-1}(1+x^2)^{-\varrho/2}\cdot\big)(1+x^2)^{\varrho/2}B(1+x^2)^{-\varrho/2}\cdot\,.\end{aligned} \tag{6.23}$$

Note that B is bounded on $L^2_\varrho(\mathbb{R}^m)$.

More general results on pseudodifferential operators with smooth symbols in weighted Sobolev spaces can be found in [31] and [38].

Finally, we proceed to

Proof of Lemma 3.4 (i). From (2.4) we have, for $u_0 \in C_0^\infty(\Gamma_0)$,

$$\begin{aligned} u(x,x_n) &= F^{-1}\exp(-x_n t(\xi))Fu_0 =: M_0 u_0\,, \quad t(\xi) := \sqrt{\xi^2 - k^2}\,, \\ \nabla_x u(x,x_n) &= F^{-1} i\xi \exp(-x_n t(\xi))Fu_0 =: M_1 u_0 = \nabla_x M_0 u_0\,, \\ \partial_n u(x,x_n) &= F^{-1}(-t(\xi))\exp(-x_n t(\xi))Fu_0 =: M_2 u_0 = -TM_0 u_0\,. \end{aligned} \tag{6.24}$$

We have to prove the estimates, for $|\varrho| < 1$ and $h > 0$,

$$\|u\|_{H^1_\varrho(U_0\setminus \bar U_h)} \le c(h,\varrho)\|u_0\|_{H^{1/2}_\varrho(\Gamma_0)}\,, \quad u_0 \in C_0^\infty(\Gamma_0)\,,$$

or equivalently, with $m = n-1$,

$$\int_0^h \int_{\mathbb{R}^m} (1+x^2)^\varrho \sum_{j=0}^2 |M_j u_0|^2\, dx\, dx_n \le c(h,\varrho)\, \|(1+x^2)^{\varrho/2} u_0\|^2_{H^{1/2}(\mathbb{R}^m)}\,. \tag{6.25}$$

This was proved in [11] for $\varrho = 0$ by taking Fourier transform. To verify (6.25) for $\varrho \ne 0$, it is then sufficient to show that the commutators

$$M_j - (1+x^2)^{\varrho/2} M_j (1+x^2)^{-\varrho/2}\cdot\,, \quad j = 0,1,2\,,$$

are uniformly bounded on $L^2(\mathbb{R}^m)$ with respect to $x_n \in (0,h)$; compare the proof of Lemma 3.3 (ii). We can write (cf. relation (6.23))

$$\begin{aligned} AM_0 - (1+x^2)^{\varrho/2} AM_0 (1+x^2)^{-\varrho/2} \cdot &= A\big(M_0 - (1+x^2)^{\varrho/2} M_0 (1+x^2)^{-\varrho/2} \cdot \big) \\ &+ \big(A - (1+x^2)^{\varrho/2} A (1+x^2)^{-\varrho/2} \cdot \big)(1+x^2)^{\varrho/2} M_0 (1+x^2)^{-\varrho/2} \cdot \,, \end{aligned} \tag{6.26}$$

where A is one of the operators $\partial_j = \partial/\partial x_j$, $1 \le j \le m$, and T. Therefore it is enough to prove the uniform boundedness of

$$M_0 : L^2_\varrho(\mathbb{R}^m) \to L^2_\varrho(\mathbb{R}^m)\,, \quad |\varrho| < 1\,, \tag{6.27}$$

$$M_0 - (1+x^2)^{\varrho/2} M_0 (1+x^2)^{-\varrho/2} \cdot : L^2(\mathbb{R}^m) \to H^1(\mathbb{R}^m)\,, \tag{6.28}$$

since $\partial_j, T : H^1(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ are bounded, the commutators $\partial_j - (1+x^2)^{\varrho/2}\partial_j(1+x^2)^{-\varrho/2}\cdot$ are obviously bounded on $L^2(\mathbb{R}^m)$, while the commutator $T - (1+x^2)^{\varrho/2} T (1+x^2)^{-\varrho/2}\cdot$ is bounded there by Lemma 3.2.

