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Differential, energetic, and metric formulations for rate-independent processes

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Abstract

We consider different solution concepts for rate-independent systems. This includes energetic solutions in the topological setting and differentiable, local, parametrized and BV solutions in the Banach-space setting. The latter two solution concepts rely on the method of vanishing viscosity, in which solutions of the rate-independent system are defined as limits of solutions of systems with small viscosity. Finally, we also show how the theory of metric evolutionary systems can be used to define parametrized and BV solutions in metric spaces.

1 Introduction

In these notes we want to give an overview of the recently developed theory for rate-independent systems. Such systems are used to model hysteresis, dry friction, elastoplasticity, magnetism, and phase transformation, and they are characterized by the fact that the changes of the state are driven solely by changes of the loading. More specifically, if the loading profile is applied with a factor α faster to the system, then rescaling the solution with the same factor α gives again a solution.

General energy-driven systems, also called generalized gradient systems, are characterized by a triple $(\mathcal{Z}, \mathcal{I}, \mathcal{R})$ where $\mathcal{Z}$ is the state space and $\mathcal{I} : [0,T]\times\mathcal{Z} \to \mathbb{R}_\infty \overset{\text{def}}{=} \mathbb{R} \cup \{\infty\}$ is the energy functional. We use $\mathcal{Z}$ to denote a general topological state space, but we use $\boldsymbol{Z}$ if it is a Banach space. For simplicity we restrict the introduction to the latter case. The dissipation potential $\mathcal{R} : \boldsymbol{Z}\times\boldsymbol{Z} \to [0,\infty]$ allows us to write the evolution equation in the form

$$0 \in \partial_{\dot{z}}\mathcal{R}(z,\dot{z}) + \overline{\partial}_z\mathcal{I}(t,z) \quad \subset \boldsymbol{Z}^*, \tag{1}$$

where $\overline{\partial}_z$ denotes a suitable subgradient of $\mathcal{I}(t,\cdot)$, while $\partial_{\dot{z}}\mathcal{R}(z,\cdot)$ denotes the convex subdifferential of $\mathcal{R}(z,\cdot)$. The generalized gradient system $(\boldsymbol{Z}, \mathcal{I}, \mathcal{R})$ is rate independent if $\mathcal{R}(z,\cdot)$ is positively homogeneous of degree 1, since this implies $\partial_v\mathcal{R}(z,\alpha v) = \partial_v\mathcal{R}(z,v)$ for all $\alpha > 0$. We then call $(\boldsymbol{Z}, \mathcal{I}, \mathcal{R})$ a rate-independent system, shortly RIS. Hence, system (1) is necessarily nonsmooth. In fact, the convex subdifferential $\partial_v\mathcal{R}(z,\cdot) : \boldsymbol{Z} \rightrightarrows \boldsymbol{Z}^*$ is not continuous and set-valued.

However, the main difference to the usually studied generalized gradient flows is that $\mathcal{R}(z,\cdot)$ has at most linear growth, and we cannot guarantee continuity of the solutions $z : [0,T] \to \boldsymbol{Z}$. Thus, there is a need to study the question under what conditions we can guarantee absolute continuity, in such a way that (1) makes sense, see Section 4.3. In fact, this is only true under strong convexity assumptions, and we mainly discuss the question, how the strong differential form should be weakened to allow for solutions with jumps.

To motivate the main structures of the different solution concepts for RIS, we start from the Fenchel equivalence

$$\eta \in \partial_v\mathcal{R}(z,v) \quad \Longleftrightarrow \quad \mathcal{R}(z,v) + \mathcal{R}^*(z,\eta) \leq \langle \eta, v\rangle,$$

where $\mathcal{R}^*(z,\cdot)$ is the Legendre-Fenchel transform of $\mathcal{R}(z,\cdot)$. While the statement on the left-hand side of this equivalence is a force balance, the statement on the fight-hand side is given in terms of energy rates. Using $-\eta = \xi(t) \in \overline{\partial}_z\mathcal{I}(t,z(t))$ and a chain rule, we find that (1) is equivalent to the scalar, upper energetic inequality

$$\begin{aligned}&\mathcal{I}(T,z(T))+\int_0^T \mathcal{R}(z(t),\dot z(t))+\mathcal{R}^*(z(t),-\xi(t))\,\mathrm{d}t\\&\le \mathcal{I}(0,z(0))+\int_0^T \partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t.\end{aligned}\tag{2}$$

The particularity of RIS is that $\mathcal{R}^*(z,-\xi)$ only takes the two values 0 and ∞, viz. $\mathcal{R}^*(z,-\xi)=0$ if and only if $0\in\partial_v\mathcal{R}(z,0)+\xi$. Thus, the energetic inequality (2) can be rewritten in terms of two conditions

$$\text{local stability}\qquad 0\in\partial_v\mathcal{R}(z(t),0)+\overline{\partial}_z\mathcal{I}(T,z(t))\ \text{a.e. in }[0,T],\tag{3a}$$

$$\text{energy inequality}\qquad \begin{aligned}&\mathcal{I}(T,z(T))+\mathrm{Diss}_{\mathcal{R}}(z,[0,T])\\&\le \mathcal{I}(0,z(0))+\int_0^T \partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t,\end{aligned}\tag{3b}$$

where $\mathrm{Diss}_{\mathcal{R}}(z,[r,t])=\int_r^t \mathcal{R}(z(s),\dot z(s))\,\mathrm{d}s$ is the energy dissipated during the time interval $[r,t]$.

The local stability condition is a purely static concept and does not involve any time dependence, which shows that RIS are very close to static systems. In particular, if the loading does not change on a time interval $[t_1,t_2]$, then the solution may also be constant. Relation (3b) is a simple scalar energy inequality, which in fact should hold as an identity and also for all times $t\in[0,T]$ and not just for $t=T$. In all the different solution concepts discussed below we have these two different principles, namely (i) a static stability condition and (ii) an energy inequality. However, a crucial point in the definitions of solutions to RIS is always that the stability condition and the energy inequality interact in such a way that the stability condition implies a lower energy estimate on all subintervals of $[0,T]$, which together with the upper energy estimate (3b) provides energy balance on all subintervals.

These arguments apply to all our notions of solutions except for the *local solutions*, which ask for local stability and an upper energy estimate like (3b) on each subinterval $[r,t]\subset[0,T]$. This notion was introduced in [ToZ09], and it turns out that all solutions considered here fall into this class. In fact, we distinguish two important concepts, namely *energetic solutions* and *BV solutions*. The former were introduced in [MiT99, MTL02] and surveyed in [Mie05]. This notion is essentially the same as the notion of *irreversible quasistatic evolution* introduced and studied in [DaT02, FrL03, DFT05, FrG06] in the context of crack or damage evolution. Since these solutions allow for jumps, the infinitesimal dissipation potential $\mathcal{R}$ is replaced by the more general *dissipation distance* $\mathcal{D}:\boldsymbol{Z}\times\boldsymbol{Z}\to[0,\infty]$, which is obtained via

$$\begin{aligned}\mathcal{D}(z_0,z_1)\overset{\text{def}}{=}\inf\Big\{\int_{r=0}^1\mathcal{R}(z(r),\dot z(r))\,\mathrm{d}r\ \Big|\ &z\in\mathrm{W}^{1,1}([0,1];\boldsymbol{Z}),\\&z(0)=z_0,\ z(1)=z_1\Big\}.\end{aligned}$$

The local stability condition (3a) is replaced by the global stability condition (S), and the energy balance (E) is obtained from (3b) by replacing the dissipation functional $\mathrm{Diss}_{\mathcal{R}}(z,[r,t])$ by

$$\begin{aligned}\mathrm{Diss}_{\mathcal{D}}(z,[r,t])\overset{\text{def}}{=}\sup\Big\{\sum_{j=1}^N\mathcal{D}(z(s_{j-1}),z(s_j))\ \Big|\ &N\in\mathbb{N},\\&r\le s_0<s_1<\dots<s_N\le t\Big\},\end{aligned}$$

see Definition 3.1.

The notion of *BV solutions*, introduced in [MRS09b, MRS09a], is quite different from energetic solutions, since BV solutions jump as late as possible while energetic solutions jump as soon as possible, cf. Example 2.3. The BV solutions are constructed via the so-called *vanishing-viscosity limit* by adding a small viscosity to (1), namely

$$0\in\partial_{\dot z}\mathcal{R}(z^\varepsilon,\dot z^\varepsilon)+\varepsilon\mathbb{V}\dot z^\varepsilon+\overline{\partial}_z\mathcal{I}(t,z^\varepsilon)\quad\subset\boldsymbol{Z}^*,\tag{4}$$

and studying the limits of the *viscosity approximations* z^ε for $\varepsilon \to 0$, see Sections 4.5, 4.6, and 5.3. Hence, BV solutions contain the set of *approximable solutions*, which are defined as all limit points of this procedure, see [DD*08, KMZ08, ToZ09, KZM09]. While the notion of approximable solutions is simply defined by all possible limits, the set of BV solutions is characterized by the local stability condition (3a) and an energy estimate using a dissipation functional $\mathrm{Diss}_{\mathfrak{p},\mathcal{J}}$ that is supplemented by additional terms involving the viscous effects in jumps. The new structure is the *vanishing-viscosity contact potential*

$$\mathfrak{p}(z,v,\xi) \overset{\text{def}}{=} \inf\{\, \mathcal{R}_\varepsilon(z,v) + \mathcal{R}_\varepsilon^*(z,\xi) \mid \varepsilon > 0 \,\},$$

where $\mathcal{R}_\varepsilon(z,v) = \mathcal{R}_0(z,v) + \frac{\varepsilon}{2}\langle \mathbb{V}v, v\rangle$ is the sum of the rate-independent dissipation potential $\mathcal{R}_0$ and the small viscosity term $\frac{\varepsilon}{2}\langle \mathbb{V}v, v\rangle$. The supplemented dissipation distance then reads

$$\begin{aligned}\varDelta(t,z_0,z_1) \overset{\text{def}}{=} \inf\Big\{ & \textstyle\int_{r=0}^1 \mathfrak{p}(\widehat{z}(r), \dot{\widehat{z}}(r), -\mathrm{D}\mathcal{J}(t,\widehat{z}(r)))\,\mathrm{d}r \;\Big| \\ & \widehat{z} \in \mathrm{W}^{1,1}([0,1];\boldsymbol{Z}),\ \widehat{z}(0)=z_0,\ \widehat{z}(1)=z_1 \Big\}.\end{aligned}$$

A useful tool for the understanding the vanishing-viscosity limit is the notion of *parametrized solutions*, which was studied in [EfM06, MiZ09]. Here the solutions $z^\varepsilon : [0,T] \to \boldsymbol{Z}$ are parametrized in the extended state space $\boldsymbol{Z}_T \overset{\text{def}}{=} [0,T]\times\boldsymbol{Z}$ such that $(t,z) = (\tau^\varepsilon(s), Z^\varepsilon(s))$ with $\dot\tau^\varepsilon(s) + \langle \mathbb{V}\dot Z^\varepsilon(s), \dot Z^\varepsilon(s)\rangle^{1/2} = 1$ for a.a. $s \in [0, S^\varepsilon]$. Under suitable conditions, see Section 4.4 and 5.2, it is then possible to show that S^ε stays bounded and that the parametrized curves converge to a limit $(\tau, Z) : [0,S] \to \boldsymbol{Z}_T$, namely the desired parametrized solution. Moreover, from this we obtain in a natural way BV solutions by taking any $z : [0,T] \to \boldsymbol{Z}$, such that for all t there exists $s \in [0,S]$ such that $(t, z(t)) = (\tau(s), Z(s))$.

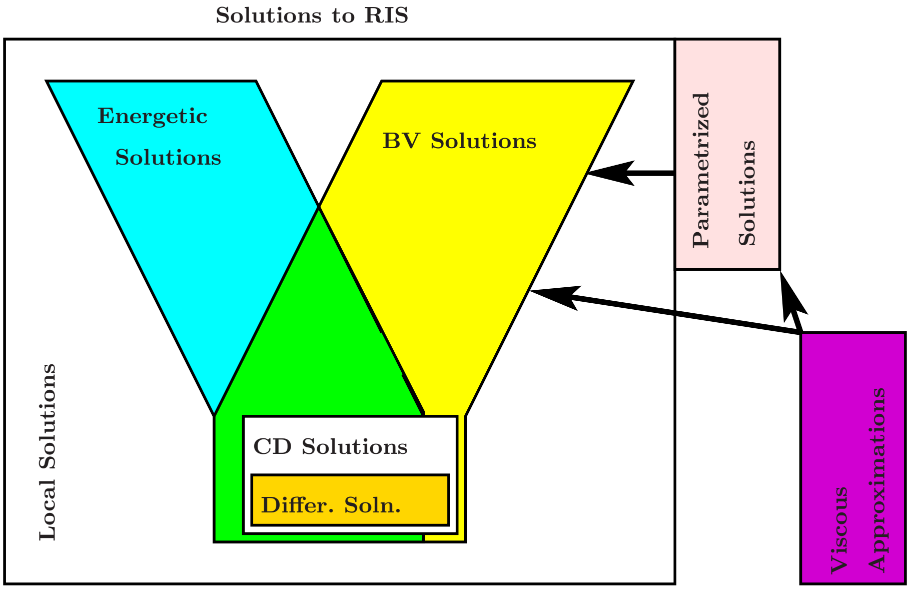

Figure 1 Overview on the different solution types for RIS

Figure 1 summarizes the solution types we are discussing in this work. We emphasize that all these notions satisfy the natural conditions for multivalued evolutionary systems, namely the concatenation and restriction property. In these notes we concentrate on the main ideas and techniques for the different solution concepts for RIS. Thus, we refrain from giving an overview of the whole theory and application of RIS, which can be found in the forthcoming monography [MiR09b].

In particular, we refer to

[Mie04, DD*07, DD*08, GaL09, Fia09] for RIS describing the evolution of microstructure via Young-measure valued internal variables. For variational characterizations of solutions for RIS we refer to [MiO08, MiS08, CoO08, Ste09].

2 Basics of rate-independent systems

2.1 Definition of rate independence

Since we develop quite abstract notions of solutions, we give a definition of rate independence that does not use differential equations. RIS occur as limit problems in many physical and mechanical systems, if the interesting time scales are much longer than the intrinsic time scales of the system. RIS are sometimes also called quasi-static systems, however, the term “quasistatic” is often used in a more general sense, namely if the inertial terms in a system are neglected but viscous effects might still be present.

This survey only considers systems which satisfy the following exact definition of rate independence. The definition is formulated in terms of input functions $\ell : [t_1, t_2] \to \mathcal{X}$ and the set $\mathcal{O}([t_1, t_2], q_0, \ell)$ of possible output functions $q : [t_1, t_2] \to \mathcal{Q}$ with $q(t_1) = q_0$. The usage of input and output functions is necessary, since RIS have no own dynamics; they rather respond to changes in the input.

Definition 2.1. *A input-output $\mathcal{H}$ is called a* rate-independent system *with input data $q_0 \in \mathcal{Q}$ and $\ell \in \mathrm{F}_0([t_1, t_2]; \mathcal{X})$, if the output set $\mathcal{O}([t_1, t_2], q_0, \ell) \subset \mathrm{F}_1([t_1, t_2]; \mathcal{Q}) \cap \{q(t_1) = q_0\}$ (where F_0 and F_1 denote suitable function spaces) satisfies, for all strictly monotone and continuous time reparametrizations $\alpha : [t_1, t_2] \to [t_1^*, t_2^*]$ with $\alpha(t_1) = t_1^*$ and $\alpha(t_2) = t_2^*$, the relation*

$$q \in \mathcal{O}([t_1, t_2], q_0, \ell) \quad \Longleftrightarrow \quad q \circ \alpha \in \mathcal{O}([t_1^*, t_2^*], q_0, \ell \circ \alpha).$$

We call the system a multi-valued evolutionary system, *if the following additional conditions hold:*

$$\begin{aligned}
&\textit{Concatenation:}\ \widehat{q} \in \mathcal{O}([t_1, t_2], q_1, \ell),\ \widetilde{q} \in \mathcal{O}([t_2, t_3], \widehat{q}(t_2), \ell),\\
&\qquad \Longrightarrow\ q \in \mathcal{O}([t_1, t_3], q_1, \ell) \text{ with } q(t) = \begin{cases} \widehat{q}(t) \text{ for } t \in [t_1, t_2],\\ \widetilde{q}(t) \text{ for } t \in [t_2, t_3],\end{cases}\\
&\textit{Restriction:}\quad t_1 \le t_2 < t_3 \le t_4 \text{ and } q \in \mathcal{O}([t_1, t_4], q_1, \ell)\\
&\qquad \Longrightarrow\ q|_{[t_2, t_3]} \in \mathcal{O}([t_2, t_3], q(t_2), \ell).
\end{aligned}$$

To compare this notion with the RIS $(\boldsymbol{Z}, \mathcal{I}, \mathcal{R})$ discussed previously, we let $\mathcal{I}(t, z) = \mathcal{U}(z) - \langle \ell(t), z\rangle$ and find the equation

$$0 \in \partial_v \mathcal{R}(z(t), \dot{z}(t)) + \overline{\partial}\mathcal{U}(z(t)) - \ell(t),$$

where we explicitly see the input ℓ. Assuming an existence result for this differential inclusion, it is then clear that the concatenation and restriction properties hold.

RIS are used to model hysteresis, which is often associated with memory effects (cf. [Vis94, BrS96, Kre99]). Here we take a different approach using suitable internal variables that carry all memory information. This fact is the content of the concatenation property. If on $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ we consider a RIS we can easily obtain a system with memory by association with (q_0, ℓ) the output set $\mathcal{O}_{\mathcal{Y}}([t_1, t_2], q_0, \ell)$ given by

$$\{\, y \in \mathrm{F}_1([t_1,t_2];\mathcal{Y}) \mid \exists\, z \in \mathrm{F}_1([t_1,t_2];\mathcal{Z})\colon (y,z) \in \mathcal{O}([t_1,t_2],q_0,\ell)\,\}.$$

Clearly, the concatenation and the restriction properties are then lost.

2.2 Differentiable formulations and the decomposition into elastic and dissipative parts

The typical situations, which are the basis of this work, are differential inclusions of the form

$$0 \in \partial_{\dot q}\mathcal{R}(q,\dot q) + \mathrm{D}_q\mathcal{E}(t,q), \tag{1}$$

where the state q lies in a Banach space $\boldsymbol{Q}$. The functional $\mathcal{R} : \boldsymbol{Q}{\times}\boldsymbol{Q} \to [0,\infty]$ is called the dissipation potential and $\mathcal{E} : \boldsymbol{Q}_T \overset{\mathrm{def}}{=} [0,T]{\times}\boldsymbol{Q} \to \mathbb{R}_\infty$ the energy functional. Thus, (1) can be interpreted as a force balance in $\boldsymbol{Q}^*$ where the dissipative force $\partial_{\dot q}\mathcal{R}(q,\dot q)$ must equilibrate the potential restoring force $-\mathrm{D}_q\mathcal{E}(t,q)$. Such systems are generalizations of the so-called doubly nonlinear systems considered in [CoV90, Col92], where the special case $\mathcal{R}(q,\dot q) = \Psi(\dot q)$ is studied.

In many applications the state space $\boldsymbol{Q}$ decomposes into two parts, namely an elastic part $\boldsymbol{Y}$ and a dissipative part $\boldsymbol{Z}$, i.e. we have $q = (y,z)$ and $\boldsymbol{Q} = \boldsymbol{Y}{\times}\boldsymbol{Z}$. The distinction comes about because the functional $\mathcal{R}$ depends only on the z-component as follows:

$$\mathcal{R}(q,\dot q) = \mathcal{R}(z,\dot z) \quad \text{and} \quad \mathcal{R}(z,\dot z) = 0 \;\Rightarrow\; \dot z = 0. \tag{2}$$

In that case (1) takes the form of a coupled system, namely

$$0 = \mathrm{D}_y\mathcal{E}(t,y,z), \qquad 0 \in \partial_{\dot z}\mathcal{R}(z,\dot z) + \mathrm{D}_z\mathcal{E}(t,y,z). \tag{3}$$

Hence, the two components y and z need to be treated differently. In particular, often we study the reduced problem by minimizing with respect to y, viz.

$$\mathcal{I}(t,z) = \min\{\, \mathcal{E}(t,y,z) \mid y \in \boldsymbol{Y} \,\}. \tag{4}$$

Since this means that we have satisfied the first relation in (3), we are left with the reduced problem

$$0 \in \partial_{\dot z}\mathcal{R}(z,\dot z) + \mathrm{D}_z\mathcal{I}(t,z). \tag{5}$$

Moreover, we can go backward from (5). If $z : [0,T] \to \boldsymbol{Z}$ solves (5), we may choose $y : [0,T] \to \boldsymbol{Y}$ such that $y(t) \in \operatorname{Arg\,min} \mathcal{E}(t,\cdot,z(t))$, then $q : t \mapsto (y(t),z(t))$ solves (3).

These systems also include classical gradient flows of the form $G(q)\dot q = -\mathrm{D}_q\mathcal{E}(t,q)$, if we choose $\mathcal{R}(q,v) = \frac{1}{2}\langle G(q)v,v\rangle$. However, rate independence now means that the mapping $v \mapsto \partial_{\dot q}\mathcal{R}(q,v)$ is positively homogeneous of degree 0, i.e. $\partial_{\dot q}\mathcal{R}(q,\gamma v) = \partial_{\dot q}\mathcal{R}(q,v)$ for all $\gamma > 0$. We generally say that a mapping $f : \boldsymbol{X} \to Y$ is *p-homogeneous*, if it is positively homogeneous of degree p, i.e. $f(\lambda x) = \lambda^p f(x)$ for all $x \in \boldsymbol{X}$ and all $\lambda > 0$. Thus, $\mathcal{R}(q,\cdot) : \boldsymbol{Q} \to \mathbb{R}_\infty$ has to be 1-homogeneous, which implies that $\mathcal{R}(q,\cdot)$ either is identically 0 or it is not differentiable at $v = 0$.

Thus, from now on all dissipation potentials are assumed to satisfy

$$\begin{aligned} &\mathcal{R}(z,\cdot) : \boldsymbol{Z} \to [0,\infty] \text{ is convex, lower semicontinuous} \\ &\text{and satisfies } \mathcal{R}(z,0) = 0. \end{aligned} \tag{6}$$

The derivative of $\mathcal{R}$ with respect to v is the set-valued convex subdifferential

$$\partial_{\dot z}\mathcal{R}(z,v) = \{\, \eta \in \boldsymbol{Z}^* \mid \forall\, w \in \boldsymbol{Z} :\ \mathcal{R}(z,w) \geq \mathcal{R}(z,v) + \langle \eta, w{-}v\rangle \,\}.$$

If $\mathcal{R}$ is 1-homogeneous, then the triple $(\boldsymbol{Q}, \mathcal{E}, \mathcal{R})$ is called a *rate-independent system* (RIS).

Very often we also look at rate-dependent versions of (1), i.e. we consider a potential $\mathcal{R}_{\rm sl}$ such that $\mathcal{R}_{\rm sl}(q,\cdot)$ is superlinear (whence not 1-homogeneous) and still assume (6). Note that the rate-independent case can be recovered by slowing down the loading rate. In fact, if we replace t in (1) by $\varepsilon\tau$ and let $\widetilde{z}(\tau) = z(\varepsilon t)$, then $\widetilde{z}$ solves the equation

$$0 \in \partial_{\widetilde{z}'}\widetilde{\mathcal{R}}_\varepsilon(\widetilde{z}, \widetilde{z}') + \mathrm{D}_{\widetilde{z}}\mathcal{I}(\tau, \widetilde{z}), \text{ where } \widetilde{\mathcal{R}}_\varepsilon(z,v) = \frac{1}{\varepsilon}\mathcal{R}_{\rm sl}(z, \varepsilon v).$$

By classical convexity arguments we have $\widetilde{\mathcal{R}}_\varepsilon(z,v) \searrow \mathcal{R}_0(z,v)$ for $\varepsilon \to 0$, where $\mathcal{R}_0$ is 1-homogeneous again. The limit passage $\varepsilon \to 0$ for the corresponding solutions z^ε is called the vanishing-viscosity limit and will be discussed in some detail in Sections 4 and 5.

2.3 Some canonical examples

(1) The simplest example is obtained in the scalar case $z \in \boldsymbol{Z} = \mathbb{R}$ with the dissipation potential $\mathcal{R}(z,v) = |v|$ and the energy functional $\mathcal{I}(t,z) = \frac{1}{2}z^2 - \ell(t)z$. We obtain the equation

$$0 \in \mathrm{Sign}(\dot{z}) + z - \ell(t), \tag{7}$$

where Sign is the multi-valued Signum function depicted in Figure 1. We observe that we always

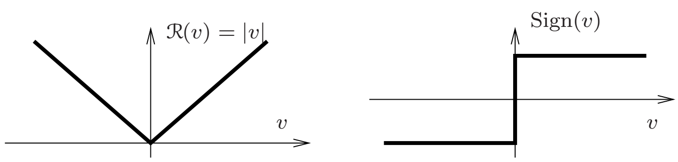

Figure 1 Multivalued signum function $\mathrm{Sign} = \partial|\cdot|$

have $|z(t) - \ell(t)| \leq 1$. Moreover, $|z(t) - \ell(t)| < 1$ implies $\dot{z}(t) = 0$, whereas $\pm\dot{z}(t) > 0$ implies $z(t) = \ell(t) \mp 1$. We obtain the so-called play operator, where q follows ℓ with a play of size 1, see Figure 2.

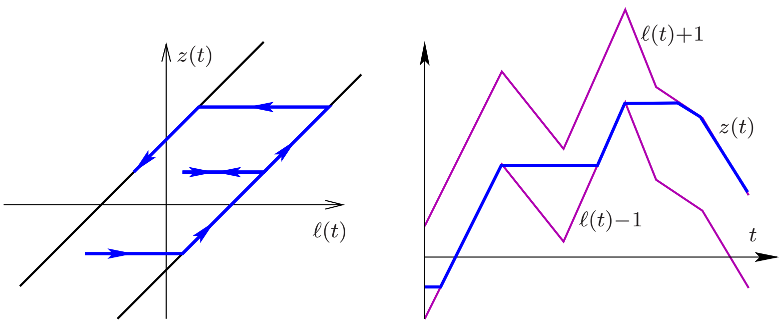

Figure 2 The play operator associated with (7).

(2) An infinite dimensional generalization leads to the most classical example of a rate-independent process formulated in a Hilbert space $\boldsymbol{Q}$ with a quadratic energy

$$\mathcal{E}(t,q) = \tfrac{1}{2}\langle \mathbb{A}q, q\rangle - \langle \ell(t), q\rangle$$

and a dissipation potential $\Psi : \boldsymbol{Q} \to [0,\infty]$. This situation is studied under the name *sweeping process*, see e.g. [Mor77, KuM98]. The differential form reads $0 \in \partial\Psi(\dot q) + \mathbb{A}q - \ell(t)$. The input ℓ is considered as the center of the moving set $C(t) = \ell(t) - \partial\Psi(0)$ and the solution needs to satisfy $\mathbb{A}q(t) \in C(t)$. Nowadays the name *play operator* is used for this process, cf. [Kre99, Vis94, BrS96].

(3) The motivation of the above sweeping process was the classical problem of *linearized elasto-plasticity*, see [Mor76, Grö78]. For a body $\Omega \subset \mathbb{R}^d$ the state q consists of the displacement $u \in \boldsymbol{Y} = \mathrm{H}^1_\Gamma(\Omega;\mathbb{R}^d) = \{\, u \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \mid u|_\Gamma = 0\,\}$ and the plastic strain tensor $z \in \mathrm{L}^2(\Omega; Z)$ with $Z = \{\, z \in \mathbb{R}^{d\times d} \mid z = z^\mathsf{T},\ \operatorname{tr} z = 0\,\}$. The total energy contains the elastic energy, the hardening energy and the external loading:

$$\mathcal{E}(t,u,z) = \int_\Omega \tfrac{1}{2}(\mathrm{e}(u)-z){:}\mathbb{C}{:}(\mathrm{e}(u)-z) + \tfrac{1}{2}z{:}\mathbb{H}{:}z - u\cdot f_\mathrm{ext}(t)\,\mathrm{d}x,$$

where $\mathrm{e}(u) = \frac{1}{2}(\nabla u + \nabla u^\mathsf{T})$ is the infinitesimal strain tensor, and $\mathbb{C}$ and $\mathbb{H}$ are positive definite fourth-order tensors for elasticity and hardening, respectively. The dissipation potential reads $\mathcal{R}(z,\dot z) = \int_\Omega \sigma_\mathrm{yield}|\dot z(x)|\,\mathrm{d}x$. The subdifferential formulation then reads

$$-\operatorname{div}\big(\mathbb{C}{:}(\mathrm{e}(u)-z)\big) = f_\mathrm{ext}, \quad 0 \in \sigma_\mathrm{yield}\operatorname{Sign}(\dot z) + \mathbb{C}{:}(z-\mathrm{e}(u)) + \mathbb{H}{:}z.$$

For more details and general small-strain models we refer to [Alb98, HaR99].

(4) Elastoplastic models with finite strain lead to highly nonlinear rate-independent models. We refer to [Mie03] for a discussion of the mathematical and mechanical background involving the associated Lie groups and to [MaM09] for existence results for energetic solutions in the the PDE context. Here we consider the simplified material-point mechanics, which applies to bodies that are deformed homogeneously.

The deformation gradient $F = \nabla\phi$ is treated as an element of the general linear group $\mathcal{Y} = \mathrm{GL}^+(\mathbb{R}^d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0\,\}$, and the plastic tensor P is taken from the special linear group $\mathcal{Z} = \mathrm{SL}(\mathbb{R}^d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F = 1\,\}$. The energy takes the form

$$\mathcal{E}(t,F,P) = W_\mathrm{elast}(F\,P^{-1}) + W_\mathrm{hard}(P) - \Sigma(t){:}F,$$

where the multiplicative decomposition of the strain tensor $F = F_\mathrm{elast}P$ gives rise to the geometric nonlinearity $F_\mathrm{elast} = F\,P^{-1}$ appearing in W_elast. Here $\Sigma(t) \in \mathbb{R}^{d\times d}$ is the applied stress. Because of plastic invariance the dissipation potential takes the form

$$\mathcal{R}(P,\dot P) = \psi(\dot P\,P^{-1}) \ \text{ with } \psi(\eta) = \sigma_\mathrm{yield}|\eta|.$$

Thus, the dissipation potential depends intrinsically on the internal state. This and the strong geometric nonlinearity give rise to solutions with jumps, cf. [Mie03, MaM09].

2.4 The basic a priori estimates

To understand the main difficulties in modeling RIS $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$, we give the basic estimates, which follow from the differential formulation (1). This part is formal and needs proper justification for the construction of solutions. At the moment we just motivate some basic concepts.

We first provide a basic property of subdifferentials of 1-homogeneous functions.

Lemma 2.2. *Let $\Psi : \boldsymbol{Q} \to [0,\infty]$ be lower semicontinuous, convex and 1-homogeneous. Then, we have*

$$\partial\Psi(v) = \{\, \eta \in K \mid \Psi(v) = \langle \eta, v\rangle \,\}, \quad \textit{where } K = \partial\Psi(0).$$

Moreover, we have the characterization

$$0 \in \partial\Psi(v) + g \quad \Longleftrightarrow \quad 0 = \Psi(v) + \langle g, v\rangle \leq \Psi(w) + \langle g, w\rangle \text{ for all } w \in \boldsymbol{Q}.$$

Using the characterization of $\partial_{\dot q}\mathcal{R}(q,\cdot)$ we see that (1) is equivalent to

$$\forall\, v \in Q: \ \langle \mathrm{D}_q\mathcal{E}(t,q), v\rangle + \mathcal{R}(q,v) \geq 0, \tag{8a}$$
$$\langle \mathrm{D}_q\mathcal{E}(t,q), \dot q\rangle + \mathcal{R}(q,\dot q) = 0. \tag{8b}$$

Differentiation $\mathcal{E}(t,q(t))$ we see that (8b) is equivalent to the energy balance

$$\mathcal{E}(t,q(t)) + \int_r^t \mathcal{R}(q(s),\dot q(s))\,\mathrm{d}s = \mathcal{E}(r,q(r)) + \int_r^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s, \tag{9}$$

where $\partial_t\mathcal{E}$ denotes the usual partial derivative with respect to time, which has the physical meaning of the power induced by the temporal changes in the system. This identity now holds for all r,t with $0 \leq r < t \leq T$.

Throughout we assume that the power $\partial_t\mathcal{E}$ can be controlled by the energy itself, namely

$$\begin{aligned}&\exists\,\lambda_{\mathcal{E}} > 0\ \forall\,(s,q) \in \mathcal{Q}_T \text{ with } \mathcal{E}(s,q) < \infty:\\ &\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t\mathcal{E}(t,q)| \leq \lambda_{\mathcal{E}}\,\mathcal{E}(t,q) \text{ for all } t \in [0,T].\end{aligned} \tag{10}$$

Using (9), (10), $\mathcal{R} \geq 0$, a Gronwall estimate gives the basic estimates

$$\mathcal{E}(t,q(t)) \leq \mathrm{e}^{\lambda_{\mathcal{E}} t}\mathcal{E}(0,q(0)) \quad \text{and} \quad \int_0^t \mathcal{R}(q(s),\dot q(s))\,\mathrm{d}s \leq \mathrm{e}^{\lambda_{\mathcal{E}} t}\mathcal{E}(0,q(0)).$$

The first estimate is useful, since we always assume that $\mathcal{E}(t,\cdot)$ is coercive. The second estimate controls the temporal behavior. First, we must take into account that $\mathcal{R}$ only controls the dissipative part z of $q = (y,z) \in \boldsymbol{Y}\times\boldsymbol{Z}$ and that this control may only be valid in a weaker norm, namely

$$\mathcal{R}(y,z,\dot y,\dot z) = \mathcal{R}(z,\dot z) \geq c_R\|\dot z\|_{\boldsymbol{X}}.$$

Second, the 1-homogeneity only provides a bound in $\mathrm{W}^{1,1}([0,T],\boldsymbol{X})$. However, since in this space the unit ball is not weakly closed, we have to work with the space $\mathrm{BV}([0,T],\boldsymbol{X})$, i.e., we have

$$\mathrm{Var}_{\boldsymbol{X}}(z,[0,T]) \leq \frac{1}{c_R}\int_0^T \mathcal{R}(q(t),\dot q(t))\,\mathrm{d}t \leq \frac{1}{c_R}\mathrm{e}^{\lambda_{\mathcal{E}} T}\mathcal{E}(0,q(0)).$$

This a priori estimate is essential to obtain temporal compactness and allows us to use a suitable version of Helly's selection principle for z. Yet, this estimate does not give control over the temporal behavior of y. Moreover, we have to be aware of the possibility of jumps, which occur in limit procedures.

The above estimates can be improved under suitable convexity assumptions. For this we use (8) as follows. We fix $\tau \in\,]0,T[$ and consider $\gamma(t) = \langle \mathrm{D}_q\mathcal{E}(t,q(t)), \dot q(\tau)\rangle + \mathcal{R}(q(t),\dot q(\tau))$. We have $\gamma(t) \geq 0$ and $\gamma(\tau) = 0$ from (8a) and (8b), respectively. Thus we conclude $\dot\gamma(\tau) = 0$, which gives

$$\langle \mathrm{D}_q^2\mathcal{E}(\tau,q(\tau))\dot q(\tau), \dot q(\tau)\rangle + \mathrm{D}_q\mathcal{R}(q(\tau),\dot q(\tau))[\dot q(\tau)] = -\langle \mathrm{D}_q\partial_t\mathcal{E}(\tau,q(\tau)), \dot q(\tau)\rangle.$$

Thus, assuming uniform convexity of $\mathcal{E}(t,\cdot): \boldsymbol{Q} \to \mathbb{R}_\infty$ and that $\mathrm{D}_q\mathcal{R}$ is sufficiently small, we obtain a bound of the type

$$\|\dot q(\tau)\|_{\boldsymbol{Q}} \leq \frac{1}{\kappa-\rho}\|\partial_t\mathrm{D}_q\mathcal{E}(\tau,q(\tau))\|_{\boldsymbol{Q}^*},$$

if the *joint-convexity condition*

$$\kappa > \rho \tag{11}$$

holds, where $\kappa > 0$ and $\rho \in [0, \kappa[$ are such that

$$\langle \mathrm{D}_q^2 \mathcal{E}(\tau, q(\tau))v, v\rangle \geq \alpha \|v\|_{\boldsymbol{Q}}^2 \quad \text{and} \quad |\mathrm{D}_q \mathcal{R}(q, v)[v]| \leq \rho \|v\|_{\boldsymbol{Q}}^2.$$

In Section 4.3 we discuss how these estimates can be used to prove the existence of differentiable solutions in the convex case.

2.5 Energetic formulation of generalized gradient flows

The starting point of the modeling are generalized gradient systems $(\boldsymbol{V}, \mathcal{I}, \mathcal{R})$, which are not necessarily rate independent. The state space $\boldsymbol{V}$ is a Hilbert space, $\mathcal{I} : \boldsymbol{V}_T \to \mathbb{R}_\infty$ is the energy functional, and the dissipation potential $\mathcal{R} : \boldsymbol{V} \times \boldsymbol{V} \to [0, \infty]$ satisfies (6). The evolution equation is given in the form

$$0 \in \partial_{\dot{z}} \mathcal{R}(z, \dot{z}) + \mathrm{D}_z \mathcal{I}(t, z).$$

The classical gradient flow is obtained, if $\mathcal{R}$ is given in terms of a Riemannian tensor $G(z) : \boldsymbol{V} \to \boldsymbol{V}^*$, which is symmetric and positive definite, viz. $\mathcal{R}(z, v) = \frac{1}{2}\langle G(z)v, v\rangle$. Then, we have

$$0 = G(z)\dot{z} + \mathrm{D}_z \mathcal{I}(t, z) \;\Leftrightarrow\; \dot{z} = -\nabla \mathcal{I}(t, z) = -G(z)^{-1} \mathrm{D}_z \mathcal{I}(t, z).$$

Rate-independent systems are special generalized gradient flows, namely those for which $\mathcal{R}(z, \cdot)$ is 1-homogeneous, i.e. $\mathcal{R}(z, \alpha v) = \alpha \mathcal{R}(z, v)$ for $\alpha > 0$. Before going into more detail we discuss some equivalent formulations of generalized gradient flows.

Let $\boldsymbol{V}$ be a Hilbert space and $F : \boldsymbol{V} \to \mathbb{R}_\infty$ a proper, lower semicontinuous, and convex function. Its Legendre-Fenchel transform $F^* : \boldsymbol{V}^* \to \mathbb{R}_\infty$ is defined via $F^*(\xi) = \sup_{v \in \boldsymbol{V}} \langle \xi, v\rangle - F(v)$. The Fenchel equivalences for subdifferentials read

$$\xi \in \partial F(v) \quad \Leftrightarrow \quad v \in \partial F^*(\xi) \quad \Leftrightarrow \quad F(v) + F^*(\xi) = \langle \xi, v\rangle. \tag{12}$$

Moreover, by the definition of F^* we always have the lower bound

$$F(v) + F^*(\xi) \geq \langle \xi, v\rangle \quad \text{for all } v \in \boldsymbol{V} \text{ and } \xi \in \boldsymbol{V}^*. \tag{13}$$

Using the equivalences in (12), our subdifferential equations can be written in three equivalent ways:

Force balance	$0 \in \partial_{\dot{z}} \mathcal{R}(z(t), \dot{z}(t)) + \xi(t) \subset \boldsymbol{V}^*$;	(14a)
Rate equation	$\dot{z}(t) \in \partial_\xi \mathcal{R}^*(z(t), -\xi(t)) \subset \boldsymbol{V}$;	(14b)
Energy balance	$\mathcal{R}(z(t), \dot{z}(t)) + \mathcal{R}^*(z(t), -\xi(t)) = \langle -\xi(t), \dot{z}(t)\rangle \in \mathbb{R}$;	(14c)

where in all three formulations the additional condition $\xi(t) \in \overline{\partial}_z \mathcal{I}(t, z(t)) \subset \boldsymbol{V}^*$ is imposed. The last relation can be combined with a chain rule, namely

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}(t, z(t)) = \langle \xi(t), \dot{z}(t)\rangle + \partial_t \mathcal{I}(t, z(t)). \tag{15}$$

Applying the chain rule and integrating (14c) over the time interval $[0, T]$ we obtain the *integral inequality*

$$\begin{aligned} &\mathcal{I}(T, z(T)) + \textstyle\int_0^T \mathcal{R}(z(t), \dot{z}(t)) + \mathcal{R}^*(z(t), -\xi(t))\,\mathrm{d}t \\ &\leq \mathcal{I}(0, z(0)) + \textstyle\int_0^T \partial_t \mathcal{I}(t, z(t))\,\mathrm{d}t \quad \text{with } \xi(t) \in \overline{\partial}_z \mathcal{I}(t, z(t)), \end{aligned} \tag{16}$$

which holds in fact as an equality. We call (16) the *energetic formulation* of the problem defined via (14). The point of importance is that already the integral inequality (16) is equivalent to the fact that the three formulations in (14) hold a.e. in $[0,T]$. To see this we use the lower estimate (13) and the chain rule (15) to obtain

$$\begin{aligned}&\langle -\xi(t), \dot z(t)\rangle \leq \mathcal{R}(z(t),\dot z(t)) + \mathcal{R}^*(z(t),-\xi(t)) \text{ a.e. on } [0,T],\\ &\textstyle\int_0^T \mathcal{R}(z(t),\dot z(t)) + \mathcal{R}^*(z(t),-\xi(t))\,\mathrm{d}t \leq \int_0^T \langle -\xi(t),\dot z(t)\rangle\,\mathrm{d}t,\end{aligned}$$

which immediately implies (14c) a.e.

