Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

BV solutions and viscosity approximations of rate-independent systems

Alexander Mielke [1,2], Riccarda Rossi [3], and Giuseppe Savaré[4]

submitted: October 16, 2009

[1] Weierstraß-Institut
für Angewandte Analysis und Stochastik
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: mielke@wias-berlin.de

[2] Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin-Adlershof
Germany

[3] Dipartimento di Matematica
Università di Brescia
Via Valotti 9
25133 Brescia
Italy
E-Mail: riccarda.rossi@ing.unibs.it

[4] Dipartimento di Matematica "F. Casorati"
Università di Pavia
Via Ferrata 1
27100 Pavia
Italy
E-Mail: giuseppe.savare@unipv.it

No. 1451
Berlin 2009

2000 *Mathematics Subject Classification.* 49Q20, 58E99.

Key words and phrases. Doubly nonlinear differential inclusions, generalized gradient flows, viscous regularization, vanishing-viscosity limit, vanishing-viscosity contact potential, parametrized solutions.

A.M. has been partially supported by DFG, Research Unit FOR 797 "MicroPlast", Mi 459/5-1. R.R. and G.S. have been partially supported by a MIUR-PRIN'06 grant for the project "Variational methods in optimal mass transportation and in geometric measure theory".

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. In the nonconvex case solutions of rate-independent systems may develop jumps as a function of time. To model such jumps, we adopt the philosophy that rate independence should be considered as limit of systems with smaller and smaller viscosity. For the finite-dimensional case we study the vanishing-viscosity limit of doubly nonlinear equations given in terms of a differentiable energy functional and a dissipation potential which is a viscous regularization of a given rate-independent dissipation potential.

The resulting definition of 'BV solutions' involves, in a nontrivial way, both the rate-independent and the viscous dissipation potential, which play a crucial role in the description of the associated jump trajectories.

We shall prove a general convergence result for the time-continuous and for the time-discretized viscous approximations and establish various properties of the limiting BV solutions. In particular, we shall provide a careful description of the jumps and compare the new notion of solutions with the related concepts of energetic and local solutions to rate-independent systems.

1. INTRODUCTION

Rate-independent evolutions occur in several contexts. We refer the reader to [32] and the forthcoming monograph [39] for a survey of rate-independent modeling and analysis in a wide variety of applications, which may pertain to very different and far-apart branches of mechanics and physics. Rate-independent systems present very distinctive common features, because of their hysteretic character [54, 24]. Driven by external loadings on a time scale much slower than their internal scale, such systems respond to changes in the external actions invariantly for time-rescalings. Thus, they in fact show (almost) no intrinsic time-scale. This kind of behavior is encoded in the simplest, but still significant, example of rate-independent evolution, namely the doubly nonlinear differential inclusion

$$\partial\Psi_0(u'(t)) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \quad \text{in } X^* \quad \text{for a.a. } t \in (0,T). \tag{DN$_0$}$$

For the sake of simplicity, we will consider here the case when X is a finite dimensional linear space, $\mathcal{E} : [0,T] \times X \to \mathbb{R}$ an energy functional ($\mathrm{D}\mathcal{E}$ denoting the differential of $\mathcal{E}$ with respect to the variable $u \in X$), and $\Psi_0 : X \to [0,+\infty)$ is a convex, nondegenerate, dissipation potential, hereafter supposed *positively homogeneous of degree* 1. Thus, (DN$_0$) is invariant for time-rescalings, rendering the system rate independence.

Since the range K^* of $\partial\Psi_0$ is a proper subset of X^*, when $\mathcal{E}(t,\cdot)$ is not strictly convex one cannot expect the existence of an absolutely continuous solution of (DN$_0$). Over the past decade, this fact has motivated the development of suitable notions of weak solutions to (DN$_0$). In the mainstream of [18, 35, 44], the present paper aims to contribute to this issue. Relying on the *vanishing-viscosity approach*, we shall propose the notion of BV solution to (DN$_0$) and thoroughly analyze it.

To better motivate the use of vanishing viscosity and highlight the features of the concept of BV solution, in the next paragraphs we shall briefly recall the other main weak solvability notions for (DN$_0$). For the sake of simplicity, we shall focus on the particular case

$$\Psi_0(v) = \|v\|, \qquad \text{for some norm } \|\cdot\| \text{ on } X. \tag{1.1}$$

Energetic and local solutions. The first attempt at a rigorous weak formulation of (DN_0) goes back to [40] and the subsequent [42, 41], which advanced the notion of *global energetic solution* to the rate-independent system (DN_0). In the simplified case (1.1), this solution concept consists of the following relations, holding for all $t \in [0,T]$:

$$\text{(S)} \qquad \forall\, z \in X: \qquad \mathcal{E}_t(u(t)) \leq \mathcal{E}_t(z) + \|z - u(t)\|,$$

$$\text{(E)} \qquad \mathcal{E}_t(u(t)) + \mathrm{Var}(u;[0,t]) = \mathcal{E}_0(u(0)) + \int_0^t \partial_t \mathcal{E}_s(u(s))\,\mathrm{d}s\,.$$

The energy identity (E) balances at every time $t \in [0,T]$ the dissipated energy $\mathrm{Var}(u;[0,t])$ (the latter symbol denotes the total variation of the solution $u \in \mathrm{BV}([0,T];X)$ on the interval $[0,t]$), with the stored energy $\mathcal{E}_t(u(t))$, the initial energy, and the work of the external forces. On the other hand, (S) is a stability condition, for it asserts that the change from the current state $u(t)$ to another state z brings about a gain of potential energy smaller than the dissipated energy. Since the competitors for $u(t)$ range in the whole space X, (S) is in fact a *global stability* condition.

The global energetic formulation (S)–(E) only involves the (assumedly smooth) power of the external forces $\partial_t\mathcal{E}$, and is otherwise *derivative-free.* Thus, it is well suited to jumping solutions. Furthermore, as shown in [27, 32], it is amenable to analysis in very general ambient spaces, even with no underlying linear structure. Because of its flexibility, this concept has been exploited in a variety of applicative contexts, like, for instance, shape memory alloys [42, 37, 5], crack propagation [15, 14, 17], elastoplasticity [29, 30, 31, 20, 10, 11, 28], damage in brittle materials [38, 6, 52, 33], delamination [23], ferroelectricity [43], and superconductivity [50].

On the other hand, in the case of nonconvex energies condition (S) turns out to be a strong requirement, for it may lead the system to change instantaneously in a very drastic way, jumping into very far-apart energetic configurations (see, for instance, [30, Ex. 6.1], [21, Ex. 6.3], and [35, Ex. 1]). On the discrete level, global stability is reflected in the global minimization scheme giving raise to approximate solutions by time-discretization. Indeed, for a fixed time-step $\tau > 0$, inducing a partition $\{0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T\}$ of the interval $[0,T]$, one constructs *discrete solutions* $(\mathrm{U}^n_\tau)_{n=1}^N$ of (S)–(E) by setting $\mathrm{U}^0_\tau := u_0$ and then solving recursively the variational incremental scheme

$$(\mathrm{IP}_0) \qquad U^n_\tau \in \operatorname*{Argmin}_{\mathrm{U}\in X} \left\{ \|\mathrm{U} - \mathrm{U}^{n-1}_\tau\| + \mathcal{E}_{t_n}(\mathrm{U}) \right\} \qquad \text{for } n = 1,\ldots,N.$$

However, a scheme based on *local minimization* would be preferable, both in view of numerical analysis and from a modeling perspective, see the discussions in [30, Sec. 6] and, in the realm of crack propagation, [16, 45, 26].