By taking Fourier transform, the uniform boundedness of (6.27) is equivalent to the estimates

$$\|m(x_n,\xi)\, v\|_{H^\varrho(\mathbb{R}^m)} \le c(h,\varrho) \|v\|_{H^\varrho(\mathbb{R}^m)}\,, \quad v \in C_0^\infty(\mathbb{R}^m)\,, \quad x_n \in (0,h)\,, \tag{6.29}$$

where $m(x_n,\xi) = \exp(-x_n t(\xi))$. Consider a decomposition $t = t^{(0)} + t^{(1)}$ as in (6.5), with $a = 1$, $t^{(0)} = \chi t$, $t^{(1)} = (1-\chi)t$ and a cut-off function χ vanishing near $|\xi| = k$, so that $t^{(0)}$ is a smooth symbol. We introduce the multiplication operators $\mathcal{M} = m(x_n,\xi)\cdot = \mathcal{M}^{(1)}\mathcal{M}^{(0)}$, $\mathcal{M}^{(j)} = \exp(-x_n t^{(j)}(\xi))\cdot$, $x_n \in (0,h)$, and check the uniform boundedness of

$$\mathcal{M}^{(0)} : H^\varrho(\mathbb{R}^m) \to H^\varrho(\mathbb{R}^m)\,, \quad |\varrho| \le 1\,, \tag{6.30}$$

$$\mathcal{M}^{(1)} : H^\varrho(\mathbb{R}^m) \to H^\varrho(\mathbb{R}^m)\,, \quad |\varrho| < 1\,. \tag{6.31}$$

Since we have, for $x_n \in (0,h)$ and $m_0(x_n,\xi) = \exp(-x_n t^{(0)}(\xi))$,

$$|m_0(x_n,\xi)| \le c(h)\,, \quad |\nabla_\xi m_0(x_n,\xi)| \le c(h)(1+\xi^2)^{-1/2} \quad \text{on} \quad \mathbb{R}^m\,, \tag{6.32}$$

the norm of (6.30) is bounded by some constant $c(h)$. To get a bound for (6.31), we write

$$\exp(-x_n t^{(1)}) = \sum_{j\ge 0} x_n^j (-t^{(1)})^j / j! \tag{6.33}$$

and apply the proof of Theorem 6.2 (ii) to estimate the norm of (6.31) by

$$\sum_{j\ge 0} h^j c(\varrho)^j / j! = \exp(hc(\varrho))\,.$$

This finishes the proof of (6.27).

To prove the uniform boundedness of (6.28), we write

$$\begin{aligned} M_0 - (1+x^2)^{\varrho/2} M_0 (1+x^2)^{-\varrho/2} \cdot &= M^{(1)}\big(M^{(0)} - (1+x^2)^{\varrho/2} M^{(0)} (1+x^2)^{-\varrho/2} \cdot \big) \\ &+ \big(M^{(1)} - (1+x^2)^{\varrho/2} M^{(1)} (1+x^2)^{-\varrho/2} \cdot \big)(1+x^2)^{\varrho/2} M^{(0)} (1+x^2)^{-\varrho/2} \cdot \,, \end{aligned} \tag{6.34}$$

where $M^{(j)} = F^{-1} \exp(-x_n t^{(j)}(\xi)) F$, $j = 0,1$. By (6.32) and Remark 6.6 (iii), we obtain the uniform boundedness of

$$M^{(0)} - (1+x^2)^{\varrho/2} M^{(0)} (1+x^2)^{-\varrho/2} \cdot : L^2(\mathbb{R}^m) \to H^1(\mathbb{R}^m)\,.$$

Moreover, $M^{(1)}$ is obviously bounded on $H^1(\mathbb{R}^m)$ since its symbol is uniformly bounded. In view of (6.30) and the isometry $F : L^2_\varrho(\mathbb{R}^m) \to H^\varrho(\mathbb{R}^m)$, it is then sufficient to verify that the last commutator in (6.34) is uniformly bounded from $L^2(\mathbb{R}^m)$ into $H^1(\mathbb{R}^m)$, and for this it is enough to show the uniform boundedness of

$$N^{(1)} = M^{(1)} - I : L^2_\varrho(\mathbb{R}^m) \to H^1_\varrho(\mathbb{R}^m)\,, \quad |\varrho| < 1\,, \quad x_n \in (0,h)\,, \tag{6.35}$$

where I is the identity operator, and the symbol of $N^{(1)}$ is $n_1(x_n, \xi) = \exp(-x_n t^{(1)}(\xi)) - 1$. Taking Fourier transform and using Remark 6.6 (ii), (6.35) is then equivalent to the uniform boundedness of

$$n_1(x_n, \xi) \cdot : H^{\varrho}(\mathbb{R}^m) \to H_1^{\varrho}(\mathbb{R}^m)\,, \quad |\varrho| < 1\,. \tag{6.36}$$

Consider the multiplication operator $\mathcal{N}^{(1)} = (1+\xi^2)^{1/2} n_1(x_n, \xi)\cdot$. Then, using relation (6.33), estimate (6.31) can be proved (in the same way) for $\mathcal{N}^{(1)}$ in place of $\mathcal{M}^{(1)}$; recall that $t^{(1)}$ has compact support. This easily implies (6.36), which finishes the proof of (6.28). Thus Lemma 3.4 (i) is proven. $\square$

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