A major advantage of the formulation as an integral inequality is seen for parameter-dependent dissipation potentials. Then, defining the functional

$$\mathcal{M}_\varepsilon(z,v,\eta) = \int_0^T \mathcal{R}_\varepsilon(z(t),\dot z(t)) + \mathcal{R}_\varepsilon^*(z(t),\eta(t))\,\mathrm{d}t,$$

we want to pass to the limit $\varepsilon \to 0$ in equations of the type

$$\mathcal{I}(T,z^\varepsilon(T)) + \mathcal{M}_\varepsilon(z^\varepsilon,\dot z^\varepsilon,-\xi^\varepsilon) \leq \mathcal{I}(0,z^\varepsilon(0)) + \int_0^T \partial_t\mathcal{I}(t,z^\varepsilon(t))\,\mathrm{d}t.$$

If the solutions z^ε converge to a limit z, we can use then Γ-limit arguments to find a limit version of an integral inequality for z in the form

$$\mathcal{I}(T,z(T)) + \mathcal{M}_0(z,\dot z,-\xi) \leq \mathcal{I}(0,z(0)) + \int_0^T \partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t,$$

In this limit passage it is important to maintain the subdifferential property, i.e. we need a closedness of the subdifferential in the following form:

$$\xi^\varepsilon(t) \in \overline{\partial}_z\mathcal{I}(t,z^\varepsilon(t),\ z^\varepsilon \rightsquigarrow z,\ \xi^\varepsilon \rightsquigarrow \xi \quad \Rightarrow \quad \xi(t) \in \overline{\partial}_z\mathcal{I}(t,z(t)).$$

Moreover, the limit functional $\mathcal{M}_0$ has to be such that it still interacts properly with a suitable chain rule, which allows us then to obtain the opposite inequality.

If $\mathcal{R}(z,\cdot)$ is 1-homogeneous, the generalized gradient system $(\boldsymbol{V},\mathcal{I},\mathcal{R})$ is a RIS. Then, $\mathcal{R}^*$ has a very specific form, namely

$$\mathcal{R}^*(z,\xi) = \begin{cases} 0 \ \text{ for } \xi \in K(z), \\ \infty \text{ otherwise,}\end{cases} \qquad \text{where } K(z) = \partial_v\mathcal{R}(z,0) \subset \boldsymbol{V}^*.$$

Thus, the term $\int_0^T \mathcal{R}^*(z(t),-\xi(t))\,\mathrm{d}t$ contributes to the right-hand side in (16) either the value 0 or the value ∞, where the latter case would violate the validity. Hence, the term only acts as a side condition asking that $-\xi(t) \in K(z(t))$ for a.a. $t \in [0,T]$. In conclusion, (16) can be rewritten as follows

$$\begin{aligned}&z(t) \in \mathcal{S}_{\mathrm{loc}}(t) \overset{\mathrm{def}}{=} \{\, z{\in}\boldsymbol{Z} \mid 0{\in}\partial_v\mathcal{R}(z(t),0){+}\overline{\partial}_z\mathcal{I}(t,z)\,\} \text{ a.e. on } [0,T],\\ &\textstyle\mathcal{I}(T,z(T)) + \int_0^T \mathcal{R}(z(t),\dot z(t))\,\mathrm{d}t \leq \mathcal{I}(0,z(0)) + \int_0^T \partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t.\end{aligned} \tag{17}$$

The first line constitutes a local stability condition that does not involve any time derivative and thus is a purely static condition, while the second condition is the usual upper energy estimate. As before, the chain rule applied to $t \mapsto \mathcal{I}(t,z(t))$ implies that the two lines provide an exact energy balance.

The problem with RIS is that in general we cannot expect solutions to be absolutely continuous with respect to time. Hence we derive notions of solutions that allow solutions with jumps. It is a common feature of all these notions that they consist of a static stability condition and an energy inequality, which we often formulate directly as an energy balance.

2.6 Solution concepts in the one-dimensional case

Here we introduce the different solutions concepts for a very trivial situation, namely the case $\boldsymbol{Q} = \boldsymbol{Z} = \mathbb{R}$, i.e. we work directly with the reduced functional $\mathcal{I}$. The aim is to discuss their mutual relations already in this easy context, where functional analytical questions do not yet show up. We let

$$\mathcal{R}(z,\dot z) = \begin{cases} r_+(z)\dot z & \text{for } \dot z \geq 0, \\ r_-(z)|\dot z| & \text{for } \dot z \leq 0; \end{cases} \qquad \text{and} \quad \mathcal{I}(t,z) = \mathcal{U}(z) - \ell(t)z, \tag{18}$$

where $r_+, r_-\mathrm{BC}(\mathbb{R})$, $r_\pm(z) \geq \rho > 0$, and the function ℓ will be specified in the different examples. With the local dissipation metric $\mathcal{R}$ we associate the dissipation distance $\mathcal{D}$ defined via

$$\mathcal{D}(z_0,z_1) = \begin{cases} \int_{z_0}^{z_1} r_+(z)\,\mathrm{d}z & \text{for } z_0 \leq z_1, \\ \int_{z_1}^{z_0} r_-(z)\,\mathrm{d}z & \text{for } z_0 \geq z_1. \end{cases}$$

All definitions for solutions are for general $\mathcal{I}$, but for examples we use (18) with the nonconvex potential $\mathcal{U}$ and the initial datum z_0 given via

$$\mathcal{U}(z) = \begin{cases} \frac12 (z{+}4)^2 & \text{for } z \leq -2, \\ 4{-}\frac12 z^2 & \text{for } |z| \leq 2, \\ \frac12 (z{-}4)^2 & \text{for } z \geq 2; \end{cases} \qquad \text{and} \quad z_0 = -5. \tag{19}$$

If $\ell : [0,T] \to \mathbb{R}$ is specified, the RIS $(\boldsymbol{Z}, \mathcal{I}, \mathcal{R})$ is fully given.

We now introduce the main solution concepts, which are somewhat less involved in the present one-dimensional setting. Note that in all cases the solution $z : [0,T] \to \mathbb{R}$ is defined for *all* t, while some conditions need to hold only a.e. in $[0,T]$.

(1) A *differential solution* $z : [0,T] \to \mathbb{R}$ is defined via $z \in \mathrm{W}^{1,1}([0,T])$ and

$$0 \in \partial\mathcal{R}(z(t),\dot z(t)) + \mathrm{D}_z\mathcal{I}(t,z(t)) \quad \text{for a.a. } t \in [0,T].$$

(2) A *CD solution* $z{:}[0,T] \to \mathbb{R}$ (for ‘C’ontinuous ‘D’issipation) is defined via

$$\begin{array}{ll} \text{cont. dissipation} & t \mapsto \mathrm{Diss}_{\mathcal{D}}(z,[0,t]) \text{ is continuous,} \\ \text{local stability} & 0 \in \partial\mathcal{R}(z(t),0) + \mathrm{D}_z\mathcal{I}(t,z(t)) \quad \text{for a.a. } t \in [0,T]; \\ \text{energy balance} & \forall\, t \in [0,T] : \\ & \mathcal{E}(t,z(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[0,t]) = \mathcal{E}(0,z(0)) - \int_0^t \dot\ell(s)z(s)\,\mathrm{d}s, \end{array}$$

where $\mathrm{Diss}_{\mathcal{D}}(z,[r,t]) = \sup \sum_{j=1}^N \mathcal{D}(z(t_{j-1}),z(t_j))$ with the supremum taken over all finite partitions $r \leq t_0 < t_1 < \cdots y < t_{N-1} < t_N \leq t$.

(3) A *local solution* $z : [0,T] \to \mathbb{R}$ is defined via

$$\begin{array}{ll} \text{local stability} & 0 \in \partial\mathcal{R}(z(t),0) + \mathrm{D}_z\mathcal{I}(t,z(t)) \ \text{ for a.a. } t \in [0,T]; \\ \text{energy inequality} & \forall\, r,t \text{ with } 0 \le r < t \le T: \\ & \mathcal{E}(t,z(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[r,t]) \le \mathcal{E}(r,z(r)) - \int_r^t \dot{\ell}(s) z(s)\,\mathrm{d}s. \end{array}$$

(4) An *energetic solution* $z : [0,T] \to \mathbb{R}$ is defined via, for all $t \in [0,T]$,

$$\begin{array}{ll} \text{global stability} & \mathcal{I}(t,z(t)) \le \mathcal{I}(t,\widetilde{z}) + \mathcal{D}(z(t),\widetilde{z}) \text{ for all } \widetilde{z} \in Z; \\ \text{energy balance} & \mathcal{E}(t,z(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[0,t]) \le \mathcal{E}(0,z(0)) + \int_0^t \partial_t \mathcal{I}(s,z(s))\,\mathrm{d}s. \end{array}$$

(5) An *approximable solution* $z : [0,T] \to \mathbb{R}$ is defined as a pointwise limit of a sequence $(z^{\varepsilon_k})_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ of solutions z^ε of the viscous problems

$$0 \in \partial\mathcal{R}(\dot{z}^\varepsilon) + \varepsilon \dot{z}^\varepsilon + \mathrm{D}_z\mathcal{I}(t,z^\varepsilon(t)) \ \text{ for a.a. } t \in [0,T].$$

Approximable solutions are also called *vanishing-viscosity solutions.*

(6) A pair $(\widehat{t},\widehat{z}) : [0,S] \to [0,T]\times\mathbb{R}$ is defined to be a *parametrized solution*, if $(\widehat{t},\widehat{z}) \in \mathrm{W}^{1,1}([0,S],\mathbb{R}^2)$ and if for a.a. $s \in [0,S]$ we have

$$\begin{array}{rl} \text{(i)} & \widehat{t}(0) = 0,\ \widehat{t}(S) = T,\ \widehat{t}\,'(s) \ge 0, \\ \text{(ii)} & \widehat{t}\,'(s) + |\widehat{z}\,'(s)| = 1, \\ \text{(iii)} & 0 \in \partial_{\widehat{z}'}\widehat{\mathcal{R}}(\widehat{z}(s),\widehat{z}\,'(s)) + \mathrm{D}_z\mathcal{I}(\widehat{t}(s),\widehat{z}(s)), \end{array}$$

where $\widehat{\mathcal{R}}(z,v) = \mathcal{R}(z,v)$ for $|v| \le 1$ and ∞ otherwise.

(7) A *BV solution* $z : [0,T] \to \mathbb{R}$ is defined via $z \in \mathrm{BV}([0,T])$ and

$$\begin{array}{ll} \text{(i)} & 0 \in \partial_{\dot{z}}\mathcal{R}(z(t),0) + \mathrm{D}_z\mathcal{I}(t,z(t)) \ \text{ a.e. in } [0,T]; \\ \text{(ii)} & \text{for all } t \in [0,T] \text{ we have} \end{array}$$
$$\mathcal{I}(t,z(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[0,t]) + \mathrm{Jmp}_{\mathcal{I}}(z,[0,t]) = \mathcal{I}(0,z(0)) + \int_0^t \partial_s\mathcal{I}(s,z(s))\,\mathrm{d}s,$$

where $\mathrm{Jmp}_{\mathcal{I}}$ is defined via the jump set $J(z) \overset{\mathrm{def}}{=} \{\, t \mid z \text{ not contin. in } t \,\}$ as

$$\begin{aligned} \mathrm{Jmp}_{\mathcal{I}}(z,[r,t]) \overset{\mathrm{def}}{=} & \varDelta(r,z(r),z(r^+)) + \varDelta(t,z(t^-),z(t)) \\ & + \textstyle\sum_{s\in J(z)\cap]r,t[} \varDelta(s,z(s^-),z(s)) + \varDelta(s,z(s),z(s^+)), \end{aligned}$$

where $\varDelta(t,z_0,z_1) = |\int_{z_0}^{z_1} \mathrm{dist}(-\mathrm{D}_z\mathcal{I}(t,z),\partial_{\dot{z}}\mathcal{R}(z,0))\,\mathrm{d}z|$ and $z(t^\pm)$ denotes one-sided limits, see (23).

For the general definitions for these solutions types we refer to Definition 4.5 for differentiable, CD, and local solutions, Definition 3.1 for energetic solutions, Definitions 4.11 and 5.4 for parametrized solutions, and Definitions 4.21, 4.26, and 5.7 for BV solutions.

The above definition (7) for BV solutions is very implicit, but it highlights the similarity to the other solutions concepts in relying on a (i) static stability concept and (ii) an energy balance. The discussions in Section 4.5 shows that (ii) asks that along jumps from $z(t^-)$ to $z(t^+)$ the driving force $\mathrm{D}_z\mathcal{I}(t,z)$ is sufficiently large, e.g. for $z \in [z(t^+),z(t^-)]$ we must have $-\mathrm{D}_z\mathcal{I}(t,z) \le -r_-(z)$ and $\varDelta(t,z(t^-),z(t^+)) = \int_{z(t^+)}^{z(t^+)} \mathrm{D}_z\mathcal{I}(t,z) - r_-(z)\,\mathrm{d}z$.

We now comment on the relation between the different solution concepts. The first fact is that the notion of local solutions includes all the others. Differential solutions may not exist, but if they do then they are also BV solutions. All approximable solutions are BV solutions, but the opposite is in general not true.

If energetic solutions are differentiable, then they coincide with differential solutions. If energetic solutions have jumps, then they jump as soon as possible, whereas BV solutions jump as late as possible. So these two solutions types should be seen as two opposite extremes in the set of all local solutions. For both these extremes we have a rather complete existence theory, see Sections 3 and 4.5, respectively.

Parametrized solutions are special, since they are defined as curves in the extended space $\boldsymbol{Z}_T \stackrel{\text{def}}{=} [0,T]\times\boldsymbol{Z}$ given in arclength parametrization. In fact, they are in correspondence to BV solutions. Under suitable technical assumptions, the latter can be turned into parametrized solutions by filling in the jumps and arclength parametrization. Vice versa, every parametrized solution generates a BV solution via $\sigma(t) = \inf\{\, s \in [0,S] \mid \widehat{t}\,(s) = t\,\}$ and $z(t) = \widehat{z}(\sigma(t))$.

The following examples show that these notions are genuinely different. In Examples 2.3–2.6 we have $(\boldsymbol{Z}, \mathcal{I}, \mathcal{R})$ as defined in (18) and (19) with $r_+ = r_- \equiv 1$, but ℓ changes from case to case. In Example 2.7 we consider varying $r_\pm$.

Example 2.3. We consider $(\mathbb{R}, \mathcal{I}, \mathcal{R})$ *according to* (18) *with* $r_+ = r_- \equiv 1$, *and* $\ell(t) = t$ *for all* $t \geq 0$. *We claim that the approximable, the parametrized, and the BV solutions on* $[0,\infty[$ *are essentially unique and coincide. However, the unique energetic solution is different. Moreover, we show that there is a uncountable family of different local solutions. With direct calculations, one sees that the energetic solutions take the form*

$$z(t) = t-5 \quad \text{for } t \in [0,1[\,, \quad z(1) \in \{-4, 4\}, \quad \text{and} \quad z(t) = t+3 \quad \text{for } t > 1.$$

Choose any $t_* \in [1,3]$ *and any* $z_* \in \big[3+t_*, 3+t_*+\min\{2, 4\sqrt{t_*-1}\}\big]$. *Then,*

$$z(t) = \begin{cases} t-5 \text{ for } t \in [0, t_*[\,, \\ \ \ z_* \ \ \text{for } t \in [t_*, z_*-3], \\ t+3 \text{ for } t \geq z_*-3, \end{cases}$$

is a local solution. Note that the starting point of the jump at $z(t_*^-) = t_*-5$ *can be chosen in a full interval. Moreover, for a fixed* $t_* \in\,]1,3]$ *we still have the possibility to choose the ending point* $z_* = z(t_*^+)$ *of the jump in a full interval. All the other solution types essentially lead (up to definition in one point) to the same solution. The approximable and BV solutions read*

$$z(t) = \begin{cases} t-5 \text{ for } t \in [0,3[\,, \\ \ \ z_* \ \ \text{for } t = 3, \\ t+3 \text{ for } t > 3, \end{cases}$$

where $z_* \in [-2, 6]$ *is arbitrary. The associated arclength-parametrized solution takes the form*

$$\big(\widehat{t}(s), \widehat{z}(s)\big) = \begin{cases} \ \ \big(\frac{s}{2}, \frac{s}{2}-5\big) & \text{for } s \in [0,6], \\ \ \ (3, s-8) & \text{for } s \in [6,14], \\ \big(\frac{s}{2}-4, \frac{s}{2}-1\big) & \text{for } s \geq 14. \end{cases}$$

Example 2.4. We take $(\mathbb{R}, \mathcal{I}, \mathcal{R})$ *as in Example 2.3 but with* $\ell(t) = \min\{t, 4-t\}$ *and obtain the differential solution* z_{diff}, *which is different from the energetic solution* z_{energ}, *namely*

$$z_{\text{diff}}(t) = \begin{cases} t{-}5 \text{ for } t \in [0,2], \\ -3 \text{ for } t \in [2,4], \\ 1{-}t \text{ for } t \geq 4; \end{cases} \qquad z_{\text{energ}}(t) = \begin{cases} t{-}5 \text{ for } t \in [0,1], \\ t{+}3 \text{ for } t \in\,]1,2[\,, \\ 5 \quad \text{for } t \in [2,4], \\ 9{-}t \text{ for }]4,5[\,, \\ 1{-}t \text{ for } t \geq 5. \end{cases}$$

Thus, even the existence of a differentiable solution does not guarantee that this is also the energetic solution.

Example 2.5. In this example we show that not all BV solutions are approximable solutions. Again, $(\mathbb{R}, \mathcal{I}, \mathcal{R})$ is as in Example 2.3 but with $\ell(t) \overset{\text{def}}{=} \min\{t, 6{-}t\}$, i.e., the loading reduces exactly when the solution reaches the jump point. It is easy to see that there are two different BV solutions: z_1, which jumps at $t = 3$, and z_2, which does not jump. We have

$$z_1(t) = \begin{cases} t{-}5 \;\text{ for }\; t \in [0,3[\,, \\ \;6 \quad \text{for }\; t \in\,]3,5]\,, \\ 11{-}t \text{ for }\; t \in [5,9], \\ 3{-}t \;\text{ for } t > 9; \end{cases} \qquad z_2(t) = \begin{cases} t{-}5 \text{ for } t \in [0,3], \\ -2 \text{ for } t \in [3,5], \\ 3{-}t \text{ for } t \geq 5. \end{cases}$$

For $\varepsilon > 0$ the viscous solution q^ε of the differential inclusion

$$0 \in \operatorname{Sign}(\dot{z}) + \varepsilon \dot{z} + \mathcal{U}'(z) - \ell(t), \quad z(0) = -5,$$

is unique and can be found by matching solutions of linear ODEs. We find

$$z^\varepsilon(t) = \begin{cases} t{-}5{+}\varepsilon(\mathrm{e}^{-t/\varepsilon}{-}1) & \text{for } \; t \in [0,3], \\ z_*^\varepsilon & \text{for } \; t \in [3, t_*^\varepsilon], \\ 3{-}t{+}\varepsilon(\mathrm{e}^{-(t-t_*^\varepsilon)/\varepsilon}{-}1) & \text{for } \; t \geq t_*^\varepsilon, \end{cases}$$

where $z_*^\varepsilon = q^\varepsilon(3-) \lesssim -2$ and $t_*^\varepsilon = 3 - z_*^\varepsilon \gtrsim 5$. Thus, we have $z^\varepsilon(t) \to z_2(t)$ for every $t \geq 0$ as $\varepsilon \downarrow 0$. Hence, z_2 is a vanishing-viscosity solution, whereas z_1 is not. As a general principle, we conjecture that viscosity slows down solutions and thus approximable solutions tend to avoid jumps if there is a choice.

Example 2.6. Here, we study the parameter dependence of solutions under the loading

$$\ell_\delta(t) = \min\{t, 6{+}2\delta{-}t\} \quad \text{for } t \geq 0,$$

where δ is a small parameter. In the case $\delta = 0$ we have two BV solutions z_1 and z_2, see Example 2.5. But only z_2 is an approximable solution. For $0 < \delta < 1$ there is only one BV solution, which is then also the unique approximable solution, namely

$$z^\delta(t) = \begin{cases} t{-}5 \text{ for } t \in [0,3[\,, \\ t{+}3 \text{ for } t \in\,]3, 3{+}\delta]\,, \\ 6{+}\delta \text{ for } t \in [3{+}\delta, 5{-}\delta]. \end{cases}$$

Taking the limit $\delta \to 0^+$ we see that the pointwise limit of the approximable solutions z^δ is z_1, which is not an approximable solution for $\delta = 0$. Thus, the set of approximable solutions is not upper semicontinuous with respect to variations of the data.

Example 2.7. Here, $\mathcal{I}(t,\cdot)$ is uniformly convex, in fact even quadratic, but $\mathcal{R}$ depends on z such that the joint convexity condition (11) does not hold. As a consequence we obtain solutions with jumps. For $\gamma > 0$ we let

$$\mathcal{I}(t,z) = \frac{1}{2}z^2 - tz \quad \text{and} \quad \mathcal{R}(z,\dot{z}) = \mu(z)|\dot{z}| \text{ with } \mu(z) = \max\{1, \min\{2-\gamma z, 3\}\}.$$

For $z \in [-1/\gamma, 1/\gamma]$ the joint convexity condition $\alpha > \rho$ (see (11)) holds for $\rho = \gamma < 1 = \alpha$. Thus, for $\gamma > 1$ solutions have to jump across the region $[-1/\gamma, 1/\gamma]$, since there are no locally stable points. We start from the initial condition $z(0) = z_0 = -3$. Then, the energetic solution z_{energ} and the BV solution z_{BV} are different, namely

$$z_{\text{energ}}(t) = \begin{cases} t-3 & \text{for } t \in [0,2[\,, \\ t-1 & \text{for } t > 2; \end{cases} \qquad z_{\text{BV}}(t) = \begin{cases} t-3 & \text{for } t \in [0, 3-1/\gamma[\,, \\ t-1 & \text{for } t > 3-1/\gamma. \end{cases}$$

2.7 Infinite-dimensional examples

Here we provide the simplest and most canonical infinite-dimensional example of a RIS $(\mathcal{Z},\mathcal{I},\mathcal{R})$ including also a viscosity term. It is used in each of the abstract sections to discuss the different solutions concepts. We first give an abstract Banach-space setting and afterwards present a special case which connects the theory to a particular PDE.

Example 2.8 (Standard semilinear example). We consider a Banach space $\boldsymbol{X}$ and two Hilbert spaces $\boldsymbol{Z}$ and $\boldsymbol{V}$, which are densely and continuously embedded as follows:

$$\boldsymbol{Z} \Subset \boldsymbol{V} \subset \boldsymbol{X},$$

where "$\Subset$" denotes compact embedding. The different Banach spaces and their norms $\|\cdot\|_{\boldsymbol{Z}}$, $\|\cdot\|_{\boldsymbol{V}}$, and $\|\cdot\|_{\boldsymbol{X}}$ are associated with the energy functional, the viscous dissipation, and the rate-independent dissipation, respectively. We further assume that there are symmetric, bounded linear operators $\mathbb{A} \in \mathscr{L}(\boldsymbol{Z},\boldsymbol{Z}^*)$ and $\mathbb{V} \in \mathscr{L}(\boldsymbol{V},\boldsymbol{V}^*)$, which are invertible with bounded inverses. Without loss of generality (after choosing an equivalent Hilbert norm) one may assume that they equal the corresponding Riesz isomorphisms.

The problem under investigation is the doubly nonlinear equation

$$\begin{aligned} &0 \in \partial\Psi(\dot{z}(t)) + \varepsilon\mathbb{V}\dot{z}(t) + \mathrm{D}_z\mathcal{I}(t,z(t)) \\ &\text{with } \mathcal{I}(t,z) = \tfrac{1}{2}\langle\mathbb{A}z,z\rangle + \Phi(z) - \langle\ell(t),z\rangle. \end{aligned} \tag{20}$$

Here, $\Phi \in \mathrm{C}^2(\boldsymbol{Z};\mathbb{R})$ is a non-quadratic potential of lower order in a sense to be made precise below. The function $\ell : [0,T] \to \boldsymbol{V}^*$ is the loading. We assume that there exists $c, C > 0$, an interpolation exponent $\theta \in\,]0,1[$, and a growth exponent $q \geq 0$, such that for all $v,z,w \in \boldsymbol{Z}$ we have

$$\boldsymbol{Z} \Subset \boldsymbol{V} \subset \boldsymbol{X} \text{ with dense embeddings;} \tag{21a}$$

$$\|v\|_{\boldsymbol{V}} \leq C\|v\|_{\boldsymbol{X}}^{\theta}\|v\|_{\boldsymbol{Z}}^{1-\theta}; \tag{21b}$$

$$c\|z\|_{\boldsymbol{Z}}^2 \leq \langle\mathbb{A}z,z\rangle \leq C\|z\|_{\boldsymbol{Z}}^2,\ \|v\|_{\boldsymbol{V}}^2 = \langle\mathbb{V}v,v\rangle; \tag{21c}$$

$$\Psi : \boldsymbol{V} \to [0,\infty[\text{ convex, 1-homogeneous, } c\|v\|_{\boldsymbol{X}} \leq \Psi(v) \leq C\|v\|_{\boldsymbol{X}}; \tag{21d}$$

$$\Phi(z) \geq 0,\ \Phi : \boldsymbol{Z} \to \mathbb{R} \text{ is weakly continuous;} \tag{21e}$$

$$\mathrm{D}\Phi \in \mathrm{C}^1(\boldsymbol{Z};\boldsymbol{V}^*),\ \|\mathrm{D}^2\Phi(z)v\|_{\boldsymbol{V}^*} \leq C(1+\|z\|_{\boldsymbol{Z}})^q\|v\|_{\boldsymbol{Z}}; \tag{21f}$$

$$\ell \in \mathrm{W}^{1,p}([0,T],\boldsymbol{V}^*) \text{ for some } p \geq 2. \tag{21g}$$

Condition (21f) on $\mathrm{D}\Phi$ can be weakened by replacing $\boldsymbol{V}^*$ with an interpolation space $[\boldsymbol{V}^*,\boldsymbol{Z}^*]_\eta$, $\eta \in]0,1[$, see [MiZ09]. We stay with $\boldsymbol{V}^*$ for notational simplicity. We introduce additional Hilbert spaces

$$\boldsymbol{Z}_1 \stackrel{\text{def}}{=} \{\, z \in \boldsymbol{Z} \mid \mathbb{A}z \in \boldsymbol{V}^* \,\} \qquad \text{with } \|z\|_1 \stackrel{\text{def}}{=} \|\mathbb{A}z\|_{\boldsymbol{V}^*}, \tag{22a}$$

$$\boldsymbol{Z}_{-1} \stackrel{\text{def}}{=} \overline{\boldsymbol{V}}^{\|\cdot\|_{-1}} \qquad \text{with } \|z\|_{-1} \stackrel{\text{def}}{=} \|\mathbb{V}z\|_{\boldsymbol{Z}^*}. \tag{22b}$$

We obtain a scale of four Hilbert spaces

$$\boldsymbol{Z}_1 \Subset \boldsymbol{Z} \Subset \boldsymbol{V} \Subset \boldsymbol{Z}_{-1}, \quad \text{and } \mathbb{A}\boldsymbol{Z}_1 = \boldsymbol{V}^*, \quad \mathbb{V}\boldsymbol{Z}_{-1} = \boldsymbol{Z}^*,$$

with dense and compact embeddings. Moreover, the scale is equally spaced in the sense of interpolation, namely $[\boldsymbol{Z}_1,\boldsymbol{V}]_{1/2} = \boldsymbol{Z}$ and $[\boldsymbol{Z},\boldsymbol{Z}_{-1}]_{1/2} = \boldsymbol{V}$. (If we compare to classical evolution triples $V \subset H \cong H^* \subset V^*$ with a linear selfadjoint, positive definite operator $A : D(A) \subset H \to H$ and $V = D(A^{1/2})$, we obtain the corresponding scale $D(A) \subset D(A^{1/2}){=}V \subset H \cong H^* \subset V^*$.)

This abstract setting can be applied to specific problems involving PDEs as they occur in modeling of hysteretic materials, like in magnetism, elastoplasticity, ferroelectricity, or shape-memory alloys. We refer to [Mie05, Mie06] for surveys on these applications. For the simplest application we consider a smooth and bounded domain $\Omega \subset \mathbb{R}^d$ and let

$$\boldsymbol{Z} = \mathrm{H}^1_0(\Omega) \Subset \boldsymbol{V} = \mathrm{L}^2(\Omega) \subset \boldsymbol{X} = \mathrm{L}^1(\Omega).$$

We have $\boldsymbol{V}^* = \mathrm{L}^2(\Omega) \Subset \boldsymbol{Z}^* = \mathrm{H}^{-1}(\Omega)$, and the operators $\mathbb{A}$ and $\mathbb{V}$ are given by $-\Delta$ and id, respectively. This leads to the additional spaces $\boldsymbol{Z}_1 = \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ and $\boldsymbol{Z}_{-1} = \mathrm{H}^{-1}(\Omega)$. The functionals take the form

$$\Psi(v) = \int_\Omega |v(x)|\,\mathrm{d}x, \quad \mathcal{I}(t,z) = \int_\Omega \tfrac{1}{2}|\nabla z(x)|^2 + \phi(z(x)) + f(t,x)z(x)\,\mathrm{d}x,$$

where $f \in \mathrm{W}^{1,p}([0,T],\mathrm{L}^2(\Omega))$ defines the loading. The function $\phi \in \mathrm{C}^2(\mathbb{R};\mathbb{R})$ is assumed to satisfy $0 \leq \phi(s) \leq C(1{+}|s|)^q$ with $q < \infty$ for $d \leq 2$ and $q < 2d/(d{-}2)$ for $d \geq 3$. Further, we assume $|\phi'(s)| \leq C(1{+}|s|)^{q/2}$ and $|\phi''(s)| \leq C(1{+}|s|)^{q/d}$. Then, all the conditions of the abstract theory are satisfied, cf. [MiZ09].

3 Energetic solutions

In this section we consistently work with a state space $\mathcal{Q} = \mathcal{Y}{\times}\mathcal{Z}$ with states $q = (y,z)$, which has its reason in applications in continuum mechanics, where the z-component is dissipative while the y-component is not, cf. [Mie06]. Whenever possible we write q instead of (y,z) to shorten notation. The state space $\mathcal{Q}$ is equipped with a Hausdorff topology, and we denote by $q_k \xrightarrow{\mathcal{Q}} q$, $y_k \xrightarrow{\mathcal{Y}} y$ and $z_k \xrightarrow{\mathcal{Z}} z$ the corresponding convergence of sequences. Throughout it is sufficient to consider sequential closedness, compactness and continuity. For notational convenience, we will not write this explicitly.

A main tool for the analysis of such systems is the interplay between the full RIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$ and its reduced version $(\mathcal{Z},\mathcal{I},\mathcal{D})$, where the *reduced energy* $\mathcal{I} : [0,T]{\times}\mathcal{Z} \to \mathbb{R}_\infty$ is defined in (4). We define energetic solutions to $(\mathcal{Q},\mathcal{E},\mathcal{D})$ and $(\mathcal{Z},\mathcal{I},\mathcal{D})$ in such a way that each solution $q = (y,z)$ for the former system gives rise to a solution z for the latter. Vice versa each solution z for $(\mathcal{Z},\mathcal{I},\mathcal{D})$

can be made to a solution $q = (y, z)$ by a suitable choice of y. We emphasize that it is not enough to choose an arbitrary $y(t) \in \operatorname{Arg\,min} \mathcal{E}(t, \cdot, z(t))$, since further restrictions arise.

At first glance, it might seem reasonable to first consider the reduced system $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ and establish an existence theory there and then derive the desired existence result for the full problem $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$. However, it turns out that in the reduction process certain natural properties (like differentiability in t) are lost. To compensate for that, stronger assumptions would be necessary, which can be avoided by working on the full system instead. Thus we present the existence theory for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, which is also more natural in material modeling, cf. [Mie06].

3.1 Abstract setup of the problem

The first ingredient of the energetic formulation is the *dissipation distance* $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0, \infty]$, which is an extended quasi-distance. Here 'extended' means that the value ∞ is allowed and 'quasi' means that we do not ask for symmetry. The following conditions are the main assumptions on $\mathcal{D}$.

Extended quasi-distance:

$$\text{(i)}\quad \forall\, z_1, z_2, z_3 \in \mathcal{Z} :\ \mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3), \tag{D1}$$
$$\text{(ii)}\quad \forall\, z_1, z_2 \in \mathcal{Z} :\ \mathcal{D}(z_1, z_2) = 0 \iff z_1 = z_2;$$

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0, \infty] \text{ is lower semicontinuous.} \tag{D2}$$

Here (D1) says that $\mathcal{D}$ is a distance except for the symmetry and the fact that the value ∞ is allowed. Relation (i) is the triangle inequality, and (ii) is the positivity. The unsymmetry is needed in many applications like in elastoplasticity or damage.

For curves $z : [0, T] \to \mathcal{Z}$ we define the *dissipation functional* $\mathrm{Diss}_{\mathcal{D}}$ via

$$\begin{aligned}\mathrm{Diss}_{\mathcal{D}}(z, [s, t]) \overset{\text{def}}{=} \sup \big\{ \textstyle\sum_{j=1}^{N} \mathcal{D}(z(t_{j-1}), z(t_j)) \ \big|\ N \in \mathbb{N},\\ s = t_0 < t_1 < \cdots < t_N{=}t \,\big\}.\end{aligned} \tag{1}$$

Further we define the following set of functions:

$$\mathrm{BV}_{\mathcal{D}}([0, T], \mathcal{Z}) \overset{\text{def}}{=} \{\, z : [0, T] \to \mathcal{Z} \mid \mathrm{Diss}_{\mathcal{D}}(z, [0, T]) < \infty \,\}. \tag{2}$$

The functions are defined everywhere and changing them at one point may increase the dissipation. Moreover, the dissipation is additive:

$$\mathrm{Diss}_{\mathcal{D}}(z, [r, t]) = \mathrm{Diss}_{\mathcal{D}}(z, [r, s]) + \mathrm{Diss}_{\mathcal{D}}(z, [s, t]) \ \text{ for all } r < s < t. \tag{3}$$

Later on, we sometimes use the notation $\mathcal{D}(q_0, q_1)$ instead of $\mathcal{D}(z_0, z_1)$ where $q_j = (y_j, z_j)$. This slight abuse of notation never leads to confusion, since $\mathcal{D}$ as a function on $\mathcal{Q} = \mathcal{Y}\times\mathcal{Z}$ still satisfies all assumptions but one has to remember that $\mathcal{D}$ satisfies the positivity (D1) only on $\mathcal{Z}$ and not on $\mathcal{Q}$.

The second ingredient is the energy-storage functional $\mathcal{E} : \mathcal{Q}_T \to \mathbb{R}_\infty$. Here $t \in [0, T]$ plays the role of a (very slow) process time which changes the loading. The following conditions form the basic assumptions on $\mathcal{E}$:

Compactness of sublevels:

$$\forall\, t \in [0, T] :\ \mathcal{E}(t, \cdot) : \mathcal{Q} \to \mathbb{R}_\infty \text{ has compact sublevels;} \tag{E1}$$

Energetic control of power:

$$\exists\,\lambda_{\mathcal{E}}>0\ \forall\,(t,q) \text{ with } \mathcal{E}(t,q)<\infty: \\ \mathcal{E}(\cdot,q)\in \mathrm{C}^1([0,T]) \ \text{ and } \ \big|\partial_s\mathcal{E}(s,q)\big| \leq \lambda_{\mathcal{E}}\,\mathcal{E}(s,q) \text{ for all } s\in[0,T]. \tag{E2}$$

Condition (E2) implies $\operatorname{dom}\mathcal{E}=[0,T]\times\operatorname{dom}\mathcal{E}(0,\cdot)$, i.e. $\operatorname{dom}\mathcal{E}(t,\cdot)$ is independent of t. From (E2) and Gronwall's inequality we easily derive

$$\mathcal{E}(t,q)\leq \mathcal{E}(s,q)\,\mathrm{e}^{\lambda_{\mathcal{E}}|t-s|} \ \text{ and } \ |\partial_t\mathcal{E}(t,q)|\leq \lambda_{\mathcal{E}}\,\mathcal{E}(s,q)\,\mathrm{e}^{\lambda_{\mathcal{E}}|t-s|}. \tag{4}$$

Most typically, $\mathcal{Q}$ is a closed, convex and bounded subset of a reflexive Banach space (like $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^m)$ or $\mathrm{L}^p(\Omega,\mathbb{R}^m)$ with $p\in(1,\infty)$) equipped with its weak topology $\mathcal{T}$. Then, lower semicontinuity of $\mathcal{E}$ and $\mathcal{D}$ in $(\mathcal{Q},\mathcal{T})$ is the same as the classical weak lower semicontinuity in the calculus of variations.

Definition 3.1 (Energetic solution). A function $q=(y,z):[0,T]\to\mathcal{Q}=\mathcal{Y}\times\mathcal{Z}$ is called an *energetic solution* of the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if $t\mapsto\partial_t\mathcal{E}(t,q(t))$ is integrable and if the *global stability (S)* and the *energy equality (E)* hold for all $t\in[0,T]$:

(S) $\quad q(t)\in\mathcal{S}(t)$;

(E) $\quad \mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(z,[0,t])=\mathcal{E}(0,q(0))+\int_0^t\partial_t\mathcal{E}(\tau,q(\tau))\,\mathrm{d}\tau$.

Condition (S) means global stability, because the *set* $\mathcal{S}(t)$ *of stable states at time* t is defined such that all $\widehat{q}\in\mathcal{Q}$ are considered as competitors:

$$\mathcal{S}(t)\overset{\text{def}}{=}\{\,q\in\mathcal{Q}\mid \mathcal{E}(t,q)<\infty,\ \mathcal{E}(t,q)\leq\mathcal{E}(t,\widehat{q})+\mathcal{D}(q,\widehat{q}) \text{ for all } \widehat{q}\in\mathcal{Q}\,\}. \tag{5}$$

We shortly call $\mathcal{S}(t)$ the *stability set* at time t. The properties of the stability sets turn out to be crucial for deriving existence results.

The definition of energetic solutions is such that we obtain a rate-independent multi-valued evolutionary system in the sense of Definition 2.1, in particular we have the concatenation and the restriction property. It is clear that the stability condition (S) has the restriction and concatenation property. To see that (E) also shares these conditions we define $\mathfrak{E}_r(t)=\mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(q,[r,t])-\int_r^t\partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s$. Then, (E) simply states that the function $\mathfrak{E}$ is equal to the constant value $\mathfrak{E}(r)$ on the whole interval. This constancy certainly remains true after restriction. When concatenating two solutions q_1 and q_2, the condition $q_1(t_2)=q_2(t_2)$ guarantees that the two constants are the same.

Rate independence manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that q is a solution to $(\mathcal{Q},\mathcal{E},\mathcal{D})$ if and only if the reparametrized curve $\widetilde{q}:t\mapsto q(\alpha(t))$, where $\dot{\alpha}>0$, is a solution to $(\mathcal{Q},\widetilde{\mathcal{E}},\mathcal{D})$ with $\widetilde{\mathcal{E}}(t,q)=\mathcal{E}(\alpha(t),q)$. In particular, the stability (S) is a static concept, and the energy balance (E) is rate-independent, since the dissipation defined via (1) is scale-invariant.

Before discussing the question of existence of solutions we want to point out, that the concept of energetic solutions provides a priori bounds on the solutions. For the time-continuous problem these bounds are easy to derive, and the main structure becomes more transparent. Of course, similar estimates are crucial in the time-discrete setting. Using the assumption (E2) the energy balance (E) gives

$$\mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(z,[r,t])\leq\mathcal{E}(r,q(r))+\int_r^t\lambda_{\mathcal{E}}\,\mathcal{E}(s,q(s)\,\mathrm{d}s \tag{6}$$

for $0 \leq r < t \leq T$. Omitting the dissipation and applying Gronwall's lemma yield $\mathcal{E}(t, q(t)) \leq \mathcal{E}(0, q(0))\,\mathrm{e}^{\lambda_\mathcal{E} t}$. Inserting this into (6) we estimate the dissipation via $\mathrm{Diss}_\mathcal{D}(z, [0, T]) \leq \mathcal{E}(0, q(0))\mathrm{e}^{\lambda_\mathcal{E} T}$, since $\mathcal{E}(t, q(t)) \geq 0$ by (E2).

3.2 The time-incremental minimization problem

The most natural approach to solve (S)&(E) is via time discretization using the fact that incremental problems exist, which are minimization problems. It is then possible to find their solutions as global minimizers of certain lower semicontinuous functionals on $\mathcal{Q}$. For this we make use of the lower semicontinuity assumptions (D2) and (E1).