As pointed out in [16], local minimization may be enforced by perturbing the variational scheme (IP_0) with a term, modulated by a *viscosity* parameter ε, which penalizes the squared distance from the previous step $\mathrm{U}^{n-1}_{\tau,\varepsilon}$

$$(\mathrm{IP}_\varepsilon) \qquad \mathrm{U}^n_{\tau,\varepsilon} \in \operatorname*{Argmin}_{\mathrm{U}\in X} \left\{ \|\mathrm{U} - \mathrm{U}^{n-1}_{\tau,\varepsilon}\| + \varepsilon \frac{|\mathrm{U} - \mathrm{U}^{n-1}_{\tau,\varepsilon}|^2}{\tau} + \mathcal{E}_{t_n}(\mathrm{U}) \right\} \quad \text{for } n = 1,\ldots,N\,,$$

and depends on a second norm $|\cdot|$, typically Hilbertian, on the space X. In a infinite-dimensional setting, one may think of $X = L^2(\Omega)$, with Ω a domain in $\mathbb{R}^d$, $d \geq 1$, and $\|\cdot\|$, $|\cdot|$ the L^1 and L^2 norms, respectively. Notice that, on the time-continuous level, (IP_0) corresponds to the *viscous doubly nonlinear* equation

$$(\mathrm{DN}_\varepsilon) \qquad \begin{aligned} &\partial\Psi_\varepsilon(u_\varepsilon'(t)) + \mathrm{D}\mathcal{E}_t(u_\varepsilon(t)) \ni 0 \quad \text{in } X^* \quad \text{for a.a. } t \in (0,T), \\ &\text{with} \quad \Psi_\varepsilon(v) = \|v\| + \frac{\varepsilon}{2}|v|^2 \end{aligned}$$

(see [9, 8] for the existence of solutions $u_\varepsilon \in \mathrm{AC}([0,T];X)$). Then, the idea would be to consider the solutions to (DN_0) arising in the passage to the limit, in the discrete scheme $(\mathrm{IP}_\varepsilon)$, as ε and τ tend to 0 *simultaneously*, keeping $\tau \ll \varepsilon$. One can guess that, at least formally, this procedure should be equivalent to considering the limit of the solutions to $(\mathrm{DN}_\varepsilon)$ as $\varepsilon \downarrow 0$.

Vanishing viscosity has by now become an established selection criterion for mechanically feasible weak solvability notions of rate-independent evolutions. We refer the reader to [25] for rate-independent problems with convex energies and discontinuous inputs, and, in more specific applied contexts, to [12] for elasto-plasticity with softening, to [19] for general material models with nonconvex elastic energies, the recent [13] for cam-clay non-associative plasticity, and [53, 21, 22] for crack propagation. Since the energy functionals involved in such applications are usually nonsmooth and nonconvex, the passage to the limit mostly relies on lower semicontinuity arguments. Let us illustrate the latter in the prototypical case $(\mathrm{DN}_\varepsilon)$. The key observation is that $(\mathrm{DN}_\varepsilon)$ is equivalent (see the discussion in Section 2.4) to the ε-*energy identity*

$$(1.2) \qquad \begin{aligned} \mathcal{E}_t(u_\varepsilon(t)) + \int_0^t \Big(\|u_\varepsilon'(s)\|\,\mathrm{d}s + \frac{\varepsilon}{2}|u_\varepsilon'(s)|^2 + \frac{1}{2\varepsilon}\mathrm{dist}_*\big(-\mathrm{D}\mathcal{E}_s(u_\varepsilon(s)), K^*\big)^2 \Big)\mathrm{d}s \\ = \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u_\varepsilon(s))\mathrm{d}s \end{aligned}$$

for all $t \in [0,T]$, where the term

$$(1.3) \qquad \mathrm{dist}_*\big(-\mathrm{D}\mathcal{E}_t(u(t)), K^*\big) := \min_{z\in K^*} |-\mathrm{D}\mathcal{E}_t(u(t)) - z|_*, \quad \text{with} \quad K^* = \big\{ z \in X^* : \|z\|_* \leq 1 \big\},$$

measures the distance with respect to the dual norm $|\cdot|_*$ of $-\mathrm{D}\mathcal{E}_t(u(t))$ from the set K^*. The term defined in (1.3) is penalized in (1.2) by the coefficient $1/2\varepsilon$. Thus, passing to the limit in (1.2) as $\varepsilon \downarrow 0$, one finds

$$\mathrm{dist}_*(-\mathrm{D}\mathcal{E}_t(u(t)), K^*) = 0 \qquad \text{for a.a.}\, t \in (0,T)\,.$$

Hence,

$$(1.4) \qquad -\mathrm{D}\mathcal{E}_t(u(t)) \in K^*, \quad \text{i.e.} \quad \|-\mathrm{D}\mathcal{E}_t(u(t))\|_* \leq 1 \qquad \text{for a.a.}\, t \in (0,T)\,,$$

which is a *local version* of the global stability (S). Furthermore, (1.2) yields, via lower-semicontinuity, the *energy inequality*

$$(1.5) \qquad \mathcal{E}_t(u(t)) + \mathrm{Var}(u;[0,t]) \leq \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T]\,.$$

Conditions (1.4)–(1.5) give raise to the notion of *local solution* of the rate-independent system (DN_0).

While the local stability (1.4) is more physically realistic than (S), its combination with the energy inequality (1.5) turns out to provide an unsatisfactory description of the solution at jumps (see the discussion in [35, Sec. 5.2] and Remark 2.8 later on). In order to capture the jump dynamics, the energetic behavior of the system in a jump regime has to be revealed. From this perspective, it seems to be crucial to recover from (1.2), as $\varepsilon \downarrow 0$, an *energy identity*, rather than an energy inequality. Thus, the passage to the limit has to somehow keep track of the limit of the term

$$\int_0^t \left(\frac{\varepsilon}{2}|u_\varepsilon'(s)|^2 + \frac{1}{2\varepsilon}\mathrm{dist}_*\big(-\mathrm{D}\mathcal{E}_s(u_\varepsilon(s)), K^*\big)^2\right)\mathrm{d}s\,,$$

which in fact encodes the contribution of the viscous dissipation $\frac{\varepsilon}{2}|u_\varepsilon'|^2$, completely missing in (1.5).

BV solutions. Moving from these considerations, it is natural to introduce the *vanishing viscosity contact potential* (which is related to the bipotential discussed in [7], see Section 3) induced by Ψ_ε, i.e. the quantity

$$\begin{aligned}\mathfrak{p}(v,w) &:= \inf_{\varepsilon>0}\Big(\Psi_\varepsilon(v) + \Psi_\varepsilon^*(w)\Big) = \inf_{\varepsilon>0}\left(\|v\| + \frac{\varepsilon}{2}|v|^2 + \frac{1}{2\varepsilon}\mathrm{dist}_*^2(w,K^*)\right)\\ &= \|v\| + |v|\,\mathrm{dist}_*(w,K^*) \quad \text{for } v\in X,\ w\in X^*\,.\end{aligned} \tag{1.6}$$

Then, the ε-energy identity (1.2) yields the inequality

$$\mathcal{E}_t(u_\varepsilon(t)) + \int_0^t \mathfrak{p}\left(u_\varepsilon'(s), -\mathrm{D}\mathcal{E}_s(u_\varepsilon(s))\right)\mathrm{d}s \le \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u_\varepsilon(s))\mathrm{d}s\,, \tag{1.7}$$

see Section 3.1. Passing to the limit in (1.7), in Theorem 4.10 we shall prove that, up to a subsequence, the solutions (u_ε) of the viscous equation (DN_ε) converge, as $\varepsilon \downarrow 0$, to a curve $u \in \mathrm{BV}([0,T];X)$ satisfying the local stability (1.4) and the following *energy inequality*

$$\mathcal{E}_t(u(t)) + \mathrm{Var}_{\mathfrak{p},\mathcal{E}}(u;[0,t]) \le \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s\,. \tag{1.8}$$

Without going into details (see Definition 3.4 later on), we may point out that (1.8) features a notion of (pseudo)-total variation (denoted by $\mathrm{Var}_{\mathfrak{p},\mathcal{E}}$) induced by the vanishing viscosity contact potential $\mathfrak{p}$ (1.6) and the energy $\mathcal{E}$. The main novelty is that a BV-curve obeying the local stability condition (1.4) always satisfies the opposite inequality in (1.8), thus yielding the energy balance

$$\mathcal{E}_t(u(t)) + \mathrm{Var}_{\mathfrak{p},\mathcal{E}}(u;[t,t]) = \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s\,. \tag{$\mathrm{E}_{\mathfrak{p},\mathcal{E}}$}$$

In fact, $\mathrm{Var}_{\mathfrak{p},\mathcal{E}}$ provides a finer description of the dissipation $\Delta_{\mathfrak{p},\mathcal{E}}$ of u, along a jump between two values u_- and u_+ at time t: it involves not only the quantity $\|u_+ - u_-\|$ related to the dissipation potential (1.1), but also the viscous contribution induced by the vanishing viscosity contact potential $\mathfrak{p}$ through the formula

$$\begin{aligned}\Delta_{\mathfrak{p},\mathcal{E}}(t;u_-,u_+) := \inf\Big\{ &\int_{r_0}^{r_1} \mathfrak{p}(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r)))\,\mathrm{d}r :\\ &\vartheta\in \mathrm{AC}([r_0,r_1];X),\ \vartheta(r_0)=u_-,\ \vartheta(r_1)=u_+\Big\}.\end{aligned} \tag{1.9}$$

By a rescaling technique, it is possible to show that, in a jump point, the system may switch to a *viscous behavior*, which is in fact reminiscent of the viscous approximation (DN_ε). In particular, when the jump point is of viscous type, the infimum in (1.9) is attained and the states u_- and u_+ are connected by some transition curve $\vartheta : [r_0, r_1] \to X$, fulfilling the *viscous* doubly nonlinear equation

$$\partial\Psi_0(\vartheta'(r)) + \vartheta'(r) + \mathrm{D}\mathcal{E}_t(\vartheta(r)) \ni 0 \quad \text{in } X^* \quad \text{for a.a. } r \in (r_0, r_1)$$

(in the case the norm $|\cdot|$ is Euclidean and we use its differential to identify X with X^*). The combination of (1.4) and (1.8) yields the notion of BV *solution* to the rate-independent system $(X, \mathcal{E}, \mathfrak{p})$. This concept was first introduced in [35], in the case the ambient space X is a finite-dimensional *manifold* $\mathcal{X}$, and both the rate-independent and the viscous approximating dissipations depend on one single Finsler distance on $\mathcal{X}$. In this paper, while keeping to a Banach framework, we shall considerably broaden the class of rate-independent and viscous dissipation functionals, cf. Remark 2.4. Moreover, the notion of BV solution shall be presented here in a more compact form than in [35], amenable to a finer analysis and, hopefully, to further generalizations.

Let us now briefly comment on our main results. First of all, we are going to show in Theorems 4.3, 4.6, and 4.7 that the concept of BV rate-independent evolution completely encompasses the solution behavior in both a purely rate-independent, non-jumping regime, and in jump regimes, where the competition between dry-friction and viscous effects is highlighted. Indeed, from (1.4) and (1.8) it is possible to deduce suitable energy balances at jumps (cf. conditions (J_{BV}) in Theorem 4.3).

Then, in Theorem 4.10 we shall prove that, along a subsequence, the viscous approximations arising from (DN_ε) converge as $\varepsilon \downarrow 0$ to a BV solution. Next, our second main result, Theorem 4.11, states that, up to a subsequence, also the discrete solutions $\mathrm{U}_{\tau,\varepsilon}$ constructed via the ε-discretization scheme (IP_ε) converge to a BV solution $u \in \mathrm{BV}([0,T];X)$ of (DN_0) as $\varepsilon \downarrow 0$ and $\tau \downarrow 0$ simultaneously, provided that the respective convergence rates are such that

$$\lim_{\varepsilon,\tau\downarrow 0} \frac{\varepsilon}{\tau} = +\infty\,.$$

Finally, in Section 5 we shall develop a different approach to BV solutions, via the rescaling technique advanced in [18] and refined in [35, 44]. The main idea is to suitably reparametrize the approximate viscous curves (u_ε) in order to capture, in the vanishing viscosity limit, the viscous transition paths at jumps points. This leads to performing an asymptotic analysis as $\varepsilon \downarrow 0$ of the graphs of the functions u_ε, in the extended phase space $[0,T] \times X$. For every $\varepsilon > 0$ the graph of u_ε can be parametrized by a couple of functions $(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon)$, t_ε being the (strictly increasing) rescaling function and $\mathsf{u}_\varepsilon := u_\varepsilon \circ \mathsf{t}_\varepsilon$ the rescaled solution. In Theorem 5.6 we assert that, up to a subsequence, the functions $(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon)$ converge as $\varepsilon \downarrow 0$ to a *parametrized rate-independent solution*. By the latter terminology we mean a curve $(\mathsf{t}, \mathsf{u}) : [0, \mathsf{S}] \to [0,T] \times X$ fulfilling

$$\begin{aligned} &\mathsf{t} : [0,\mathsf{S}] \to [0,T] \text{ is nondecreasing,} \\ &\mathsf{t}'(s) + \|\mathsf{u}'(s)\| > 0 \quad \text{for a.a. } s \in (0,\mathsf{S}), \end{aligned} \tag{1.10a}$$

$$\left.\begin{array}{rcl} \mathsf{t}'(s) > 0 & \Longrightarrow & \|-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))\| \leq 1, \\ \|\mathsf{u}'(s)\| > 0 & \Longrightarrow & \|-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))\| \geq 1 \end{array}\right\} \quad \text{for a.a. } s \in (0,\mathsf{S})\,, \tag{1.10b}$$

and the *energy identity*

$$\text{(1.10c)}\quad \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)) - \partial_t\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))\,\mathsf{t}'(s) \\ = -\|\mathsf{u}'(s)\| - |\mathsf{u}'(s)|\mathrm{dist}_*(-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)),K^*) \quad \text{for a.a. } s\in(0,\mathsf{S}),$$

As already pointed out in [18, 35], like the notion of BV solution, relations (1.10) as well comprise both the purely rate-independent evolution as well as the viscous transient regime at jumps. The latter regime in fact corresponds to the case $-\mathrm{D}\mathcal{E}_{\mathsf{t}}(\mathsf{u}) \notin K^*$: the system does not obey the local stability constraint (1.4) any longer, and switches to viscous behavior, see also Remark 5.7 later on.

As a matter of fact, Theorem 5.8 shows that parametrized rate-independent solutions may be viewed as the "continuous counterpart" to BV evolutions. With a suitable transformation, it is possible to associate with every parametrized rate-independent solution a BV one, and conversely. One advantage of the parametrized notion is that it avoids the technicalities related to BV functions. Hence, it is for instance more easily amenable to a stability analysis (cf. [35, Rmk. 6]). Furthermore, in [44] a highly refined vanishing viscosity analysis has been developed, with this reparametrization technique, in the infinite-dimensional (L^1, L^2)-framework, where $(\mathrm{DN}_\varepsilon)$ is replaced by a general quasilinear evolutionary PDE.