For the time discretization we use the notation $\mathrm{Part}([r, s])$ for all finite partitions of the interval $[r, s] \subset \mathbb{R}$, i.e.

$$\mathrm{Part}([r, s]) \overset{\text{def}}{=} \{\, (t_0, t_1, ..., t_N) \mid r = t_0 < t_1 < ... < t_N = r \,\}. \tag{7}$$

For a partition $\Pi \in \mathrm{Part}([r, s])$, we define N_Π as the number of subintervals and $\phi(\Pi)$ as its *fineness*, namely as the length of its largest interval:

$$\phi(\Pi) \overset{\text{def}}{=} \max\{\, t_k - t_{k-1} \mid k = 1, \ldots, N_\Pi \,\}. \tag{8}$$

Note that $\phi(\Pi) = 2 \max_{t \in [0,T]} \mathrm{dist}(t, \Pi)$. In particular, always $\mathrm{dist}(t, \Pi) \leq \phi(\Pi)$. Having fixed a partition $\Pi = (t_0, t_1, ..., t_N) \in \mathrm{Part}([0, T])$, we seek for some q_k, $k = 1, ..., N_\Pi$, which approximate the solution q at t_k, i.e., $q_k \approx q(t_k)$.

Our energetic approach has the major advantage that the values q_k can be found incrementally via the *incremental minimization problem*

$$(\mathrm{IMP}^\Pi) \qquad \begin{array}{l} \text{For } q_0 \in \mathcal{S}(0) \subset \mathcal{Q} \text{ find } q_1, \ldots, q_N \in \mathcal{Q} \text{ such that} \\ q_k \text{ minimizes } q \mapsto \mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q). \end{array} \tag{9}$$

We briefly write $q_k \in \mathrm{Arg\,min}\{\, \mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q) \mid q \in \mathcal{Q} \,\}$, where "Arg min" denotes the set of all minimizers. The following result shows that (IMP^Π) is intrinsically linked to (S)&(E). Without any smallness assumptions on the time steps, the solutions of (IMP^Π) satisfy properties, which are closely related to (S)&(E).

Proposition 3.2 (Estimates for the incremental problem). *Let* (D1) *and* (E2) *hold. All solutions of (IMP^Π) from* (9) *satisfy the following properties:*
(i) For $k = 1, \ldots, N_\Pi$ we have that q_k is stable at time t_k, i.e., $q_k \in \mathcal{S}(t_k)$.
(ii) With $e_j = \mathcal{E}(t_j, q_j)$ and $\delta_k = \mathcal{D}(z_{k-1}, z_k)$ we have, for $k = 1, \ldots, N_\Pi$,

$$\int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s \leq e_k - e_{k-1} + \delta_k \leq \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\,\mathrm{d}s. \tag{10}$$

(iii) If (D2) *and* (E1) *hold additionally, then solutions of (IMP^Π) exist.*

Proof. Ad (i). The stability follows from minimization properties of the solutions and the triangle inequality. For all $\widehat{q} \in \mathcal{Q}$ we have

$$\begin{aligned} &\mathcal{E}(t_k, \widehat{q}) + \mathcal{D}(z_k, \widehat{z}) = \mathcal{E}(t_k, \widehat{q}) + \mathcal{D}(z_{k-1}, \widehat{z}) + \mathcal{D}(z_k, \widehat{z}) - \mathcal{D}(z_{k-1}, \widehat{z}) \\ &\geq \mathcal{E}(t_k, q_k) + \mathcal{D}(z_{k-1}, z_k) + \mathcal{D}(z_k, \widehat{z}) - \mathcal{D}(z_{k-1}, \widehat{z}) \geq \mathcal{E}(t_k, q_k). \end{aligned}$$

Ad (ii). The first estimate is deduced from $q_{k-1} \in \mathcal{S}(t_{k-1})$ as follows:

$$\begin{aligned}
&\mathcal{E}(t_k, q_k) + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, q_{k-1}) \\
&= \mathcal{E}(t_{k-1}, q_k) + \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, q_{k-1}) \\
&\geq \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s.
\end{aligned}$$

Since $q_k \in \operatorname{Arg\,min}_{\mathcal{Q}} \mathcal{E}(t_k, q) + \mathcal{D}(z_{k-1}, z)$, the second estimate follows via

$$\begin{aligned}
&\mathcal{E}(t_k, q_k) - \mathcal{E}(t_{k-1}, q_{k-1}) + \mathcal{D}(z_{k-1}, z_k) \\
&\leq \mathcal{E}(t_k, q_{k-1}) - \mathcal{E}(t_{k-1}, q_{k-1}) + \mathcal{D}(z_{k-1}, z_{k-1}) \;=\; \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\,\mathrm{d}s.
\end{aligned}$$

Ad (iii). The minimizers are constructed inductively. In the k-th step, q_{k-1} is known and any minimizer y has to satisfy $\mathcal{J}_k(y) \overset{\text{def}}{=} \mathcal{E}(t_k, q) + \mathcal{D}(z_{k-1}, z) \leq \mathcal{E}(t_k, q_{k-1}) = \mathcal{J}_k(q_{k-1})$, since $q = q_{k-1}$ is a candidate. Using $\mathcal{D} \geq 0$ it suffices to minimize the lower semicontinuous functional $\mathcal{J}_k$ on the compact sublevel $\mathcal{E}(t_k, \cdot) \leq \mathcal{E}(t_k, q_{k-1})$. Hence, Weierstraß' extremum principle provides the existence of a minimizer q_k. ☐

Now we use assumption (E2) to obtain a priori bounds on the energy and the dissipation for the solution of (IMP$^{\Pi}$). Combining (E2), (4) and the upper estimate in (ii) of Proposition 3.2 give

$$e_k + \delta_k \;\leq\; e_{k-1} + e_{k-1}\big(\mathrm{e}^{\lambda_{\mathcal{E}}(t_k - t_{k-1})} - 1\big) \;=\; e_{k-1}\,\mathrm{e}^{\lambda_{\mathcal{E}}(t_k - t_{k-1})}. \tag{11}$$

Using $\delta_k \geq 0$ induction over k leads to

$$e_k \leq e_0 \prod_{j=1}^{k} \mathrm{e}^{\lambda_{\mathcal{E}}(t_j - t_{j-1})} = e_0\,\mathrm{e}^{\lambda_{\mathcal{E}} t_k} \text{ for } k = 1, \ldots, N_{\Pi}. \tag{12}$$

Summing (11) from $k = 1$ to n we find, after cancellations and using (12),

$$\begin{aligned}
e_n + \textstyle\sum_{j=1}^{n} \delta_j &\leq e_0 + \textstyle\sum_{j=1}^{n} e_{j-1}\,(\mathrm{e}^{\lambda_{\mathcal{E}}(t_j - t_{j-1})} - 1) \\
&\leq e_0 + e_0 \textstyle\sum_{1}^{n} (\mathrm{e}^{\lambda_{\mathcal{E}} t_j} - \mathrm{e}^{\lambda_{\mathcal{E}} t_{j-1}}) \;=\; e_0\,\mathrm{e}^{\lambda_{\mathcal{E}} t_n}.
\end{aligned}$$

For each incremental solution $(q_k)_{k=1,\ldots,N}$ of (IMP$^{\Pi}$) associated with a partition $\Pi = (t_0, t_1, \ldots, t_N) \in \mathrm{Part}([0, T])$, we define the piecewise constant interpolant $\underline{q}^{\Pi}$ with

$$\underline{q}^{\Pi}(t) \overset{\text{def}}{=} q_{k-1} \text{ for } t \in [t_{k-1}, t_k[\text{ and } k = 1, \ldots, N, \text{ and } \underline{q}^{\Pi}(T) \overset{\text{def}}{=} q_N, \tag{13}$$

which is continuous from the right.

Corollary 3.3. *Assume that* (D1) *and* (E2) *hold and let* $\Pi \in \mathrm{Part}([0, T])$. *Then, for any solution* $(q_k)_{k=0,\ldots,N_{\Pi}}$ *of (IMP*$^{\Pi}$*) the interpolant* $\underline{q}^{\Pi} : [0, T] \to \mathcal{Q}$ *satisfies the following relations.*

(i) *(S)*$_{discr}$ *For* $t \in \Pi$ *we have* $\underline{q}^{\Pi}(t) \in \mathcal{S}(t)$.

(ii) *(E)*$_{discr}$ *For* $s, t \in \Pi$ *with* $s < t$ *we have the energy estimate*

$$\mathcal{E}(t, \underline{q}^{\Pi}(t)) + \mathrm{Diss}_{\mathcal{D}}(\underline{z}^{\Pi}, [s, t]) \leq \mathcal{E}(s, \underline{q}^{\Pi}(s)) + \int_s^t \partial_\tau \mathcal{E}(\tau, \underline{q}^{\Pi}(\tau))\,\mathrm{d}\tau.$$

(iii) *For all* $t \in [0, T]$ *we have the a priori estimate*

$$\mathcal{E}(t, \underline{q}^{\Pi}(t)) + \mathrm{Diss}_{\mathcal{D}}(\underline{z}^{\Pi}, [0, t]) \leq \mathrm{e}^{\lambda_{\mathcal{E}} t}\, \mathcal{E}(0, q_0).$$

3.3 Statement of the main existence result

The existence theory developed below is based on the incremental minimization problem (IMP$^\Pi$) and the a priori estimates derived above. Choosing a sequence of partitions whose fineness tends to 0, we obtain a sequence of approximations and need to extract a suitable subsequence that converges. This can be done for the z-component only, since the dissipation provides an a priori estimate of BV-type, which allows for an application of a suitable version of Helly's selection principle as stated in Theorem 3.13.

Since the y-component allows for no control of the temporal oscillations, it has to be handled differently. We could use a technique developed in [DFT05, FrM06], which chooses additional subsequences for each $t \in [0,T]$ and thus is relying on the axiom of choice. Instead, we rely on a metrizability assumption of the underlying topology, which guarantees the existence of measurable solutions. This idea uses the fact that for stable states $q = (y,z)$ the energy $\mathcal{E}(t,\cdot)$ depends only on the component z. In particular, for the reduced functional $\mathcal{I}$ defined in (4) we have

$$\mathcal{I}(t,z) = \mathcal{E}(t,y,z) \quad \text{for all } (y,z) \in \mathcal{S}(t). \tag{14}$$

We also define the *reduced power* via

$$\mathcal{P}_{\mathrm{red}}(t,z) \stackrel{\text{def}}{=} \sup\{\, \partial_t \mathcal{E}(t,y,z) \mid y \in \operatorname*{Arg\,min}_{y\in\mathcal{Y}} \mathcal{E}(t,\cdot,z)\,\}. \tag{15}$$

The important new observation is that along all energetic solutions the reduced power $\mathcal{P}_{\mathrm{red}}(t,z)$ is realized, see (18).

After having identified a subsequence and a limit function, it is necessary to show that this limit is an energetic solution. For this we need further conditions on the functionals $\mathcal{E}$ and $\mathcal{D}$ expressing a certain compatibility between these two functionals. To define these conditions, we introduce the notion of a *stable sequence* $(t_k,q_k)_{k\in\mathbb{N}}$ via

$$\sup_{k\in\mathbb{N}} \mathcal{E}(t_k,q_k) < \infty \quad \text{and} \quad \forall\, k \in \mathbb{N}:\ q_k \in \mathcal{S}(t_k). \tag{16}$$

The *compatibility conditions* between $\mathcal{E}$ and $\mathcal{D}$ rely on convergent stable sequences and read as follows:

$$\forall \text{ stable sequences } (t_k,q_k)_{k\in\mathbb{N}} \text{ with } (t_k,q_k) \stackrel{\mathcal{Q}_T}{\to} (t,q) \text{ it holds:}$$
$$\partial_t\mathcal{E}(t,q) = \lim_{k\to\infty} \partial_t \mathcal{E}(t,q_k), \tag{C1}$$
$$q \in \mathcal{S}(t). \tag{C2}$$

Condition (C2) is called the *closedness of the stability set.* Condition (C1) is called *conditioned continuity of the power of the external forces.* Note that in the limit in (C1) the time is fixed to t, although $q_k \in \mathcal{S}(t_k)$. These central conditions are discussed more in detail in Section 3.5.

We now state our existence result for energetic solutions. After preparing a few intermediate results, the proof is completed on pp. 28–29 below.

Theorem 3.4 (Existence of energetic solutions). *Assume that $\mathcal{E}$ and $\mathcal{D}$ satisfy the assumptions* (D1)–(D2), (E1)–(E2), *and the compatibility conditions* (C2) *and* (C1). *Further assume that*

$$\textit{the topology of } \mathcal{Q} \textit{ restricted to compact sets is separable metrizable.} \tag{17}$$

(i) Then, for each $q_0 \in \mathcal{S}(0)$ there exists an energetic solution $q = (y,z) : [0,T] \to \mathcal{Q}$ to the RIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$ with $q(0) = q_0$. Moreover, $q : [0,T] \to \mathcal{Q}$ is measurable and

$$\partial_t\mathcal{E}(t,y(t),z(t)) = \mathcal{P}_{\mathrm{red}}(t,z(t)) \textit{ for a.a. } t \in [0,T]. \tag{18}$$

(ii) If $\Pi^l \in \mathrm{Part}([0,T])$ is a sequence of partitions with fineness $\phi(\Pi^l) \to 0$ for $l \to \infty$, and $\underline{q}^{\Pi_l}$ is the interpolant of any solution of the associated (IMP$^{\Pi^l}$), then there exist a subsequence $q_k = q^{\Pi_{l_k}}$ and a solution $\widetilde{q} = (\widetilde{y}, \widetilde{z})$ to the initial-value problem $(\mathcal{Q}, \mathcal{E}, \mathcal{D}, q_0)$ such that the following holds:

$$\forall\, t \in [0,T]:\ z_k(t) \stackrel{\mathcal{Z}}{\to} \widetilde{z}(t); \tag{19a}$$
$$\forall\, t \in [0,T]:\ \mathrm{Diss}_{\mathcal{D}}(z_k, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(\widetilde{z}, [0,t]); \tag{19b}$$
$$\forall\, t \in [0,T]:\ \mathcal{E}(t, q_k(t)) \to \mathcal{E}(t, \widetilde{q}(t)); \tag{19c}$$
$$\partial_t \mathcal{E}(\cdot, q_k(\cdot)) \to \partial_t \mathcal{E}(\cdot, \widetilde{q}(\cdot)) \quad in\ \ \mathrm{L}^1((0,T)). \tag{19d}$$

(iii) If additionally the functional $\mathcal{E}$ is such that for each stable point $q = (y,z) \in \mathcal{S}(t)$ the functional $\mathcal{E}(t, \cdot, z)$ has the unique minimizer y, then the convergence in (19a) can be improved to $q_k(t) \stackrel{\mathcal{Q}}{\to} \widetilde{q}(t)$.

We also provide an easy applicable version of the existence result, where we strengthen the assumptions considerably, but still allow for a big variety of applications. By making $\mathcal{D}$ continuous on $\mathcal{Z}$, it is possible to decouple the assumptions on $\mathcal{E}$ and $\mathcal{D}$ completely, and the compatibility conditions (C2) and (C1) can be established easily.

Theorem 3.5 (Simplified existence result for energetic solutions). *Assume that $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfy (D1), (E1), (E2), (17) as well as the following conditions:*

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty[\ \ \textit{is continuous}; \tag{20}$$
$$\begin{aligned} &\exists\, C_E^* > 0\ \forall\, q \in \mathcal{Q} \textit{ with } \mathcal{E}(0,q) < \infty : \\ &\mathcal{E}(\cdot, q) \in \mathrm{C}^1([0,T]),\ |\partial_t \mathcal{E}(t,q) - \partial_t \mathcal{E}(s,q)| \le C_E^* |t-s| \mathcal{E}(0,q). \end{aligned} \tag{21}$$

Then, all assumptions of the main existence result Theorem 3.4 are fulfilled and, hence, existence of energetic solutions to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ is guaranteed for all initial conditions $q(0) = q_0 \in \mathcal{S}(0)$.

Proof. Clearly, (20) implies (D2). To establish the compatibility conditions we use Corollary 3.9, since our present assumptions (20) and (21) are exactly the conditions (28) and (30) imposed there. □

3.4 Jump conditions for energetic solutions

Here we discuss some basic properties of solutions.

Let $q : [0,T] \to \mathcal{Q}$ be an energetic solution to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$. First, we exploit the energy balance to show that q satisfies simple a priori estimates for the energy and the dissipation. For this, we use that (E) holds for all intervals $[s,t]$. Omitting the nonnegative dissipation in (E) and employing (4) in the power term give $\mathcal{E}(t, q(t)) \le \mathcal{E}(s, q(s))\, \mathrm{e}^{\lambda_{\mathcal{E}}(t-s)}$ for all $0 \le s < t \le T$. Inserting this into the right-hand side of the energy balance yields

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(q, [s,t]) \le \mathcal{E}(s, q(s))\, \mathrm{e}^{\lambda_{\mathcal{E}}(t-s)} \quad \text{for } 0 \le s < t \le T. \tag{22}$$

Second, we derive a simple lemma, which implies continuity a.e. in $[0,T]$ of the z component. For $f : [a,b] \to Y$ we use the following definition of *one-sided limits*:

$$\begin{aligned} f(t^+) &\stackrel{\text{def}}{=} \lim_{h\to 0^+} f(t{+}h), \quad f(b^+) \stackrel{\text{def}}{=} f(b), \\ f(t^-) &\stackrel{\text{def}}{=} \lim_{h\to 0^+} f(t{-}h), \quad f(a^-) \stackrel{\text{def}}{=} f(a). \end{aligned} \tag{23}$$

To analyze the behavior at jump points, first note that $\mathrm{Diss}_{\mathcal{D}}(z,[0,T])<\infty$ implies that $\delta : t\mapsto \mathrm{Diss}_{\mathcal{D}}(z,[0,t])$ has at most a countable number of jump points. At a continuity point of δ we have

$$\mathcal{D}(z(t-\varepsilon),z(t))+\mathcal{D}(z(t),z(t+\varepsilon))\leq \mathrm{Diss}_{\mathcal{D}}(z,[t-\varepsilon,t+\varepsilon])\to 0 \text{ for } \varepsilon\to 0.$$

Because $\{\, z(t-\varepsilon) \mid 0<\varepsilon<\varepsilon_0 \,\}$ lies in a compact sublevel, we may assume $z(t-\varepsilon_j) \overset{\mathcal{Z}}{\to} z_*$. By the lower semicontinuity (D2) we find $\mathcal{D}(z_*,z(t))\leq \liminf_{j\to\infty}\mathcal{D}(z(t-\varepsilon_j),z(t))=0$. Using (D1) we conclude $z_*=z(t)$. By uniqueness of the limit we find $z(t^-)=\lim z(t-\varepsilon)=z(t)$. Similarly, we have $z(t^+)=z(t)$. Hence, we conclude that $z:[0,T]\to\mathcal{Z}$ is continuous at every continuity point of δ. Moreover, at every jump point of δ the left-hand and right-hand limits $z(t^-)$ and $z(t^+)$ exist. In general, the three values $z(t^-)$, $z(t)$, and $z(t^+)$ may be different.

Since the jump conditions are most easily formulated in terms of the reduced system $(\mathcal{Z},\mathcal{I},\mathcal{D})$ we define the *reduced stability sets*

$$\widehat{\mathcal{S}}(t)\overset{\mathrm{def}}{=}\{\, z\in\mathcal{Z} \mid \mathcal{I}(t,z)<\infty,\ \forall \widetilde{z}\in\mathcal{Z}:\ \mathcal{I}(t,z)\leq\mathcal{I}(t,\widetilde{z})+\mathcal{D}(z,\widetilde{z})\,\}. \tag{24}$$

Lemma 3.6 (Jump conditions for energetic solutions). *Assume that* (D1)*,* (D2)*,* (E1)*,* (E2) *and* (C2) *hold. Let $q=(y,z):[0,T]\to\mathcal{Q}$ be an energetic solution to $(\mathcal{Q},\mathcal{E},\mathcal{D})$. Then, for all $t\in[0,T]$ we have the relations*

$$\mathcal{I}(t,z(t))+\mathcal{D}(z(t^-),z(t))=\mathcal{I}(t,z(t^-)), \tag{25a}$$
$$\mathcal{I}(t,z(t^+))+\mathcal{D}(z(t),z(t^+))=\mathcal{I}(t,z(t)), \tag{25b}$$
$$\mathcal{I}(t,z(t^-))=\lim_{\tau\to t^-}\mathcal{I}(\tau,z(\tau)), \tag{25c}$$
$$\mathcal{I}(t,z(t^+))=\lim_{\tau\to t^+}\mathcal{I}(\tau,z(\tau)), \tag{25d}$$
$$\mathcal{D}(z(t^-),z(t))+\mathcal{D}(z(t),z(t^+))=\mathcal{D}(z(t^-),z(t^+)). \tag{25e}$$

Moreover, we have $z(t^-),z(t),z(t^+)\in\widehat{\mathcal{S}}(t)\subset\mathcal{Z}$ for all $t\in[0,T]$.

Proof. We consider only the first statement for $t>0$, since the second works analogously for $t<T$. We subtract the energy balance for $\tau<t$ from that of t and use (3) to obtain $\mathcal{I}(t,z(t))+\mathrm{Diss}_{\mathcal{D}}(z,[\tau,t]) = \mathcal{E}(\tau,z(\tau))+\int_\tau^t \mathcal{P}_{\mathrm{red}}(s,z(s))\,\mathrm{d}s$. Passing to the limit $\tau\to t^-$, the last term disappears, and we find $\mathcal{I}(t,z(t))+\mathcal{D}(z(t^-),z(t))=\lim_{\tau\to t^-}\mathcal{I}(\tau,z(\tau))$.

We claim $\mathcal{I}(t,z(t^-))=\lim_{\tau\to t^-}\mathcal{I}(\tau,z(\tau))$. By the lower semicontinuity (E1) we know $\mathcal{I}(t,z(t^-))\leq \liminf_{\tau\to t^-}\mathcal{I}(\tau,z(\tau))$ as $z(\tau)\overset{\mathcal{Z}}{\to} z(t^-)$. The opposite inequality follows from stability of $z(\tau)$ with respect to $z(t^-)$, namely $\mathcal{I}(\tau,z(\tau))\leq\mathcal{I}(\tau,z(t^-))+\mathcal{D}(z(\tau),z(t^-))$. Using $\mathcal{D}(z(\tau),z(t^-))\leq \lim_{s\to t^-}\mathrm{Diss}_{\mathcal{D}}(z,[\tau,s])$ we obtain $\mathcal{D}(z(\tau),z(t^-))\to 0$ for $\tau\to t^-$. This implies $\limsup_{\tau\to t^-}\mathcal{I}(\tau,z(\tau))\leq\mathcal{I}(t,z(t^-))+0$, and (25a) and (25c) are established. Assertions (25b) and (25d) are obtained analogously.

To establish the last statement fix $t\in\,]0,T]$ and consider $q_n=q(t-\frac{1}{n})\in\mathcal{S}(t-\frac{1}{n})$. Using (E1) and (C2) there exists a convergent subsequence such that $q_{n_m}\overset{\mathcal{Q}}{\to}(\widetilde{y},z(t^-))\in\mathcal{S}(t)$, i.e., $z(t^-)\in\widehat{\mathcal{S}}(t)$. Analogously, we show $z(t^+)\in\widehat{\mathcal{S}}(t)$.

To establish (25e) it suffices to show "$\leq$", since the triangle inequality (D1) implies "$\geq$". For this we use $z(t^-)\in\widehat{\mathcal{S}}(t)$ and test with $z(t^+)$ to obtain $\mathcal{I}(t,z(t^-))\leq\mathcal{I}(t,z(t^+))+\mathcal{D}(z(t^-),z(t^+))$. Inserting (25a) and (25b) the desired estimate follows. $\square$

3.5 On compatibility conditions (C1) *and* (C2)

The central condition that makes the whole theory working is the conditioned closedness of the stability set (C2). For this, the interplay of the chosen topology and the properties of $\mathcal{E}$ and $\mathcal{D}$ are essential. The main philosophy of this condition is that stable sequences behave better than usual sequences.

In many applications the power continuity (C1) is really a condition on $\mathcal{E}$ alone, namely if $\partial_t\mathcal{E} : \{\, (t,q) \mid \mathcal{E}(t,q) \leq E \,\} \to \mathbb{R}$ is continuous for all $E > 0$. Such cases typically occur if the space $\mathcal{Q}$ is a reflexive Banach space equipped with the weak topology and if the loading of the problem is lower order or even linear. However, there are also important applications where the full generality of (C1) is needed, in particular, if $\mathcal{D}$ is not continuous.

A fairly general way of establishing the crucial closedness condition (C2) is given in terms of finding a *joint recovery sequence* $(\widetilde{q}_l)_{l\in\mathbb{N}}$:

$$\begin{aligned}&\forall \text{ stab.seq. } (t_l,q_l) \overset{\mathcal{Q}_T}{\to} (t,q)\ \forall\, \widetilde{q}\in\mathcal{Q}\ \exists\, \widetilde{q}_l \overset{\mathcal{Q}}{\to} \widetilde{q} :\\ &\limsup_{l\to\infty} \big(\mathcal{E}(t_l,\widetilde{q}_l)+\mathcal{D}(q_l,\widetilde{q}_l)-\mathcal{E}(t_l,q_l)\big) \leq \mathcal{E}(t,\widetilde{q})+\mathcal{D}(q,\widetilde{q})-\mathcal{E}(t,q).\end{aligned} \tag{26}$$

We also provide two stronger conditions, namely

$$\begin{aligned}&\forall \text{ stab.seq. } (t_l,q_l) \overset{\mathcal{Q}_T}{\to} (t,q)\ \forall\, \widetilde{q}\in\mathcal{Q}\ \exists\, \widetilde{q}_l \overset{\mathcal{Q}}{\to} \widetilde{q} :\\ &\limsup_{l\to\infty} \big(\mathcal{E}(t_l,\widetilde{q}_l)+\mathcal{D}(q_l,\widetilde{q}_l)\big) \leq \mathcal{E}(t,\widetilde{q})+\mathcal{D}(q,\widetilde{q});\end{aligned} \tag{27}$$

$$\left.\begin{aligned}&q_k \overset{\mathcal{Q}}{\to} q,\ \widetilde{q}_k \overset{\mathcal{Q}}{\to} \widetilde{q} \text{ and}\\ &\sup_{k\in\mathbb{N}} \big(\mathcal{E}(t,q_k)+\mathcal{E}(t,\widetilde{q}_k)\big) < \infty\end{aligned}\right\} \implies \mathcal{D}(q_k,\widetilde{q}_k) \to \mathcal{D}(q,\widetilde{q}). \tag{28}$$

Since the following results are straightforward, we refer to [MiR09b] for a full proof.

Proposition 3.7 (Sufficient conditions for (C2)**).** *Assume* (E1).

(i) If for each stable sequence $(t_l,q_l) \overset{\mathcal{Q}_T}{\to} (t,q)$ *there exists a sequence* $(\widetilde{q}_l)_{l\in\mathbb{N}}$ *such that* $\limsup_{l\to\infty} \mathcal{E}(t_l,\widetilde{q}_l)+\mathcal{D}(q_l,\widetilde{q}_l) \leq \mathcal{E}(t,q)$*, then the energy converges along stable sequences, i.e.*

$$\forall\ \textit{stab.seq. } (t_l,q_l) \overset{\mathcal{Q}_T}{\to} (t,q) : \quad \mathcal{E}(t_l,q_l) \to \mathcal{E}(t,q). \tag{29}$$

In particular, (27) *implies* (29).

(ii) We have the implications (28) $\implies$ (27) $\implies$ (26) $\implies$ (C2).

(iii) If (E2) *holds additionally, then the conditions* (26) *and* (27) *remain the same if* $\mathcal{E}(t_l,\cdot)$ *is replaced by* $\mathcal{E}(t,\cdot)$.

Concerning the conditioned continuity of the power, we mention that often the case is considered that $\mathcal{Y}$ is a weakly closed subset of a reflexive Banach space $\boldsymbol{Y}$ equipped with the weak topology. Moreover the energy takes the form $\mathcal{E}(t,q) = \Phi(q) - \langle \ell(t), y\rangle$, where $\ell \in \mathrm{W}^{1,1}([0,T],\boldsymbol{Y}^*)$; then it is easy to establish (C1) even without using the stability.

The following abstract result establishes the continuity of the power (C1) under more general conditions. It purely relies on semicontinuity properties, is independent of a linear structure, and goes back to an idea in [DFT05] for showing that the stresses in nonlinear elasticity converge weakly, if the functions y^n as well as the energy converge. The following result is an abstract and much simpler version of this fact, see [Mie05, Prop. 5.6] for the proof.

Proposition 3.8 (Sufficient conditions for (C1)**).** *If $\mathcal{E}$ satisfies* (E1)–(E2)*, then* (30) *implies* (31)*:*

$$\begin{aligned}&\forall E>0\ \forall\varepsilon>0\ \exists\delta>0\ \forall t_1,t_2\in[0,T]\ \forall q\in\mathcal{Q}:\\ &\mathcal{E}(0,q)\le E,\ |t_2-t_1|\le\delta\implies \big|\partial_t\mathcal{E}(t_1,q)-\partial_t\mathcal{E}(t_2,q)\big|\le\varepsilon,\end{aligned}\tag{30}$$

$$\left.\begin{aligned}&(t_m,q_m)\overset{\mathcal{Q}_T}{\to}(t,q)\ \textit{and}\\ &\mathcal{E}(t_m,q_m)\to\mathcal{E}(t,q)<\infty\end{aligned}\right\}\implies\ \partial_t\mathcal{E}(t,q_m)\to\partial_t\mathcal{E}(t,q).\tag{31}$$

Together with Proposition 3.7 we obtain the following result.

Corollary 3.9. *Assume that* (D1), (D2), (E1), (E2), (27)*, and* (30) *hold. Then both compatibility conditions* (C1) *and* (C2) *are satisfied.*

3.6 Proof of Theorem 3.4

The proof follows the main steps in [FrM06], however it includes a new argument, namely a precise characterization of the power, see (18) and Proposition 3.11. This approach allows us to simplify the assumptions considerably in the case that $\mathcal{Q}$ satisfies the metrizability condition (17).

The proof of the main existence result makes extensive use of the reduced energy functional $\mathcal{I}:\mathcal{Z}_T\to\mathbb{R}_\infty$, since the stability condition (S) as well as the energy balance (E) can be formulated easily for the reduced RIS $(\mathcal{Z},\mathcal{I},\mathcal{D})$. For this recall the reduced stability set $\widehat{\mathcal{S}}(t)$ defined in (24). Clearly, we have $q=(y,z)\in\mathcal{S}(t)$ if and only if $z\in\widehat{\mathcal{S}}(t)$ and $y\in\operatorname{Arg\,min}\mathcal{E}(t,\cdot,z)$. The only difficulty in reducing from $\mathcal{Q}$ to $\mathcal{Z}$ is that in general $\mathcal{I}(\cdot,z)$ is no longer differentiable. Thus, we define energetic solutions $z:[0,T]\to\mathcal{Z}$ of the reduced RIS $(\mathcal{Z},\mathcal{I},\mathcal{D})$ via the reduced power $\mathcal{P}_{\mathrm{red}}$ defined in (15) as follows:

$$\begin{aligned}&(\mathrm{S})_{\mathrm{red}}\quad z(t)\in\widehat{\mathcal{S}}(t),\\ &(\mathrm{E})_{\mathrm{red}}\quad \mathcal{I}(t,z(t))+\mathrm{Diss}_{\mathcal{D}}(z,[0,t])=\mathcal{I}(0,z(0))+\textstyle\int_0^t\mathcal{P}_{\mathrm{red}}(s,z(s))\,\mathrm{d}s.\end{aligned}\tag{32}$$

Because of (18) each energetic solution $q=(y,z):[0,T]\to\mathcal{Q}$ gives rise to a reduced energetic solution $z:[0,T]\to\mathcal{Z}$ for $(\mathcal{Z},\mathcal{I},\mathcal{D})$. The next lemma shows that the opposite is also true. Each solution z for $(\mathcal{Z},\mathcal{I},\mathcal{D})$ can be made into a full solution $q=(y,z)$ for the RIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$ by selecting a suitable measurable y-component.

Lemma 3.10 (Selection of the y-component). *Assume that* (D1), (D2), (E1), (E2), (C2), (C1)*, and* (17) *are satisfied. Let $z:[0,T]\to\mathcal{Z}$ be measurable with $\mathrm{Diss}_{\mathcal{D}}(z,[0,T])+\sup_{t\in[0,T]}\mathcal{I}(t,z(t))<\infty$ and $z(t)\in\widehat{\mathcal{S}}(t)$ for each $t\in[0,T]$. Then there exists a measurable function $y:[0,T]\to\mathcal{Y}$ such that for all $t\in[0,T]$ we have*

$$(y(t),z(t))\in\mathcal{S}(t)\quad\textit{and}\quad \mathcal{P}_{\mathrm{red}}(t,z(t))=\partial_t\mathcal{E}(t,y(t),z(t)).$$

Proof. Our proof is based on a variant of Filippov's selection theorem, which we use here with the complete measure space $([0,T],\mathfrak{S},\mu)$ with $\mathfrak{S}=$ the σ-algebra of the Lebesgue measurable subsets and $\mu=\mathfrak{L}^1(\cdot)$ the one-dimensional Lebesgue measure. For given $(t,z)\in\mathcal{Z}_T$ we define $M(t,z)\overset{\mathrm{def}}{=}\operatorname{Arg\,min}\{\,\mathcal{E}(t,\widetilde{y},z)\mid\widetilde{y}\in\mathcal{Y}\,\}$. For the given measurable $z:[0,T]\to\mathcal{Z}$ we compose a set-valued mapping $G:[0,T]\rightrightarrows\mathcal{Y}$ via

$$G(t) \stackrel{\text{def}}{=} M(t, z(t^-)) \cup M(t, z(t)) \cup M(t, z(t^+)) \subset Y \subset \mathcal{Y},$$

where Y is a compact subset of $\mathcal{Y}$, which exists due to (E1). Using assumption (17), we know that the topology on Y is complete, separable and metrizable.

Using (E1) each $M(t, z)$ is nonempty, and hence each $G(t)$ is nonempty. Employing (C2) we show that the graph $\mathrm{Gr}(G) = \{\, (t, y) \mid y \in G(t)\,\}$ is closed in $[0, T]\times Y$ and hence is measurable. Indeed, consider $(t_k, y_k) \in \mathrm{Gr}(G)$ with $t_k \to t_*$ and $y_k \to y_*$, then there exists $z_k = z(t^\nu_k)$ with $t^\nu_k \in \{t_k, t^-_k, t^+_k\}$ such that $y_k \in M(t_k, z_k)$. Using the last statement in Lemma 3.6 (which is valid for every measurable $z : [0, T] \to \mathcal{Z}$ with $z(t) \in \widehat{\mathcal{S}}(t)$ and not only for energetic solutions), we conclude $(y_k, z_k) \in \mathcal{S}(t_k)$. After taking a subsequence (not relabeled) we may assume $z_k \stackrel{\mathcal{Z}}{\to} z_*$ and (C2) provides $(y_*, z_*) \in \mathcal{S}(t_*)$, which implies $y_* \in M(t_*, z_*)$. Moreover, (t_*, z_*) lies in the closure of $\mathrm{Gr}(z) \subset \mathcal{Z}_T$, which means $z_* = z(t^\nu_*)$ with $t^\nu_k \in \{t_k, t^-_k, t^+_k\}$. Thus, we have established $y_* \in G(t_*)$ as desired.

The set-valued mapping $F : [0, T] \rightrightarrows Y$ is defined via $F(t) = M(t, z(t))$. Clearly, $F(t)$ is nonempty and closed for each t. Since z is continuous outside an at most countable set $J(z) \subset [0, T]$, we have $F(t) = G(t)$ for $t \in [0, T]\backslash J(z)$. Thus, F is a measurable set-valued mapping as well.

We now define the function $g : \mathrm{Gr}(F) \to \mathbb{R}$ via

$$g(t, y) \stackrel{\text{def}}{=} \partial_t\mathcal{E}(t, y, z(t)) - \mathcal{P}_{\mathrm{red}}(t, z(t)) \text{ for } t \in [0, T] \text{ and } y \in F(t).$$

For fixed $t \in [0, T]$ the function $g(t, \cdot) : F(t) \to \mathbb{R}$ is continuous because of (C1). Since $\mathcal{E}$ has compact sublevels by (E1), g is Borel-measurable. Moreover, $z : [0, T] \to \mathcal{Z}$ is Borel-measurable, since it is continuous except for a countable number of points. Thus, for each $h \geq 0$ the functions $\gamma_h : [0, T{-}h]\times Y \to \mathbb{R}; (t, y) \mapsto \mathcal{E}(t{+}h, y, z(t))$ are $(\mathfrak{L}\otimes\mathfrak{B}(Y), \mathfrak{B}(\mathbb{R}))$-measurable. Since g is the difference of the pointwise limit of the measurable difference quotients $\frac{1}{h}(\gamma_h - \gamma_0)$ and $t \mapsto \mathcal{P}_{\mathrm{red}}(t, z(t))$ (which is measurable), it is measurable as well. Hence, the restriction of g to $\mathrm{Gr}(F)$ is measurable.

Next we show that for each t there exists $y \in F(t)$ with $g(t, y) = 0$. Indeed, by (E1) the set $M(t, z(t)) = \operatorname{Arg\,min} \mathcal{E}(t, \cdot, z(t))$ is a nonempty compact set. Choose a sequence $(y_m)_m$ approaching the supremum in the definition (15) of the reduced power, viz., $\mathcal{P}_{\mathrm{red}}(t, z(t)) = \sup\{\, \partial_t\mathcal{E}(t, \widetilde{y}, z(t)) \mid \widetilde{y} \in M(t, z(t))\,\}$. Taking a subsequence, we may assume $y_{m_n} \stackrel{\mathcal{Y}}{\to} y_* \in M(t, z(t))$. Since $(y_{m_n}, z(t)) \in \mathcal{S}(t)$ we have a stable sequence, and (C1) gives $\mathcal{P}_{\mathrm{red}}(t, z(t)) = \lim_{n\to\infty} \partial_t\mathcal{E}(t, y_{m_n}, z(t)) = \partial_t\mathcal{E}(t, y_*, z(t))$, as desired.

We are now able to apply Filippov's theorem (cf. [AuF90, Thm. 8.2.9+10] and obtain the desired measurable selection $y : [0, T] \to Y$ with $y(t) \in F(t)$ and $g(t, y(t)) = 0$. □

We next present a lower energy estimate that is valid for all stable processes. The fact that stability implies such a lower energy estimate was first observed in [MiT04]. Here we use a stronger version that replaces the work of the external forces on the right-hand side by the integral of the reduced power $\mathcal{P}_{\mathrm{red}}$. Since the left-hand side in (33) does not depend on the y-component of the stable process (see (14)), it is clear that the lower bound on the right-hand side should also be expressible in terms of z alone. It is this seemingly simple observation that allowed us to simplify the assumptions of the main existence result.

Proposition 3.11 (Lower energy estimate). *Assume that* (D1)*,* (D2)*,* (E1)*,* (E2)*,* (C2)*,* (C1)*, and* (17) *hold. Let $q = (y, z) : [0, T] \to \mathcal{Q}$ be measurable with $\sup_{t\in[0,T]} \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0, T]) < \infty$, and $q(t) \in \mathcal{S}(t)$ for all $t \in [0, T]$. Then, for all $0 \leq r < s \leq T$ we have the lower energy inequality*

$$\begin{aligned} &\mathcal{E}(s, q(s)) + \mathrm{Diss}_{\mathcal{D}}(z, [r, s]) - \mathcal{E}(r, q(r)) \\ &\geq \int_r^s \mathcal{P}_{\mathrm{red}}(t, z(t))\,\mathrm{d}t \geq \int_r^s \partial_t\mathcal{E}(t, q(t))\,\mathrm{d}t. \end{aligned} \tag{33}$$

Proof. We use the fact that the left-hand side is independent of the y-component, viz. (14). By the stability of $q(r)$ and $q(s)$ it can be written as $\mathcal{I}(s,z(s))+\mathrm{Diss}_{\mathcal{D}}(z,[r,s])-\mathcal{I}(r,z(r))$. Thus, it suffices to show the first inequality in (33), since the second one follows directly from the definition (15) of the reduced power, namely $\partial_t\mathcal{E}(t,q(t))\leq \mathcal{P}_{\mathrm{red}}(t,z(t))$ for all $t\in[0,T]$. By Lemma 3.10 we may choose y such that we have equality.

Hence we assume now that q satisfies this equality, i.e. $\partial_t\mathcal{E}(t,q(t))=\mathcal{P}_{\mathrm{red}}(t,z(t))$. Take any partition $\Pi=(t_0,t_1,\ldots,t_N)\in\mathrm{Part}([r,s])$. For each $t_{j-1}\in\Pi$ we use $q(t_{j-1})\in\mathcal{S}(t_{j-1})$ to obtain $\mathcal{E}(t_{j-1},q(t_{j-1}))\leq\mathcal{E}(t_{j-1},q(t_j))+\mathcal{D}(q(t_{j-1}),q(t_j))$, which is the same as

$$\mathcal{E}(t_j,q(t_j))+\mathcal{D}(z(t_{j-1}),z(t_j))-\mathcal{E}(t_{j-1},q(t_{j-1}))\geq\big(\mathcal{E}(t_j,q(t_j))-\mathcal{E}(t_{j-1},q(t_j))\big).$$

Summing over $j\in\{1,\ldots,N_\Pi\}$ we find

$$\begin{aligned}&\mathcal{E}(s,q(s))+\mathrm{Diss}_{\mathcal{D}}(z,[r,s])-\mathcal{E}(r,q(r))\\&\geq\mathcal{E}(s,q(s))+\textstyle\sum_{j=1}^{N_\Pi}\mathcal{D}(z(t_{j-1}),z(t_j))-\mathcal{E}(r,q(r))\\&\geq\textstyle\sum_{j=1}^{N_\Pi}\big(\mathcal{E}(t_j,q(t_j))-\mathcal{E}(t_{j-1},q(t_j))\big)=\sum_{j=1}^{N_\Pi}\int_{t_{j-1}}^{t_j}\partial_t\mathcal{E}(\tau,q(t_j))\,\mathrm{d}\tau\\&=\textstyle\int_r^s\partial_t\mathcal{E}(\tau,\overline{q}^\Pi(\tau))\,\mathrm{d}\tau,\end{aligned}$$

where $\overline{q}^\Pi$ is the left-continuous interpolant with $\overline{q}^\Pi(t)=q(t_j)$ for $t\in\,]t_{j-1},t_j]$. Since the partition Π was arbitrary, we can apply Lemma 3.12 to obtain the desired result. $\square$

Lemma 3.12. *Let the conditions* (E2), (C1) *and the metrizability condition* (17) *hold. Moreover, assume that* $q:[0,T]\to\mathcal{Q}$ *is measurable, and there is a* $C>0$ *such that for all* $t\in[0,T]$ *we have* $\mathcal{E}(t,q(t))\leq C$ *and* $q(t)\in\mathcal{S}(t)$. *Then, for all* $r,s\in[0,T]$ *with* $r<s$ *we have*

$$\sup_{\Pi\in\mathrm{Part}([r,s])}\int_r^s\partial_t\mathcal{E}(\tau,\overline{q}^\Pi(\tau))\,\mathrm{d}\tau\geq\int_s^t\partial_\tau\mathcal{E}(\tau,q(\tau))\,\mathrm{d}t.$$

Proof. Since each function $t\mapsto\mathcal{E}(t+h,q(t))$ is measurable, the power $\tau\mapsto\partial_\tau\mathcal{E}(\tau,q(\tau))$ is measurable as well, because it is a pointwise limit of measurable difference quotients. Moreover, there is a constant $c_0>0$ such that $|\partial_t\mathcal{E}(t,\overline{q}^\Pi(t))|\leq c_0\lambda_{\mathcal{E}}$.

Using (17) we may apply Lusin's theorem to q, which takes values in a compact set since the energy is bounded. For $\varepsilon>0$ we find a compact set $K\subset[r,s]$ with

$$c_0\mathfrak{L}^1([r,s]\setminus K)\lambda_{\mathcal{E}}<\varepsilon,\quad\text{and}\quad q|_K:K\to\mathcal{Q}\text{ is continuous.}\tag{34}$$

This implies $\int_r^s\partial_t\mathcal{E}(t,\overline{q}^\Pi(t))\,\mathrm{d}t\geq\int_K\partial_t\mathcal{E}(t,\overline{q}^\Pi(t))\,\mathrm{d}t-\varepsilon$ for all partitions Π.

We now construct a sequence of partitions $(\Pi_n)_n$ that allows us to prove the assertion. Let $t_0^n=r$ and define the other points inductively, namely as long as $t_j^n<s$ we set

$$t_{j+1}^n=\begin{cases}\max\{\,t\in K\mid t_j^n<t\leq t_j^n+\frac{1}{n}\,\}\text{ if }K\cap\big]t_j^n,t_j^n+\frac{1}{n}\big]\neq\emptyset,\\ \qquad\min\{t_j^n+\frac{1}{n},s\}\qquad\qquad\text{else.}\end{cases}$$

On the one hand, there cannot be two adjacent intervals that are small: if $t_{j+1}^n<t_j^n+\frac{1}{n}$, then $K\cap\big]t_{j+1}^n,t_j^n+\frac{1}{n}\big]$ is empty. Now, if $t_{j+1}^n<s$, then t_{j+2}^n exceeds $\min\{t_j^n+\frac{1}{n},s\}$. Hence, Π_n has at most $2(s-r)n+1$ intervals, and by construction the fineness satisfies $\phi(\Pi_n)\leq\frac{1}{n}$. On the other hand, the choice of the nodes in Π_n is such that for $t\in K$ we always have $\overline{\tau}^\Pi(t)\in K$ as well. Indeed, $t_{j+1}^n\in\Pi\backslash K$ occurs only, if $\big]t_j^n,t_{j+1}^n\big]$ has empty intersection with K. Thus, we have shown $\overline{\tau}^{\Pi_n}(t)\in K$ and $\overline{\tau}^{\Pi_n}(t)\to t^+$ for $n\to\infty$ for all $t\in K$.

Recall $\overline{q}^{\Pi_n}(t) = q(\overline{\tau}^{\Pi_n}(t))$ and use the stability of q to conclude that $(\overline{\tau}^{\Pi_n}(t), \overline{q}^{\Pi_n}(t))_{n\in\mathbb{N}}$ is a stable sequence converging to $(t, q(t))$ because of (34). Exploiting (C1) we find $\partial_t\mathcal{E}(t, \overline{q}^{\Pi_n}(t)) \to \partial_t\mathcal{E}(t, q(t))$ and $\lim_{n\to\infty}\int_K \partial_t\mathcal{E}(t, \overline{q}^{\Pi_n}(t))\,\mathrm{d}t = \int_K \partial_t\mathcal{E}(t, q(t))\,\mathrm{d}t$ by Lebesgue's dominated convergence theorem. In summary we have

$$\begin{aligned}
&\sup_{\Pi}\textstyle\int_r^s \partial_t\mathcal{E}(t, \overline{q}^{\Pi}(t))\,\mathrm{d}t \geq \limsup_{n\to\infty}\int_r^s \partial_t\mathcal{E}(t, \overline{q}^{\Pi_n}(t))\,\mathrm{d}t\\
&\geq -\varepsilon + \limsup_{n\to\infty}\textstyle\int_K \partial_t\mathcal{E}(t, \overline{q}^{\Pi_n}(t))\,\mathrm{d}t = -\varepsilon + \int_K \partial_t\mathcal{E}(t, q(t))\,\mathrm{d}t\\
&\geq -2\varepsilon + \textstyle\int_r^s \partial_t\mathcal{E}(t, q(t))\,\mathrm{d}t.
\end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, this is the desired result. □

The existence theory developed below builds on the (IMP$^{\Pi}$) and a priori estimates. The general strategy for constructing solutions to (S)&(E) is to choose a sequence of partitions Π^m with $\phi(\Pi^m) \to 0$, to extract a convergent subsequence of $(z^l)_{l\in\mathbb{N}}$ of $(z^{\Pi^m})_{m\in\mathbb{N}}$, and then to show that the limit $z : [0,T] \to \mathcal{Z}$ solves (S)&(E). The existence of a convergent subsequence is guaranteed by the following version of Helly's selections principle, see [MaM05, MRS08] for a full proof.

Theorem 3.13 (Generalized version of Helly's selection principle). *Let $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ satisfy* (D1) *and* (D2) *and let $\mathcal{K}$ be a (sequentially) compact subset of $\mathcal{Z}$. Then, for every sequence $(z^l)_{l\in\mathbb{N}}$ with $z^l : [0,T] \to \mathcal{K}$ and $\sup_{l\in\mathbb{N}} \mathrm{Diss}_{\mathcal{D}}(z^l, [0,T]) \leq \infty$, there exist a subsequence $(z^{l_n})_{n\in\mathbb{N}}$ and functions $z_\infty : [0,T] \to \mathcal{K}$ and $\delta_\infty : [0,T] \to [0,C]$ such that the following holds:*

(i) $\delta_{l_n}(t) \stackrel{\text{def}}{=} \mathrm{Diss}_{\mathcal{D}}(z^{l_n}, [0,t]) \to \delta_\infty(t)$ *for all* $t \in [0,T]$*;*

(ii) $z_{l_n}(t) \stackrel{\mathcal{Z}}{\to} z_\infty(t)$ *for all* $t \in [0,T]$*;*

(iii) $\mathrm{Diss}_{\mathcal{D}}(z_\infty, [t_0,t_1]) \leq \delta_\infty(t_1) - \delta_\infty(t_0)$ *for all* $0 \leq t_0 < t_1 \leq T$.

We are now ready to prove the main existence result stated in Theorem 3.4.

Proof (of Theorem 3.4). We divide the proof into 6 steps.

Step 1: A priori estimates. We choose an arbitrary sequence of partitions Π^m whose fineness $f_m = \phi(\Pi^m)$ tends to 0. The time-incremental minimization problems (IMP$^{\Pi}$) are solvable and the piecewise constant interpolants $\underline{q}^m = (\underline{y}^m, \underline{z}^m) : [0,T] \to \mathcal{Q}$ defined in (13) satisfy the a priori estimates

$$\mathrm{Diss}_{\mathcal{D}}(\underline{z}^m, [0,T]) \leq C_{\mathcal{D}} \quad\text{and}\quad \forall\, t \in [0,T] :\ \mathcal{E}(t, \underline{q}^m(t)) \leq C_{\mathcal{E}},$$

where $C_{\mathcal{D}}$ and $C_{\mathcal{E}}$ are given explicity in Corollary 3.3.

Step 2: Selection of subsequences. Our version of Helly's selection principle in Theorem 3.13 allows us to select a subsequence of $(\underline{z}^m)_{m\in\mathbb{N}}$ that converges pointwise and that makes the dissipation converge as well. Moreover, the functions $P^m : t \mapsto \partial_t\mathcal{E}(t, \underline{q}^m(t))$ form an equibounded sequence in $\mathrm{L}^1((0,T))$. Thus, by choosing a further subsequence $(\underline{q}^{m_k})_{k\in\mathbb{N}}$ we may assume the following convergence properties for $k \to \infty$, where we write q_k as shorthand for $\underline{q}^{m_k}$ and p_k for P^{m_k}:

$$\begin{aligned}
&p_k \rightharpoonup p_{\text{weak}} \text{ in } \mathrm{L}^1((0,T)),\\
&\forall\, t \in [0,T] :\ \delta_k(t) \stackrel{\text{def}}{=} \mathrm{Diss}_{\mathcal{D}}(z_k, [0,t]) \to \delta(t) \text{ and } z_k(t) \stackrel{\mathcal{Z}}{\to} z(t).
\end{aligned}$$

Since the limit $z : [0,T] \to \mathcal{Z}$ satisfies $\mathrm{Diss}_{\mathcal{D}}(z, [0,T]) \leq \delta(T) \leq C_{\mathcal{D}} < \infty$, we know that z is measurable and that it satisfies the energetic bound $\mathcal{I}(t, z(t)) \leq C_{\mathcal{E}}$. Thus, Lemma 3.10 provides a measurable $y : [0,T] \to \mathcal{Y}$ such that

$$y(t) \in \operatorname{Arg\,min} \mathcal{E}(t, \cdot, z(t)) \quad\text{and}\quad \partial_t\mathcal{E}(y(t), z(t)) = \mathcal{P}_{\text{red}}(t, z(t)), \tag{35}$$

and $y(0) = y_0$, where $q_0 = (y_0, z_0)$ is the given initial value with $q_0 \in \mathcal{S}(0)$. By construction $z(0) = z^m(0) = z_0$ such that $y(0) = y_0$ is an admissible choice satisfying the first relation in (35) but not necessarily the second.

Step 3: Stability of the limit function. We use the compatibility condition (C2). For fixed $t \in [0, T]$ we define τ_k to be the largest value in $\Pi^{m_k} \cap [0, t]$ such that $q_k(t) = q_k(\tau_k)$. Then, $q_k(t) \in \mathcal{S}(\tau_k)$, $\tau_k \le t$, and $\tau_k \to t$. By choosing a further subsequence, if necessary, we obtain $q_{k_l}(t) \xrightarrow{\mathcal{Q}} \widetilde{q} = (\widetilde{y}, z(t))$. In particular, $(\tau_{k_l}, q_{k_l}(t))_{l\in\mathbb{N}}$ forms a convergent stable sequence. Now, (C2) yields $\widetilde{q} \in \mathcal{S}(t)$, whence $\widetilde{y} \in \operatorname{Arg\,min} \mathcal{E}(t, \cdot, \widetilde{z}(t))$. However, this also implies $q(t) = (y(t), z(t)) \in \mathcal{S}(t)$, since for all $\widehat{q} = (\widehat{y}, \widehat{z}) \in \mathcal{Q}$ we have $\mathcal{E}(t, q(t)) = \mathcal{I}(t, z(t)) = \mathcal{E}(t, \widetilde{q}) \le \mathcal{E}(t, \widehat{q}) + \mathcal{D}(z(t), \widehat{z})$.

Step 4: Upper energy estimate. We define the functions

$$
\begin{aligned}
e_k(t) &\overset{\text{def}}{=} \mathcal{E}(t, q_k(t)), \quad \delta_k(t) \overset{\text{def}}{=} \mathrm{Diss}_{\mathcal{D}}(z_k, [0, t]), \quad e_\infty(t) \overset{\text{def}}{=} \liminf_{k\to\infty} e_k(t),\\
E(t) &\overset{\text{def}}{=} \mathcal{E}(t, q(t)), \qquad \Delta(t) \overset{\text{def}}{=} \mathrm{Diss}_{\mathcal{D}}(z, [0, t]), \qquad \delta_\infty(t) \overset{\text{def}}{=} \lim_{k\to\infty} \delta_k(t),\\
w_k(t) &\overset{\text{def}}{=} \textstyle\int_0^t \partial_s \mathcal{E}(s, q_k(s))\,\mathrm{d}s = \int_0^t p_k(s)\,\mathrm{d}s,\\
W(t) &\overset{\text{def}}{=} \textstyle\int_0^t \partial_s \mathcal{E}(s, q(s))\,\mathrm{d}s = \int_0^t \mathcal{P}_{\mathrm{red}}(s, z(s))\,\mathrm{d}s,
\end{aligned}
$$

where by construction $e_k(0) = E(0) = e_\infty(0)$. Employing Corollary 3.3(ii) and the boundedness of $\partial_t\mathcal{E}$ by $C_1\lambda_{\mathcal{E}}$ (use (E2) and Step 1) give $e_k(t) + \delta_k(t) \le E(0) + w_k(t) + C_1\lambda_{\mathcal{E}}\phi(\Pi^{m_k})$. Since $\mathcal{E}$ and $\mathrm{Diss}_{\mathcal{D}}$ are lower semicontinuous (cf. Theorem 3.13(iii)) and since by weak convergence we have $w_\infty(t) = \lim_{k\to\infty} w_k(t) = \int_0^t p_{\mathrm{weak}}(s)\,\mathrm{d}s$, the limit $k \to \infty$ leads to

$$
E(t) + \Delta(t) \le e_\infty(t) + \delta_\infty(t) \le E(0) + w_\infty(t) = E(0) + \textstyle\int_0^t p_{\mathrm{weak}}(s)\,\mathrm{d}s. \tag{36}
$$

The next step is now to relate p_{weak} and $\mathcal{P}_{\mathrm{red}}(\cdot, z(\cdot))$ using the compatibility condition for the power (C1). As in Step 3 we choose a subsequence of $(q_k(t))_k$ such that $\mathcal{S}(\tau_{k_l}) \ni q_{k_l}(t) \xrightarrow{\mathcal{Q}} \widetilde{q}$ and $p_{k_l}(t) \to p_{\sup}(t) \overset{\text{def}}{=} \limsup_{k\to\infty} p_k(t)$. Thus, (C1) is applicable and we find

$$
p_{k_l}(t) = \partial_t\mathcal{E}(t, q_{k_l}(t)) \to \partial_t\mathcal{E}(t, \widetilde{q}) = p_{\sup}(t) \le \mathcal{P}_{\mathrm{red}}(t, z(t)),
$$

where the latter estimate follows from $\widetilde{q} = (\widetilde{y}, z(t)) \in \mathcal{S}(t)$ (use (C2)) and the definition of $\mathcal{P}_{\mathrm{red}}$. Fatou's lemma gives $w_\infty(t) \le \int_0^t p_{\sup}(s)\,\mathrm{d}s$, and we conclude the upper energy estimate

$$
E(t) + \Delta(t) \le e_\infty(t) + \delta_\infty(t) \le E(0) + w_\infty(t) \le E(0) + W(t).
$$

Step 5: Lower energy estimate. Because of our construction of the function $q : [0, T] \to \mathcal{Q}$, we are able to apply Proposition 3.11 and obtain the lower energy estimate $E(t) + \Delta(t) = \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0, t]) \ge \mathcal{E}(0, z(0)) + \int_0^t \partial_s\mathcal{E}(s, q(s))\,\mathrm{d}s = E(0) + W(t)$. Thus, we have shown that the limit function $q : [0, T] \to \mathcal{Q}$ satisfies stability and energy balance for all times, whence it is an energetic solution.

Step 6: Improved convergence. Finally we show that the convergences (19) stated at the end of the theorem hold. The convergence (19a) is already shown. The lower and upper energy estimate imply $E(0)+W(t) \le E(t) + \Delta(t) \le e_\infty(t)+\delta_\infty(t) \le E(0)+ \int_0^t p_{\mathrm{weak}}\,\mathrm{d}s \le E(0)+ \int_0^t p_{\sup}\,\mathrm{d}s \le E(0)+W(t)$. Hence, all inequalities are in fact equalities. Using $E(t) \le e_\infty(t)$, $\Delta(t) \le \delta_\infty(t)$, and $p_{\mathrm{weak}} \le p_{\sup} \le \mathcal{P}_{\mathrm{red}}$ we conclude $\Delta(t) = \delta_\infty(t)$ and $E(t) = e_\infty(t)$, which proves the convergence statements (19b) and (19c). Moreover, we also find $p_{\mathrm{weak}}(t) = p_{\sup}(t) = \mathcal{P}_{\mathrm{red}}(t, z(t))$ a.e. in $[0, T]$. Since the weak limit and the pointwise limsup of the sequence p_k coincide, the strong convergence (19d) holds, cf. [FrM06, Prop. A2]. Thus, Theorem 3.4 is proved. □

3.7 Γ-convergence for sequences of rate-independent systems

We now consider sequences of rate-independent systems $((\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k))_{k\in\mathbb{N}}$ and study the question under what assumptions energetic solutions $q_k : [0,T] \to \mathcal{Q}$ converge to a limit, which is an energetic solution to a limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$. As already revealed in [MRS08], this theory is still very close to the existence theory for energetic solutions above, so the proof of Γ-convergence follows essentially the same six steps of the proof of Theorem 3.4.

The notion of *Γ-convergence*, introduced by De Giorgi [Deg77], exclusively applies to functionals. It is sometimes also called *variational convergence* or *epigraph convergence*, cf. [Att84, AuF90, Dal93, Bra02]. Here we just give a brief outline that is sufficient for our purposes. We consider a metrizable topological space $\mathcal{Q}$, which means for our application that we restrict to a compact sublevel and use the metrizability assumption (17). For a sequence $(\mathcal{J}_k)_{k\in\mathbb{N}}$ of functionals $\mathcal{J}_k : \mathcal{Q} \to \mathbb{R}_\infty$ we are interested in the behavior for $k \to \infty$, which reflects the behavior of minimizers. In particular, the Γ-limit $\mathcal{J}$ is defined in such a way that if q_k minimizes $\mathcal{J}_k$ and $q_k \xrightarrow{\mathcal{Q}} q_\infty$, then q_∞ minimizes $\mathcal{J}$.

Definition 3.14 (Γ-convergence). A sequence $(\mathcal{J}_k)_{k\in\mathbb{N}}$ of functionals on a metrizable topological space $\mathcal{Q}$ *Γ-converges to* $\mathcal{J} : \mathcal{Q} \to \mathbb{R}_\infty$, written $\mathcal{J} = \Gamma\text{-}\lim\limits_{k\to\infty} \mathcal{J}_k$ or $\mathcal{J}_k \xrightarrow{\Gamma} \mathcal{J}$, if

$(\Gamma_{\inf})$ *Γ-liminf estimate:*

$$q_k \xrightarrow{\mathcal{Q}} q \quad \Longrightarrow \quad \mathcal{J}(q) \le \liminf_{k\to\infty} \mathcal{J}_k(q_k),$$

$(\Gamma_{\sup})$ *Γ-limsup estimate or "existence of recovery sequences":*

$$\forall\, \widehat{q} \in \mathcal{Q}\ \exists\, (\widehat{q}_k)_{k\in\mathbb{N}} \text{ with } \widehat{q}_k \xrightarrow{\mathcal{Q}} \widehat{q}: \quad \mathcal{J}(\widehat{q}) \ge \limsup_{k\to\infty} \mathcal{J}_k(\widehat{q}_k).$$

The sequence $(\widehat{q}_k)_{k\in\mathbb{N}}$ is called a *recovery sequence* for $\widehat{q}$ since $(\Gamma_{\inf})$ and $(\Gamma_{\sup})$ imply $\mathcal{J}_k(\widehat{q}_k) \to \mathcal{J}(\widehat{q})$, i.e., $\widehat{q}_k$ recovers the correct energy level. The following results are fundamental in the theory of Γ-convergence.

Proposition 3.15. *Under the above assumptions we have the following:*
(i) $\mathcal{J} = \Gamma\text{-}\liminf_{k\to\infty} \mathcal{J}_k$ is always lower semicontinuous.
(ii) For $\mathcal{J}, \mathcal{J}_k : \mathcal{Q} \to \mathbb{R}_\infty$ with $\mathcal{J} = \Gamma\text{-}\lim_{k\to\infty} \mathcal{J}_k$, set $\alpha = \inf_{\mathcal{Q}} \mathcal{J}$ and $\alpha_k = \inf_{\mathcal{Q}} \mathcal{J}_k$. Assume $\alpha \in \mathbb{R}$ and that there exist $\delta > 0$ and a compact set $C \subset \mathcal{Q}$ such that all sublevels $\{\, q \mid \mathcal{J}_k(q) \le \alpha+\delta \,\}$ are contained in C. Then, $\alpha_k \to \alpha$ and for each sequence q_k with $q_k \to \widetilde{q}$ and $\limsup_{k\to\infty} \mathcal{J}_k(\widetilde{q}_k) = \alpha$ we have $\mathcal{J}(\widetilde{q}) = \alpha$, i.e., $\widetilde{q}$ is a minimizer of $\mathcal{J}$. In particular, if q_k are minimizers of $\mathcal{J}_k$, we conclude that all accumulation points of $(q_k)_k$ are minimizers of $\mathcal{J}$.

It is surprising that Γ-convergence can be used in a rather easy way for energetic solutions of RIS. This is certainly due to the fact that the evolution is strongly governed by the static stability condition. Applications of Γ-convergence occur naturally in space-time discretizations ([KMR05, MiR09a, MPP09]) or homogenization ([MiT07]) of rate-independent material models. See also [GiP06, BFM08] for applications in fracture or [BMR09, Mie09] for damage.

We now list the assumptions on the rate-independent systems $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$, $k \in \mathbb{N}_\infty \overset{\text{def}}{=} \mathbb{N} \cup \{\infty\}$, which are sufficient for our convergence theory. They are in complete analogy to the assumptions in the existence theory above; however, certain assumptions need to be uniform in k, while other assumptions are only needed for the limiting system with $k = \infty$. Since we are already dealing with a sequence of problems and we have to choose subsequences several times, we need to adjust the notion of stable sequences. The stability sets $\mathcal{S}_k(t)$ are defined for $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ as in (5). A sequence $((t_l, q_{k_l}))_{l\in\mathbb{N}}$ is a *stable sequence* (abbreviated as "stab.seq." further on), if

$$q_{k_l} \in \mathcal{S}_{k_l}(t_l) \text{ for all } l \in \mathbb{N} \quad \text{and} \quad \sup_{l\in\mathbb{N}} \mathcal{E}_{k_l}(t_l, q_{k_l}) < \infty. \tag{37}$$

Note that $(q_{k_l})_{l\in\mathbb{N}}$ denotes a subsequence to indicate the index k_l for which we have stability. As in the previous sections, we say that $((t_l, \widetilde{q}_l))_{l\in\mathbb{N}}$ is a stable sequence for $(\mathcal{Q}, \mathcal{E}, \mathcal{D}_\infty)$ if $\widetilde{q}_l \in \mathcal{S}_\infty(t_l)$, and we shortly write "stab.seq.$^\infty$" in that case.

We collect all our assumptions and comment on them afterwards.

Quasi-distance: $\forall\, k \in \mathbb{N}_\infty\ \forall\, z, \widetilde{z}, \widehat{z} \in \mathcal{Z}:$
$$\mathcal{D}_k(z, \widetilde{z}) = 0 \ \Leftrightarrow\ z = \widetilde{z} \ \text{ and } \ \mathcal{D}_k(z, \widehat{z}) \le \mathcal{D}_k(z, \widetilde{z}) + \mathcal{D}_k(\widetilde{z}, \widehat{z}). \tag{38a}$$

Lower semicontinuity of $\mathcal{D}_k$:
$$\forall\, k \in \mathbb{N}_\infty : \quad \mathcal{D}_k : \mathcal{Z}\times\mathcal{Z} \to [0, \infty] \text{ is lower semicontinuous.} \tag{38b}$$

Lower Γ-limit for $\mathcal{D}_k$:
$$\forall \text{ stab.seq. } (t_l, q_{k_l}) \overset{\mathcal{Q}_T}{\to} (t, q) \text{ and } (\widetilde{t}_l, \widetilde{q}_{k_l}) \overset{\mathcal{Q}_T}{\to} (\widetilde{t}, \widetilde{q}) : \\ \mathcal{D}_\infty(q, \widetilde{q}) \le \liminf_{l\to\infty} \mathcal{D}_{k_l}(q_{k_l}, \widetilde{q}_{k_l}). \tag{38c}$$

Compactness of energy sublevels:

For all $t \in [0, T]$ and all $E \in \mathbb{R}$ we have

(i) $\forall\, k \in \mathbb{N}_\infty : \quad \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t, q) \le E \,\}$ is compact; (38d)

(ii) $\bigcup_{k=1}^\infty \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t, q) \le E \,\}$ is relatively compact.

Separability and metrizability: The topology restricted to sublevels of $\mathcal{E}(t, \cdot)$ is compact, separable and metrizable. (38e)

Uniform control of the power $\partial_t \mathcal{E}_k$:
$$\exists\, \lambda_{\mathcal{E}} > 0\ \forall\, k \in \mathbb{N}_\infty\ \forall\, (t, q) \text{ with } \mathcal{E}_k(t, q) < \infty : \\ \mathcal{E}_k(\cdot, q) \in \mathrm{C}^1([0, T]),\ |\partial_t \mathcal{E}_k(t, q)| \le \lambda_{\mathcal{E}} \mathcal{E}_k(s, q) \text{ for } s \in [0, T]. \tag{38f}$$

Lower Γ-limit for $\mathcal{E}_k$:
$$\forall \text{ stab.seq. } (t_l, q_{k_l}) \overset{\mathcal{Q}_T}{\to} (t, q) : \ \mathcal{E}_\infty(t, q) \le \liminf_{l\to\infty} \mathcal{E}_{k_l}(t_l, q_{k_l}). \tag{38g}$$

Conditioned semicontinuity of the power: $\forall\, t \in [0, T]:$
$$\forall \text{ stab.seq. } (t_l, q_{k_l}) \overset{\mathcal{Q}_T}{\to} (t, q) : \ \limsup_{l\to\infty} \partial_t \mathcal{E}_{k_l}(t, q_{k_l}) \le \partial_t \mathcal{E}_\infty(t, q), \tag{38h}$$
$$\forall \text{ stab.seq.}^\infty\ (t_l, \widetilde{q}_l) \overset{\mathcal{Q}_T}{\to} (t, q) : \ \liminf_{l\to\infty} \partial_t \mathcal{E}_\infty(t, \widetilde{q}_l) \ge \partial_t \mathcal{E}_\infty(t, q). \tag{38i}$$

Conditioned upper semicontinuity of stability sets:
$$\forall \text{ stab.seq. } (t_l, q_{k_l}) \overset{\mathcal{Q}_T}{\to} (t, q) : \quad q \in \mathcal{S}_\infty(t). \tag{38j}$$

Assumptions (38a)–(38c) mainly concern the dissipation distances $\mathcal{D}_k$: the first two correspond to the earlier conditions (D1) and (D2), whereas (38c) is the new Γ-liminf condition. Assumptions (38d)–(38g) are mainly on the stored-energy functionals $\mathcal{E}_k$: the first two correspond to the earlier (E1) and (E2), whereas (38g) is the new Γ-liminf condition. Conditions (38j) and (38h)–(38i) correspond to the compatibility conditions (C1) and (C2), respectively.

It may seem strange that we do not ask the Γ-convergences of $\mathcal{D}_k \xrightarrow{\Gamma} \mathcal{D}_\infty$ and $\mathcal{E}_k(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}_\infty(t,\cdot)$ for $k \to \infty$. In fact, we do not need this in general, because the compatibility conditions (38h), (38i), and (38j) implicitly provide the Γ-limsup estimates when restricted to the stability sets $\mathcal{S}_\infty(t)$. In fact, condition (38j) is almost identical to the compatibility condition (C2). Hence, the construction of joint-recovery sequences in Section 3.5 applies equally here. Whereas in many practical applications $\mathcal{D}_k \xrightarrow{\Gamma} \mathcal{D}_\infty$ and $\mathcal{E}_k(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}_\infty$ holds, the importance of the interlinked assumptions is that we are automatically forced to consider Γ-convergence in the intrinsic topology, namely the one induced by convergence of stable sequences.

We present two convergence results and refer to [MRS08] for the proofs. The first result concerns exact solutions q_k of the RIS $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$, and we already assume that these solutions converge. This is not a restrictive assumption, since from the proof it becomes clear that any sequence of solutions has a subsequence for which the z-component pointwise converges, and that is the only important assumption.

Theorem 3.16 ($(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ converges to $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$). *Assume that* (38) *holds and that* $q_k : [0,T] \to \mathcal{Q}$ *are energetic solutions of* $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$. *Let us further assume that, for all* $t \in [0,T]$, *we have* $q_k(t) \xrightarrow{\mathcal{Q}} q(t)$ *and* $\mathcal{E}_k(0, q_k(0)) \to \mathcal{E}_\infty(0, q(0))$ *for* $k \to \infty$. *Then* $q : [0,T] \to \mathcal{Q}$ *is an energetic solution of* $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$, *and for all* $t \in [0,T]$ *we have*

$$\begin{aligned}
&\mathcal{E}_k(t, q_k(t)) \to \mathcal{E}_\infty(t, q(t)), \quad \mathrm{Diss}_k(q_k, [0,t]) \to \mathrm{Diss}_\infty(q, [0,t]), \\
&\partial_t \mathcal{E}_k(\cdot, q_k(\cdot)) \to \partial_t \mathcal{E}_\infty(\cdot, q(\cdot)) \ \textit{in}\ \mathrm{L}^1([0,T]).
\end{aligned}$$

Next we show that even incremental solutions of $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ for a given sequence $(\Pi^k)_{k\in\mathbb{N}}$ of partitions with fineness $\phi(\Pi^k) \to 0$ have subsequences converging to solutions of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$. Thus, we do not need exact solutions of each $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ to guarantee that the limiting functions are solutions. For the partitions $\Pi^k = (0 = t_0^k, t_1^k, \ldots, t_{N_k-1}^k, t_{N_k}^k = T)$, we use the fully implicit incremental minimization problem $(\mathrm{IMP})_k^{\Pi^k}$

$$\text{Given } q_0^k \in \mathcal{Q}, \text{ for } j = 1, ..., N_k \text{ find } q_j^k \in \operatorname*{Arg\,min}_{q\in\mathcal{Q}} \Big(\mathcal{E}_k(t_j^k, q) + \mathcal{D}_k(q_{j-1}^k, q) \Big).$$

For each solution $((t_j^k, q_j^k))_{j=0,1,\ldots,N_k}$ we define the piecewise constant interpolants $\underline{q}^k : [0,T] \to \mathcal{Q}$ as in (13). The following result states the convergence of subsequences of the solutions $\underline{q}^k$ to energetic solutions of the limit system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.

Theorem 3.17 ($(\mathrm{IMP}_k^{\Pi^k})$ converges to $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$). *Let conditions* (38a)–(38j) *hold. Let the sequence of partitions* Π^k *satisfy* $\phi(\Pi^k) \to 0$, *and let the sequence of initial conditions* q_0^k *satisfy*

$$q_0^k \xrightarrow{\mathcal{Q}} q_0 \quad \textit{and} \quad \mathcal{E}_k(0, q_0^k) \to \mathcal{E}_\infty(0, q_0) \in \mathbb{R}. \tag{39}$$

Then, each (IMP)^k has at least one solution $\underline{q}^k : [0,T] \to \mathcal{Q}$ *and there exist a subsequence* $(\underline{q}^{k_l})_{l\in\mathbb{N}}$ *and a measurable, energetic solution* $q : [0,T] \to \mathcal{Q}$ *for the RIS* $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$ *with* $q(0) = q_0$, *such that (i)–(iv) hold:*

$$\begin{aligned}
&\text{(i)}\ \ \forall\, t \in [0,T] : \mathcal{E}_{k_l}(t, \underline{q}^{k_l}(t)) \to \mathcal{E}_\infty(t, q(t)), \\
&\text{(ii)}\ \ \forall\, t \in [0,T] : \mathrm{Diss}_{k_l}(\underline{q}^{k_l}, [0,t]) \to \mathrm{Diss}_\infty(q, [0,t]), \\
&\text{(iii)}\ \ \forall\, t \in [0,T] : \underline{z}^{k_l}(t) \xrightarrow{\mathcal{Z}} z(t), \\
&\text{(iv)}\ \ \partial_t \mathcal{E}_{k_l}(\cdot, \underline{q}^{k_l}(\cdot)) \to \partial_t \mathcal{E}_\infty(\cdot, q(\cdot)) \ \textit{in}\ \mathrm{L}^1([0,T]).
\end{aligned} \tag{40}$$

Moreover, any $\widetilde{q} : [0,T] \to \mathcal{Q}$ *obtained as such a limit is an energetic solution of* $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$, *if additionally* $y(t) \in \operatorname{Arg\,min} \mathcal{E}(t, \cdot, z(t))$ *for all* t *and* $\partial_t \mathcal{E}_\infty(t, y(t), z(t)) = \mathcal{P}_{\mathrm{red}}^\infty(t, z(t))$ *a.e. in* $[0,T]$.

4 Rate-independent systems in Banach spaces

In Banach spaces we have two important additional tools deriving from the linear structure. First, the functionals at hand may have differentials or subdifferentials such that it is possible to formulate force balances and rate equations rather than comparing energies, as in the energetic formulation. Second, we can employ convexity and duality methods like the Legendre-Fenchel transform as indicated in Section 2.5.

In Section 4.2 we discuss several weakened versions of the subdifferential problem

$$0 \in \partial_{\dot q}\mathcal{R}(q(t),\dot q(t)) + \varepsilon\mathbb{V}\dot z + \overline{\partial}_q\mathcal{E}(t,q(t)), \tag{1}$$

with $\varepsilon = 0$ and provide some comparison to energetic solutions. Using convexity arguments we derive temporal continuity properties in Section 4.3. Finally the vanishing-viscosity approach considers first $\varepsilon > 0$ and then the limit $\varepsilon \to 0$ is used to derive new types of solutions, namely the notion of parametrized and BV solutions.

4.1 The basic Banach-space setup

Before starting with details we explain the usage of the different Banach spaces $\boldsymbol{X}$, $\boldsymbol{V}$, $\boldsymbol{C}$, and $\boldsymbol{Q} = \boldsymbol{Y}\times\boldsymbol{Z}$. The smallest space $\boldsymbol{Q}$ is a reflexive Banach space, such that $\mathcal{E} : \boldsymbol{Q}_T \overset{\text{def}}{=} [0,T]\times\boldsymbol{Q} \to \mathbb{R}_\infty$ has bounded and weakly closed sublevels. For $\mathcal{E} : \boldsymbol{Q}_T \to \mathbb{R}_\infty$ (similarly for the reduced functional $\mathcal{I} : \boldsymbol{Z}_T \to \mathbb{R}_\infty$) we make the following assumptions, which are used without further mentioning in the sequel:

$$\exists\, c, C > 0\ \forall\,(t,q) \in \boldsymbol{Q}_T : \quad \mathcal{E}(t,q) \geq c\|q\| - C; \tag{2a}$$

$$\begin{aligned}&\exists\,\lambda_{\mathcal{E}} > 0\ \forall\,(s,q) \in \mathcal{Q}_T \text{ with } \mathcal{E}(s,q) < \infty :\\ &\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t\mathcal{E}(t,q)| \leq \lambda_{\mathcal{E}}\,\mathcal{E}(t,q) \text{ for all } t.\end{aligned} \tag{2b}$$

Then, the basic energy estimate, cf. Section 2.4, shows that all solutions and approximations of interest lie in a bounded set in $\boldsymbol{Q}$, which is useful for extracting subsequences that weakly converge in $\boldsymbol{Q}$.

The function space $\boldsymbol{X}$ is chosen to make the dissipation coercive, i.e. $\mathcal{R}(z,v) \geq c\|v\|_{\boldsymbol{X}}$. The space $\boldsymbol{C}$ is used to provide uniform convexity properties like $\langle \mathrm{D}_q^2\mathcal{E}(t,q)w,w\rangle \geq c\|w\|_{\boldsymbol{C}}^2$. Finally, the Hilbert space $\boldsymbol{V}$ is used for measuring viscosity, e.g. in the small viscosity approximation we use the dissipation potential $\mathcal{R}_\varepsilon(z,v) = \Psi(v) + \frac{\varepsilon}{2}\|v\|_{\boldsymbol{V}}^2$. Throughout we assume that the embeddings $\boldsymbol{Q} \subset \boldsymbol{C}$, $\boldsymbol{Z} \subset \boldsymbol{X}$, and $\boldsymbol{Z} \subset \boldsymbol{V}$ hold. Having in mind the PDE version of our standard Example 2.8, namely

$$0 \in \mathrm{Sign}(\dot z) + \varepsilon\mathbb{V}\dot z - \Delta z + \Phi'(z) - \ell(t,x), \quad (t,x) \in [0,T]\times\Omega, \quad z(t,\cdot)|_{\partial\Omega} = 0,$$

we have $\mathcal{R}_\varepsilon(z,v) = \|v\|_{\mathrm{L}^1} + \frac{\varepsilon}{2}\|v\|_{\mathrm{L}^2}^2$ and $\mathcal{I}(t,z) = \int_\Omega \frac{1}{2}|\nabla z|^2 + \Phi(z) - \ell(t)z\,\mathrm{d}x$. This leads to the typical choice $\boldsymbol{Z} = \mathrm{H}_0^1(\Omega) \Subset \boldsymbol{V} = \mathrm{L}^2(\Omega) \subset \boldsymbol{X} = \mathrm{L}^1(\Omega)$.

A major difficulty for rate-independent systems arising in applications in continuum mechanics is that the rate-independent norm of $\boldsymbol{X}$ is usually given in terms of a (weighted) L^1 norm. Thus, in general $\boldsymbol{X}$ is not reflexive and does not enjoy the Radon-Nikodym property, which would provide differentiability of Lipschitz functions. In principle, it would be possible to use the weak* derivative $\frac{1}{h}(z(t{+}h) - z(t)) \overset{*}{\rightharpoonup} \dot z$ in a bigger space $\boldsymbol{X}_0$ which contains $\boldsymbol{X}$ as a closed subspace and is the dual of a separable space, e.g. $\mathrm{M}(\Omega) \supset \mathrm{L}^1(\Omega)$. However, we are able to avoid this concept by using the dissipation Diss in derivative-free form. Another way to handle the missing weak closedness of $\boldsymbol{X}$ is discussed in Section 5, where $\boldsymbol{X}$ is treated as a complete metric space.

Example 4.1. A typical example for $\boldsymbol{X} = \mathrm{L}^1(\mathbb{R})$ is obtained by the functions $z(t) = \chi_{[\alpha(t),\beta(t)]}$, where $\alpha, \beta \in \mathrm{W}^{1,1}([0,T])$ with $\alpha \leq \beta$ a.e. Letting $f(t) = |\dot\alpha(t)|+|\dot\beta(t)|$ we obtain $\|z(t_2)-z(t_1)\|_{\mathrm{L}^1} \leq |\alpha(t_2)-\alpha(t_1)| + |\beta(t_2)-\beta(t_1)| \leq \int_{t_1}^{t_2} f(t)\,\mathrm{d}t$. Hence z lies in $\mathrm{AC}([0,T],\mathrm{L}^1(\mathbb{R}))$ but not in $\mathrm{W}^{1,1}([0,T],\mathrm{L}^1(\mathbb{R}))$, since $\dot z(t) = \dot\beta(t)\delta_{\beta(t)} - \dot\alpha(t)\delta_{\alpha(t)}$ is a Radon measure but not in $\mathrm{L}^1(\mathbb{R})$.