Generalizations and future developments. So far we have focused on dissipation functionals of the type (1.1) and $\Psi_\varepsilon(v) = \|v\| + \frac{\varepsilon}{2}|v|^2$ as in $(\mathrm{DN}_\varepsilon)$ for expository reasons only, in order to highlight the main variational argument leading to the notion of BV solution. Indeed, the analysis developed in this paper is targeted to a general

$$\text{positively 1-homogeneous, convex dissipation } \Psi_0 : X \to [0,+\infty),$$

(cf. (2.1)), and considers a fairly wide class of approximate viscous dissipation functionals Ψ_ε, defined by conditions $(\Psi.1)$–$(\Psi.3)$ in Section 2.3. Furthermore, at the price of just technical complications, our results could be extended to the case of a *Finsler-like* family of dissipation functionals $\Psi_0(u,\cdot)$, depending on the state variable $u \in X$, and satisfying uniform bounds and Mosco-continuity with respect to u, see [35, Sect. 2] and [47, Sect. 6, 8].

The extension to *infinite-dimensional* ambient spaces and *nonsmooth* energies is crucial for application of the concept of BV solution to the PDE systems modelling rate-independent evolutions in continuum mechanics. A first step in this direction is to generalize the known existence results for doubly nonlinear equations, driven by a viscous dissipation, to nonconvex and nonsmooth energy functionals in infinite dimensions. As shown in [48, 47], in the nonsmooth and nonconvex case one can replace the energy differential $\mathrm{D}\mathcal{E}_t$ with a suitable notion of *subdifferential* $\partial\mathcal{E}_t$. Accordingly, instead of continuity of $\mathrm{D}\mathcal{E}_t$, one asks for closedness of the multivalued subdifferential $\partial\mathcal{E}_t$ in the sense of graphs. These ideas shall be further advanced in the forthcoming work [36]. Therein, exploiting techniques from nonsmooth analysis, we shall also tackle energies which do not depend smoothly on time (this is relevant for rate-independent applications, see e.g. [22] and [25]).

On the other hand, the requirement that the ambient space is finite-dimensional could be replaced by suitable compactness (of the sublevels of the energy) and reflexivity assumptions on the ambient space X. The latter topological requirement in fact ensures that X has the so-called Radon-Nikodým property, i.e. that absolutely continuous curves with values in X are almost everywhere differentiable. The vanishing viscosity analysis in spaces which do not enjoy this property requires a subtler approach, involving metric arguments (see e.g. [47, Sect. 7]), or ad-hoc stronger estimates [44]. See also [34] for some preliminary approaches to BV solutions for PDE problems.

Plan of the paper. Section 2 is devoted to an extended presentation of energetic and local solutions to rate-independent systems. In particular, after fixing the setup of the paper in Section 2.1, in Sec. 2.2 we recall the definition of global energetic solution, show its differential characterization and the related variational time-incremental scheme. We develop the vanishing-viscosity approach in Secs. 2.3 and 2.4, thus arriving at the notion of local solution (see Section 2.5), which also admits a differential characterization.

In Section 3 we introduce the concept of vanishing viscosity contact potential and thoroughly analyze its properties, as well as the induced (pseudo)-total variation. With these ingredients, in Sec. 4 we present the notion of BV solution. We show that BV rate-independent evolutions admit, too, a differential characterization, and, in Sec. 4.2, that they provide a careful description of the energetic behavior of the system. Then, in Section 4.3, we state our main results on BV solutions.

While Section 5 is focused on the alternative notion of parametrized rate-independent solutions, the last Sec. 6 contains some technical results which lie at the core of our theory.

2. Global energetic versus local solutions, and their viscous regularizations

In this section, we will briefly recall the notion of *energetic solutions* and show that their viscous regularizations give raise to *local solutions.*

2.1. Rate-independent setting: dissipation and energy functionals.

We let

$$(X, \|\cdot\|_X) \quad \text{be a finite-dimensional normed vector space,}$$

endowed with a gauge function Ψ_0, namely a

$$\text{non-degenerate, positively 1-homogeneous, convex dissipation } \Psi_0 : X \to [0, +\infty), \tag{2.1}$$

i.e. Ψ_0 satisfies $\Psi_0(v) > 0$ if $v \neq 0$, and

$$\Psi_0(v_1 + v_2) \le \Psi_0(v_1) + \Psi_0(v_2), \quad \Psi_0(\lambda v) = \lambda \Psi_0(v) \quad \text{for every } \lambda \ge 0,\ v, v_1, v_2 \in X.$$

In particular, there exists a constant $\eta > 0$ such that

$$\eta^{-1}\|v\|_X \le \Psi_0(v) \le \eta \|v\|_X \quad \text{for every } v \in X.$$

Since Ψ_0 is 1-homogeneous, its subdifferential $\partial\Psi_0 : X \rightrightarrows X^*$ can be characterized by

$$\partial\Psi_0(v) := \Big\{ w \in X : \langle w, z\rangle \le \Psi_0(z) \text{ for every } z \in X, \quad \langle w, v\rangle = \Psi_0(v) \Big\} \subset X^*; \tag{2.2}$$

$\partial\Psi_0$ takes its values in the convex set $K^* \subset X^*$, given by

$$K^* = \partial\Psi_0(0) := \big\{ w \in X^* : \langle w, z\rangle \le \Psi_0(z) \quad \forall\, z \in X \big\} \supset \partial\Psi_0(v) \quad \text{for every } v \in X, \tag{2.3}$$

which enjoys some useful (and well-known, see e.g. [46]) properties. For the reader's convenience we list them here:

K1. K^* is the proper domain of the Legendre transform Ψ_0^* of Ψ_0, since

$$\Psi_0^*(w) = \mathrm{I}_{K^*}(w) = \begin{cases} 0 & \text{if } w \in K^*, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

K2. Ψ_0 is the support function of K^*, since

$$\Psi_0(v) = \sup_{w \in K^*} \langle w, v \rangle \quad \text{for every } v \in X, \tag{2.5}$$

and K^* is the polar set of the unit ball $K := \{v \in X : \Psi_0(v) \le 1\}$ associated with Ψ_0.

K3. K^* is the unit ball of the support function Ψ_{0*} of K:

$$K^* = \{w \in X^* : \Psi_{0*}(w) \le 1\}, \quad \text{with} \quad \Psi_{0*}(w) = \sup_{v \in K} \langle w, v \rangle = \sup_{v \ne 0} \frac{\langle w, v \rangle}{\Psi_0(v)}. \tag{2.6}$$

K4. In the even case (i.e., when $\Psi_0(v) = \Psi_0(-v)$ for all $v \in X$), we have that Ψ_0 is an equivalent norm for X, Ψ_{0*} is its dual norm, K and K^* are their respective unit balls.

Further, we consider a *smooth* energy functional

$$\mathcal{E} \in \mathrm{C}^1([0,T] \times X),$$

which we suppose bounded from below and with energy-bounded time derivative

$$\exists\, C > 0\ \forall\, (t,u) \in [0,T] \times X : \qquad \mathcal{E}_t(u) \ge -C, \qquad |\partial_t \mathcal{E}_t(u)| \le C\left(1 + \mathcal{E}_t(u)^+\right), \tag{2.7}$$

where $(\cdot)^+$ denotes the positive part. The rate-independent system associated with the energy functional $\mathcal{E}$ and the dissipation potential Ψ_0 can be formally described by the *rate-independent doubly nonlinear* differential inclusion

$$\partial \Psi_0(u'(t)) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \quad \text{in } X^* \quad \text{for a.a. } t \in (0,T). \tag{DN$_0$}$$

As already mentioned in the Introduction, for nonconvex energies solutions to (DN$_0$) may exhibit discontinuities in time. The first weak solvability notion for (DN$_0$) is the concept of *(global) energetic solution* to the rate-independent system (DN$_0$) (see [42, 40, 41] and the survey [32]), which we recall in the next section.