The linear Banach space structure allows for the usage of subdifferentials. For the dissipation potential $\mathcal{R}$ we always use the convex subdifferential $\partial_v\mathcal{R}(z,\cdot)$ for the convex function $\mathcal{R}(z,\cdot) : \boldsymbol{X} \to \mathbb{R}_\infty$. For the energy functional $\mathcal{E}(t,\cdot) : \boldsymbol{Q} \to \mathbb{R}_\infty$ there are several possible choices of subdifferentials. For simplicity we restrict to the limiting subdifferential used in [RoS06], also called Mordukhovich differential. It is a suitable closure of the Fréchet subdifferential

$$\partial_q^{\mathrm{Fr}}\mathcal{E}(t,q) \overset{\mathrm{def}}{=} \{\, \eta \in \boldsymbol{Q}^* \mid \mathcal{E}(t,q{+}w) \geq \mathcal{E}(t,q) + \langle \eta, w\rangle + o(\|w\|_{\boldsymbol{Q}})_{z\to 0} \,\}$$

and is given in the form

$$\begin{aligned}\overline{\partial}_q\mathcal{E}(t,q) \overset{\mathrm{def}}{=} \{\, \eta \in \boldsymbol{Q}^* \mid \exists\, (q_n,\eta_n)_{n\in\mathbb{N}} :\ & \eta_n \in \partial_q^{\mathrm{Fr}}\mathcal{E}(t,q_n),\ q_n \rightharpoonup q \text{ in } \boldsymbol{Q},\\ & \eta_n \rightharpoonup \eta \text{ in } \boldsymbol{Q}^*,\ \sup\nolimits_{n\in\mathbb{N}} \mathcal{E}(t,q_n) < \infty \,\}.\end{aligned}$$

For the sum of a convex function $\mathcal{J}_1$ and a C^1 function $\mathcal{J}_2$ we have the sum rule

$$\overline{\partial}(\mathcal{J}_1{+}\mathcal{J}_2)(q) = \overline{\partial}\mathcal{J}_1(q) + \mathrm{D}\mathcal{J}_2(q). \tag{3}$$

Of course, most definitions and some of the results can be transferred to other subdifferentials, but this would complicate the presentation unnecessarily.

The difficulty in finding a suitable notion of subdifferential lies in the two opposite requirements. First, we want the subdifferential to be sufficiently large, such that it has good closure properties. If approximations satisfy $\eta_n(t) \in \overline{\partial}_q\mathcal{E}(t,q_n(t))$ a.e. in $[0,T]$, $q_n \rightsquigarrow q$, and $\eta_n \rightsquigarrow \eta$, we want to be able to conclude $\eta(t) \in \overline{\partial}_q\mathcal{E}(t,q(t))$ a.e. in $[0,T]$. Second, we want the subdifferential to be not too big, such that we still can show a counterpart to the classical chain rule $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) = \langle \mathrm{D}_q\mathcal{E}(t,q(t)), \dot q(t)\rangle + \partial_t\mathcal{E}(t,q(t))$.

As a start, we define a suitable generalized chain rule, which will be useful in the following sections. Here $\boldsymbol{Y}$ is a general Banach space, which can play the role of $\mathbb{R}{\times}\boldsymbol{Z}$ for time-dependent functionals.

Definition 4.2 (Chain rules). Let $\boldsymbol{Y}$ be a Banach space and $\mathcal{J} : \boldsymbol{Y} \to \mathbb{R}_\infty$ a functional with (sub)differential $\overline{\partial}\mathcal{J} : \boldsymbol{Y} \rightrightarrows \boldsymbol{Y}^*$. We say that the triple $(\boldsymbol{Y}, \mathcal{J}, \overline{\partial}\mathcal{J})$ satisfies the *chain-rule equality*, if for all $y \in \mathrm{W}^{1,1}([0,T];\boldsymbol{Y})$ and all measurable $\eta : [0,T] \to \boldsymbol{Y}^*$ the following holds:

$$\begin{aligned}&\text{if } \sup\nolimits_{t\in[0,T]} \mathcal{J}(y(t)) < \infty,\ \int_0^T \|\dot y(t)\|_{\boldsymbol{Y}}\|\eta(t)\|_{\boldsymbol{Y}^*}\,\mathrm{d}t < \infty,\\ &\qquad \text{and } \eta(t) \in \overline{\partial}\mathcal{J}(y(t)) \text{ a.e. in } [0,T],\\ &\text{then } t \mapsto \mathcal{J}(y(t)) \text{ is absolutely continuous and}\\ &\qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(y(t)) = \langle \eta(t), \dot y(t)\rangle \text{ a.e. in } [0,T].\end{aligned} \tag{4}$$

We say that $(\boldsymbol{Y}, \mathcal{J}, \overline{\partial}\mathcal{J})$ satisfies the *chain-rule inequality*, if for all $y \in \mathrm{W}^{1,1}([0,T];\boldsymbol{Y})$ and all measurable $\eta : [0,T] \to \boldsymbol{Y}^*$ we have

$$\begin{aligned}&\text{if } \sup_{t\in[0,T]} \mathcal{J}(y(t)) < \infty,\ \int_0^T \|\dot y(t)\|_{\boldsymbol{Y}}\|\eta(t)\|_{\boldsymbol{Y}^*}\,\mathrm{d}t < \infty,\\ &\quad\text{and } \eta(t) \in \overline{\partial}\mathcal{J}(y(t)) \text{ a.e. in } [0,T],\\ &\text{then } \int_0^T \langle \dot y(t), \eta(t)\rangle\,\mathrm{d}t \ \ge\ \mathcal{J}(y(0)) - \mathcal{J}(y(T)).\end{aligned} \tag{5}$$

The chain rule holds for functionals $\mathcal{E} \in \mathrm{C}^1(\mathcal{Q}_T)$, but also in much more general situations, see e.g. [RoS06]. We apply the chain rules to the space $\boldsymbol{Y} = \mathbb{R}\times\boldsymbol{V}$, and we always assume classical differentiability of $\mathcal{I}$ with respect $t \in [0,T]$. Hence we have to be careful only with respect to $\overline{\partial}_z\mathcal{I}$ and usually split off the work of the external forces $\int_0^T \partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t$.

We now provide a few more examples for the Banach-space setting. They are rather degenerate and will serve as counterexamples.

Example 4.3. We let $\Omega = \,]0,1[\, \subset \mathbb{R}^1$, $\boldsymbol{X} = \mathrm{L}^1(\Omega)$, and $\boldsymbol{Z} = \mathrm{L}^2(\Omega)$. Hence, $\boldsymbol{Z}$ is not compactly embedded in $\boldsymbol{X}$. The functionals are

$$\begin{aligned}&\mathcal{I}_\alpha(t,z) = \int_\Omega \Phi_\alpha(z(x)) - (t+x)z(x)\,\mathrm{d}x,\\ &\mathcal{R}_\varepsilon(z,\dot z) = \int_\Omega |\dot z(x)| + \tfrac{\varepsilon}{2}|\dot z(t)|^2\,\mathrm{d}x, \quad \text{and } z_0(x) = 0.\end{aligned}$$

We consider the case $\alpha \in [2,\infty]$ and

$$\Phi_\alpha(z) = \tfrac{1}{\alpha}|z|^\alpha \text{ for } |z| \le 1 \ \text{ and } \Phi_\alpha(z) = \tfrac12|z|^2 + \tfrac1\alpha - \tfrac12 \text{ for } |z| \ge 1.$$

The point of this example is that we are able to calculate the solution by solving uncoupled scalar differential inclusions for each $x \in \Omega$, namely

$$0 \in \mathrm{Sign}(\dot z(t,x)) + \varepsilon\dot z(t,x) + \Phi'_\alpha(z(t,x)) - t - x, \quad z(0,x) = z_0(x) = 0. \tag{6}$$

Because of $\Phi'_\alpha(0) = 0$ we see that z_0 is locally stable at $t = 0$.

For $\varepsilon = 0$ we obtain the solution $z_\alpha(t,x) = M_\alpha(\max\{0, t+x-1\})$ where $M_\alpha(\sigma) = \sigma^{1/(\alpha-1)}$ for $\sigma \in [0,1]$ and $M_\alpha(\sigma) = \sigma$ for $\sigma \ge 1$. For $\alpha = 2$ the abstract theory of uniformly convex problems gives $z \in \mathrm{C}^{\mathrm{Lip}}([0,T], \mathrm{L}^2(\Omega))$. However, using the explicit formula

$$\dot z_\alpha(t,x) = \begin{cases} 0 & \text{for } t+x < 1,\\ \frac{1}{\alpha-1}(t+x-1)^{-\frac{\alpha-2}{\alpha-1}} & \text{for } 1 < t+x < 2,\\ 1 & \text{for } t+x > 2,\end{cases}$$

we find $\dot z_\alpha \in \mathrm{L}^\infty([0,T], \mathrm{L}^p(\Omega))$ whenever $p > \frac{\alpha-1}{\alpha-2}$. In particular, $p = 1$ is always possible, and we are able to write down the abstract differential inclusion

$$0 \in \partial\mathcal{R}_0(\dot z_\alpha(t)) + \mathrm{D}_z\mathcal{I}_\alpha(t, z_\alpha(t)) \quad \subset \mathrm{L}^2(\Omega) = \boldsymbol{Z}^*, \tag{7}$$

since $\partial\mathcal{R}_0(\dot z_\alpha(t)) \in \mathrm{L}^\infty(\Omega) = \boldsymbol{X}^* \subset \boldsymbol{Z}^*$.

The limit case $\alpha \to \infty$ is more interesting, because it leads to the degenerate convex limit potential $\Phi_\infty(z) = \frac12\max\{0, |z|^2-1\}$ and the limit solution z with

$$z(t,x) = \begin{cases} 0 & \text{for } t+x < 1,\\ 1 & \text{for } 1 < t+x \le 2,\\ t+x-1 & \text{for } t+x \ge 2;\end{cases} \qquad \dot z(t) = \begin{cases} \delta_{1-t} & \text{for } t \in \,]0,1[\,,\\ \chi_{]2-t,1[} & \text{for } t \in \,]1,2[\,,\\ 1 & \text{for } t > 2\end{cases}$$

Thus, we have $z \in \mathrm{C}^{\mathrm{Lip}}([0,T],\boldsymbol{X})$ with $\boldsymbol{X} = \mathrm{L}^1(\Omega)$, but the derivative only exists in the weak* sense, namely $\dot z \in \mathrm{L}^\infty_{\mathrm{w}*}([0,T],\mathrm{M}(\Omega))$ with $\mathrm{M}(\Omega) = \mathrm{C}_0(\Omega)^*$. We are no longer able to give any sense to the subdifferential inclusion (7), since after extending $\mathcal{R} : \boldsymbol{X} \to \mathbb{R}_\infty$ to $\mathcal{R}_\circ : \mathrm{M}(\Omega) \to \mathbb{R}_\infty$ via the weak* lower semicontinuous hull, the subdifferential $\partial\mathcal{R}_\circ(\dot z_\alpha)$ needs to be treated in $\mathrm{M}(\Omega)^*$ which is no longer comparable to $\boldsymbol{Z}^*$. Thus, we do not have a *differential solution*, but z is still a *CD solution* in the sense of Definition 4.5.

For completeness and later reference, we also give the viscous approximations for the case $\alpha = \infty$. The solution z^ε reads

$$z^\varepsilon(t,x) = U^\varepsilon(t{+}x) \quad \text{with } U^\varepsilon(\tau) = \begin{cases} 0 & \text{for } \tau \le 1, \\ (\tau{-}1)/\varepsilon & \text{for } 1 \le \tau \le 1{+}\varepsilon, \\ 1 & \text{for } 1{+}\varepsilon \le \tau \le 2, \\ \tau{-}1{+}\varepsilon \mathrm{e}^{-(\tau-2)/\varepsilon}{-}\varepsilon & \text{for } \tau \ge 2. \end{cases}$$

Obviously, we have $z^\varepsilon(t,\cdot) \to z(t,\cdot)$ in $\boldsymbol{Z}$. For each $\varepsilon > 0$ we have $z^\varepsilon \in \mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{L}^p(\Omega))$. However, the Lipschitz bound blows up for $\varepsilon > 0$ if $p > 1$. Only in the case $p = 1$ we obtain a uniform bound.

Example 4.4. This example uses exactly the same function spaces as the previous Example 4.3, but the potential defining $\mathcal{I}$ is replaced by the nonconvex function $\Phi_{\mathrm{nc}} : z \mapsto \mathcal{U}(z)$ with $\mathcal{U}$ from (19). For the viscous problem (6) with the initial condition $z_0(x) = -4$ we obtain the unique solution $z^\varepsilon(t,x) = V^\varepsilon(t{+}x)$ with

$$V^\varepsilon(\tau) = \begin{cases} -4 & \text{for } \tau \in [0,1{+}\varepsilon], \\ \tau{-}5{-}\varepsilon & \text{for } \tau \in [1{+}\varepsilon, 3{+}\varepsilon], \\ 2\varepsilon \mathrm{e}^{(\tau-3-\varepsilon)/\varepsilon}{-}\tau{+}1{-}\varepsilon & \text{for } \tau \in [3{+}\varepsilon, 3{+}\varepsilon{+}\delta_\varepsilon], \\ \tau{+}3{-}\varepsilon{-}(4{+}\delta_\varepsilon)\mathrm{e}^{-(\tau-3-\varepsilon-\delta_\varepsilon)/\varepsilon} & \text{for } \tau \ge 3{+}\varepsilon{+}\delta_\varepsilon, \end{cases}$$

where $\delta_\varepsilon = \varepsilon \log(2/\varepsilon) + O(\varepsilon)$ for $\varepsilon \to 0$. For $\varepsilon \to 0$, the solutions $z^\varepsilon(t,\cdot)$ converge to z strongly in $\boldsymbol{Z} = \mathrm{L}^2(\Omega)$, where $z(t,x) = V^0(t{+}x)$ a.e. in Ω, with $V^0(\tau) = \max\{-4, \tau{-}5\}$ for $\tau < 3$ and $V^0(\tau) = \tau{+}3$ for $\tau > 3$.

As in the previous example, the functions z^ε have a uniform Lipschitz bound (namely 8), if the values are considered in $\boldsymbol{X} = \mathrm{L}^1(\Omega)$. However, the total length in $\boldsymbol{Z} = \mathrm{L}^2(\Omega)$ tends to ∞ like $1/\sqrt{\varepsilon}$. More important is the fact, that the limit function $z \in \mathrm{C}^{\mathrm{Lip}}([0,5];\boldsymbol{X}) \cap \mathrm{C}^0([0,5];\boldsymbol{Z})$ does not satisfy the simple energy balance. All quantities can be calculated explicitly, and we find

$$\mathcal{I}(t,z(t)) + \mathrm{Diss}_\Psi(z,[0,t]) + \varrho(t) = \mathcal{I}(0,z(0)) + \textstyle\int_0^t \partial_s\mathcal{I}(s,z(s))\,\mathrm{d}s$$

with $\varrho(t) = \max\{0, 16\min\{t{-}2, 1\}\}$. This means that in the time interval $t \in [2,3]$, which is exactly where z jumps from -2 to $+6$, there is an additional limit dissipation from the "infinitesimal viscous jumps". However, these jumps are distributed continuously in time such that z remains continuous in time as a function with values in $\boldsymbol{Z} = \mathrm{L}^2(\Omega)$.

This example will be reconsidered in Example 4.27 to highlight the difference between strong and weak BV solutions.

4.2 Differential, CD, and local solutions

To define solutions concepts that avoid derivatives we use the special feature of the subdifferential for 1-homogeneous dissipation potentials given in Lemma 2.2, i.e. the rate-independent differential inclusion (1) is equivalent to

$$q(t) \in \mathcal{S}_{\rm loc}(t) = \{\, q = (y,z) \in \mathcal{Q} \mid 0 \in \partial_v \mathcal{R}(z,\dot z) + \overline{\partial}_q \mathcal{E}(t,q) \,\},$$
$$\mathcal{E}(t,q(t)) + \int_0^t \mathcal{R}(z,\dot z)\,\mathrm{d}s = \mathcal{E}(0,z(0)) + \int_0^t \partial_s \mathcal{E}(s,z(s))\,\mathrm{d}s.$$

However, the point is that the derivative of z in the energy balance can be replaced by using the dissipation functional $\mathrm{Diss}_{\mathcal{D}}$. This leads to a derivative-free formulation. Thus, we consider compatible pairs $(\mathcal{R},\mathcal{D})$ which satisfy

$$\exists\, c_R, C_R > 0\ \forall\, z \in \boldsymbol{Z},\ v \in \boldsymbol{X} : c_R \|v\|_{\boldsymbol{X}} \le \mathcal{R}(z,v) \le C_R \|v\|_{\boldsymbol{X}}, \tag{8a}$$
$$\forall\, z,v \in \mathcal{Z} : \lim_{\varepsilon \to 0^+} \tfrac{1}{\varepsilon} \mathcal{D}(z, z{+}\varepsilon v) = \mathcal{R}(z,v), \tag{8b}$$
$$\forall\, z \in \mathrm{W}^{1,1}([r,t],\boldsymbol{X}) \cap \mathrm{C}_{\rm w}([r,t],\boldsymbol{Z}) : \quad \textstyle\int_r^t \mathcal{R}(z(s),\dot z(s))\,\mathrm{d}s = \mathrm{Diss}_{\mathcal{D}}(z,[r,t]). \tag{8c}$$

As usual we assume the Banach-space structure $\boldsymbol{Q} = \boldsymbol{Y} {\times} \boldsymbol{Z}$ with continuous and dense embedding $\boldsymbol{Z} \subset \boldsymbol{X}$. Then, $\partial_{\dot z}\mathcal{R}(z,v) \subset \boldsymbol{X}^* \subset \boldsymbol{Z}^*$ can also be embedded into $\boldsymbol{Q}^*$ by putting 0 in the component $\boldsymbol{Y}^*$. The main problem in the notions of solutions to RIS arises from the fact, that we want to allow for solutions $q = (y,z) : [0,T] \to \boldsymbol{Y}{\times}\boldsymbol{Z}$ which are not necessarily differentiable, maybe not even continuous. In this section, discontinuity is only allowed for local solutions, while the other solution concepts ask for weak continuity.

Definition 4.5 (Differentiable, CD, and local solutions). Consider $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$ with compatible $\mathcal{D}$. A function $q = (y,z) : [0,T] \to \boldsymbol{Q} = \boldsymbol{Y}{\times}\boldsymbol{Z}$ is called
(i) a *differential solution* to $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$, if $q \in \mathrm{W}^{1,1}([0,T],\boldsymbol{Q})$ and

$$0 \in \begin{pmatrix} 0 \\ \partial_{\dot z}\mathcal{R}(z(t),\dot z(t)) \end{pmatrix} + \overline{\partial}_q \mathcal{E}(t,q(t)) \quad \text{for a.a. } t \in [0,T]; \tag{9}$$

(ii) a *semi-differential solution* to $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$, if (9) holds, $q \in \mathrm{C}_{\rm w}([0,T],\boldsymbol{Q})$, and $z \in \mathrm{W}^{1,1}([0,T],\boldsymbol{X})$;
(iii) a *CD solution* (for 'C'ontinuous 'D'issipation) to $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$, if $t \mapsto \mathrm{Diss}_{\mathcal{D}}(q,[0,t])$ is continuous and if for all $t \in [0,T]$ we have

$$0 \in \begin{pmatrix} 0 \\ \partial_{\dot z}\mathcal{R}(z(t),0) \end{pmatrix} + \overline{\partial}_q \mathcal{E}(t,q(t)) \quad \text{and} \tag{10a}$$
$$\mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(q,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_s \mathcal{E}(s,q(s))\,\mathrm{d}s; \tag{10b}$$

(iv) a *local solution* to $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$, if

$$0 \in \begin{pmatrix} 0 \\ \partial_{\dot z}\mathcal{R}(z(t),0) \end{pmatrix} + \overline{\partial}_q \mathcal{E}(t,q(t)) \quad \text{for a.a. } t \in [0,T] \ \text{ and} \tag{11a}$$
$$\mathcal{E}(t_2,q(t_2)) + \mathrm{Diss}_{\mathcal{D}}(q,[t_1,t_2]) \le \mathcal{E}(t_1,q(t_1)) + \int_{t_1}^{t_2} \partial_s \mathcal{E}(s,q(s))\,\mathrm{d}s \quad \text{for all } 0 \le t_1 < t_2 \le T. \tag{11b}$$

Referring to Definition 2.1 we note that the solution types in (i)–(iv) define possibly multi-valued evolutionary systems in the sense that the concatenation and restriction properties hold. The restriction property will fail, if we replace (11b) by the weaker global energy inequality with $t_1 = 0$ and $t_2 = T$. The point is that the local stability condition is not strong enough to provide a lower energy estimate unless additional continuity properties are assumed.

The notion of differentiable solutions can be rewritten as the following evolutionary quasi-variational inequality, which is often used in the literature, see e.g. [Kre99, BKS04]:

$$\begin{aligned} &\forall_{\text{a.a.}} t \in [0,T]\ \exists\, \eta(t) \in \boldsymbol{X}^* \cap \overline{\partial}_q \mathcal{E}(t,q(t))\ \forall\, v \in \boldsymbol{X} : \\ &\quad \langle \eta(t), v - \dot z(t)\rangle_{\boldsymbol{X}^*\times\boldsymbol{X}} + \mathcal{R}(z(t),v) - \mathcal{R}(z(t),\dot z(t)) \geq 0. \end{aligned} \tag{12}$$

The important fact about the definitions of CD and local solutions is that they do not assume any differentiability of the solution. To see that the notions are genuinely different, we refer to Example 4.3, where for $\mathcal{I}_\infty$ we have a solution $z \in \mathrm{C}([0,T],\boldsymbol{Z}) \cap \mathrm{C}^{\mathrm{Lip}}([0,T],\boldsymbol{X})$ which does not lie in $\mathrm{W}^{1,1}([0,T],\boldsymbol{X})$. Thus, we have a CD solution which is not a differential solution.

If a suitable chain-rule condition holds for $\mathcal{E}$, we see that the above notions of solutions are ordered from strong to weak and that we are able to go backward, if the solutions have the appropriate temporal behavior.

Proposition 4.6. *Assume that* $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$ *and* $\mathcal{D}$ *satisfy* (8) *and the chain rule* (4)*. Then,* $q \in \mathrm{W}^{1,1}([0,T],\boldsymbol{Q})$ *is a differential solution if and only if it is a CD solution.*

Proof. Assume that q is a differential solution. Obviously, (10a) holds by Lemma 2.2, which characterizes the subdifferentials of 1-homogeneous functionals. To establish the energy balance, take a measurable selection η with $\eta(t) \in \overline{\partial}_q\mathcal{E}(t,q(t))$ a.e. in $[0,T]$. Then, $\eta = (0,\zeta)$ with $\zeta \in \mathrm{L}^\infty([0,T],\boldsymbol{Z}^*)$, because (8a) implies $\|\zeta(t)\|_{\boldsymbol{X}^*} \leq C_R$ and $\boldsymbol{X}^* \subset \boldsymbol{Z}^*$ continuously. The same lemma also gives $\mathcal{R}(z(t),\dot z(t)) = -\langle \zeta(t), \dot z(t)\rangle$. Using the chain rule (4), integration, and (8c) give (10b).

If q is a CD solution, we choose $\eta = (0,\zeta) \in \overline{\partial}_q\mathcal{E}(t,q(t))$ according to (10a), which implies $-\zeta(t) \in \mathcal{R}(z(t),0)$. Using (10b), (8c), and (4) gives $\mathcal{R}(z(t),\dot z(t)) = -\langle \zeta(t),\dot z(t)\rangle$. With Lemma 2.2 we obtain the subdifferential formulation (9). □

Under reasonable assumptions it is possible to show that all local solutions that are also continuous are in fact CD solutions. For this one needs to show that local stability together with continuity provides a lower energy estimate.

The next result shows that continuous energetic solutions are in fact CD solutions.

Proposition 4.7. *Assume that* $(\boldsymbol{Q},\mathcal{E},\mathcal{R})$ *satisfies* (8)*. Then, an energetic solution* q *with* $q \in \mathrm{C_w}([0,T],\boldsymbol{Q})$ *is a CD solution.*

Proof. Since the energy balance is common in both formulations it suffices to show the local stability. For $t \in [0,T]$ let $\mathcal{J}(\widetilde q) = \mathcal{E}(t,\widetilde q) + \mathcal{D}(z(t),\widetilde z)$. From the global stability (S) (see Definition 3.1) we see that $(y,w) = q(t)$ is the global minimizer, which implies $0 \in \overline{\partial}\mathcal{J}(q(t))$. Moreover, with (8a) and (8b) we find $\overline{\partial}\mathcal{J}(q(t)) = \overline{\partial}\mathcal{E}(t,q(t)) + (0,\partial_{\dot z}\mathcal{R}(z(t),0))^\mathsf{T}$, which provides the desired result. □

As a corollary we obtain a first existence result for CD solutions.

Theorem 4.8. *Assume that* $\boldsymbol{Z}$ *is compactly embedded into* $\boldsymbol{X}$*, and that* $\mathcal{R}(z,v) = \Psi(v)$*, with* $\Psi : \boldsymbol{X} \to [0,\infty[$ *being coercive and strongly continuous. Moreover, let* $\mathcal{E} : [0,T] \to \boldsymbol{Q} \to \mathbb{R}_\infty$ *satisfy* (E1) *and* (E2) *with respect to the weak topology on* $\boldsymbol{Q}$*. Moreover, assume*

$$\begin{aligned} &\forall\, t \in [0,T]:\ \ \mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty \textit{ is strictly convex;} \\ &\exists\, C_E^*\ \forall\, q \textit{ with } \mathcal{E}(0,q) < \infty:\ \ \mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T]), \\ &|\partial_t\mathcal{E}(t,q) - \partial_t\mathcal{E}(s,q)| \leq C_E^* |t-s|\,\mathcal{E}(0,q). \end{aligned}$$

Then, for each $q_0 \in \mathcal{S}(0)$ *there exists an energetic solution* q *that is also a CD solution to* $(\boldsymbol{Q},\mathcal{E},\Psi)$ *with* $q(0) = q_0$*.*

Proof. We first employ Theorem 3.5, which provides the existence of an energetic solution. Using the strict convexity of $\mathcal{E}(t,\cdot)$ we conclude that $y(t)$ is uniquely defined from $z(t)$ as minimizer of $\mathcal{E}(t,\cdot,z(t))$. Moreover, the reduced functional $\mathcal{I}$ with $\mathcal{I}(t,z) = \min_{y\in \boldsymbol{Y}} \mathcal{E}(t,y,z)$ is still strictly convex with respect to z. Thus, the mappings $\mathcal{J}_{t,\widehat{z}} : z \mapsto \mathcal{I}(t,z) + \Psi(z-\widehat{z})$ are strictly convex as well. Lemma 3.6 states that the left limit $z(t^-)$ (in the strong $\boldsymbol{X}$-topology) is globally stable as well. Hence, it is a minimizer of $\mathcal{J}_{t,z(t^-)}$. By the jump relations (25), $z(t)$ is a minimizer as well, which now must coincide with $z(t^-)$ by continuity. Similarly, $z(t^+)$ must coincide with $z(t)$. Hence we have shown $z(t^-) = z(t) = z(t^+)$, which implies strong continuity in $\boldsymbol{X}$ and weak continuity in $\boldsymbol{Z}$.

By uniqueness of the minimizers of $\mathcal{E}(t,\cdot,z(t))$ we also obtain $y(t^-) = y(t) = y(t^+)$, where limits are taken weakly in $\boldsymbol{Y}$. We use here that weak limits of minimizers are minimizers because of the weak lower semicontinuity. Thus, $q \in \mathrm{C_w}([0,T],\boldsymbol{Q})$ is established, and Proposition 4.7 provides the CD solution. □

4.3 Systems with convexity properties

The temporal continuity results from the previous section can be improved under stronger convexity assumptions. For this we introduce a possibly larger function space $\boldsymbol{C}$ such that $\boldsymbol{Q}$ is continuously embedded into $\boldsymbol{C}$. We consider general energetic solutions to $(\boldsymbol{Q},\mathcal{E},\mathcal{D})$ under the following uniform α-convexity condition:

$$\begin{aligned} &\exists\, \alpha\geq 2,\, E>0\ \exists\, c_*>0\ \forall\, \theta\in[0,1]\ \forall\, t,q_0,q_1 \text{ with } \mathcal{E}(t,q_0),\mathcal{E}(t,q_1)\leq E:\\ &\quad \mathcal{E}(t,q_\theta)+\mathcal{D}(z_0,z_\theta)+c_*\theta(1-\theta)\|q_1-q_0\|_{\boldsymbol{C}}^\alpha\\ &\qquad \leq (1-\theta)\big(\mathcal{E}(t,q_0)+\mathcal{D}(z_0,z_0)\big)+\theta\big(\mathcal{E}(t,q_1)+\mathcal{D}(z_0,z_1)\big), \end{aligned} \tag{13}$$

where $q_\theta = (1-\theta)q_0+\theta q_1$. Here $\alpha = 2$ is the case of classical uniform convexity. Clearly, this property is satisfied, if $\mathcal{E}(t,\cdot)$ is uniformly α-convex and $\mathcal{D}:(z_0,z_1)\mapsto \Psi(z_1-z_0)$ is merely convex. As a second assumption we need that the power $\partial_t\mathcal{E}$ is Lipschitz or Hölder continuous with respect to the same norm $\|\cdot\|_{\boldsymbol{C}}$, namely,

$$\begin{aligned} &\exists\, \beta\in\,]0,1]\ \ \forall\, E>0\ \exists\, C_*>0\ \forall\, t,q_0,q_1 \text{ with } \mathcal{E}(t,q_0),\mathcal{E}(t,q_1)\leq E:\\ &\quad |\partial_t\mathcal{E}(t,q_1)-\partial_t\mathcal{E}(t,q_0)|\leq C_*\|q_1-q_0\|_{\boldsymbol{C}}^\beta. \end{aligned} \tag{14}$$

Before stating the main time-regularity result, we emphasize that in smooth cases the convexity condition (13) with $\alpha=2$ and $\boldsymbol{C}=\boldsymbol{Q}$ is essentially the same as the joint convexity condition (11). In fact, for $\mathcal{E}\in\mathrm{C}^2$ and $\mathcal{R}(\cdot,\widehat{z})\in\mathrm{C}^1$ condition (13) implies

$$\langle \mathrm{D}^2\mathcal{E}(t,q)\widehat{q},\widehat{q}\rangle + \mathrm{D}_z\mathcal{R}(z,\widehat{z})[\widehat{z}] \geq 2c_*\|\widehat{q}\|_{\boldsymbol{C}}^2 \text{ for all } \widehat{q}\in\boldsymbol{Q}.$$

The following result is taken from [ThM09, Tho09], where more details and some applications are given.

Theorem 4.9 (Lipschitz and Hölder continuity). *Let $(\boldsymbol{Q},\mathcal{E},\mathcal{D})$ be a RIS satisfying the power control* (E2), (13) *and* (14). *Then, for every energetic solution $q:[0,T]\to\boldsymbol{Q}$ there exists $C>0$ such that*

$$\|q(t)-q(s)\|_{\boldsymbol{C}} \leq C|t-s|^{1/(\alpha-\beta)} \quad \textit{for all } s,t\in[0,T].$$

Proof. We choose $E = \sup\{\,\mathcal{E}(t,q(t)) \mid t\in[0,T]\,\}$ and obtain $c_*,C_*>0$ from (13) and (14). Exploiting the uniform α-convexity we derive an *improved stability estimate*. Indeed, using $q(s)\in\mathcal{S}(s)$ and (13) with $q_0=q(s)$ and $q_1=q(t)$ where $s<t$ we obtain

$$\begin{aligned}\mathcal{E}(s,q(s)) &\leq \mathcal{E}(s,q_\theta)+\mathcal{D}(z_0,z_\theta) \qquad (\text{where } q_\theta=(1-\theta)q(s)+\theta q(t)\,)\\ &\leq (1-\theta)\mathcal{E}(s,q(s))+\theta\big(\mathcal{E}(s,q(t))+\mathcal{D}(z(s),z(t))\big)-c_*\theta(1-\theta)\|q(t)-q(s)\|_{\boldsymbol{C}}^\alpha.\end{aligned}$$

Subtracting $\mathcal{E}(s,q(s))$, dividing by θ, and taking the limit $\theta\to 0^+$ lead to

$$\mathcal{E}(s,q(s))+c_*\|q(t)-q(s)\|_{\boldsymbol{C}}^\alpha \leq \mathcal{E}(s,q(t))+\mathcal{D}(z(s),z(t)), \tag{15}$$

which is the desired improved stability estimate. Employing the dissipation estimate $\mathcal{D}(z(s),z(t))\leq \mathrm{Diss}_{\mathcal{D}}(z,[s,t])$ and the energy balance we obtain

$$\begin{aligned}&c_*\|q(t)-q(s)\|_{\boldsymbol{C}}^\alpha \leq \mathcal{E}(s,q(t))+\mathcal{D}(z(s),z(t))-\mathcal{E}(s,q(s))\\ &\leq \mathcal{E}(s,q(t))-\mathcal{E}(t,q(t))+\mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(z,[s,t])-\mathcal{E}(s,q(s))\\ &=\textstyle\int_s^t \partial_r\mathcal{E}(r,q(r))-\partial_r\mathcal{E}(r,q(t))\,\mathrm{d}r \leq \int_s^t C_*\|q(t)-q(r)\|_{\boldsymbol{C}}^\beta\,\mathrm{d}r.\end{aligned}$$

Letting $\eta(\tau)=\int_{t-\tau}^t\|q(t)-q(r)\|_{\boldsymbol{C}}^\beta\,\mathrm{d}r$ for $\tau\in[0,t-s]$ leads to $\eta'(\tau)\leq (C_*\eta(\tau)/c_*)^{\beta/\alpha}$. Since $\eta(0)=0$ we find $\eta(\tau)\leq C_1\tau^{\alpha/(\alpha-\beta)}$ and, thus,

$$\|q(t)-q(s)\|_{\boldsymbol{C}}=\eta'(t-s)^{1/\beta}\leq \big(\tfrac{C_*}{c_*}\eta(t-s)\big)^{1/\alpha}\leq \big(\tfrac{C_*C_1}{c_*}\big)^{1/\alpha}(t-s)^{1/(\alpha-\beta)},$$

where C_1 depends only on C_*, c_*, α, and β. This is the desired result. □

Since $\boldsymbol{Q}\subset\boldsymbol{C}$ and $\boldsymbol{Q}$ is reflexive, the solutions studied in Theorem 4.9 lie in $\mathrm{C_w}([0,T];\boldsymbol{Q})$ and thus are CD solutions, cf. Proposition 4.7. If $\alpha=2$, $\beta=1$, and $\boldsymbol{C}=\boldsymbol{Q}$ we even obtain differential solutions.

Corollary 4.10. *Let the energetic system $(\boldsymbol{Q},\mathcal{E},\mathcal{D})$ satisfy (13) and (14) with $\alpha=2$ and $\beta=1$ and some reflexive space $\boldsymbol{C}$ such that $\boldsymbol{Q}$ embeds into $\boldsymbol{C}$ continuously.*

(A) If the space $\boldsymbol{C}$ equals $\boldsymbol{Q}$, then every energetic solution is a differential solution.

(B) If there exists $C>0$ such that $\|v\|_{\boldsymbol{X}}\leq C\|(0,v)\|_{\boldsymbol{C}}$ for all $v\in\boldsymbol{Z}$, then every energetic solution is a semi-differential solution.

Proof. Theorem 4.9 gives $q\in\mathrm{C^{Lip}}([0,T],\boldsymbol{C})$, which implies $\dot q\in\mathrm{L}^\infty([0,T],\boldsymbol{C})$ by reflexivity of $\boldsymbol{C}$. Now, (A) follows by employing Propositions 4.6 and 4.7.

Part (B) follows similarly using that $q\in\mathrm{L}^\infty([0,T],\boldsymbol{Q})$ and that $q\in\mathrm{C^{Lip}}([0,T],\boldsymbol{C})$ implies $q\in\mathrm{C_w}([0,T],\boldsymbol{Q})$ and $z\in\mathrm{W}^{1,\infty}([0,T],\boldsymbol{X})$. □

We finally mention some results where *existence of solutions* is established in cases of uniform convexity and smoothness properties, and thus are independent of any compactness arguments. In [MiT04, Thm. 7.1] the case $\mathcal{E}\in\mathrm{C}^3(\boldsymbol{Q}_T)$ satisfying (E2) and the uniform convexity $\langle \mathrm{D}_z^2\mathcal{E}(t,q)w,w\rangle\geq\alpha\|w\|_{\boldsymbol{Q}}^2$ and $\Psi\in\mathrm{C}^0(\boldsymbol{Q})$ being 1-homogeneous and convex (no coercivity needed) is studied. It is shown that the RIS $(\boldsymbol{Q},\mathcal{E},\Psi)$ has for each stable initial condition a unique differential solution q, i.e., we have

$$0\in\partial\Psi(\dot q(t))+\mathrm{D}_q\mathcal{E}(t,q(t)) \quad \text{for a.a. } t\in[0,T]. \tag{16}$$

Clearly the joint convexity condition (11) holds, since $\rho=0$. In [KuM98] existence was derived for the case of quadratic energies in a Hilbert space $\boldsymbol{H}$, namely $\mathcal{I}(t,z,)=\frac12(z|z)-(\ell(t)|z)$, where $(\cdot|\cdot)$ denotes the scalar product in $\boldsymbol{H}$. The dissipation distance is formulated in terms of the sets $K(z)=\partial_{\dot z}\mathcal{R}(z,0)$. The joint convexity now reads

$$\exists\,\rho\in\,]0,1[\;\;\forall\, z_1,z_2\in\boldsymbol{H}:\; d_{\mathrm{Hausdorff}}(K(z_1),K(z_2))\leq\rho\|z_1-z_2\|_{\boldsymbol{H}}.$$

The definition of $\mathcal{I}$ gives $\kappa = 1$, and it is easy to see that the last condition implies $|\mathcal{R}(z_1, v)-\mathcal{R}(z_2, v)| \leq \rho\|z_1-z_2\|_{\boldsymbol{H}}\|v\|_{\boldsymbol{H}}$, but it is unclear whether the existence result of [KuM98] holds under this weaker condition.

Finally we want to comment on the possibility to use convexity and smoothness to obtain *uniqueness results*. We follow the simpler result in [MiT04] and refer to [BKS04] and [MiR07] for generalizations. We again consider (16) with the same specifications for $(\boldsymbol{Q}, \mathcal{E}, \Psi)$ as there. Comparing two differential solutions q_1 and q_1, we can use the monotonicity of $\partial\Psi$ and obtain, for a.a. $t \in [0, T]$, the estimate

$$\mu(t) \stackrel{\text{def}}{=} \langle \mathrm{D}_q\mathcal{E}(t, q_1(t))-\mathrm{D}_q\mathcal{E}(t, q_2(t)), \dot{q}_1(t)-\dot{q}_2(t)\rangle \leq 0. \tag{17}$$

For $\gamma(t) \stackrel{\text{def}}{=} \langle \mathrm{D}_q\mathcal{E}(t, q_1(t))-\mathrm{D}_q\mathcal{E}(t, q_2(t)), q_1(t)-q_2(t)\rangle$ uniform 2-convexity gives $\gamma(t) \geq \kappa\|q_1(t)-q_2(t)\|^2$. Moreover,

$$\dot{\gamma}(t) = \mu(t) + \langle \mathrm{D}_q^2\mathcal{E}(t, q_1)\dot{q}_1-\mathrm{D}_q^2\mathcal{E}(t, q_2)\dot{q}_2, q_1(t)-q_2(t)\rangle + \tau(t),$$

where $\tau(t) = \langle \partial_t\mathrm{D}_q\mathcal{E}(t, q_1)-\partial_t\mathrm{D}_q\mathcal{E}(t, q_2), q_1-q_2\rangle$. The smoothness $\mathcal{E} \in \mathrm{C}^3$ gives $|\tau(t)| \leq C\|q_1-q_2\|^2 \leq C\gamma/\alpha$. Subtracting $2\mu(t) \leq 0$ and rearranging the terms we find

$$\dot{\gamma} \leq \langle \xi_1, \dot{q}_1\rangle + \langle \xi_2, \dot{q}_2\rangle + C\gamma/\alpha$$

where $\xi_j = \mathrm{D}_q\mathcal{E}(t, q_{3-j})-\mathrm{D}_q\mathcal{E}(t, q_j)-\mathrm{D}_q^2\mathcal{E}(t, q_j)(q_{3-j}-q_j)$. From $q_j \in \mathrm{C}^{\mathrm{Lip}}([0, T], \boldsymbol{Q})$ we know that $\|\dot{q}_j\|$ is bounded, and differentiability of $\mathcal{E}$ implies $\|\xi_j\| \leq C\|q_1-q_2\|^2 \leq C\gamma/\kappa$. Thus, $\dot{\gamma} \leq C_*\gamma$ and Gronwall's lemma provide

$$\|q_1(t) - q_2(t)\|_{\boldsymbol{Q}} \leq C\|q_1(0) - q_2(0)\|_{\boldsymbol{Q}}\mathrm{e}^{C_*t},$$

which gives the desired uniqueness result.

In [BKS04, MiR07] the uniqueness results are generalized to RIS in a Hilbert space $\boldsymbol{H}$ with a dissipation potential $\mathcal{R}$ depending on $z \in \boldsymbol{H}$. The key is to use the auxiliary function $\mathcal{B}(z, \xi) = \sup\{\, \langle \xi, v\rangle - \frac{1}{2}\mathcal{R}(q, v) \mid v \in \boldsymbol{H} \,\}$, where $\frac{1}{2}-\mathcal{B}\big(z(t), -\mathrm{D}_z\mathcal{I}(t, z)\big)$ measures the distance of $\mathrm{D}_z\mathcal{I}(t, z)$ from the boundary of $\partial\mathcal{R}(z, 0)$. Under restrictive assumption it is then possible to derive estimates of the type $\Gamma(t) \leq C\mathrm{e}^{C_*t}\Gamma(0)$ for the combined quantity

$$\Gamma(t) \stackrel{\text{def}}{=} \sqrt{\gamma(t)} + \Big|\mathcal{B}\big(z_1(t), -\mathrm{D}_z\mathcal{I}(t, z_1(t))\big) - \mathcal{B}\big(z_2(t), -\mathrm{D}_z\mathcal{I}(t, z_2(t))\big)\Big|.$$

4.4 Parametrized solutions via the vanishing-viscosity approach

A main challenge in modeling rate-independent processes is the appearance of jumps. Since rate independence is a limit for systems under vanishing loading rates, we expect solutions of rate-independent systems to occur as limits of systems with relaxation times that are very small compared to the changes in the loading. Thus, we expect the solutions to occur as pointwise limits of time-continuous solutions. In particular, in a nonconvex situation solutions change slowly by following the loading for most of the time, but in-between there are sudden transitions from one stable regime to another one.

Here we define notions of solutions that are associated with the so-called *vanishing-viscosity approach*. For rate-independent processes this was proposed in [EfM06] and further analyzed in [MRS09b, KZM09, MiZ09, MRS09a]. In particular, we use the idea of arclength parametrization of solutions developing jumps, which was established earlier for systems with dry friction and small viscosity, see [MMG94, MS*95, Bon96, GMM98].

For simplicity we restrict the presentation in this and the following sections to the reduced case, since the vanishing-viscosity approach does not help to control the non-dissipative component $y \in \boldsymbol{Y}$ in the limit. Reasonable theories could be obtained in the case that $\mathcal{E}(t,\cdot,z)$ has a unique minimizer $y = Y(t,z)$ such that for the reduced energy functional $\mathcal{I}(t,z) = \mathcal{E}(t,Y(t,z),z)$ the power $\partial_t \mathcal{I}(t,z) = \partial_t \mathcal{E}(t,Y(t,z),z)$ is well defined.

For superlinear dissipation potentials $\mathcal{R}_{\rm sl}$ we define the rescaled potential $\mathcal{R}_\varepsilon(z,v) = \frac{1}{\varepsilon}\mathcal{R}_{\rm sl}(z,\varepsilon v)$, where the small parameter $\varepsilon > 0$ is the quotient obtained from dividing the time scale induced by the loading by the relaxation time due to viscosity (which is inverse to the viscosity). For parametrized solutions, the dissipation potential $\mathcal{R}$ must have the special form

$$\mathcal{R}_\varepsilon(z,v) = \mathcal{R}(z,v) + \tfrac{\varepsilon}{2}\langle \mathbb{V}(z)v, v\rangle,$$

where $\mathcal{R}$ is the rate-independent part and $\frac{\varepsilon}{2}\langle \mathbb{V}(z)v,v\rangle$ the small viscous part. Thus, we are led to study the differential inclusion

$$0 \in \partial_{\dot z}\mathcal{R}(z(t),\dot z(t)) + \varepsilon \mathbb{V}(z(t))\dot z(t) + \overline{\partial}_z \mathcal{I}(t,z(t)), \quad z(0) = z_0. \tag{18}$$

We simplify further by assuming that $\mathbb{V}$ is independent of z and denote by $\boldsymbol{V}$ the Hilbert space with norm $\|w\|_{\boldsymbol{V}} = \big(\langle \mathbb{V}w, w\rangle\big)^{1/2}$, i.e., $\mathbb{V} : \boldsymbol{V} \to \boldsymbol{V}^*$ is a norm-preserving bijection.

Under reasonable assumptions (see e.g., [CoV90, Col92, Rou05, RMS08]) one obtains solutions z^ε of (18) satisfying the energy identity

$$\mathcal{I}(t,z^\varepsilon(t)) + \int_0^t \mathcal{R}(z^\varepsilon(s),\dot z^\varepsilon(s)) + \varepsilon\|\dot z^\varepsilon(s)\|_{\boldsymbol{V}}^2 \,\mathrm{d}s = \mathcal{I}(0,z_0) + \int_0^t \partial_s \mathcal{I}(s,z^\varepsilon(s))\,\mathrm{d}s.$$

Using the coercivity $\mathcal{R}(z,w) \geq c_0\|w\|_{\boldsymbol{X}}$ and the coercivity of $\mathcal{I}$ provides the a priori estimates

$$\mathrm{Var}_{\boldsymbol{X}}(z^\varepsilon,[0,T]) + \varepsilon\|\dot z^\varepsilon\|^2_{\mathrm{L}^2([0,T];\boldsymbol{V})} + \|z^\varepsilon\|_{\mathrm{L}^\infty([0,T];\boldsymbol{Z})} \leq C. \tag{19}$$

Clearly for $\varepsilon > 0$ the solutions satisfy $z^\varepsilon \in \mathrm{H}^1([0,T];\boldsymbol{V}) \subset \mathrm{C}^0([0,T];\boldsymbol{V})$. Thus, solutions are not able to jump over potential barriers as it is the case for energetic solutions. Even in the limit $\varepsilon \to 0$ the potential barriers remain active and delay possible jumps.

These a priori bounds allow us to take the limit $\varepsilon \to 0$, if we assume that the embedding $\boldsymbol{Z} \subset \boldsymbol{X}$ is compact. Using the bound on the variation in $\boldsymbol{X}$, we are then able to apply Helly selection principle (cf. Thm. 3.13)) to extract a subsequence $(z^{\varepsilon_n})_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ such that $z^{\varepsilon_n}(t) \rightharpoonup z(t)$ in $\boldsymbol{Z}$ for some limit $z : [0,T] \to \boldsymbol{Z}$. This limit is then called a $\mathbb{V}$-*approximable solution* of the RIS $(\boldsymbol{Z},\mathcal{I},\mathcal{R})$, cf. [KMZ08, ToZ09, Cag09].

We proceed further by deriving equations that characterize such limit solutions. For this we introduce the concept of parametrized solutions that should be seen as a helpful, intermediate tool in understanding the limit procedure. The idea for resolving jumps in rate-independent systems is to consider the graph of the viscous solutions in the extended phase space $\boldsymbol{Z}_T$ and to study the limit of the whole graph. The advantage is that jumps do not shrink to a single point at jump time t, but provide a jump curve lying in the plane $\{t\}\times\boldsymbol{Z}$. In [EfM06] it was observed that the scaling invariance of RIS can be effectively used for parametrizing these graphs.

For a viscous solution $z^\varepsilon : [0,T] \to \boldsymbol{Z}$ we consider the graph

$$\mathrm{Graph}(z^\varepsilon) \overset{\mathrm{def}}{=} \{\, (t,z^\varepsilon(t)) \mid t \in [0,T] \,\} \subset \boldsymbol{Z}_T.$$

We use an arclength parametrization that is based on the viscous norm, namely $s = \sigma^\varepsilon(t) = t + \int_0^t \|z^\varepsilon(r)\|_{\boldsymbol{V}}\,\mathrm{d}r$, which has the inverse $t = \tau^\varepsilon(s)$. The choice of the $\boldsymbol{V}$-norm is crucial to maintain the structure of a generalized gradient flow. Introducing the rescaled function $Z^\varepsilon(s) = z^\varepsilon(\tau^\varepsilon(s))$ we observe that it is a solution of the transformed problem

$$\left.\begin{array}{l}\tau(0)=0,\ Z(0)=z_0, \qquad \dot\tau(s)+\|\dot Z(s)\|_{\boldsymbol V}=1,\ \text{and}\\ 0\in\partial_{\dot z}\mathcal{R}(Z(s),\dot Z(s))+\frac{\varepsilon}{1-\|\dot Z(s)\|_{\boldsymbol V}}\mathbb{V}\dot Z(s)+\overline{\partial}_z\mathcal{I}(\tau(s),Z(s))\end{array}\right\} \tag{20}$$

for a.a. $s\in[0,S^\varepsilon]$, where $S^\varepsilon=\sigma^\varepsilon(T)$. For this, it was essential that $\partial\mathcal{R}(z,\cdot)$ is 0-homogeneous.

The main observation is that the viscous term with ε as a prefactor is again a subdifferential, namely of the potential

$$\mathcal{V}_\varepsilon(w)\overset{\text{def}}{=}\varepsilon g(\|w\|_{\boldsymbol V})\ \text{ with } g(\nu)=\begin{cases}-\log(1-\nu)-\nu & \text{for } \nu<1\\ \infty & \text{otherwise.}\end{cases}$$

Such a potential only exists because we used the norm $\|\cdot\|_{\boldsymbol V}$ for parametrizing the graph. Thus, defining the potential $\widetilde{\mathcal{R}}_\varepsilon(z,w)\overset{\text{def}}{=}\mathcal{R}(z,w)+\mathcal{V}_\varepsilon(w)$ we can rewrite the second equation in (20) in the form $0\in\partial_{\dot Z}\widetilde{\mathcal{R}}_\varepsilon(Z,\dot Z)+\overline{\partial}\mathcal{I}(\tau,Z)$. Moreover, $\widetilde{\mathcal{R}}_\varepsilon$ converges monotonously to the limit functional

$$\widetilde{\mathcal{R}}_0(z,w)\overset{\text{def}}{=}\begin{cases}\mathcal{R}(z,w) & \text{for } \|w\|_{\boldsymbol V}\le 1,\\ \infty & \text{otherwise.}\end{cases}$$

In the case $\boldsymbol V\subset\boldsymbol X$ the convergence is even a Mosco convergence, and $\widetilde{\mathcal{R}}_0(z,\cdot):\boldsymbol V\to[0,\infty]$ is weakly lower semicontinuous on $\boldsymbol V$. We let $\mathcal{R}^{\boldsymbol V}(z,w)=\widetilde{\mathcal{R}}_0(z,w)$ and observe that we have the sum rule for the subdifferentials

$$\partial_{\dot z}\mathcal{R}^{\boldsymbol V}(z,w)=\partial_{\dot z}\mathcal{R}(z,w)+\partial\mathcal{V}_0(w),\ \text{ where } \mathcal{V}_0(w)=\begin{cases}0 & \text{for } \|w\|_{\boldsymbol V}\le 1,\\ \infty & \text{otherwise.}\end{cases}$$

In the case $\boldsymbol X\subset\boldsymbol V$, the functional $\widetilde{\mathcal{R}}_0(z,\cdot)$ must be extended to $\boldsymbol V$ via ∞ outside of $\boldsymbol X$. In general, $\widetilde{\mathcal{R}}_0(z,\cdot)$ is not weakly lower semicontinuous and we set

$$\mathcal{R}^{\boldsymbol V}(z,\cdot)=\mathrm{wlsc}\widetilde{\mathcal{R}}_0(z,\cdot):w\mapsto\inf\{\liminf_{k\to\infty}\widetilde{\mathcal{R}}_0(z,w_k)\mid w_k\rightharpoonup w\text{ in }\boldsymbol V\},$$

where "wlsc" denotes the weak lower semicontinuous hull. Taking the formal limit $\varepsilon\to 0$ in (20) we are lead to the following definition.

Definition 4.11 (Parametrized solutions). Let the RIS $(\boldsymbol Z,\mathcal{I},\mathcal{R})$ and $\boldsymbol X$ be given such that (8a) holds. Moreover, let $\mathbb{V}$ and $\boldsymbol V$ be given as above.

Then, a pair $\zeta=(\tau,Z):[0,S]\to\boldsymbol Z_T$ is called a $\mathbb{V}$-*parametrized solution*, if $(\tau,Z)\in\mathrm{W}^{1,1}([0,T],\mathbb{R}\times\boldsymbol V)$ and the following equations hold:

$$\left.\begin{array}{l}\tau(0)=0,\ \tau(S)=T,\ \dot\tau(s)\ge 0,\ \dot\tau(s)+\|\dot Z(s)\|_{\boldsymbol V}=1\\ 0\in\partial_{\dot z}\mathcal{R}^{\boldsymbol V}(Z(s),\dot Z(s))+\overline{\partial}_z\mathcal{I}(\tau(s),Z(s))\end{array}\right\}\ \text{a.e. on } [0,S]. \tag{21}$$

We also say that ζ is a parametrized solution to the RIS $(\boldsymbol Z,\mathcal{I},\mathcal{R},\mathbb{V})$.

The present definition follows [EfM06] and hides the rate-independent nature by asking for a strict arclength parametrization in $\boldsymbol V_T\overset{\text{def}}{=}[0,T]\times\boldsymbol V\subset\mathbb{R}\times\boldsymbol V$, where we use the extended norm $\|(t,v)\|_{\boldsymbol V_T}=|t|+\|v\|_{\boldsymbol V}$. Following [MRS09b, MRS09a], the parametrization may be kept free by replacing the last two relations in (21) by

$$\dot\tau(s)+\|\dot Z(s)\|_{\boldsymbol V}=\alpha(s),\quad 0\in\partial_{\dot z}\mathcal{R}^{\boldsymbol V}(Z(s),\tfrac{1}{\alpha(s)}\dot Z(s))+\overline{\partial}_z\mathcal{I}(\tau(s),Z(s)),$$

where the *parametrization function* $\alpha \in \mathrm{L}^1([0,S])$ satisfies $\alpha(s)>0$ a.e. in $[0,S]$. Clearly, a rescaling of the graph does not change the problem. Moreover, a rescaling of the time dependence of $\mathcal{I}$ can be compensated by a rescaling as follows. If $\widetilde{\mathcal{I}}(\widetilde{t},z)=\mathcal{I}(\phi(\widetilde{t}),z)$ and $\zeta:[0,S]\to \boldsymbol{V}_T$ is a parametrized solution to $(\boldsymbol{Z},\mathcal{I},\mathcal{R},\mathbb{V})$ with parametrization function α, then $\widetilde{\zeta}:\widetilde{t}\mapsto \zeta(\phi(\widetilde{t}))$ is a parametrized solution to $(\boldsymbol{Z},\widetilde{\mathcal{I}},\mathcal{R},\mathbb{V})$ with parametrization function $\widetilde{\alpha}:\widetilde{t}\mapsto \alpha(\phi(\widetilde{t}))\phi'(\widetilde{t})$.

The main feature of parametrized solutions can be seen by discussing the subdifferential of $\mathcal{R}^{\boldsymbol{V}}$, which is done using the polar $\mathcal{R}^\circ(z,\cdot)$ of $\mathcal{R}(z,\cdot)$, where the polar Ψ° of a convex potential Ψ is defined via $\Psi^\circ(\xi)\overset{\mathrm{def}}{=}\sup\{\,\langle\xi,v\rangle \mid \Psi(v)\le 1\,\}$. Along the arclength parametrized solutions we can distinguish three different dynamical regimes:

Sticking: We have $\dot Z(s)=0$ and $\dot\tau(s)=1$, i.e. the potential forces $\xi(s)\in\overline{\partial}_z\mathcal{I}(\zeta(s))$ are so small that the state does not change, namely $\mathcal{R}^\circ(z(s),-\xi(s))\le 1$ or equivalently $0\in\partial\mathcal{R}(Z(s),0)+\xi(s)$.

Rate-independent slip: We have $0<\|\dot Z\|_{\boldsymbol{V}}<1$ and $0<\dot\tau(s)<1$, i.e. the state changes so slowly that the rate-independent friction is strong enough to compensate the driving force, namely $0\in\partial\mathcal{R}(Z(s),\dot Z(s))+\xi(s)$, which implies $\mathcal{R}^\circ(z(s),-\xi(s))=1$.

Viscous jump: We have $\|\dot Z\|_{\boldsymbol{V}}=1$ and $\dot\tau(s)=0$, i.e., the motion is faster than the loading scale, and the system moves in a jump-like fashion. During this jump phase the driving force $\xi(s)$ satisfies $0\in\partial\mathcal{R}(Z(s),\dot Z(s))+\lambda(s)\mathbb{V}\dot Z(s)+\xi(s)$ for some $\lambda(s)\ge 0$, which implies $\mathcal{R}^\circ(z(s),-\xi(s))\ge 1$.

From this we also find another equivalent formulation of (21), namely

$$\left.\begin{array}{l}\tau(0)=0,\ \tau(S)=T,\ \dot\tau(s)\ge 0,\ \dot\tau(s)+\|\dot Z(s)\|_{\boldsymbol{V}}=1\\ 0\in\partial_{\dot z}\mathcal{R}(Z(s),\dot Z(s))+\lambda(s)\mathbb{V}\dot Z(s)+\overline{\partial}_z\mathcal{I}(\tau(s),Z(s))\\ \lambda(s)\ge 0,\qquad \lambda(s)(1-\|\dot Z(s)\|_{\boldsymbol{V}})=0\end{array}\right\}\ \text{a.e. on }[0,S]. \tag{22}$$

Using the chain rule for $\mathcal{I}$ we obtain an energy balance in the form

$$\begin{aligned}&\mathcal{I}(\zeta(s_2))+\textstyle\int_{s_1}^{s_2}\mathcal{R}(Z(s),\dot Z(s))+\lambda(s)\|\dot Z(s)\|_{\boldsymbol{V}}^2\,\mathrm{d}s\\ &=\mathcal{I}(\zeta(s_1))+\textstyle\int_{s_1}^{s_2}\partial_t\mathcal{I}(\zeta(s))\dot\tau(s)\,\mathrm{d}s,\end{aligned} \tag{23}$$

which shows the viscous contribution $\lambda\|\dot Z\|_{\boldsymbol{V}}^2$ to the total dissipation that remains in the vanishing-viscosity limit.

Passing from arclength parametrized solutions to the limit $\varepsilon\to 0$ we may arrive at limits $\zeta:[0,S]\to\boldsymbol{Z}_T$ that are not properly parametrized but satisfy

$$\left.\begin{array}{l}\tau(0)=0,\ \dot\tau(s)\ge 0,\ \dot\tau(s)+\|\dot Z(s)\|_{\boldsymbol{V}}=\alpha(s)\in[0,1]\\ 0\in\partial_{\dot z}\mathcal{R}^{\boldsymbol{V}}(Z(s),\dot Z(s))+\overline{\partial}_z\mathcal{I}(\tau(s),Z(s))\end{array}\right\}\ \text{a.e. on }[0,S]. \tag{24}$$

The following lemma states that reparametrization of the graph then leads to an arclength parametrized solution again.

Lemma 4.12 (Reparametrization). *Assume that* $\zeta\in\mathrm{C_w}([0,S];\boldsymbol{Z}_T)$ *satisfies* (24). *Let* $\widehat\sigma(s)=\int_0^s\dot\tau(r)+\|\dot Z(r)\|_{\boldsymbol{V}}\,\mathrm{d}r$, $\widehat S=\widehat\sigma(S)$. *If* $\widehat S>0$, *then the* reparametrization

$$\widehat\zeta:\begin{cases}[0,\widehat S]\to\ \ \boldsymbol{Z}_T;\\ \quad\widehat s\ \ \mapsto \zeta(\sigma_{\min}(\widehat s))\end{cases}\qquad \textit{with } \sigma_{\min}(\widehat s)=\inf\{\,s\in[0,S]\mid\widehat\sigma(s)=\widehat s\,\} \tag{25}$$

satisfies $\widehat\zeta\in\mathrm{C^{Lip}}([0,\widehat S],\boldsymbol{V}_T)\cap\mathrm{C_w}([0,\widehat S],\boldsymbol{Z}_T)$ *and is a* $\mathbb{V}$*-parametrized solution.*

Proof. Since the result is more or less standard, we sketch the arguments and refer to [MRS09b, Lem. 4.1] and [MRS09a, Prop 6.10] for more details. Clearly, $\widetilde{\zeta}$ is well-defined, and for $0 \le \widehat{s}_1 < \widehat{s}_2 \le \widehat{S}$ we have

$$\begin{aligned}&\|\widehat{\zeta}(\widehat{s}_1)-\widehat{\zeta}(\widehat{s}_2)\|_{\boldsymbol{V}_T} \overset{\text{def}}{=} |\widehat{\tau}(\widehat{s}_1)-\widehat{\tau}(\widehat{s}_2)| + \|\widehat{Z}(\widehat{s}_1)-\widehat{Z}(\widehat{s}_2)\|_{\boldsymbol{V}}\\ &\le \|\zeta(\sigma_{\min}(\widehat{s}_1))-\zeta(\sigma_{\min}(\widehat{s}_2))\|_{\boldsymbol{V}_T} \le \widehat{\sigma}(\sigma_{\min}(\widehat{s}_2))-\widehat{\sigma}(\sigma_{\min}(\widehat{s}_1)) = \widehat{s}_2-\widehat{s}_1.\end{aligned}$$

Thus, $\dot{\widehat{\zeta}}$ exists almost everywhere, and we obtain $\|\dot{\widehat{\zeta}}(\widehat{s})\|_{\boldsymbol{V}_T} = 1$ a.e. on $[0,\widehat{S}]$ by the standard chain rule. □

The existence theory for parametrized solutions is a delicate matter depending on the choice of the space $\boldsymbol{V}$. If we choose $\mathbb{V}$ such that $\boldsymbol{X} \subset \boldsymbol{V}$, then we have $\|w\|_{\boldsymbol{V}} \le C\mathcal{R}(z,w)$, and the a priori estimates (19) imply that $S^\varepsilon = T + \int_0^T \|\dot{Z}^\varepsilon(t)\|_{\boldsymbol{V}}\,\mathrm{d}t \le C$. Thus, it is easy to extract a converging subsequence. However, it is difficult to control the convergence of $\partial\widetilde{\mathcal{R}}_\varepsilon(Z^\varepsilon,\dot{Z}^\varepsilon)$ towards $\partial\mathcal{R}^{\boldsymbol{V}}$. In the opposite case $\boldsymbol{V} \subset \boldsymbol{X}$ the convergence of the subgradients of $\partial\widetilde{\mathcal{R}}_\varepsilon$ to $\partial\mathcal{R}^{\boldsymbol{V}}$ follows easily from the Mosco convergence, but it is unclear whether the arclength S^ε of the curves stays bounded.

Example 4.13. In fact, Example 4.3 with $\alpha = \infty$ provides a case with $\boldsymbol{V} = \mathrm{L}^2(\Omega) \subset \boldsymbol{X} = \mathrm{L}^1(\Omega)$ for which $S^\varepsilon \to \infty$ and hence $\widehat{S} = 0$. For $t \in [\varepsilon, 1]$ we have

$$\|\dot{z}^\varepsilon(t)\|_{\mathrm{L}^1} = 1 \quad\text{and}\quad \|\dot{z}^\varepsilon(t)\|_{\mathrm{L}^2} = 1/\sqrt{\varepsilon}.$$

Hence, the L^2 parametrized viscous solutions $(\tau^\varepsilon, Z^\varepsilon)$ satisfy $\tau^\varepsilon(s) \to \infty$ and $Z^\varepsilon(s) \to 0$ for all $s > 0$.

In the rest of this section, we reduce the discussion to our standard Example 2.8. The main reason for this restriction is that we need to show that S^ε remains bounded. This is trivial in the finite-dimensional cases treated in [EfM06, MRS09a], where $\boldsymbol{X} = \boldsymbol{V}$. For infinite-dimensional cases with $\boldsymbol{V} \subsetneqq \boldsymbol{X}$ this problem is solved only in the semilinear setting discussed below.

The first step towards the existence theory is a general convergence result for the vanishing-viscosity limit in the case $\boldsymbol{V} \subset \boldsymbol{X}$. The proof is based on the energetic formulation of generalized gradient systems as introduced in Section 2.5. We employ the Mosco convergence of $\widetilde{\mathcal{R}}_\varepsilon$ to $\widetilde{\mathcal{R}}_0$ and the chain-rule inequality (5). Since $\widetilde{\mathcal{R}}_\varepsilon(v) = \Psi(v) + \varepsilon g(\|v\|_{\boldsymbol{V}})$ and $\Psi^*(\xi) = 0$ for $\xi \in \partial\Psi(0)$ and ∞ otherwise, the Fenchel transform reads

$$\widetilde{\mathcal{R}}^*_\varepsilon(\xi) = \min\{\, \varepsilon g^*(\tfrac{1}{\varepsilon}\|\xi-\kappa\|_{\boldsymbol{V}^*}) \mid \kappa \in \partial\Psi(0)\,\}$$

where $g^*(\varrho) = \sup_{r\ge 0} \varrho r - g(r)$. In the limit $\varepsilon \to 0$ we find the Γ-limits

$$\begin{aligned}&\widetilde{\mathcal{R}}_0(v) = \Psi(v) + \mathcal{V}_0(v), \quad\text{and}\\ &\widetilde{\mathcal{R}}^*_0(\xi) = M_\Psi^{\boldsymbol{V}}(\xi) \overset{\text{def}}{=} \min\{\, \|\xi-\kappa\|_{\boldsymbol{V}^*} \mid \kappa \in \partial\Psi(0)\,\},\end{aligned} \tag{26}$$

where $\mathcal{V}_0(v) = 0$ for $\|v\|_{\boldsymbol{V}} \le 1$ and ∞ otherwise.

Proposition 4.14 (Vanishing-viscosity limit). *Let $(\boldsymbol{Z}, \mathcal{I}, \Psi)$, $\boldsymbol{X}$, and $\boldsymbol{V}$ be given as in Example 2.8. Assume that $\zeta^\varepsilon = (\tau^\varepsilon, Z^\varepsilon) : [0,S] \to \boldsymbol{Z}_T$ satisfy (20) on $[0,S]$ with $Z^\varepsilon(0) = z_0 \in \boldsymbol{Z}$. Then there exist a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and functions $\zeta : [0,S] \in \mathrm{C}^{\mathrm{Lip}}([0,S],\boldsymbol{V})$ and $\alpha \in \mathrm{L}^\infty([0,S])$ such that (24) and the convergences $\zeta^{\varepsilon_n}(s) \rightharpoonup \zeta(s)$ in $\boldsymbol{Z}$ for all $s \in [0,S]$ and $\dot{\zeta}^{\varepsilon_n} \overset{*}{\rightharpoonup} \dot{\zeta}$ in $\mathrm{L}^\infty([0,S];\boldsymbol{V}_T)$ hold.*

If $\widehat{S} = \int_0^S \dot{\tau}(r) + \|\dot{Z}(r)\|_{\boldsymbol{V}}\,\mathrm{d}r > 0$, then the reparametrization $\widehat{\zeta} : [0,\widehat{S}] \times \boldsymbol{Z}_T$ defined in (25) is a $\mathbb{V}$-parametrized solution.

Proof. We have uniform a priori bounds in $\mathrm{C}^{\mathrm{Lip}}([0,S];\boldsymbol{V})$ and $\mathrm{L}^\infty([0,T];\boldsymbol{Z}_T)$ for ζ^ε. Thus, we find a $\zeta \in \mathrm{C}^{\mathrm{Lip}}([0,T];\boldsymbol{V}) \cap \mathrm{L}^\infty([0,T];\boldsymbol{Z}_T)$ such that

$$\begin{aligned}&\text{(i)}\ \ \zeta^\varepsilon \to \zeta \text{ in } \mathrm{C}([0,T];\boldsymbol{V}_T), \quad \text{(ii)}\ \ \dot\zeta^\varepsilon \overset{*}{\rightharpoonup} \dot\zeta \text{ in } \mathrm{L}^\infty([0,T];\boldsymbol{V}_T),\\ &\text{(iii)}\ \ Z^\varepsilon(t) \overset{*}{\rightharpoonup} Z(t) \text{ in } \boldsymbol{Z} \text{ for all } t\in[0,T].\end{aligned}$$

along a suitable subsequence (not relabeled). For (iii) note that $\zeta^\varepsilon([0,S];\boldsymbol{V}_T) \cap \mathrm{L}^\infty([0,S];\boldsymbol{Z}_T)$ implies $\zeta^\varepsilon \in \mathrm{C_w}([0,S];\boldsymbol{Z}_T)$. The uniform bound $\|Z^\varepsilon(s)\|_{\boldsymbol{Z}} \le C$ for all $\varepsilon>0$ and $s\in[0,S]$ implies that for each fixed s_* $\zeta^{\varepsilon_j}(s_*) \rightharpoonup \zeta_*$ in $\boldsymbol{Z}_T$. Since, we know $\zeta^\varepsilon(s_*) \to \zeta(s_*)$ in $\boldsymbol{V}_T$ we obtain (iii).

For $\xi^\varepsilon(s) = \mathrm{D}_z\mathcal{I}(\zeta^\varepsilon(s))$ we use (21) and the bounds on Z^ε to conclude that ξ^ε is bounded in $\mathrm{L}^\infty([0,T];\boldsymbol{Z}^*)$. Choosing a further subsequence (not relabeled) we find $\xi^\varepsilon \overset{*}{\rightharpoonup} \xi$ in $\mathrm{L}^\infty([0,T],\boldsymbol{Z}^*)$. The semilinear structure (21) of $\mathrm{D}_z\mathcal{I}(t,\cdot): \boldsymbol{Z} \to \boldsymbol{Z}^*$ implies weak continuity, which gives $\xi^{\varepsilon_n}(s) \rightharpoonup \xi(s) = \mathrm{D}_z(\zeta(s))$.

In the limit $\varepsilon \to 0$, we easily obtain the upper line of conditions in (24). To obtain the differential inclusion in the lower line, we use the equivalent formulation via the Legendre-Fenchel duality of the lower line in (20), namely

$$\begin{aligned}&\mathcal{I}(\zeta^\varepsilon(S)) + \mathcal{M}_\varepsilon(\dot Z^\varepsilon, -\xi^\varepsilon) = \mathcal{I}(0,z_0) + \textstyle\int_0^S \partial_t\mathcal{I}(\zeta^\varepsilon(s))\dot\tau^\varepsilon(s)\,\mathrm{d}s,\\ &\text{where } \mathcal{M}_\varepsilon(V,\xi) = \textstyle\int_0^S \widetilde{\mathcal{R}}_\varepsilon(V(s)) + \widetilde{\mathcal{R}}^*_\varepsilon(\xi(s))\,\mathrm{d}s.\end{aligned} \tag{27}$$

We extend $\widetilde{\mathcal{R}}^*_\varepsilon$ for $\varepsilon\in[0,1]$ to $\boldsymbol{Z}^*$ by ∞ outside of $\boldsymbol{V}^*$. By the definition (26) and direct calculation we obtain the liminf estimates

$$\begin{aligned}V^\varepsilon \rightharpoonup V \in \boldsymbol{V} &\implies \widetilde{\mathcal{R}}_0(V) \le \liminf_{\varepsilon\to0}\widetilde{\mathcal{R}}_\varepsilon(V^\varepsilon),\\ \xi^\varepsilon \rightharpoonup \xi \in \boldsymbol{Z}^* &\implies \widetilde{\mathcal{R}}^*_0(\xi) \le \liminf_{\varepsilon\to0}\widetilde{\mathcal{R}}^*_\varepsilon(\xi^\varepsilon).\end{aligned}$$

In fact, by [Att84, Sect. 3.3.1] this is equivalent to Mosco convergence of $\widetilde{\mathcal{R}}_\varepsilon$ to $\widetilde{\mathcal{R}}_0$. Hence, $\mathcal{M}_\varepsilon$ weakly Γ-converges to $\mathcal{M}_0$ on $\mathrm{L}^2([0,S];\boldsymbol{V})\times\mathrm{L}^2([0,S];\boldsymbol{Z}^*)$, and we can pass to the limit $\varepsilon\to0$ in (27) and obtain

$$\mathcal{I}(\zeta(S)) + \mathcal{M}_0(\dot Z, -\xi) \le \mathcal{I}(0,0,z_0) + \textstyle\int_0^S \partial_t\mathcal{I}(\zeta(s))\dot\tau(s)\,\mathrm{d}s,$$

where we use $\dot\tau^\varepsilon \overset{*}{\rightharpoonup} \dot\tau$ in $\mathrm{L}^\infty([0,S])$ and $\partial_t\mathcal{I}(\zeta^\varepsilon(\cdot)) \to \partial_t\mathcal{I}(\zeta(\cdot))$ in $\mathrm{L}^1([0,T])$ the linearity of $\partial_t\mathcal{I}(t,z) = -\langle\dot\ell(t),z\rangle$.

Note that $\mathcal{M}_0(\dot Z,\xi) < \infty$ implies $\xi(\cdot) \in \mathrm{L}^1([0,S];\boldsymbol{V}^*)$, since $\widetilde{\mathcal{R}}^*_0(\xi(\cdot)) \in \mathrm{L}^1([0,S])$ and $\|\xi\|_{\boldsymbol{V}^*} \le \widetilde{\mathcal{R}}^*_0(\xi) + k_0$ with $k_0 = \sup_{\kappa\in\partial\Psi(0)}\|\kappa\|_{\boldsymbol{V}^*} < \infty$. Thus, exploiting $\|\dot Z(s)\|_{\boldsymbol{V}} \le 1$ a.e., we can use the chain-rule inequality (5) for the limit function ζ and find

$$\mathcal{M}_0(\dot Z,\xi) = \textstyle\int_0^S \widetilde{\mathcal{R}}_0(\dot Z(s)) + \widetilde{\mathcal{R}}^*_0(-\xi(s))\,\mathrm{d}s \le \int_0^S \langle \dot Z(s), -\xi(s)\rangle\,\mathrm{d}s.$$

By the definition of the Legendre-Fenchel duality we have $\langle \dot Z(s), -\xi(s)\rangle \le \widetilde{\mathcal{R}}_0(\dot Z(s)) + \widetilde{\mathcal{R}}^*_0(-\xi(s))$ and conclude equalities. We obtain $0 \in \partial\widetilde{\mathcal{R}}_0(\dot Z(s)) + \xi(s)$, which gives the desired differential inclusion on the lower line in (24). $\square$

Our main existence result for parametrized solutions uses the spaces

$$\boldsymbol{Z}_1 \Subset \boldsymbol{Z} \Subset \boldsymbol{V} \Subset \boldsymbol{Z}_{-1}, \quad \text{and} \quad \boldsymbol{V} \subset \boldsymbol{X},$$

as defined in Example 2.8.

Theorem 4.15 (Parametrized solutions). *Let all assumptions of Example 2.8 and* (21) *hold for* $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$ *and the spaces* $\boldsymbol{Z} \Subset \boldsymbol{V} \subset \boldsymbol{X}$. *Then, for each* $z_0 \in \boldsymbol{Z}_1$ *there exists a parametrized solution* $\zeta = (\tau, Z) : [0, S] \to \boldsymbol{Z}_T$ *with* $Z(0) = z_0$, *which further satisfies* $\zeta \in \mathrm{C_w}([0, S]; \boldsymbol{Z}_T) \cap \mathrm{BV}([0, S], \boldsymbol{Z}_T)$.

The proof of this result relies on the convergence result established above together with an estimate on S^ε. The latter is based on higher-order a priori estimates. It is a surprising feature of rate-independent systems that certain a priori estimates are independent of the dissipation functional $\mathcal{R}$. We are in the case of translationally invariant dissipations $\mathcal{R}(z, v) = \Psi(v)$, where Ψ is assumed to be convex, 1-homogeneous, and continuous on $\boldsymbol{V}$. The three a priori estimates we will use derive from the following basic properties of Ψ:

$$
\begin{aligned}
&\text{(i)} && \langle \partial\Psi(v), v\rangle = \Psi(v),\\
&\text{(ii)} && \text{“ } \langle \mathrm{D}^2\Psi(v)[w], v\rangle = \langle \mathrm{D}^2\Psi(v)[v], w\rangle = 0 \text{ ”},\\
&\text{(iii)} && \langle \partial\Psi(v_1) - \partial\Psi(v_2), v_1 - v_2\rangle \geq 0.
\end{aligned}
$$

Here the first relation is 1-homogeneity, and the third is simply monotonicity. The middle relation was put into quotation marks, since $\mathrm{D}^2\Psi$ does not exist; however, by 0-homogeneity of $\partial\Psi$ the directional derivative $\partial\Psi(v)$ in the direction v is 0.

We first state the corresponding a priori estimates that are obtained from

$$
0 \in \partial\Psi(\dot z(t)) + \varepsilon \mathbb{V}\dot z(t) + \mathrm{D}_z\mathcal{I}(t, z(t)) \tag{28}
$$

by assuming smoothness and (i) by applying $\langle \cdot, \dot z\rangle$, (ii) by differentiation with respect to t and applying $\langle \cdot, \dot z\rangle$, and (iii) by differentiation with respect to t and applying $\langle \cdot, \ddot z\rangle$:

$$
\Psi(\dot z) + \varepsilon \|\dot z\|_{\boldsymbol{V}}^2 + \langle \mathrm{D}_z\mathcal{I}(t, z(t)), \dot z\rangle = 0, \tag{29a}
$$

$$
\varepsilon\langle \mathbb{V}\dot z, \ddot z\rangle + \langle \mathrm{D}_z^2\mathcal{I}(t, z)\dot z, \dot z\rangle + \langle \partial_t\mathrm{D}_z\mathcal{I}(t, z), \dot z\rangle = 0, \tag{29b}
$$

$$
\varepsilon \|\ddot z\|_{\boldsymbol{V}}^2 + \langle \mathrm{D}_z^2\mathcal{I}(t, z)\dot z, \ddot z\rangle + \langle \partial_t\mathrm{D}_z\mathcal{I}(t, z), \ddot z\rangle \leq 0. \tag{29c}
$$

The next result presents sufficient conditions on the solution such that these relations can be proved rigorously.

Lemma 4.16 (A priori estimates). *Let the assumptions* (21) *hold, then for each initial value* $z_0 \in \boldsymbol{Z}$ *there exists* $C > 0$ *such that for all* $\varepsilon \in\,]0, 1[$ *and all solutions* z *of* (28) *with* $z(0) = z_0$ *we have:*
(A) If $z \in \mathrm{H}^1([0, T], \boldsymbol{V}) \cap \mathrm{C_w}([0, T], \boldsymbol{Z})$, *then* (29a) *holds a.e., and we have the energy balance*

$$
\mathcal{I}(t, z(t)) + \textstyle\int_s^t \Psi(\dot z(r)) + \varepsilon \|\dot z(r)\|_{\boldsymbol{V}}^2 \,\mathrm{d}r = \mathcal{I}(s, z(s)) + \int_s^t \partial_r\mathcal{I}(r, z(r))\,\mathrm{d}r. \tag{30}
$$

(B) If $z \in \mathrm{H}^1([0, T], \boldsymbol{Z}) \cap \mathrm{H}^2([0, T], \boldsymbol{Z}_{-1})$, *then* (29b) *holds a.e.,* $\dot z \in \mathrm{C}([0, T]; \boldsymbol{V})$, *and*

$$
\varepsilon \max\nolimits_{t\in[0,T]} \|\dot z(t)\|_{\boldsymbol{V}}^2 + \textstyle\int_0^T \|\dot z(r)\|_{\boldsymbol{Z}}^2 \,\mathrm{d}r \leq C(1 + \varepsilon\|\dot z(0)\|_{\boldsymbol{V}}^2). \tag{31}
$$

(C) If $z \in \mathrm{H}^1([0, T], \boldsymbol{Z}_1) \cap \mathrm{H}^2([0, T], \boldsymbol{V})$, *then* (29c) *holds a.e.,* $\dot z \in \mathrm{C}([0, T]; \boldsymbol{Z})$, *and*

$$
\max\nolimits_{t\in[0,T]} \|\dot z(t)\|_{\boldsymbol{Z}}^2 + \varepsilon \textstyle\int_0^T \|\ddot z(r)\|_{\boldsymbol{V}}^2 \,\mathrm{d}r \leq C(1 + \|\dot z(0)\|_{\boldsymbol{Z}}^2). \tag{32}
$$

Proof. Part (A) follows simply by using the chain rule.

For Part (B) we set $g(t) = \varepsilon\mathbb{V}\dot z(t) + \mathrm{D}_z\mathcal{I}(t, z(t))$. By (21) we have $g \in \mathrm{H}^1([0, T], \boldsymbol{Z}^*)$. From (28) we have $0 \in \partial\Psi(\dot z(t)) + g(t)$, and the characterization of $\partial\Psi(v)$ in Lemma 2.2 gives

$$
\langle g(s), \dot z(t)\rangle \geq -\Psi(\dot z(t)) = \langle g(t), \dot z(t)\rangle \text{ for all } s, t \in\,]0, T]\,.
$$

The left-hand side attains its global minimum at $s=t$. If additionally $s=t$ is a point of differentiability of g, then the left-hand side is differentiable with respect to s with derivative 0 at $s=t$, i.e., $0=\langle \dot g(t), \dot z(t)\rangle$. This is the desired relation in (29b). The a priori estimate (31) follows by standard arguments using (21), cf. the proof of Proposition 4.17.