2.2. **Energetic solutions and variational incremental scheme.**

Definition 2.1 (Energetic solution)**.** A curve $u \in \mathrm{BV}([0,T];X)$ is an energetic solution of the *rate independent system* $(X, \mathcal{E}, \Psi_0)$ if for all $t \in [0,T]$ the *global stability* (S) and the *energy balance* (E) holds:

$$\forall\, z \in X : \qquad \mathcal{E}_t(u(t)) \le \mathcal{E}_t(z) + \Psi_0(z - u(t)), \tag{S}$$

$$\mathcal{E}_t(u(t)) + \mathrm{Var}_{\Psi_0}(u;[0,t]) = \mathcal{E}_0(u(0)) + \int_0^t \partial_t \mathcal{E}_s(u(s))\, \mathrm{d}s. \tag{E}$$

BV **functions.** Hereafter, we shall consider functions of bounded variation *pointwise defined in every point* $t \in [0,T]$, such that the *pointwise* total variation with respect to Ψ_0 (any equivalent norm of X can be chosen) $\mathrm{Var}_{\Psi_0}(u;[0,T])$ is finite, where

$$\mathrm{Var}_{\Psi_0}(u;[a,b]) := \sup\Big\{ \sum_{m=1}^{M} \Psi_0\big(u(t_m)-u(t_{m-1})\big) : a=t_0<t_1<\cdots<t_{M-1}<t_M=b\Big\}.$$

Notice that a function u in $\mathrm{BV}([0,T];X)$ admits left and right limits at every $t\in[0,T]$:

$$u(t_-) := \lim_{s\uparrow t} u(s), \quad u(t_+) := \lim_{s\downarrow t} u(s), \quad \text{with the convention } u(0_-):=u(0),\ u(T_+):=u(T), \tag{2.8}$$

and its *pointwise* jump set J_u is the at most countable set defined by

$$\mathrm{J}_u := \big\{t\in[0,T] : u(t_-)\neq u(t) \text{ or } u(t)\neq u(t_+)\big\} \supset \text{ess-}\mathrm{J}_u := \big\{t\in[0,T] : u(t_-)\neq u(t_+)\big\}. \tag{2.9}$$

We denote by u' the distributional derivative of u, and recall that u' is a Radon vector measure with finite total variation $|u'|$. It is well known [3] that u' can be decomposed into the sum of the three mutually singular measures

$$u' = u'_{\mathscr{L}} + u'_{\mathrm{C}} + u'_{\mathrm{J}}, \quad u'_{\mathscr{L}} = \dot u\,\mathscr{L}^1, \quad u'_{\mathrm{co}} := u'_{\mathscr{L}} + u'_{\mathrm{C}}\,. \tag{2.10}$$

Here, $u'_{\mathscr{L}}$ is the absolutely continuous part with respect to the Lebesgue measure $\mathscr{L}^1$, whose Lebesgue density $\dot u$ is the usual pointwise (and $\mathscr{L}^1$-a.e. defined) derivative, u'_{J} is a discrete measure concentrated on ess-$\mathrm{J}_u \subset \mathrm{J}_u$, and u'_{C} is the so-called Cantor part, still satisfying $u'_{\mathrm{C}}(\{t\}) = 0$ for every $t\in[0,T]$. Therefore $u'_{\mathrm{co}} = u'_{\mathscr{L}} + u'_{\mathrm{C}}$ is the diffuse part of the measure, which does not charge J_u. In the following, it will be useful to use a nonnegative and diffuse reference measure μ on $(0,T)$ such that $\mathscr{L}^1$ and u'_{C} are absolutely continuous w.r.t. μ: just to fix our ideas, we set

$$\mu := \mathscr{L}^1 + |u'_{\mathrm{C}}|. \tag{2.11}$$

With a slight abuse of notation, for every $(a,b)\subset(0,T)$ we denote by $\int_a^b \mathrm{d}\Psi_0(u'_{\mathrm{co}})$ the integral

$$\int_a^b \mathrm{d}\Psi_0(u'_{\mathrm{co}}) := \int_a^b \Psi_0\left(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}\right)\mathrm{d}\mu = \int_a^b \Psi_0(\dot u)\,\mathrm{d}\mathscr{L}^1 + \int_a^b \Psi_0\left(\frac{\mathrm{d}u'_{\mathrm{C}}}{\mathrm{d}|u'_{\mathrm{C}}|}\right)\mathrm{d}|u'_{\mathrm{C}}|. \tag{2.12}$$

Since Ψ_0 is 1-homogeneous, the above integral is independent of μ, provided u'_{co} is absolutely continuous w.r.t. μ.

Towards a differential characterization of energetic solutions. Let us first of all point out that (S) is stronger than the local stability condition

$$-\mathrm{D}\mathcal{E}_t(u(t)) \in K^* \quad \text{for every } t\in[0,T]\setminus \mathrm{J}_u, \tag{$\mathrm{S_{loc}}$}$$

which can be formally deduced from (DN_0) and (2.3). Indeed, the global stability (S) yields for every $z = u(t)+hv\in X$ and $h>0$

$$\langle -\mathrm{D}\mathcal{E}_t(u(t)), hv\rangle + o(|h|) \le \mathcal{E}_t(u(t)) - \mathcal{E}_t(u(t)+hv) \le h\Psi_0(v)$$

and therefore, dividing by h and passing to the limit as $h\downarrow 0$, one gets

$$\langle -\mathrm{D}\mathcal{E}_t(u(t)), v\rangle \le \Psi_0(v) \quad \text{for every } z\in X,$$

so that $(\mathrm{S_{loc}})$ holds. We obtain more insight into (E) by representing the Ψ_0 variation $\mathrm{Var}_{\Psi_0}(u;[a,b])$ in terms of the distributional derivative u' of u. In fact, recalling (2.11) and (2.12), we have

$$\mathrm{Var}_{\Psi_0}(u;[a,b]) := \int_a^b \mathrm{d}\Psi_0(u'_{\mathrm{co}}) + \mathrm{Jmp}_{\Psi_0}(u;[a,b]),$$

where the jump contribution $\mathrm{Jmp}_{\Psi_0}(u;[a,b])$ can be described, in terms of the quantities

$$\Delta_{\Psi_0}(v_0,v_1) := \Psi_0(v_1-v_0), \qquad \Delta_{\Psi_0}(v_-,v,v_+) := \Psi_0(v-v_-)+\Psi_0(v_+-v), \tag{2.13}$$

by

$$\mathrm{Jmp}_{\Psi_0}(u;[a,b]) := \Delta_{\Psi_0}(u(a),u(a_+)) + \Delta_{\Psi_0}(u(b_-),u(b)) + \sum_{t\in \mathrm{J}_u\cap(a,b)} \Delta_{\Psi_0}(u(t_-),u(t),u(t_+)). \tag{2.14}$$

Also notice that, as usual in rate-independent evolutionary problems, u is pointwise *everywhere* defined and the jump term $\mathrm{Jmp}_{\Psi_0}(u;[\cdot,\cdot])$ takes into account the value of u at every time $t\in J_u$. Therefore, if u is not continuous at t, this part may yield a strictly bigger contribution than the total mass of the distributional jump measure u'_{J} (which gives rise to the so-called *essential* variation).