For part (C) we use the monotonicity of $\partial\Psi$ and obtain

$$\varepsilon\|\dot z(t)-\dot z(s)\|_{\boldsymbol{V}}^2 + \langle g(t)-g(s), \dot z(t)-\dot z(s)\rangle \le 0$$

for all $s,t\in[0,T]$. By the assumptions we have $\dot z, \mathbb{V}^{-1}g \in \mathrm{H}^1([0,T],\boldsymbol{V})$, thus we can divide by $(t-s)^2$ and pass to the limit a.e. This provides (29c), and the a priori estimate (32) again follows by standard arguments using (21). $\square$

The following result relies on parabolic estimates, which essentially use the semilinear structure. For Galerkin approximations, we are able to exploit the above a priori estimates, which then survive in the limit. An essential novel feature is the derivation of an estimate that is invariant under rescaling. This estimate then allows us to derive upper bounds for S^ε. For more details and applications to quasilinear problems see [MiZ09].

Proposition 4.17. *Let assumptions of Example 2.8 and* (21) *hold for the RIS* $(\boldsymbol{Z},\mathfrak{I},\Psi)$ *and the spaces* $\boldsymbol{Z}\Subset\boldsymbol{V}\subset\boldsymbol{X}$. *Then, for each* $z_0\in\boldsymbol{Z}$ *equation* (28) *has a unique solution* $z^\varepsilon\in\mathrm{H}^1([0,T],\boldsymbol{V})\cap \mathrm{C}_{\mathrm{w}}([0,T],\boldsymbol{Z})$ *with* $z(0)=z_0$, *which satisfies* $z^\varepsilon\in\mathrm{L}^2([0,T];\boldsymbol{Z}_1)$. *If additionally* $z_0\in\boldsymbol{Z}_1$, *then there exists a constant* $C>0$ *independent of* ε *such that* z^ε *lies in* $\mathrm{H}^1([0,T],\boldsymbol{Z})$ *and satisfies the a priori estimate*

$$\int_0^T \|\dot z^\varepsilon(t)\|_{\boldsymbol{Z}}\,\mathrm{d}t \le C\Big(\varepsilon\|\dot z^\varepsilon(0)\|_{\boldsymbol{V}} + \int_0^T \Psi(\dot z^\varepsilon(t)) + \|\dot\ell(t)\|_{\boldsymbol{V}^*}\,\mathrm{d}t\Big). \tag{33}$$

Proof. We first construct solutions for $z_0\in\boldsymbol{Z}_1$, which means $\mathbb{A}z_0\in\boldsymbol{V}$. Then, from the equation we find $\dot z(0)\in\boldsymbol{V}$. We consider approximate solutions z_N by using some Galerkin projector P_N which commutes with $\mathbb{V}$ and $\mathbb{A}$, i.e., $\langle\mathbb{A}v, P_N w\rangle=\langle\mathbb{A}P_N v, w\rangle$ and $\langle\mathbb{V}v, P_N w\rangle=\langle\mathbb{V}P_N v, w\rangle$ and has finite-dimensional range $\boldsymbol{X}_N=P_N\boldsymbol{Z}=P_N\boldsymbol{V}$. (Such a P_N can be constructed using the eigenpairs (λ_j,ϕ_j) of $\mathbb{A}\phi=\lambda\mathbb{V}\phi$, which exist due to $\boldsymbol{Z}\Subset\boldsymbol{V}$.) Now take $z_N\in\mathrm{H}^1([0,T],P_N\boldsymbol{Z})$ as the unique solution of $z_N(0)=P_N z_0$ and

$$P_N^*\boldsymbol{Z}^*\ni\ 0\in P_N^*\big(\partial\Psi(\dot z_N)+\varepsilon\mathbb{V}\dot z_N+\mathbb{A}z_N+\mathrm{D}\Phi(z_N)-\ell(t)\big). \tag{34}$$

This inclusion can be inverted to $\dot z_N=M_N(-P_N(\mathbb{A}z_N+\mathrm{D}\Phi(z_N)-\ell))$, where $M_N:\boldsymbol{X}_N^*\to\boldsymbol{X}_N$ is the Lipschitz continuous inverse of the strictly monotone mapping $\boldsymbol{X}_N\ni v\mapsto P_N(\partial\Psi(v)+\varepsilon\mathbb{V}v)\subset\boldsymbol{X}_N^*$. Since on $\boldsymbol{X}_N$ the norms of $\boldsymbol{Z}$ and $\boldsymbol{V}$ are equivalent, we conclude $z_N\in\mathrm{W}^{2,p}([0,T],\boldsymbol{X}_N)$. For these finite-dimensional approximations we are now able to exploit the assumption $z_0\in\boldsymbol{Z}_1$, which gives $\|\dot z_N(0)\|_{\boldsymbol{V}}\to\|\dot z(0)\|_{\boldsymbol{V}}<\infty$. Thus, the a priori estimate (31) provides boundedness of $(z_N)_{N\in\mathbb{N}}$ in $\mathrm{H}^1([0,T],\boldsymbol{Z})$. We obtain a weakly converging subsequence (not relabeled) $z_N\rightharpoonup z$ in $\mathrm{H}^1([0,T],\boldsymbol{Z})$. It is easy to see that z is a solution of (28) with $z(0)=z_0$.

To show uniqueness we consider two solutions z_1 and z_2 in $\mathrm{H}^1([0,T],\boldsymbol{Z})$. Using (21f) and (21g) we find $z_j\in\mathrm{L}^2([0,T],\boldsymbol{Z}_1)$. Setting $w=z_1-z_2$, the monotonicity of $\partial\Psi$ gives

$$0\ge\varepsilon\|\dot w(t)\|_{\boldsymbol{V}}^2+\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\langle Aw(t),w(t)\rangle+\langle H(t)w(t),\dot w(t)\rangle$$

with $H(t)=\int_0^1\mathrm{D}^2\Phi(z_2(t)+rw(t))\,\mathrm{d}r$, where $H(t)\in\mathscr{L}(\boldsymbol{Z},\boldsymbol{V}^*)$ by (21f). Because of (21c) we may assume without loss of generality that the norm in $\boldsymbol{Z}$ is given via $\|z\|_{\boldsymbol{Z}}^2=\langle\mathbb{A}z,z\rangle$. Thus, with (21f) we obtain

$$\varepsilon\|\dot w\|_{\boldsymbol{V}}^2+\frac{\mathrm{d}}{2\mathrm{d}t}\|w\|_{\boldsymbol{Z}}^2\le C\|w\|_{\boldsymbol{Z}}\|\dot w\|_{\boldsymbol{V}}\le\varepsilon\|\dot w\|_{\boldsymbol{V}}^2+\tfrac{C}{\varepsilon}\|w\|_{\boldsymbol{Z}}^2.$$

Applying Gronwall's lemma we arrive at

$$\|z_2(t)-z_1(t)\|_{\boldsymbol{Z}} = \|w\|_{\boldsymbol{Z}} \leq \mathrm{e}^{Ct/\varepsilon}\|w(0)\|_{\boldsymbol{Z}} = \mathrm{e}^{Ct/\varepsilon}\|z_2(0)-z_1(0)\|_{\boldsymbol{Z}}, \tag{35}$$

which implies the desired uniqueness if $z_0 \in \boldsymbol{Z}_1$. Hence, we conclude that the full sequence z_N converges weakly to z.

To obtain existence and uniqueness for the general case $z_0 \in \boldsymbol{Z}$ we proceed as follows. Using the same Galerkin approach as above we obtain uniform a priori estimates for z_N in $\mathrm{H}^1([0,T],\boldsymbol{V}) \cap \mathrm{C_w}([0,T],\boldsymbol{Z})$. Note that the Gronwall estimate (35) holds for the finite-dimensional approximations as well and that the constants are independent of N. Thus, we can pass to the limit along subsequences, obtain solutions of (28) with $z(0) = z_0$, and that these solutions still satisfy the Gronwall estimate, which implies uniqueness.

It remains to establish the a priori estimate (33). For this return to the Galerkin approximation z_N, which satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{\varepsilon}{2}\|\dot z_N(t)\|^2_{\boldsymbol{V}} + \|\dot z_N(r)\|^2_{\boldsymbol{Z}} \leq C\|\dot z_N(t)\|_{\boldsymbol{Z}}\|\dot z_N(t)\|_{\boldsymbol{V}} + \|\dot\ell(t)\|_{\boldsymbol{V}^*}\|\dot z_N(t)\|_{\boldsymbol{V}}$$

a.e. on $[0,T]$, where here and in the rest of this proof the constant C may take different values but is independent of ε and N. We let $\nu = \|\dot z_N\|_{\boldsymbol{V}}$, $\zeta = \|\dot z_N(t)\|_{\boldsymbol{Z}}$, $\psi = \Psi(\dot z_N)$, and $\lambda = \|\dot\ell(t)\|_{\boldsymbol{V}^*}$, then with (21b), (21d), and $\vartheta = 1/(1{+}\theta)$ we find

$$\varepsilon\nu\dot\nu + \zeta^2 \;\leq\; C\nu^{1-\vartheta}(\psi^\theta\zeta^{1-\theta})^\vartheta\zeta + \lambda\nu \;\leq\; \tfrac12\zeta^2 + (C\psi{+}\lambda)\nu.$$

Using $\nu \leq C\zeta$ we find $\varepsilon\nu\dot\nu + \frac{1}{2C}\nu\zeta \leq (C\psi{+}\lambda)\nu$. Without loss of generality we may assume $\nu > 0$ (otherwise take $\sqrt{\nu^2{+}\delta}$, which satisfies the same estimate, and let $\delta \to 0^+$ afterwards). Dividing by $\nu > 0$ and integrating gives

$$\tfrac{1}{2C}\int_0^T \|\dot z_N(r)\|_{\boldsymbol{Z}}\,\mathrm{d}r \leq \varepsilon\|\dot z_N(0)\|_{\boldsymbol{V}} + \int_0^T C\Psi(\dot z_N(r)) + \|\dot\ell(r)\|_{\boldsymbol{V}^*}\,\mathrm{d}r.$$

Estimate (33) follows with $N \to \infty$ if we show $\lim_{N\to\infty}\int_0^T \Psi(\dot z_N(r))\,\mathrm{d}r \leq \int_0^T \Psi(\dot z(r))\,\mathrm{d}r$. Indeed, this equality follows by passing to the limit in the energy balance (30), since convergence in $\mathrm{H}^1([0,T],\boldsymbol{Z})$ implies $\mathcal{I}(t,z_N(t)) \to \mathcal{I}(t,z(t))$ and $\partial_t\mathcal{I}(t,z_N(t)) \to \partial_t\mathcal{I}(t,z(t))$ for all $t \in [0,T]$. Thus, we find

$$\int_0^T \Psi(\dot z_N(r))\,\mathrm{d}r + \int_0^T \varepsilon\|\dot z_N(r)\|^2_{\boldsymbol{V}}\,\mathrm{d}r \to \int_0^T \Psi(\dot z(r))\,\mathrm{d}r + \int_0^T \varepsilon\|\dot z(r)\|^2_{\boldsymbol{V}}\,\mathrm{d}r.$$

Together with the weak lower semicontinuity of each of the integrals we obtain the desired convergence and the proof is complete. □

We are now ready for stating the major existence result for parametrized solutions, which is obtained via the vanishing-viscosity approach. We emphasize that the result is not optimal, in the sense that we have to assume too much regularity for the initial condition. This is also seen in the a priori estimate (33), where we obtained the higher $\boldsymbol{Z}$-norm, but we actually only need the $\boldsymbol{V}$-norm. The existence result is obtained by combining Proposition 4.17 and the vanishing-viscosity theory of Proposition 4.14.

Proof (of Theorem 4.15). For each $\varepsilon \in \,]0,1[$ we can construct a solution $z^\varepsilon \in \mathrm{H}^1([0,T];\boldsymbol{Z})$ of (28) satisfying the rate-independent a priori estimate (33). The latter implies that for $\sigma^\varepsilon : [0,T] \mapsto t + \int_0^t \|\dot z^\varepsilon(r)\|_{\boldsymbol{V}}\,\mathrm{d}r$, the total length $S^\varepsilon = \sigma^\varepsilon(T)$ of the graph of z^ε in $\boldsymbol{V}_T$ stays bounded. We let $S = \limsup_{\varepsilon\to0} S^\varepsilon$ and choose a subsequence ε_j such that $S^{\varepsilon_j} \to S$.

We parametrize the graph of z^ε in $\boldsymbol{V}_T$ by arclength to obtain the Lipschitz functions $\zeta^\varepsilon = (\tau^\varepsilon, Z^\varepsilon) : [0,S] \to \boldsymbol{V}_T$, where we set $\zeta^\varepsilon(s) = \zeta^\varepsilon(S^\varepsilon)$ for $s > S^\varepsilon$. Since all functions ζ^{ε_j} have Lipschitz constant 1, the Arzelà-Ascoli theorem allows us to choose a further subsequence ζ^{ε_n} as in Proposition 4.14. The limit $\widetilde\zeta$ satisfies (24) and additionally $\widetilde\tau(S) = T$, which follows from $T = \tau^{\varepsilon_n}(S^{\varepsilon_n})$ and $|\tau^{\varepsilon_n}(S^{\varepsilon_n}){-}\tau^{\varepsilon_n}(S)| \leq |S^{\varepsilon_n}{-}S|$.

Thus we are able to do the reparametrization (25) and the desired solution ζ is constructed. It remains to show that Z lies in $\mathrm{BV}([0,S],\boldsymbol{Z})$. For this we recall the rate-independent estimate (33) for each z^ε. The right-hand side there is bounded independently of ε. For $\int_0^T \Psi(\dot z^\varepsilon(s))\,\mathrm{d}s$ this follows from the energy balance and for $\varepsilon\dot z^\varepsilon(0)$ this follows from $0 = w^\varepsilon(0)+\varepsilon\mathbb{V}\dot z^\varepsilon(0)+\mathbb{A}z_0+\mathrm{D}\Phi(z_0)-\ell(0)$, where $w^\varepsilon(0)\in\partial\Psi(\dot z^\varepsilon(0))$ is bounded in $\boldsymbol{V}^*$ by (21d). Thus, $z_0\in\boldsymbol{Z}_1$ and (21f) imply $\varepsilon\|\dot z^\varepsilon(0)\|_{\boldsymbol{V}}\le C$. Since the BV norm is scaling invariant, we have $\mathrm{Var}_{\boldsymbol{Z}}(Z^\varepsilon,[0,S^\varepsilon]) = \mathrm{Var}_{\boldsymbol{Z}}(z^\varepsilon,[0,T])\le C$. In the limit $\varepsilon_n\to 0$ this estimate still holds by lower semicontinuity of the variation. $\square$

Remark 4.18. In [MiZ09] it is additionally shown that the limit $\widetilde{\zeta}$ obtained in the middle of the above proof is nondegenerate in the sense that there exists a $\alpha_0>0$ such that $\dot{\widetilde{\tau}}(s)+\|\dot{\widetilde{Z}}(s)\|_{\boldsymbol{V}}\ge\alpha_0$. It is still an open problem, whether the arclength parametrization remains preserved in the limit $\varepsilon\to 0$. A positive result was obtained in [EfM06] under strong conditions. Moreover, the metric approach in Section 5.2 is such that the arclength parametrization is preserved.

4.5 BV solutions and optimal jump paths

The drawback of parametrized solutions is that we need to deal with functions in the extended state space $\boldsymbol{Z}_T$. Thus, it is not easy to compare this notion to all the other solution types, which are defined for functions $z:[0,T]\to\boldsymbol{Z}$ only. Thus, for each parametrized solution ζ we consider all associated projections $z:[0,T]\to\boldsymbol{Z}$ defined as follows. For $\zeta=(\tau,Z):[0,S]\to\boldsymbol{Z}_T$ with $\tau(0)=0$ and $\tau(S)=T$ and τ monotone, we define $\mathfrak{P}(\zeta)$ to be the set of functions $z:[0,T]\to\boldsymbol{Z}$ such that for all $t\in[0,T]$ there exists $s\in[0,S]$ such that $(t,z(t))=\zeta(s)=(\tau(s),Z(s))$.

We first show that such projections lead to local solutions, see (11). Then, we derive the notion of *BV solutions* in such a way that we can show that all these projections are BV solutions. Thus, we are able to study convergence of the viscous approximations $z^\varepsilon\in\mathrm{H}^1([0,T];\boldsymbol{V})$ solving (18) towards BV solutions z. Recall the a priori estimate (19), which allows us to find a limit $z\in\mathrm{BV}([0,T];\boldsymbol{X})\cap\mathrm{L}^\infty([0,T];\boldsymbol{Z})$. This limit passage was used in several applications already, see e.g. [KMZ08, Cag09, DD*08, ToZ09, KZM09]. The aim of this section is to characterize these limits, also called *approximable solutions*, as good as possible. In particular, we need to derive conditions that characterize the jumps occurring in the vanishing-viscosity limit. We follow [MRS09a], where the finite-dimensional case is treated in full detail.

We first motivate our definition of BV solutions by referring to parametrized solutions. Then, we give some more motivation by doing the vanishing-viscosity limit directly in the energetic formulation, which leads to a new central object called *vanishing-viscosity contact potential* $\mathfrak{p}$. It leads to a supplemented dissipation distance Δ in a natural way, which includes the rate-independent contributions to the dissipation as well as the contributions from the vanishing viscosity. The associated dissipation functional $\mathrm{Diss}_{\mathfrak{p},\mathcal{J}}$ then leads to the notion of BV solutions that are defined to satisfy a local stability condition and the energy balance with the new dissipation functional. Finally, we present the convergence results (i) for the vanishing-viscosity limit $z^\varepsilon\to z$ and (ii) for the time-discrete incremental approximations given by

$$z_i^h\in\operatorname*{Arg\,min}_{z\in\boldsymbol{Z}}\Big(\mathcal{I}(hi,z)+\Psi(z-z_{i-1}^h)+\tfrac{\varepsilon}{2h}\|z-z_{i-1}^N\|_{\boldsymbol{V}}^2\Big).$$

In the latter case convergence of subsequences to BV solutions follows if ε, the time-step h, and the quotient h/ε tend to 0.

We start with a few facts about BV spaces. For a Banach space $\boldsymbol{Y}$ we let

$$\mathrm{BV}([a,b];\boldsymbol{Y}) \stackrel{\text{def}}{=} \{\, y : [a,b] \to \boldsymbol{Y} \mid \mathrm{Var}_{\boldsymbol{Y}}(y,[a,b]) < \infty \,\} \text{ with}$$
$$\mathrm{Var}_{\boldsymbol{Y}}(y,[a,b]) = \sup \Big\{ \textstyle\sum_{j=1}^N \|y(t_j) - y(t_{j-1})\|_{\boldsymbol{Y}} \;\Big|\; N \in \mathbb{N}, \; a \le t_0 < t_1 < \dots < t_N \le b \Big\}.$$

As usual in evolutionary problems the functions in $\mathrm{BV}([a,b];\boldsymbol{Y})$ are defined everywhere on $[0,T]$ and the variation is sensitive to changing a function at a single point. Clearly, $\mathrm{BV}([a,b];\boldsymbol{Y})$ is a Banach space equipped with the norm $\|y\|_{\mathrm{BV}([a,b];\boldsymbol{Y})} = \|y(a)\|_{\boldsymbol{Y}} + \mathrm{Var}_{\boldsymbol{Y}}(y,[a,b])$. Moreover, $\mathrm{BV}([a,b];\boldsymbol{Y}) \subset \mathrm{L}^\infty([a,b];\boldsymbol{Y})$ with $\|y(t)\|_{\boldsymbol{Y}} \le \|y\|_{\mathrm{BV}([a,b];\boldsymbol{Y})}$ for all t and

$$\mathrm{Var}_{\boldsymbol{Y}}(y,[t_1,t_2]) + \mathrm{Var}_{\boldsymbol{Y}}(y,[t_2,t_3]) = \mathrm{Var}_{\boldsymbol{Y}}(y,[t_1,t_3]) \text{ for } y \in \mathrm{BV}([t_1,t_3];\boldsymbol{Y}).$$

Finally, for $y \in \mathrm{BV}([a,b];\boldsymbol{Y})$ and each $t \in [a,b]$ the right limit $y(t^+)$ and the left limit $y(t^-)$ (cf. (23)) exist in the strong norm topology of $\boldsymbol{Y}$.

For a given parametrized solution ζ we consider $z \in \mathfrak{P}(\zeta)$, then $\mathrm{Var}_{\boldsymbol{X}}(z,[0,T]) \le C\,\mathrm{Var}_{\boldsymbol{V}}(z,[0,T]) \le C\,\mathrm{Var}_{\boldsymbol{V}}([0,S],Z) \le CS < \infty$, since Z has Lipschitz constant less than 1. For $z \in \mathrm{BV}([0,T];\boldsymbol{X})$ we define the continuity set $C(z)$ and the jump set $J(z)$ of z by

$$C(z) = \{\, t \in [0,T] \mid z(t^-) = z(t) = z(t^+) \,\} \quad \text{and} \quad J(z) = [0,T] \setminus C(z),$$

where left and right limits exist in $\boldsymbol{X}$ and where $J(z)$ is countable.

For each $t \in J(z)$ the monotone function $\tau : [0,S] \to [0,T]$ may have plateaus $[a^t, b^t]$ with $b^t > a^t$, such that $\tau([a^t,b^t]) = \{t\}$. Outside of all these intervals we are either in the sticking regime or in rate-independent slip. Hence, there exists $\xi(s) \in \overline{\partial}_z \mathcal{I}(\zeta(s))$ with $0 \in \partial\Psi(0) + \xi(s)$. Thus, we obtain the local stability condition

$$0 \in \partial\Psi(0) + \overline{\partial}_z \mathcal{I}(t, z(t)) \quad \text{for all } t \in C(z). \tag{36}$$

Using (23) and $\lambda(s) \ge 0$ we easily see that z also satisfies the energy inequality (11b). Thus, we have proved the following result.

Corollary 4.19. *If (τ, Z) is a parametrized solution of $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$, then each $z \in \mathfrak{P}((\tau,Z))$ is a local solution, see* (11).

The important addition to make local solutions into BV solutions is the careful analysis of the jumps. For each plateau $[a^t,b^t] \subset [0,S]$ associated with $t \in J(z)$, we denote by $y^t \in \mathrm{C}^{\mathrm{Lip}}([0,1];\boldsymbol{V})$ the normalized jump curve $y^t(r) = Z(a^t + r(b^t - a^t))$. The point is that each such jump curve is an optimal curve in a specific sense. We first calculate the dissipation $D(t)$ along the jump curve associated with t, namely

$$D(t) = \int_{a^t}^{b^t} \Psi(\dot Z(s)) + M_\Psi^{\boldsymbol{V}}(-\mathrm{D}_z \mathcal{I}(t, Z(s)))\,\mathrm{d}s = \int_0^1 \mathfrak{p}(\dot y^t(r), -\mathrm{D}_z \mathcal{I}(t, y^t(r)))\,\mathrm{d}r$$

where $\mathfrak{p}(v,\xi) = \Psi(v) + \|v\|_{\boldsymbol{V}} M_\Psi^{\boldsymbol{V}}(\xi)$ with $M_\Psi^{\boldsymbol{V}}$ from (26). Here we have used that the arclength parametrization enforced $\|\dot Z(s)\|_{\boldsymbol{V}} = 1$, while the normalized jump curve satisfies $\|\dot y^t(r)\|_{\boldsymbol{V}} = b^t - a^t$. Thus, the factor in front of $M_\Psi^{\boldsymbol{V}}$ appeared because of the reparametrization only. However, the important effect is that the integrand now is 1-homogeneous in v, which reflects the rate independence nicely.

Another way to arrive at the same integrand gives the following definition.

Definition 4.20 (Vanishing-viscosity contact potential). Given a superlinear dissipation potential $\mathcal{R}_{\mathrm{sl}} : \boldsymbol{X} \to [0,\infty]$ we set $\mathcal{R}_\varepsilon(v) = \frac{1}{\varepsilon}\mathcal{R}_{\mathrm{sl}}(\varepsilon v)$. The *vanishing-viscosity contact potential* $\mathfrak{p} : \boldsymbol{X} \times \boldsymbol{X}^* \to \mathbb{R}_\infty$ is defined via

$$\mathfrak{p}(v,\xi) \stackrel{\text{def}}{=} \inf_{\varepsilon>0} \Big(\mathcal{R}_\varepsilon(v) + \mathcal{R}^*_\varepsilon(\xi) \Big) = \inf_{\varepsilon>0} \Big(\tfrac{1}{\varepsilon}\mathcal{R}_{\rm sl}(v) + \tfrac{1}{\varepsilon}\mathcal{R}^*_{\rm sl}(\xi) \Big).$$

The *contact set* $\mathfrak{p}$ is given by $\mathfrak{C}_\mathfrak{p} \stackrel{\text{def}}{=} \{\, (v,\xi) \mid \mathfrak{p}(v,\xi) = \langle \xi, v\rangle \,\}$.

For our choice $\mathcal{R}_{\rm sl}(v) = \Psi(v) + \frac{1}{2}\|v\|^2_{\boldsymbol{V}}$ we have $\mathcal{R}^*_\varepsilon(\xi) = \frac{1}{2\varepsilon} M^{\boldsymbol{V}}_\Psi(\xi)^2$ and find

$$\mathfrak{p}(v,\xi) = \Psi(v) + \|v\|_{\boldsymbol{V}} M^{\boldsymbol{V}}_\Psi(\xi).$$

The motivation for the definition of the vanishing-viscosity contact potential is obtained by the following lower bounds for the dissipation integrals

$$I_\varepsilon = \int_{t_1}^{t_2} \mathcal{R}_\varepsilon(\dot z) + \mathcal{R}^*_\varepsilon(\xi)\,\mathrm{d}t \ge \int_{t_1}^{t_2} \mathfrak{p}(\dot z, -\mathrm{D}\mathcal{I}(t,z(t)))\,\mathrm{d}t.$$

It turns out that this lower bound is sharp in the limit $\varepsilon \to 0$ along the jump curves.

We find the following important properties for $\mathfrak{p}$ and $\mathfrak{C}_\mathfrak{p}$:

rate independence	$\mathfrak{p}(\lambda v,\xi) = \lambda \mathfrak{p}(v,\xi)$,	(37a)
lower bound	$\mathfrak{p}(v,\xi) \ge \langle \xi, v\rangle$,	(37b)
separate convexity	$\mathfrak{p}(\cdot,\xi)$ is convex	(37c)
	and $\mathfrak{p}(v,\cdot)$ has convex sublevels,	
contact set	$(v,\xi) \in \mathfrak{C}_\mathfrak{p} \iff \xi \in \partial_v \mathfrak{p}(v,\xi)$	(37d)
	$\iff \begin{cases} v = 0 \text{ or} \\ v \ne 0 \text{ and } \xi \in \partial\Psi(v) + \frac{M^{\boldsymbol{V}}_\Psi(v)}{\|v\|_{\boldsymbol{V}}}\mathbb{V}v. \end{cases}$	

Using the contact potential $\mathfrak{p}$ we are now able to define a supplemented distance between points z_1 and z_2 involving both the dissipation due to Ψ and the possibly additional dissipation arising from fast viscous transitions:

$$\begin{aligned} \Delta(t,z_1,z_2) \stackrel{\text{def}}{=} \inf \Big\{ \textstyle\int_0^1 \mathfrak{p}(\dot{\widehat z}(r), -\xi(r))\,\mathrm{d}r \;\Big|\; \widehat z \in A_{\boldsymbol{V}}(z_1,z_2), \\ \xi(r) \in \overline{\partial}_z \mathcal{I}(t,\widehat z(r)) \text{ a.e. in } [0,1] \Big\} \end{aligned} \tag{38}$$

$$\text{where } A_{\boldsymbol{Y}}(z_1,z_2) \stackrel{\text{def}}{=} \{\, \widehat z \in \mathrm{C}^{\rm Lip}([0,1];\boldsymbol{Y}) \mid \widehat z(0) = z_1,\ \widehat z(1) = z_2 \,\}.$$

Note that Δ is defined with time t as a frozen parameter. Clearly, we have the triangle inequality $\Delta(t,z_0,z_2) \le \Delta(t,z_0,z_1) + \Delta(t,z_1,z_2)$ and the lower estimate $\Delta(t,z_1,z_2) \ge \Psi(z_2{-}z_1)$.

The crucial observation is that the lower estimate (37b) and the classical chain rule $\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{I}(t,z(r)) = \langle \xi(r), \dot z(r)\rangle$ imply the estimate

$$\mathcal{I}(t,z_2) + \Delta(t,z_1,z_2) \ge \mathcal{I}(t,z_1) \quad \text{for all } z_1,z_2 \in \boldsymbol{Z}. \tag{39}$$

Thus, we define *optimal jump paths* by enforcing equality in this estimate:

$$\begin{aligned} &\mathrm{OJP}(t,z_1,z_2) \stackrel{\text{def}}{=} \\ &\Big\{\, \widehat z \in A_{\boldsymbol{V}}(z_1,z_2) \;\Big|\; \Delta(t,z_1,z_2) = \mathcal{I}(t,z_1){-}\mathcal{I}(t,z_2) = \mathfrak{p}(\dot{\widehat z}(r), -\xi(r)) \\ &\qquad\qquad \text{and } \xi(r) \in \overline{\partial}_z \mathcal{I}(t,\widehat z(r)) \text{ for a.a. } r \in [0,1] \,\Big\}. \end{aligned}$$

Clearly, these equalities imply that a.e. along the whole jump path the lower bound (37b) has to be an equality, i.e., the solution must lie in the contact set, viz. $(\dot z(r), -\xi(r)) \in \mathfrak{C}_\mathfrak{p}$, which is again equivalent to $0 \in \partial\mathfrak{p}(\cdot, -\xi(r))(\dot z(r)) + \xi(r)$ and implies (see (37d))

$$0 \in \partial\Psi(\dot z(r)) + \lambda(r)\mathbb{V}\dot z(r) + \xi(r) \text{ with } \lambda(r) \geq 0 \text{ and } \Psi^\circ(-\xi(r)) \geq 1. \tag{40}$$

For the definition of BV solutions we use the associated supplemented dissipation functional $\mathrm{Diss}_{\mathfrak{p},\mathcal{J}}$ defined on functions $z \in \mathrm{BV}([0,T];\boldsymbol{X}) \cap \mathrm{L}^\infty([0,T];\boldsymbol{Z})$. It takes into account the rate-independent friction via Ψ and the viscous friction at possible jumps:

$$\begin{aligned}
\mathrm{Diss}_{\mathfrak{p},\mathcal{I}}(z,[t_1,t_2]) &= \mathrm{Cont}_\Psi(z,[t_1,t_2]) + \Delta(t_1, z(t_1), z(t_1^+)) + \Delta(t_2, z(t_2^-), z(t_2)) \\
&\quad + \textstyle\sum_{t\in J(z)} \big(\Delta(t, z(t^-), z(t)) + \Delta(t, z(t), z(t^+))\big), \\
\mathrm{Cont}_\Psi(z,[t_1,t_2]) &= \mathrm{Diss}_\Psi(z,[t_1,t_2]) - \Psi(z(t_1^+) - z(t_1)) - \Psi(z(t_2) - z(t_2^-)) \\
&\quad - \textstyle\sum_{t\in J(z)} \big(\Psi(z(t) - z(t^-)) + \Psi(z(t^+) - z(t))\big).
\end{aligned} \tag{41}$$

Thus, $\mathrm{Diss}_{\mathfrak{p},\mathcal{I}}(z,[t_1,t_2])$ consists of the classical dissipation $\mathrm{Diss}_\Psi(z,[t_1,t_2])$ on continuous parts of z, while at jumps the integration of $\mathfrak{p}(\dot z, -\xi) = \Psi(\dot z) + \|\dot z\|_{\boldsymbol{V}} M_\Psi^{\boldsymbol{V}}(-\xi)$ along jump paths contains the rate-independent dissipation (which may be strictly larger than $\Psi(z_2 - z_1)$) and the viscous contributions via $\|\dot z\|_{\boldsymbol{V}} M_\Psi^{\boldsymbol{V}}(-\xi)$.

Definition 4.21 (BV solutions). A function $z \in \mathrm{BV}([0,T];\boldsymbol{X})$ is called a *BV solution* of the RIS $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$, if $z \in \mathrm{L}^\infty([0,T];\boldsymbol{Z})$ and (42) hold:

$$\forall\, t \in C(z):\; z(t) \in \mathcal{S}_{\mathrm{loc}}(t) \stackrel{\mathrm{def}}{=} \{\, z \in \boldsymbol{Z} \mid 0 \in \partial\Psi(0) + \overline{\partial}_z \mathcal{I}(t,z)\,\}; \tag{42a}$$

$$\forall\, t\in[0,T]\colon \mathcal{I}(t,z(t)) + \mathrm{Diss}_{\mathfrak{p},\mathcal{I}}(z,\,[0,t]) = \mathcal{I}(0,z(0)) + \int_0^t \partial_\tau \mathcal{I}(\tau, z(\tau))\,\mathrm{d}\tau. \tag{42b}$$

The function z is called a *connectable BV solution*, if additionally the following holds:

$$\forall\, t \in J(z)\; \exists\, \widehat z^{\,t} \in \mathrm{OJP}(t, z(t^-), z(t^+))\; \exists\, r^t \in [0,1]:\; z(t) = \widehat z^{\,t}(r^t). \tag{42c}$$

Note that the energy inequality (11b) in the definition of local solutions differs from the energy identity (42b) exactly by replacing $\Delta(t, z(t^-), z(t^+))$ by $\Psi(z(t^+) - z(t^-))$. This also allows us to give precise formulae for the energy drop at jump points, which are in full analogy to (25) for energetic solutions:

$$\begin{aligned}
&\mathcal{I}(t, z(t)) + \Delta(t, z(t^-), z(t)) = \mathcal{I}(t, z(t^-)), \\
&\mathcal{I}(t, z(t^+)) + \Delta(t, z(t), z(t^+)) = \mathcal{I}(t, z(t)), \\
&\mathcal{I}(t, z(t^-)) = \lim_{\tau\to t^-} \mathcal{I}(\tau, z(\tau)), \\
&\mathcal{I}(t, z(t^+)) = \lim_{\tau\to t^+} \mathcal{I}(\tau, z(\tau)), \\
&\Delta(t, z(t^-), z(t)) + \Delta(t, z(t), z(t^+)) = \Delta(t, z(t^-), z(t^+)).
\end{aligned} \tag{43}$$

The existence of optimal jump paths is not needed for general BV solutions. If the RIS has the property that the infimum in the definition of $\Delta(t, z_1, z_2)$ is attained for all locally stable $z_1, z_2 \in \boldsymbol{Z}$ (i.e., $0 \in \partial\Psi(0) + \overline{\partial}_z \mathcal{I}(t, z_j)$), then every BV solution is also connectable, as we may concatenate the optimal jump paths from $z(t^-)$ to $z(t)$ and from $z(t)$ to $z(t^+)$ to obtain an optimal jump path from $z(t^-)$ to $z(t^+)$.

The following corollary states that all BV solutions are local solutions, which is an easy consequence of the definitions, and that parametrized solutions give rise to connectable BV solutions. This also provides an existence result by employing Theorem 4.15.

Corollary 4.22. *Let the RIS $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$ be given.*

(A) If $(\tau, Z) \in \mathrm{C}^{\mathrm{Lip}}([0,T]; \boldsymbol{V}_T)$ *is a parametrized solution, then every* $z \in \mathfrak{P}((\tau, Z))$ *is a connectable BV solution.*

(B) Let the RIS $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$ *be the standard Example 2.8 and* $z_0 \in Z_1$*. Then, there exists a connectable BV solution* z *with* $z(0) = z_0$*.*

(C) Every BV solution is a local solution, cf. (11).

We now use the advantage that BV solutions are defined as functions from the time interval $[0, T]$ into the state space $\boldsymbol{Z}$ like the viscous approximations. Thus, the natural question is how the solutions z^ε converge to BV solutions. This question was first answered in [MRS09a] for the finite-dimensional setting. Here we give a similar result for our standard Example 2.8.

Theorem 4.23 (Pointwise convergence to BV solutions). *Let the RIS* $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$ *and the spaces* $\boldsymbol{Z}_1 \Subset \boldsymbol{Z} \Subset \boldsymbol{V} \Subset Z_{-1} \subset \boldsymbol{X}$ *be given as in Example 2.8. Choose any* $z_0 \in \boldsymbol{Z}_1$ *and consider viscous approximations* $z^\varepsilon \in \mathrm{H}^1([0,T]; \boldsymbol{V}) \cap \mathrm{C}_{\mathrm{w}}([0,T]; \boldsymbol{Z})$ *solving* (20) *with* $z^\varepsilon(0) = z_0$*. Then, there exists a subsequence* $(z^{\varepsilon_n})_{n\in\mathbb{N}}$ *with* $\varepsilon_n \to 0$ *and a limit function* $z \in \mathrm{BV}([0,T]; \boldsymbol{V})$ *such that*

$$\begin{aligned} &\forall\, t \in [0,T]: \quad z^{\varepsilon_n}(t) \rightharpoonup z(t) \quad \text{in } \boldsymbol{Z}; \\ &z \text{ is a connectable BV solution to } (\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V}). \end{aligned}$$

Moreover, any pointwise limit z *of a subsequence of* $(z^\varepsilon)_{\varepsilon>0}$ *is a BV solution.*

The last statement shows that in this case *all approximable solutions* are BV solutions.

Proof. The result follows by using the convergence of parametrized solutions once again. Given the sequence z^ε we define the associated parametrized solution $\zeta^\varepsilon \in \mathrm{C}^{\mathrm{Lip}}([0, S^\varepsilon]; \boldsymbol{V}_T)$. As in the proof of Theorem 4.15 we obtain a subsequence $(\zeta^{\varepsilon_j})_{j\in\mathbb{N}}$ that converges pointwise to a limit $\zeta = (\tau, Z)$ (which is possibly not arclength parametrized). For this ζ we consider all $\widetilde{z} : [0,T] \to \boldsymbol{Z}$ with $\widetilde{z} \in \mathfrak{P}(\zeta)$. The value of $\widetilde{z}$ is uniquely defined at all $t \in [0,T] \setminus P$, where the countable set P is the image under τ of the plateaus of τ.

We first claim that $z^{\varepsilon_j}(t_*) \rightharpoonup z(t_*)$ for all $t_* \in [0,T] \setminus P$. To show this we use that τ^{ε_j} converges to τ uniformly on $[0,T]$. Since t_* is not in the image of a plateau of τ, the pseudo-inverse $\sigma : t \mapsto \min\{\, s \mid \tau(s) = t\,\}$ is continuous in t_*. Moreover, there is a unique s_* with $\tau(s_*) = t_*$ and $s_* = \sigma(t_*)$. Thus, for each $\delta > 0$ there exists $\rho_\delta > 0$ such that $|\sigma(t) - \sigma(t_*)| \leq \delta$ for $|t - t_*| \leq \rho_\delta$. In particular, there is a j_δ such that $\|\tau^{\varepsilon_j} - \tau\|_{\mathrm{L}^\infty} \leq \rho_\delta$ for $j \geq j_\delta$. Now choose $s_j \in [0, S]$ such that $\tau^{\varepsilon_j}(s_j) = t_* = \tau(s_*)$, then $|\tau(s_j) - \tau(s_*)| = |\tau(s_j) - \tau^{\varepsilon_j}(s_j) \leq \rho_\delta$ and we obtain $|s_j - s_*| \leq \delta$ for $j \geq j_\delta$. With this we find

$$z(t_*) - z^{\varepsilon_j}(t_*) = Z(\sigma(t_*)) - Z^{\varepsilon_j}(s_*) + Z^{\varepsilon_j}(s_*) - Z^{\varepsilon_j}(s_j) = Z(s_*) - Z^{\varepsilon_j}(s_*) + w_j,$$

where $\|w_j\|_{\boldsymbol{V}} \leq |s_j - s_*| \leq \delta$. Since $Z^{\varepsilon_j}(s_*) \rightharpoonup Z(s_*)$ in $\boldsymbol{Z}$, we obtain $\|z^{\varepsilon_j}(t_*) - z(t_*)\|_{\boldsymbol{V}} \to 0$, and the desired weak convergence follows by the a priori bound in the Hilbert space $\boldsymbol{Z}$.

It remains to establish pointwise convergence in the possible jump points. However, P is countable, so we may choose a further subsequence to obtain weak pointwise convergence for all $t \in [0,T]$. Proceeding as above, it is not difficult to show that the limit $z(t_\circ)$ for $t_\circ \in P$ satisfies $z(t_\circ) = Z(s_\circ)$, where $s_\circ$ lies in a plateau $[a^\circ, b^\circ]$ on which $\tau(s) = t_\circ$ holds. Clearly, the pointwise limit satisfies $z \in \mathrm{BV}([0,T]; \boldsymbol{V})$ and it is a connectable BV solution.

To show that every pointwise limit point must be a BV solution, take any pointwise converging sequence (z^{ε_n}) and denote the limit by z. By choosing a further subsequence as above, we obtain a BV solution $\widehat{z}$ as limit of the further subsequence. Obviously, $z = \widehat{z}$, and we are done. $\square$

We want to emphasize that in general the convergence result may fail in the infinite-dimensional setting. For this we refer to Example 4.4, where the pointwise limit z of the viscous approximations

z^ε is a local solution but not a BV solution. In fact, the solution $z : [0,T] \to \boldsymbol{Z} = \boldsymbol{V}$ is strongly continuous, i.e. there are no jumps, but the energy balance does not hold because of missing viscous contributions. We heal this remedy in the next subsection by introducing the notion of weak BV solutions.

In general it is not clear whether BV solutions can be completed to parametrized solutions. We need an additional condition, which is certainly satisfied in the finite-dimensional setting discussed in [MRS09a], namely

$$\begin{aligned} &\exists\, C>0\ \forall\, t\in[0,T]\ \forall\, z_1,z_2\in\boldsymbol{Z} \text{ with } 0\in\partial\Psi(0)+\overline{\partial}_z\mathcal{I}(t,z_j):\\ &y\in \mathrm{OJP}(t,z_1,z_2) \implies \mathrm{Var}_{\boldsymbol{V}}(y,[0,1])\le C\varDelta(t,z_1,z_2). \end{aligned} \tag{44}$$

We conjecture that this condition also holds for the standard Example 2.8, which is suggested by (33), where the term involving $\dot\ell$ disappears and, at least formally, $\dot z(0)=0$, since z_1 is locally stable.

Proposition 4.24. *If the RIS $(\boldsymbol{Z},\mathcal{I},\Psi,\mathbb{V})$ satisfies (44), then for each connectable BV solution $z\in \mathrm{BV}([0,T];\boldsymbol{V})$ there is a parametrized solution $(\tau,Z)\in \mathrm{C}^{\mathrm{Lip}}([0,T];\boldsymbol{V}_T)$ such that $z\in\mathfrak{P}((\tau,Z))$.*

Proof. The graph of z has finite length $T+\mathrm{Var}_{\boldsymbol{V}}(z,[0,T])$ in $\boldsymbol{V}_T$. For each $t\in J(z)$, we now add the graphs $Y^t \overset{\mathrm{def}}{=} \{\,(t,y^t(r)) \mid r\in[0,1]\,\}$ of the optimal jump paths. By assumption (44) we know that the length of this curve is bounded by $C\varDelta(t,z(t^-),z(t^+))$. Using the energy balance (42b) we see that the total length of all these added curves is finite. By construction $G(z)\cup\bigcup_{t\in J(z)}Y^t$ is a connected curve of finite length in $\boldsymbol{V}_T$ that can be reparametrized with respect to arclength, which provides the function $\zeta=(\tau,Z):[0,S]\to\boldsymbol{V}_T$.

We still have to show that ζ satisfies (21). The conditions in the upper line are trivial. For $s\in[0,S]$ such that $\zeta(s)\in Y^t$ we have $\dot\tau(s)=0$ and the differential inclusion in (21) holds because of (40). In the other case we have local stability, namely $0=\partial\Psi(0)+\xi(s)$ with $\xi(s)\in\overline{\partial}_z\mathcal{I}(\zeta(s))$. However, in these points the energy balance together with the chain rule provides $0=\Psi(\dot Z(s))+\langle\xi(s),\dot Z(s)\rangle$, which implies the desired differential inclusion by Lemma 2.2. □

We conclude this subsection with a result concerning time discretizations. Time-incremental minimization techniques are the central tools in generalized gradient flows as well as for energetic solutions, see Section 3.2. In our vanishing-viscosity approach, we are especially interested in the interaction between the smallness of the time steps and the smallness of the viscosity. It turns out that BV solutions are easily obtained by a joint limit. For simplicity we again study a RIS of the form $(\boldsymbol{Z},\mathcal{I},\Psi,\mathbb{V})$ for small viscosity ε. We also discretize the time interval in the form $\varPi=(t_0,t_1,\dots,t_{N_\varPi})\in\mathrm{Part}([0,T])$ with fineness $\phi(\varPi)=\max\{\,t_k{-}t_{k-1}\mid k=1,\dots,N_\varPi\,\}$, see (8). The incremental minimization problem for the viscous problem reads as follows:

$$z_k\in\mathrm{Arg\,min}\{\,\mathcal{I}(t_k,z)+\Psi(z{-}z_{k-1})+\frac{\varepsilon}{t_k{-}t_{k-1}}\|z{-}z_{k-1}\|_{\boldsymbol{V}}^2 \mid z\in\boldsymbol{Z}\,\}.$$

We denote by $\underline{z}^{\varPi,\varepsilon}:[0,T]\to\boldsymbol{Z}$ the piecewise constant interpolant, see (13). Then the following result was proved in [MRS09a] for the finite-dimensional setting. A corresponding convergence result of incremental problems to parametrized solutions was also obtained in [EfM06]. We expect that a similar result holds in infinite dimensions, in particular for our standard Example 2.8.

Theorem 4.25 (Convergence of viscous time discretizations). *Assume that all Banach spaces are finite-dimensional, that $\mathcal{I}\in\mathrm{C}^1(\boldsymbol{Z}_T)$ and that the RIS $(\boldsymbol{Z},\mathcal{I},\Psi,\mathbb{V})$ satisfies the standard coercivity assumptions. Take any sequence $(\varPi_n)_{n\in\mathbb{N}}$ of partitions and any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of viscosities such that*

$$\phi(\varPi_n)\to0,\quad \varepsilon_n\to0,\quad \frac{\varepsilon_n}{\phi(\varPi_n)}\to\infty. \tag{45}$$

For an initial value $z_0 \in \boldsymbol{Z}$ *construct* $\underline{z}^n \overset{\text{def}}{=} \underline{z}^{\Pi_n,\varepsilon_n} : [0,T] \to \boldsymbol{Z}$. *Then, there exist a subsequence* $(\underline{z}^{n_l})_{l\in N}$ *and a BV solution* z *for* $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$ *such that*

$$\forall\, t \in [0,T] : z^{n_l}(t) \rightharpoonup z(t) \text{ in } \boldsymbol{Z} \text{ for } l \to \infty.$$

Moreover, any pointwise limit of a subsequence of $(\underline{z}^n)_{n\in\mathbb{N}}$ *is a BV solution.*

4.6 Weak BV solutions and time-varying dissipation distances

Here we introduce a slightly more complicated rate-independent dissipation which allows us to define a weaker version of BV solutions. As a generalization to the definition of $\Delta(t, z_0, z_1)$ in (38) we now define a dissipation distance $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}$ in the extended phase space $\boldsymbol{Z}_T$, i.e. $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}} : \boldsymbol{Z}_T \times \boldsymbol{Z}_T \to [0,\infty]$. It measures the minimal dissipation needed for moving from a point z_0 to a point z_1 during the time interval $[t_0, t_1]$. For $t_1 < t_0$ we set $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_0, z_1, t_1, z_1) = \infty$ and for $t_0 < t_1$ we let

$$\begin{aligned} &\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_0, z_0, t_1, z_1) \\ &\overset{\text{def}}{=} \inf \Big\{ \int_0^1 \mathfrak{p}(\dot{Z}(r), -\xi(r))\,\mathrm{d}r \;\Big|\; (\tau, Z) \in A_{\boldsymbol{V}_T}((t_0, z_0), (t_1, z_1)), \\ &\qquad\qquad \dot{\tau}(r) \geq 0 \text{ and } \xi(r) \in \overline{\partial}_z \mathcal{I}(\tau(r), Z(r)) \text{ a.e. in } [0,1] \Big\}. \end{aligned} \tag{46}$$

Clearly, we have $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t, z_0, t, z_1) = \Delta(t, z_0, z_1)$, $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_0, z_0, t_1, z_1) \geq c\Psi(z_1 - z_0)$ and the triangle inequality

$$\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_0, z_0, t_2, z_2) \leq \widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_0, z_0, t_1, z_1) + \widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_1, z_1, t_2, z_2).$$

The associated dissipation functional for curves $z : [0,T] \to \boldsymbol{Z}$ reads

$$\begin{aligned} \mathrm{Diss}^{\mathcal{I}}_{\mathfrak{p}}(z, [r,s]) = \sup \Big\{ &\sum_{k=1}^{N} \widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_{k-1}, z(t_{k-1}), t_k, z(t_k)) \;\Big|\; N \in \mathbb{N}, \\ &r \leq t_0 < t_1 < \cdots < t_{N-1} < t_N \leq s \Big\}. \end{aligned}$$

From the definitions, it is immediate that this dissipation is greater or equal to $\mathrm{Diss}_{\mathfrak{p},\mathcal{I}}$ defined in (41). With the new dissipation functional $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}$ we are now able to define a new notion of BV solution.

Definition 4.26 (Weak BV solutions). Let the RIS $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$ be given. A function $z \in \mathrm{BV}([0,T]; \boldsymbol{X}) \cap \mathrm{L}^\infty([0,T]; \boldsymbol{Z})$ is called *weak BV solution*, if

$$\forall\, t \in C(z) : \; z(t) \in \mathcal{S}_{\mathrm{loc}}(t) \overset{\text{def}}{=} \{\, z \in \boldsymbol{Z} \mid 0 \in \partial\Psi(0) + \overline{\partial}_z \mathcal{I}(t,z) \,\}; \tag{47a}$$

$$\forall\, t \in [0,T] : \mathcal{I}(t, z(t)) + \mathrm{Diss}^{\mathcal{I}}_{\mathfrak{p}}(z, [0,t]) = \mathcal{I}(0, z(0)) + \int_0^t \partial_\tau \mathcal{I}(\tau, z(\tau))\,\mathrm{d}\tau. \tag{47b}$$

The following example shows that $\mathrm{Diss}^{\mathcal{I}}_{\mathfrak{p}}(z, [0,T])$ may be bigger than $\mathrm{Diss}_{\mathfrak{p},\mathcal{I}}(z, [0,T])$ even for continuous curves. Moreover, it states that the vanishing-viscosity limit in Example 4.4 provides a weak BV solution that is not a BV solution.

Example 4.27. We reconsider the setting of Example 4.4 where

$$\boldsymbol{X} = \mathrm{L}^1(\Omega), \quad \boldsymbol{V} = \boldsymbol{Z} = \mathrm{L}^2(\Omega) \text{ with } \Omega =]0,1[\,,$$
$$\mathfrak{p}(v,\xi) = \|v\|_{\mathrm{L}^1} + \|v\|_{\mathrm{L}^2}\|\max\{|\xi(\cdot)|-1,0\}\|_{\mathrm{L}^2}, \quad \mathcal{I}(t,z) = \int_\Omega \mathcal{U}(z) - (t+x)z\,\mathrm{d}x.$$

The vanishing-viscosity limit z obtained there jumps from -2 to 6 along the line $t+x = 3$. It can be shown that for $2 \leq r < t \leq 3$ we have

$$\begin{aligned} 9(t-r) &= \Psi(z(t)-z(r)) = \mathrm{Diss}_{\mathfrak{p},\mathcal{J}}(z,[r,t]) \\ &< \widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(r,z(r),t,z(t)) = \mathrm{Diss}^{\mathcal{I}}_{\mathfrak{p}}(z,[r,t]) = 25(t-r). \end{aligned}$$

We should interpret the additional dissipation as a limiting effect of uncountably many infinitesimal jumps.

Considering the additional term ϱ in the energy balance in Example 4.4, we see that z fails to be a BV solution, however it is a weak BV solution.

The notion of weak BV solutions is close to the notion of energetic solutions, if we replace the standard dissipation distance $\mathcal{D}$ by the extended dissipation distance $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}$. However, global stability with respect to $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}$ is useless, since by construction of $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}$ the points are globally stable. This is in agreement with the fact, that in principle every point z_0 should be possible as an initial condition. Nevertheless, weak BV solutions can be obtained via a similar incremental minimization problem. Given a partition $(0 = t_0, t_1, \ldots, t_N = T)$ and an initial state $z_0 \in \boldsymbol{Z}$, find $z_1, \ldots, z_N$ via

$$z_k \in \mathrm{Arg\,min}\{\, \mathcal{I}(t_k,z) + \widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}(t_{k-1},z_{k-1},t_k,z) \mid z \in \boldsymbol{Z}\,\} \cap \mathcal{S}_{\mathrm{loc}}(t_k).$$

The additional constraint $z_k \in \mathcal{S}_{\mathrm{loc}}(t_k)$ is added to avoid solutions that linger too long in jump paths. We expect that the intersection of Argmin and S_{loc} is always nonempty, since for any point in Argmin we can start an optimal jump path that ends in a point in the intersection and still lies in Argmin because of the definition of $\widehat{\mathcal{D}}^{\mathcal{I}}_{\mathfrak{p}}$.

5 Metric formulations

Here we discuss the notions of parametrized and BV solutions in a metric setting by generalizing the ideas from Banach spaces to general metric spaces. Thus, the theory becomes more general in the sense that, as in Section 3 for energetic solutions, we dispose of the linear structure. However, in contrast to the previous section we have to restrict to the case that the viscous dissipations is proportional to the square of the rate-independent dissipation, i.e., the theory of this section includes all the results of the previous one, if we assume $\mathcal{R}_\varepsilon(v) = \Psi(v) + \frac{\varepsilon}{2}\Psi(v)^2$. Thus, the standard Example 2.8 can not be treated with the methods developed below.

The theory is based on the abstract approach to evolutionary problems in general metric spaces. We refer to [Amb95, AGS05, RMS08] for general reading and to [MRS09b] for more details on the results presented here.

5.1 Metric velocity, slope, and evolution

In this section we recall results presented in [AGS05], starting from a complete metric space $(\mathcal{Z},\mathcal{D})$, where we again use the letter $\mathcal{Z}$ (instead of $\boldsymbol{Z}$) to indicate that $\mathcal{Z}$ does not need a linear structure. Moreover, we now assume that $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty[$ is a true metric distance, i.e. in addition to the assumptions above it is also symmetric and assumes only finite value. (It is expected that the theory can be generalized to extended quasi-distances, cf. [RMS08].)

A curve $z:[0,T]\to\mathcal{Z}$ is called *absolutely continuous* (written as $z\in\mathrm{AC}([0,T];\mathcal{Z})$), if $\exists\ m\in\mathrm{L}^1([0,T])$ such that $\mathcal{D}(q(r),q(t))\le\int_r^t m(s)\,\mathrm{d}s$ for $0\le r<t\le T$.

Theorem 5.1 (Metric velocity). *If* $z\in\mathrm{AC}([0,T],\mathcal{Z})$*, then for a.a.* $t\in[0,T]$ *the* metric velocity $|\dot z|(t)\overset{\text{def}}{=}\lim_{h\to0}\frac1h\mathcal{D}(z(t),z(t{+}h))$ *exists. Moreover,* $|\dot z|(t)\le m(t)$ *a.e. and* $\mathrm{Diss}_{\mathcal{D}}(z,[r,t])=\int_r^t|\dot z|(s)\,\mathrm{d}s$.

The dot $\dot{}$ and the norm $|\cdot|$ in the notation $|\dot z|\in\mathrm{L}^1([0,T])$ are used only to indicate that the metric velocity relates to the norm of a velocity in the classical case. In fact, if $\mathcal{Z}=\boldsymbol{Z}$, $\mathcal{D}$ is defined via a rate-independent dissipation potential $\mathcal{R}$ in the sense of (8), and $z\in\mathrm{W}^{1,1}([0,T),\boldsymbol{Z})$, then $|\dot z|(t)=\mathcal{R}(z(t),\dot z(t))$.

We emphasize that the metric concept is even useful in the case of non-reflexive Banach spaces $\boldsymbol{X}$ which we then equip with the distance induced by the norm. For instance, we may consider $\mathcal{Z}=\mathrm{L}^1(\mathbb{R})$ with $\mathcal{D}(z_0,z_1)=\|z_1{-}z_0\|_{\mathrm{L}^1}$. In Example 4.1 the curve z lies in $z\in\mathrm{AC}([0,T];\mathrm{L}^1(\mathbb{R}))$ but not in $\mathrm{W}^{1,1}([0,T];\mathrm{L}^1(\mathbb{R}))$. The metric velocity exists, namely $|\dot z|(t)=|\dot\alpha(t)|+|\dot\beta(t)|$.

For a functional $\mathcal{J}:\mathcal{Z}\to\mathbb{R}$ we define the *metric slope* $|\partial\mathcal{J}|_*(z)$ of $\mathcal{J}$ in the point z via

$$|\partial\mathcal{J}|_*(z)\overset{\text{def}}{=}\limsup_{\widetilde z\to z}\frac{\max\{\mathcal{J}(z)-\mathcal{J}(\widetilde z),0\}}{\mathcal{D}(z,\widetilde z)}.$$

For functionals $\mathcal{I}:\mathcal{Z}_T\to\mathbb{R}$ we write with a slight abuse of notation

$$|\partial\mathcal{I}|_*(t,z)\overset{\text{def}}{=}|\partial\mathcal{I}(t,\cdot)|_*(z).$$

Again the sign ∂ and the dual norm $|\cdot|_*$ in the notation $|\partial\mathcal{J}|_*$ are formal only and should indicate that in the classical case the metric slope relates to the (dual) norm of (sub)gradient. In fact, if $\mathcal{J}\in\mathrm{C}^1(\mathcal{Z})$ and $\mathcal{D}$ is given via $\mathcal{R}$ as above, then $|\partial\mathcal{J}|_*(z)=\mathcal{R}^\circ(z,-\mathrm{D}\mathcal{J}(z))$, where $\mathcal{R}^\circ(z,\xi)=\sup\{\,\langle\xi,v\rangle\mid\mathcal{R}(z,v)\le1\,\}$.

As a major assumption on our RIS $(\mathcal{Z},\mathcal{I},\mathcal{D})$ we impose the following chain-rule inequality:

$$\begin{aligned}&z{\in}\mathrm{AC}([0,T];\mathcal{Z}),\ t{\mapsto}|\partial\mathcal{I}|_*(t,z(t))\text{ measurable, }\int_0^T|\dot z|(t)|\partial\mathcal{I}|_*(t,z(t))\,\mathrm{d}t<\infty\\ &\Longrightarrow\quad t\mapsto\mathcal{I}(t,z(t))\text{ is absolutely continuous on }[0,T]\text{ and}\\ &\qquad\quad\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z(t))+|\dot z|(t)|\partial\mathcal{I}|_*(t,z(t))\ge\partial_t\mathcal{I}(t,z(t)).\end{aligned}\tag{1}$$

Clearly, such a chain-rule inequality holds in the classical setting since

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z(t))-\partial_t\mathcal{I}(t,z(t))=\langle\mathrm{D}_z\mathcal{I}(t,z),\dot z\rangle\ge-\mathcal{R}(z,\dot z)\,\mathcal{R}^\circ(z,-\mathrm{D}\mathcal{I}(t,z)).$$

For a lower semicontinuous, convex function $\psi:[0,\infty[\ \to[0,\infty[$ we can now define metric solutions to the evolutionary systems $(\mathcal{Z},\mathcal{I},\mathcal{D},\psi)$ in analogy to the energetic formulations in Section 2.5.

Definition 5.2 (Metric evolution, ψ-gradient flow). Let $(\mathcal{Z},\mathcal{I},\mathcal{D})$ satisfy the chain-rule inequality (1). A function $z\in\mathrm{AC}([0,T],\mathcal{Z})$ is called a *metric solution* of the *ψ-gradient flow*, also called a *metric evolutionary system* (in the sense of De Giorgi) $(\mathcal{Z},\mathcal{I},\mathcal{D},\psi)$ if

$$\begin{aligned}&\mathcal{I}(T,z(T))+\textstyle\int_0^T\psi(|\dot z|(t)){+}\psi^*(|\partial\mathcal{I}|_*(t,z(t)))\,\mathrm{d}t\\ &\le\mathcal{I}(0,z(0))+\textstyle\int_0^T\partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t.\end{aligned}\tag{2}$$

As in Section 2.5 the chain-rule inequality implies that (2) is equivalent to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z(t)) + \psi(|\dot z|(t)) + \psi^*(|\partial\mathcal{I}|_*(t,z(t))) = \partial_t\mathcal{I}(t,z(t)) \quad \text{a.e. on } [0,T].$$

So far, the theory of metric evolutionary systems relies heavily on the absolute continuity of z. In the case of RIS we would like to choose $\psi(\nu)=\nu$, which led to $\psi^*(\xi)=\chi_{[0,1]}(\xi)$. Then, any approximation procedure leads to a priori estimates for $|\dot z|$ in $\mathrm{L}^1([0,T])$ only, which would not be enough to pass to the limit since jumps may develop.

5.2 Parametrized metric solutions

As in the Banach-space setting we introduce viscous approximations via $\psi_\varepsilon(\nu)=\nu+\frac{\varepsilon}{2}\nu^2$. The Legendre transform is $\psi_\varepsilon^*(\xi)=\frac{1}{2\varepsilon}\max\{\xi{-}1,0\}^2$. The associated metric evolutionary system reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t,z^\varepsilon)+\psi_\varepsilon(|\dot z^\varepsilon|(t))+\psi_\varepsilon^*(|\partial\mathcal{I}|_*(t,z^\varepsilon(t))) \le \partial_t\mathcal{I}(t,z^\varepsilon(t)), \quad z^\varepsilon(0)=z_0. \tag{3}$$

Using the standard energy bounds we find the a priori estimates

$$\mathcal{I}(t,z^\varepsilon(t)) \le \mathrm{e}^{Ct}\mathcal{I}(0,z_0), \qquad \int_0^T \psi_\varepsilon(|\dot z^\varepsilon|(t))+\psi_\varepsilon^*(|\partial\mathcal{I}|_*(t,z^\varepsilon(t)))\,\mathrm{d}t \le \mathrm{e}^{CT}\mathcal{I}(0,z_0).$$

The classical existence theory in [Amb95, AGS05] works for the cases $\psi(\nu)=\nu^p/p$, whereas [RMS08] treats the case of general convex and superlinear ψ, covering our case $\psi_\varepsilon(\nu)=\nu+\nu^2/\varepsilon$. Thus, for each $\varepsilon>0$ we obtain a solution $z^\varepsilon\in\mathrm{AC}([0,T];\mathcal{Z})$ with the estimates

$$\||\dot z^\varepsilon|\|_{\mathrm{L}^1([0,T])} \le C \quad \text{and} \quad \||\dot z^\varepsilon|\|_{\mathrm{L}^2([0,T])} \le C/\sqrt{\varepsilon}. \tag{4}$$

As in Section 4.4 we introduce an arclength parametrization of the graph of z^ε, namely $\{\,(t,z^\varepsilon(t)) \mid t\in[0,T]\,\}\subset\mathcal{Z}_T$ via $s^\varepsilon(t)=t+\int_0^t|\dot z^\varepsilon|(r)\,\mathrm{d}r$ and the inverse functions $t=\tau^\varepsilon(s)$. The good message here is that we immediately have an upper bound for the total length S^ε by using (4). Thus, we define the arclength parametrized solutions as

$$\zeta^\varepsilon(s)=(\tau^\varepsilon(s),Z^\varepsilon(s))\in\mathcal{Z}_T \ \text{ with } Z^\varepsilon(s)=z^\varepsilon(\tau^\varepsilon(s)).$$

We introduce the function

$$M_\varepsilon(\alpha,\nu,\xi)=\begin{cases}\alpha\psi_\varepsilon(\nu/\alpha)+\alpha\psi_\varepsilon^*(\xi) & \text{for } \alpha>0,\\ \infty & \text{otherwise,}\end{cases}$$

which explicitly means $M_\varepsilon(\alpha,\nu,\xi)=\nu+\frac{\varepsilon}{2\alpha}\nu^2+\frac{\alpha}{2\varepsilon}\max\{\xi{-}1,0\}^2$ for $\alpha>0$.

The rescaling of (3) leads to the relations

$$\left.\begin{array}{l}\tau^\varepsilon(0)=0,\ \tau^\varepsilon(S^\varepsilon)=T,\ \dot\tau^\varepsilon(s)\ge 0,\ \dot\tau^\varepsilon(s)+|\dot Z^\varepsilon|(s)=1,\\ \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\zeta^\varepsilon(s))+M_\varepsilon(\dot\tau^\varepsilon(s),|\dot Z^\varepsilon|(s),|\partial\mathcal{I}|_*(\zeta^\varepsilon(s)))\le\partial_t\mathcal{I}(\zeta^\varepsilon(s))\dot\tau^\varepsilon(s)\end{array}\right\} \tag{5}$$

a.e. on $[0,S^\varepsilon]$. On $[0,\infty[^3$ it is easy to see that the functions M_ε converge to

$$M_0:(\alpha,\nu,\xi)\mapsto\begin{cases}\nu\max\{\xi,1\} & \text{for } \alpha=0,\\ \nu+\chi_{[0,1]}(\xi) & \text{for } \alpha>0,\end{cases}$$

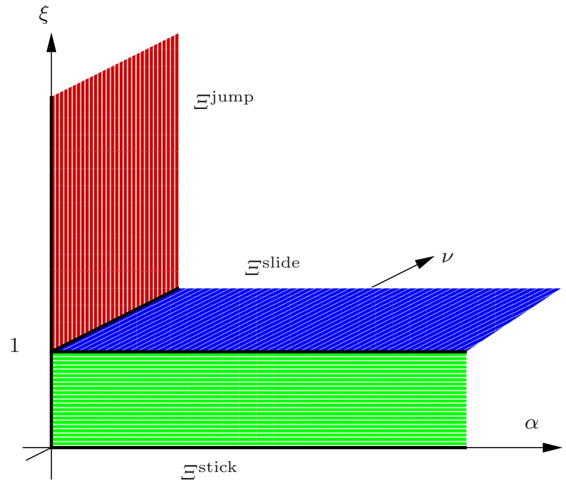

Figure 1 The three different regimes for parametrized metric solutions.

in the sense of Γ-convergence. In fact, we have the following liminf estimate for the associated functionals, which is an application of Ioffe's theorem [Iof77], see [MRS09b, Lem. 3.1]. It uses convexity in (α, ν) and monotonicity in ξ.

Proposition 5.3 (Lower semicontinuity).
Let $\mathcal{M}_\varepsilon(\alpha,\nu,\xi) = \int_0^S M_\varepsilon(\alpha(s),\nu(s),\xi(s))\,\mathrm{d}s$ *for* $\varepsilon \geq 0$ *and assume that the sequence* $((\alpha^\varepsilon,\nu^\varepsilon,\xi^\varepsilon))_{\varepsilon>0}$ *satisfies* $\alpha^\varepsilon \rightharpoonup \widehat{\alpha}$ *and* $\nu^\varepsilon \rightharpoonup \widehat{\nu}$ *in* $\mathrm{L}^1([0,S])$ *and* $\widehat{\xi}(s) \leq \liminf_{\varepsilon\to 0^+} \xi^\varepsilon(s)$ *a.e. on* $[0,S]$*, then we have* $\mathcal{M}_0(\widehat{\alpha},\widehat{\nu},\widehat{\xi}) \leq \liminf_{\varepsilon\to 0} \mathcal{M}_\varepsilon(\alpha^\varepsilon,\nu^\varepsilon,\xi^\varepsilon)$.

We now define parametrized metric solutions by using the functions M_0 in place of the small viscosity functions M_ε. Without loss of generality we only consider arclength parametrized solutions.

Definition 5.4 (Parametrized metric solutions). A function $\zeta = (\tau, Z) : [0,S] \to \mathcal{Z}_T$ is called *parametrized metric solution* of the RIS $(\mathcal{Z},\mathcal{I},\mathcal{D})$, if $\zeta \in \mathrm{C}^{\mathrm{Lip}}([0,S];\mathcal{Z}_T)$ and if for a.a. $s \in [0,S]$ we have

$$\begin{aligned} &\tau(0)=0,\ \ \tau(S)=T,\ \ \dot\tau(s)\geq 0,\ \ \dot\tau(s)+|\dot Z|(s)=1,\\ &\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\zeta(s)) + M_0(\dot\tau(s),|\dot Z|(s),|\partial\mathcal{I}|_*(\zeta(s)) \leq \partial_t\mathcal{I}(\zeta(s))\dot\tau(s). \end{aligned} \tag{6}$$

Since $M_0(\alpha,\nu,\xi) \geq \nu\xi$, the *contact set* Ξ of M_0 plays a central role:

$$\Xi \overset{\text{def}}{=} \{\, (\alpha,\nu,\xi) \mid M_0(\alpha,\nu,\xi) = \nu\xi \,\}$$

The explicit form of M_0 gives three distinct regimes: $\Xi = \Xi^{\text{stick}} \cup \Xi^{\text{slide}} \cup \Xi^{\text{jump}}$ with

$$\begin{aligned} \Xi^{\text{stick}} &= \{\, (\alpha,0,\xi) \mid \alpha \geq 0,\ \xi \leq 1 \,\},\\ \Xi^{\text{slide}} &= \{\, (\alpha,\nu,1) \mid \alpha,\nu \geq 0 \,\},\\ \Xi^{\text{jump}} &= \{\, (0,\nu,\xi) \mid \nu \geq 0,\ \xi \geq 1 \,\}, \end{aligned}$$

see Figure 1. Using the chain-rule inequality (1) and the last line of (6) we conclude that for a parametrized metric solution we have

$$(\dot\tau(s), |\dot Z|(s), |\partial\mathcal{I}|_*(\zeta(s))) \in \Xi \quad \text{for a.a. } s \in [0,S].$$

This leads to an alternative equivalent definition for parametrized metric solutions, which highlights these different regimes more clearly.

Lemma 5.5. *A function $\zeta = (\tau, Z) : [0, S] \to \mathcal{Z}_T$ is a parametrized metric solution of the RIS $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ if and only if for a.a. $s \in [0, S]$ we have*

$$\begin{aligned}
&\dot\tau(s) \geq 0 \quad \textit{and} \quad \dot\tau(s) + |\dot Z|(s) = 1;\\
&\dot\tau(s) > 0 \implies |\partial\mathcal{I}|_*(\zeta(s)) \leq 1;\\
&|\dot Z|(s) > 0 \implies |\partial\mathcal{I}|_*(\zeta(s)) \geq 1;\\
&\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\zeta(s)) + |\dot Z|(s)|\partial\mathcal{I}|_*(\zeta(s)) = \partial_s\mathcal{I}(\zeta(s))\dot\tau(s).
\end{aligned}$$

The last condition in the above lemma says that if a solution moves, then, it moves along a gradient flow curve. Moreover, integrating the last relation we obtain the energy balance

$$\begin{aligned}
&\mathcal{I}(\zeta(s)) + \int_0^s |\dot Z|(s) + \gamma(s)\,\mathrm{d}s = \mathcal{I}(0, z_0) + \int_0^s \partial_r\mathcal{I}(\zeta(r))\,\mathrm{d}r\\
&\text{with } \gamma(s) = |\dot Z|(s)\max\{|\partial\mathcal{I}|_*(\zeta(s)){-}1, 0\},
\end{aligned} \tag{7}$$

which is analogous to (23) for parametrized solutions in the Banach-space setting. Thus, we again see an explicit term arising in jump paths that is needed to lead to a correct energy balance.

The next result shows that under quite general assumptions we have convergence of parametrized solutions in the vanishing-viscosity limit and thus obtain existence of parametrized metric solutions. In contrast to the Banach-space setting we have much more general initial conditions and we are able to show that the arclength parametrization is inherited by the limit. In the additional conditions on $\mathcal{I}$ we use the topology on $\mathcal{Z}$ that is induced by the metric $\mathcal{D}$.

Theorem 5.6 (Vanishing viscosity, parametrized metric solutions). *Let the RIS $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ satisfy the assumptions from above and*

$$\begin{aligned}
&\mathcal{I}(t, \cdot) : \mathcal{Z} \to \mathbb{R} \textit{ has sequentially compact sublevels},\\
&\partial_t\mathcal{I} : \mathcal{Z}_T \to \mathbb{R} \textit{ is continuous},\\
&|\partial\mathcal{I}|_* : \mathcal{Z}_T \to [0, \infty] \textit{ is lower semicontinuous.}
\end{aligned} \tag{8}$$

Choose any $z_0 \in \mathcal{Z}$. Then, for any family of parametrized metric solutions $\zeta^\varepsilon : [0, S] \to \mathcal{Z}_T$ of (5) with $\zeta^\varepsilon(0) = (0, z_0)$ there exist a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ and a limit function $\zeta = (\tau, Z) \in \mathrm{AC}([0, S]; \mathcal{Z}_T)$ such that

$$\begin{aligned}
&\zeta \textit{ is a parametrized metric solution with } \zeta(0) = (0, z_0),\\
&\zeta^{\varepsilon_k} \to \zeta \textit{ in } \mathrm{C}^0([0, T]; \mathcal{Z}_T);\\
&\dot\tau^{\varepsilon_k} \overset{*}{\rightharpoonup} \dot\tau,\ |\dot Z^{\varepsilon_k}| \overset{*}{\rightharpoonup} |\dot Z| \textit{ in } \mathrm{L}^\infty([0, S]).
\end{aligned}$$

Proof. Since the functions ζ^ε have the uniform Lipschitz bound 1 and all ζ^ε lie in a sequentially compact sublevel of $\mathcal{I}$, we can apply the Arzelà-Ascoli theorem and obtain a sequence converging uniformly. Since $\dot\tau^\varepsilon$ and $|\dot Z^{\varepsilon_k}|$ are bounded by 1, we may also assume the weak* convergence to limits μ and η, respectively. Obviously, $\mu{+}\eta \equiv 1$ and it is easy to see that $\mu = \dot\tau$. From

$$\begin{aligned}
\mathcal{D}(Z(s_1), Z(s_2)) &= \lim\nolimits_k \mathcal{D}(Z^{\varepsilon_k}(s_1), Z^{\varepsilon_k}(s_2))\\
&\leq \liminf\nolimits_k \textstyle\int_{s_1}^{s_2} |\dot Z^{\varepsilon_k}|(s)\,\mathrm{d}s = \int_{s_1}^{s_2} \eta(s)\,\mathrm{d}s,
\end{aligned}$$

we find $|\dot Z|(s) \leq \eta(s)$ a.e. on $[0, S]$. From the assumptions in (8) we obtain

$$\mathcal{I}(\zeta(s)) \leq \liminf_k \mathcal{I}(\zeta^{\varepsilon_k}(s)),\ \ \partial_t\mathcal{I}(\zeta(s)) = \lim_k \partial_t\mathcal{I}(\zeta^{\varepsilon_k}(s)),$$
$$\xi(s) = |\partial\mathcal{I}|_*(\zeta(s)) \leq \liminf_k \xi^k(s),\ \text{ where } \xi^k(s) = |\partial\mathcal{I}|_*(\zeta^{\varepsilon_k}(s)).$$

We can now pass to the limit in the integrated version of the last line in (5) and, using Proposition 5.3, we obtain

$$\mathcal{I}(\zeta(S)) + \mathcal{M}_0(\dot\tau, \eta, |\partial\mathcal{I}|_*(\zeta(\cdot))) \leq \mathcal{I}(0, z_0) + \int_0^S \partial_t\mathcal{I}(\zeta(s))\dot\tau(s)\,\mathrm{d}s. \tag{9}$$

For the convergence in the last integral note that the integrand is the product of a strongly and a weakly converging sequence.

In the following estimates we first use the chain-rule-inequality (1), then $\nu\xi \leq M_0(\alpha,\nu,\xi)$, next $|\dot Z| \leq \eta$ and monotonicity of $M_0(\alpha,\cdot,\xi)$, and finally (9):

$$\begin{aligned}
&\mathcal{I}(\zeta(0)) - \mathcal{I}(\zeta(S)) + \textstyle\int_0^S \partial_t\mathcal{I}(\zeta(s))\dot\tau(s)\,\mathrm{d}s \leq \int_0^S |\dot Z|(s)|\partial\mathcal{I}|_*(\zeta(s))\,\mathrm{d}s\\
&\leq \mathcal{M}_0(\dot\tau, |\dot Z|, |\partial\mathcal{I}|_*(\zeta(\cdot))) \leq \mathcal{M}_0(\dot\tau, \eta, |\partial\mathcal{I}|_*(\zeta(\cdot)))\\
&\leq \mathcal{I}(\zeta(0)) - \mathcal{I}(\zeta(S)) + \int_0^S \partial_t\mathcal{I}(\zeta(s))\dot\tau(s)\,\mathrm{d}s.
\end{aligned}$$

We conclude that all estimates are equalities. In particular, we find the second line of (6), and the strict monotonicity of $M_0(\alpha,\cdot,\xi)$ implies $|\dot Z|(s) = \eta(s)$ a.e., which implies $\dot\tau + |\dot Z|(s) = 1$ a.e. $\square$

5.3 Metric BV solutions

Like in the Banach-space setting it is desirable to define a solution concept that is closely associated with parametrized solutions, but that avoids the artificial arclength parametrization. Our definition follows the spirit of Section 4.5, which is simpler than the original approach in [MRS09b]. The function space $\mathrm{BV}([0,T];\mathcal{Z})$ contains functions $z : [0,T] \to \mathcal{Z}$ defined everywhere and such that $\mathrm{Var}_{\mathcal{D}}(z,[0,T]) = \mathrm{Diss}_{\mathcal{D}}(z,[0,T]) < \infty$. As before, left and right limits $z(t^-)$ and $z(t^+)$ exist for each $t \in [0,T]$. Moreover, the set of jump times $J(z)$ is well defined and countable.

We again define a supplemented dissipation measure $\varDelta_M(t,\cdot) : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ via

$$\varDelta_M(t,z_0,z_1) \overset{\mathrm{def}}{=} \inf\{\, \textstyle\int_0^1 M_0(0, |\dot{\widehat z}|(r), |\partial\mathcal{I}|_*(t,\widehat z(r)))\,\mathrm{d}r \mid \widehat z \in A(z_0,z_1)\,\},$$
$$\text{where } A(z_0,z_1) \overset{\mathrm{def}}{=} \{\, \widehat z \in \mathrm{AC}([0,1];\mathcal{Z}) \mid \widehat z(0) = z_0,\ \widehat z(1) = z_1 \,\}.$$

Definition 5.7 (Metric BV solutions). Let the metric RIS $(\mathcal{Z},\mathcal{I},\mathcal{D})$ be given. Then a function $z \in \mathrm{BV}([0,T];\mathcal{Z})$ is called *metric BV solution* if (10a) and (10b) hold:

$$\forall\, t \in C(z):\ \ |\partial\mathcal{I}|_*(t,z(t)) \leq 1; \tag{10a}$$
$$\forall\, t{\in}[0,T] : \mathcal{I}(t,z(t)){+}\mathrm{Diss}_{\mathcal{D},\mathcal{I}}(z,[0,t]) = \mathcal{I}(0,z(0)){+}\int_0^t \partial_\tau\mathcal{I}(\tau,z(\tau))\,\mathrm{d}\tau. \tag{10b}$$

The metric BV solution z is called *connectable*, if additionally

$$\forall\, t \in J(z)\ \exists\, \widehat z^{\,t} \in \mathrm{OJP}(t,z(t^-),z(t^+))\ \exists\, r^t \in [0,1]:\ \ z(t) = \widehat z^{\,t}(r^t). \tag{10c}$$

Again the set of optimal jump paths is defined via $\mathrm{OJP}(t,z_1,z_2) \overset{\mathrm{def}}{=}$

$$\big\{ z \in A(z_1,z_2) \;\big|\; \Delta_M(t,z_1,z_2) = \mathcal{I}(t,z_1) - \mathcal{I}(t,z_2) \\ \qquad\qquad = M_0(0, |\dot z|(r), |\partial\mathcal{I}|_*(t,z(r))) \text{ for a.a. } r \in [0,1] \,\big\}.$$

In [MRS09b] the following two results concerning existence, convergence and time discretization are derived.

Theorem 5.8 (Convergence and existence of metric BV solutions). *Assume that* $(\mathcal{Z},\mathcal{I},\mathcal{D})$ *satisfies the assumption of Theorem 5.6. Then, for each* $z_0 \in \mathcal{Z}$ *with* $\mathcal{I}(0,z_0) < \infty$ *there exists a metric BV solution* z *with* $z(0) = z_0$*.*

Moreover, if $z^\varepsilon \in \mathrm{AC}([0,T];\mathcal{Z})$ *are solutions of the viscous metric evolutionary system* (3) *with* $z^\varepsilon(0) = z_0$*, then there exist a subsequence* $\varepsilon_k \to 0$ *and a metric BV solution* z *such that* $z^{\varepsilon_k}(t) \to z(t)$ *for all* $t \in [0,T]$*.*

For a partition $\Pi = (0 = t_0, t_1, \ldots, t_N = T)$, the time-incremental minimization problems with small viscosity read

$$z_j \in \operatorname{Arg\,min}\{\, \mathcal{I}(t_k,z) + \mathcal{D}(z_{k-1},z) + \tfrac{\varepsilon}{2(t_k - t_{k-1})}\mathcal{D}(z_{k-1},z)^2 \mid z \in \mathcal{Z} \,\}.$$

Defining the piecewise constant interpolants $\underline{z}^{\Pi_\varepsilon}$ as before, we obtain the following result.

Theorem 5.9 (Convergence of time discretizations). *Assume that the RIS* $(\mathcal{Z},\mathcal{I},\mathcal{D})$ *satisfies the assumption of Theorem 5.6. Assume further that for sequence* Π_k *and* ε_k *we have*

$$\varepsilon_k \to 0, \quad \phi(\Pi_k) \to 0, \quad \varepsilon_k/\phi(\Pi_k) \to \infty.$$

Then there exist a subsequence (not relabeled) and a metric BV solution z *such that* $\underline{z}^{\Pi_k,\varepsilon_k}(t) \to z(t)$ *for all* $t \in [0,T]$*.*

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