The following result provides an equivalent characterization of energetic solutions: besides the global stability condition (S), it involves a BV formulation of the differential inclusion (DN_0) (cf. the *subdifferential formulation* of [41]) and a *jump condition* at any jump point of u.

Proposition 2.2. *A curve $u\in \mathrm{BV}([0,T];X)$ satisfying the global stability condition (S) is an energetic solution of the rate-independent system $(X,\mathcal{E},\Psi_0)$ if and only if it satisfies the differential inclusion*

$$\partial\Psi_0\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{d\mu}(t)\Big) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \quad \textit{for } \mu\textit{-a.e. } t\in[0,T], \qquad \mu := \mathscr{L}^1 + |u'_{\mathrm{C}}|, \tag{$\mathrm{DN_{0,BV}}$}$$

and the jump conditions

$$\begin{gathered}\mathcal{E}_t(u(t)) - \mathcal{E}_t(u(t_-)) = -\Delta_{\Psi_0}(u(t_-),u(t)), \quad \mathcal{E}_t(u(t_+)) - \mathcal{E}_t(u(t)) = -\Delta_{\Psi_0}(u(t),u(t_+)),\\ \mathcal{E}_t(u(t_+)) - \mathcal{E}_t(u(t_-)) = -\Delta_{\Psi_0}(u(t_-),u(t_+)).\end{gathered} \tag{$\mathrm{J_{ener}}$}$$

for every $t\in \mathrm{J_u}$ (recall convention (2.8) in the case $t=0,T$).

We shall simply sketch the proof, referring to the arguments for the forthcoming Proposition 2.7 for all details.

Proof. By the additivity property of the total variation $\mathrm{Var}_{\Psi_0}(u;[\cdot,\cdot])$, (E) yields for every $0\le t_0<t_1\le T$

$$\mathrm{Var}_{\Psi_0}(u;[t_0,t_1]) + \mathcal{E}_{t_1}(u(t_1)) = \mathcal{E}_{t_0}(u(t_0)) + \int_{t_0}^{t_1} \partial_t\mathcal{E}_t(u(t))\,\mathrm{d}t\,. \tag{E'}$$

Arguing as in the proof of Proposition 2.7 later on, one can see that the global stability (S) and (E') yield the differential inclusion $(\mathrm{DN_{0,BV}})$ and conditions $(\mathrm{J_{ener}})$.

Conversely, repeating the arguments of Proposition 2.7 one can verify that ($\mathrm{DN}_{0,\mathrm{BV}}$) and ($\mathrm{J}_{\mathrm{ener}}$) imply (E). $\square$

Incremental minimization scheme. Existence of energetic solutions can be proved by solving a minimization scheme, which is also interesting as construction of an effective approximation of the solutions.

For a given time-step $\tau > 0$ we consider a uniform partition (for simplicity) $0 = t_0 < t_1 < \cdots < t_{N-1} < T \le t_N$, $t_n := n\tau$, of the time interval $[0,T]$, and an initial value $\mathrm{U}_\tau^0 \approx u_0$. In order to find good approximations of $\mathrm{U}_\tau^n \approx u(t_n)$ we solve the incremental minimization scheme

$$(\mathrm{IP}_0) \qquad \text{find } \mathrm{U}_\tau^1, \cdots, \mathrm{U}_\tau^N \quad \text{such that} \quad \mathrm{U}_\tau^n \in \operatorname*{Argmin}_{\mathrm{U}\in X} \Big\{ \Psi_0(\mathrm{U} - \mathrm{U}_\tau^{n-1}) + \mathcal{E}_{t_n}(\mathrm{U}) \Big\}.$$

Setting

$$\overline{\mathrm{U}}_\tau(t) := \mathrm{U}_\tau^n \quad \text{if } t \in (t_{n-1}, t_n], \tag{2.15}$$

it is possible to find a suitable vanishing sequence of step sizes $\tau_k \downarrow 0$ (see, e.g., [41, 32] for all calculations), such that

$$\exists \lim_{k\to+\infty} \overline{\mathrm{U}}_{\tau_k}(t) =: u(t) \quad \text{for every } t \in [0,T],$$

and u is an energetic solution of (DN_0).

2.3. **Viscous approximations of rate-independent systems.** In the present paper we want to study a different approach to approximate and solve (DN_0): the main idea is to replace the linearly growing dissipation potential Ψ_0 with a suitable *convex and superlinear* "viscous" regularization $\Psi_\varepsilon : X \to [0,+\infty)$ of Ψ_0, depending on a "small" parameter $\varepsilon > 0$ and "converging" to Ψ_0 in a suitable sense as $\varepsilon \downarrow 0$. Solving the doubly nonlinear differential inclusion (we use the notation $\dot u$ for the time derivative when u is absolutely continuous)

$$(\mathrm{DN}_\varepsilon) \qquad \partial\Psi_\varepsilon(\dot u_\varepsilon(t)) + \mathrm{D}\mathcal{E}_t(u_\varepsilon(t)) \ni 0 \quad \text{in } X^* \quad \text{for a.a. } t \in (0,T),$$

one can consider the sequence (u_ε) as a good approximation of the solution u of (DN_0) as $\varepsilon \downarrow 0$.

There is also a natural discrete counterpart to (DN_ε), which regularizes the incremental minimization problem (IP_0). We simply substitute Ψ_0 by Ψ_ε in (IP_0), recalling that now the time-step τ should explicitly appear, since Ψ_ε is not 1-homogeneous any longer. The viscous incremental problem is therefore

$$(\mathrm{IP}_\varepsilon) \qquad \text{find } \mathrm{U}_{\tau,\varepsilon}^1, \cdots, \mathrm{U}_{\tau,\varepsilon}^N \quad \text{such that} \quad \mathrm{U}_{\tau,\varepsilon}^n \in \operatorname*{Argmin}_{\mathrm{U}\in X} \Big\{ \tau\Psi_\varepsilon\Big(\frac{\mathrm{U} - \mathrm{U}_{\tau,\varepsilon}^{n-1}}{\tau}\Big) + \mathcal{E}_{t_n}(\mathrm{U}) \Big\}.$$

Setting as in (2.15)

$$\overline{\mathrm{U}}_{\tau,\varepsilon}(t) := \mathrm{U}_{\tau,\varepsilon}^n \quad \text{if } t \in (t_{n-1}, t_n],$$

one can study the limit of the discrete solutions when $\tau \downarrow 0$ and $\varepsilon \downarrow 0$, under some restriction on the behavior of the quotient ε/τ (see Theorem 4.11 later on).

The choice of the viscosity approximation Ψ_ε. Here we consider the particular case when the potential Ψ_ε can be obtained starting from a given

$$(\Psi.1)\quad \text{convex function } \Psi : X \to [0,+\infty) \quad \text{such that} \quad \Psi(0)=0, \qquad \lim_{\|v\|_X\uparrow+\infty} \frac{\Psi(v)}{\|v\|_X} = +\infty,$$

by the canonical rescaling

$$(\Psi.2)\qquad \Psi_\varepsilon(v) := \varepsilon^{-1}\Psi(\varepsilon v) \quad \text{for every } v \in X,\ \varepsilon > 0,$$

and Ψ_ε is linked to Ψ_0 by the relation

$$(\Psi.3)\qquad \Psi_0(v) = \lim_{\varepsilon\downarrow 0} \Psi_\varepsilon(v) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\Psi(\varepsilon v) \quad \text{for every } v \in X.$$

Remark 2.3. Notice that, by convexity of Ψ and the fact that $\Psi(0) = 0$, the map $\varepsilon \mapsto \varepsilon^{-1}\Psi(\varepsilon v)$ is nondecreasing for all $v \in X$. Hence,

$$(2.16)\qquad \Psi_0(v) \le \Psi_\varepsilon(v) \quad \text{for all } v \in X, \text{ for all } \varepsilon > 0.$$

Furthermore, by the coercivity condition $(\Psi.1)$,

$$\partial\Psi_\varepsilon(v) := \partial\Psi(\varepsilon v) \quad \text{is a surjective map.}$$

Here are some examples, showing that $(\Psi.2)$ still provides a great flexibility and covers several interesting cases.

Example 2.4.

Ψ_0-viscosity: The simplest example, still absolutely non trivial [35], is to consider

$$(2.17)\qquad \begin{gathered} \Psi(v) := \Psi_0(v) + \frac{1}{2}\big(\Psi_0(v)\big)^2, \quad \Psi_\varepsilon(v) := \Psi_0(v) + \frac{\varepsilon}{2}\big(\Psi_0(v)\big)^2, \\ \partial\Psi_\varepsilon(v) = \Big(1 + \varepsilon\Psi_0(v)\Big)\partial\Psi_0(v). \end{gathered}$$

A similar regularization can be obtained by choosing a real convex and superlinear function $F_V : [0,+\infty) \to [0,+\infty)$, with $F_V(0) = F_V'(0) = 0$, and setting

$$(2.18)\qquad \Psi(v) := \Psi_0(v) + F_V(\Psi_0(v)) = F(\Psi_0(v)), \quad \text{with } F(r) := r + F_V(r).$$

Quadratic or p-viscosity induced by a norm $\|\cdot\|$: The most interesting case involves an arbitrary norm $\|\cdot\|$ on X and considers for $p > 1$

(2.19)

$$\Psi(v) = \Psi_0(v) + \frac{1}{p}\|v\|^p, \quad \Psi_\varepsilon(v) = \Psi_0(v) + \frac{\varepsilon^{p-1}}{p}\|v\|^p, \quad \partial\Psi_\varepsilon(v) = \partial\Psi_0(v) + \varepsilon^{p-1}J_p(v),$$

where J_p is the p-duality map associated with $\|\cdot\|$. In particular, if $\|\cdot\|$ is a Hilbertian norm and $p = 2$, then J_2 is the Riesz isomorphism and we can choose $J_2(v) = v$ by identifying X with X^*. Hence, $(\mathrm{DN}_\varepsilon)$ reads

$$\partial\Psi_\varepsilon(\dot u_\varepsilon(t)) + \varepsilon\dot u_\varepsilon(t) + \mathrm{D}\mathcal{E}_t(u_\varepsilon(t)) \ni 0 \quad \text{in } X^* \qquad \text{for a.a. } t \in (0,T),$$

and the incremental problem $(\mathrm{IP}_\varepsilon)$ looks for $U^n_{\tau,\varepsilon}$ which recursively minimizes

$$U \mapsto \Psi_0(U - U^{n-1}_{\tau,\varepsilon}) + \frac{\varepsilon}{2\tau}\|U - U^{n-1}_{\varepsilon,\tau}\|^2 + \mathcal{E}_{t_n}(U).$$

This is the typical situation which motivates our investigation.

Additive viscosity: More generally, we can choose a convex "viscous" potential $\Psi_V : X \to [0,+\infty)$ satisfying

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\Psi_V(\varepsilon v) = 0, \quad \lim_{\lambda\uparrow+\infty} \lambda^{-1}\Psi_V(\lambda v) = +\infty \quad \text{for all } v \in X, \tag{2.20}$$

and set

$$\Psi(v) := \Psi_0(v)+\Psi_V(v), \quad \Psi_\varepsilon(v) := \Psi_0(v)+\varepsilon^{-1}\Psi_V(\varepsilon v), \quad \partial\Psi_\varepsilon(v) = \partial\Psi_0+\partial\Psi_V(\varepsilon v). \tag{2.21}$$

2.4. **Viscous energy identity.** Since Ψ has a superlinear growth, the results of [9, 8] ensure that for every $\varepsilon > 0$ and initial datum $u_0 \in X$ there exists at least one solution $u_\varepsilon \in \mathrm{AC}([0,T];X)$ to equation (DN_ε), fulfilling the Cauchy condition $u_\varepsilon(0) = u_0$.

In order to capture its asymptotic behavior as $\varepsilon \downarrow 0$, we split equation (DN_ε) in a simple system of two conditions, involving an auxiliary variable $w_\varepsilon : [0,T] \to X^*$ and a scalar function $p_\varepsilon : [0,T] \to \mathbb{R}$

$$\partial\Psi_\varepsilon(\dot u_\varepsilon(t)) \ni w_\varepsilon \qquad \text{for a.a. } t \in (0,T)\,, \tag{2.22a}$$

$$\mathrm{D}\mathcal{E}_t(u_\varepsilon(t)) = -w_\varepsilon(t), \qquad \partial_t\mathcal{E}_t(u_\varepsilon(t)) = -p_\varepsilon(t) \qquad \text{for all } t \in [0,T]. \tag{2.22b}$$

Denoting by $\Psi^*, \Psi^*_\varepsilon$ the conjugate functions of Ψ and Ψ_ε, we have

$$0 = \Psi^*(0) \le \Psi^*(\xi) < +\infty, \qquad \Psi^*_\varepsilon(\xi) = \varepsilon^{-1}\Psi^*(\xi) \quad \text{for every } \xi \in X^*. \tag{2.23}$$

Due to (2.16), there holds

$$\Psi^*_\varepsilon(\xi) \le \Psi^*_0(\xi) \quad \text{for all } v \in X,\ \varepsilon > 0. \tag{2.24}$$

The classical characterization of the subdifferential of Ψ_ε yields that the first condition (2.22a) is equivalent to

$$\Psi_\varepsilon(\dot u_\varepsilon(t)) + \Psi^*_\varepsilon(w_\varepsilon(t)) = \langle w_\varepsilon(t), \dot u_\varepsilon(t)\rangle \qquad \text{for a.a. } t \in (0,T)\,. \tag{2.25}$$

On the other hand, the chain rule for the C^1 functional $\mathcal{E}$ shows that along the absolutely continuous curve u_ε

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_t(u_\varepsilon(t)) = \langle \mathrm{D}\mathcal{E}_t(u_\varepsilon(t)), \dot u_\varepsilon(t)\rangle + \partial_t\mathcal{E}_t(u_\varepsilon(t)) = -\langle w_\varepsilon(t), \dot u_\varepsilon(t)\rangle - p_\varepsilon(t) \ \text{ for a.a. } t \in (0,T). \tag{2.26}$$

Thus, if $w_\varepsilon(t) = -\mathrm{D}\mathcal{E}_t(u_\varepsilon(t))$, equation (2.22a) is equivalent to the *energy identity*

$$\int_{t_0}^{t_1} \Big(\Psi_\varepsilon(\dot u_\varepsilon(r)) + \Psi^*_\varepsilon(w_\varepsilon(r)) + p_\varepsilon(r)\Big)\,\mathrm{d}r + \mathcal{E}_{t_1}(u_\varepsilon(t_1)) = \mathcal{E}_{t_0}(u_\varepsilon(t_0)), \tag{2.27}$$

for every $0 \le t_0 \le t_1 \le T$.

Remark 2.5 (The role of Ψ^*_ε). In the general, *additive-viscosity* case (see (2.21)), when $\Psi(v) = \Psi_0(v) + \Psi_V(v)$ the inf-sup convolution formula yields

$$\Psi^*_\varepsilon(\xi) = \inf_{\substack{\xi_1+\xi_2=\xi \\ \xi_1,\xi_2\in X^*}} \left\{ \mathrm{I}_{K^*}(\xi_1) + \frac{1}{\varepsilon}\Psi^*_V(\xi_2)\right\} = \varepsilon^{-1}\min_{z\in K^*}\Psi^*_V(\xi - z).$$

In particular, when $\Psi_V(\xi) := \frac{1}{2}|v|^2$ for some norm $|\cdot|$ of X, one finds

$$\Psi^*_\varepsilon(\xi) = \frac{1}{2\varepsilon}\min_{z\in K^*}|\xi - z|^2_*,$$

where $|\cdot|_*$ is the dual norm of $|\cdot|$. Thus, for all $\xi \in X^*$ the functional $\Psi_\varepsilon^*(\xi)$ is the squared distance of ξ from K^*, with respect to $|\cdot|_*$. This shows that, in the viscous regularized equation (DN$_\varepsilon$), the (local) stability condition $w(t) = -\mathrm{D}\mathcal{E}_t(u(t)) \in K^*$ has been replaced by the contribution of the penalizing term

$$\frac{1}{2\varepsilon}\int_0^T \min_{z\in K^*} |-\mathrm{D}\mathcal{E}_t(u_\varepsilon(t)) - z|_*^2 \,\mathrm{d}t$$

in the energy identity (2.27).

2.5. **Pointwise limit of viscous approximations and local solutions.** Using (2.7), it is not difficult to show that the viscous solutions u_ε of (DN$_\varepsilon$) satisfy the *a priori* bound

$$\int_0^T \Big(\Psi_\varepsilon(\dot u_\varepsilon(t)) + \Psi_\varepsilon^*(w_\varepsilon(t))\Big)\,\mathrm{d}t \le C, \quad \text{with } w_\varepsilon(t) = -\mathrm{D}\mathcal{E}_t(u_\varepsilon(t)) \text{ for all } t \in [0,T]. \tag{2.28}$$

Therefore, Helly's compactness theorem shows that, up to the extraction of a suitable subsequence, the sequence (u_ε) pointwise converges to a BV curve u. From the convergence $w_\varepsilon(t) \to w(t) = -\mathrm{D}\mathcal{E}_t(u(t))$ as $\varepsilon \downarrow 0$ and the fact that for all $t \in [0,T]$

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}\Psi^*(w_\varepsilon(t)) \overset{(2.23)}{\ge} \Psi_0^*(w(t)) = \mathrm{I}_K^*(w(t)) = \begin{cases} 0 & \text{if } w(t) \in K^*, \\ +\infty & \text{otherwise,}\end{cases} \tag{2.29}$$

we infer that the limit curve u satisfies the (local) stability condition (S$_{\mathrm{loc}}$). On the other hand, passing to the limit in (2.27) one gets the energy inequality

$$\mathcal{E}_{t_1}(u(t_1)) + \mathrm{Var}_{\Psi_0}(u;[t_0,t_1]) \le \mathcal{E}_{t_0}(u(t_0)) + \int_{t_0}^{t_1} \partial_t\mathcal{E}_t(u(t))\,\mathrm{d}t \quad \text{for } 0 \le t_0 < t_1 \le T. \tag{E$'_{\mathrm{ineq}}$}$$

The above discussion motivates the concept of *local solution* (see also [35, Sec. 5.2] and the references therein).

Definition 2.6 (Local solutions)**.** A curve $u \in \mathrm{BV}([0,T];X)$ is called a *local solution* of the rate independent system $(X,\mathcal{E},\Psi_0)$ if it satisfies the *local stability* condition

$$-\mathrm{D}\mathcal{E}_t(u(t)) \in K^* \quad \text{for every } t \in [0,T]\setminus \mathrm{J}_u, \tag{S$_{\mathrm{loc}}$}$$

and the *energy dissipation inequality* (E$'_{\mathrm{ineq}}$).

Local solutions admit the following differential characterization.

Proposition 2.7 (Differential characterization of local solutions)**.** *A curve* $u \in \mathrm{BV}([0,T];X)$ *is a* local solution *of the rate independent system* $(X,\mathcal{E},\Psi_0)$ *if and only if it satisfies the* BV *differential inclusion*

$$\partial\Psi_0\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{d\mu}(t)\Big) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \quad \text{for } \mu\text{-a.e. } t \in [0,T], \quad \mu := \mathscr{L}^1 + |u'_{\mathrm{C}}|, \tag{DN$_{0,\mathrm{BV}}$}$$

and the jump inequalities

$$\begin{gathered}\mathcal{E}_t(u(t)) - \mathcal{E}_t(u(t_-)) \le -\Delta_{\Psi_0}(u(t_-),u(t)), \quad \mathcal{E}_t(u(t_+)) - \mathcal{E}_t(u(t)) \le -\Delta_{\Psi_0}(u(t),u(t_+)), \\ \mathcal{E}_t(u(t_+)) - \mathcal{E}_t(u(t_-)) \le -\Delta_{\Psi_0}(u(t_-),u(t_+)),\end{gathered} \tag{J$_{\mathrm{local}}$}$$

at each jump time $t \in \mathrm{J_u}$.

Proof. Notice that at every point $t \in (0,T)$ where $\mathrm{d}u'_{\mathrm{co}}(t)/\mathrm{d}\mu = 0$, the differential inclusion $(\mathrm{DN}_{0,\mathrm{BV}})$ reduces to the local stability condition $(\mathrm{S}_{\mathrm{loc}})$. In the general case, $(\mathrm{DN}_{0,\mathrm{BV}})$ follows by differentiation of $(\mathrm{E}'_{\mathrm{ineq}})$. Indeed, the latter procedure provides the following inequality between the distributional derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_t(u(t))$ of the map $t \mapsto \mathcal{E}_t(u(t))$ and the Ψ_0-total variation measure $\Psi_0(u'_{\mathrm{co}}) := \Psi_0\big(\mathrm{d}u'_{\mathrm{co}}/\mathrm{d}\mu\big)\mu$ for $\mu := u'_{\mathrm{C}} + \mathscr{L}^1$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_t(u(t)) + \Psi_0(u'_{\mathrm{co}}) - \partial_t\mathcal{E}_t(u(t))\mathscr{L}^1 \le 0\,. \tag{2.30}$$

Applying the chain rule formula for the composition of the C^1 functional $\mathcal{E}$ and the BV curve u (see [2] and [3, Thm. 3.96]) and taking into account the fact that u'_{co} and u'_{J} are mutually singular, we obtain from (2.30) that

$$\Big\langle -\mathrm{D}\mathcal{E}_t(u(t)), \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}\Big\rangle\mu \ge \Psi_0(u'_{\mathrm{co}}) = \Psi_0\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}\Big)\mu\,. \tag{2.31}$$

Combining (2.31) with the local stability condition $(\mathrm{S}_{\mathrm{loc}})$, in view of the characterization (2.2) of $\partial\Psi_0$ and of (2.3) we finally conclude $(\mathrm{DN}_{0,\mathrm{BV}})$. Localizing $(\mathrm{E}'_{\mathrm{ineq}})$ around a jump point t we get the inequalities $(\mathrm{J}_{\mathrm{local}})$.

Conversely, let us suppose that a BV curve u satisfies $(\mathrm{DN}_{0,\mathrm{BV}})$ and $(\mathrm{J}_{\mathrm{local}})$. The local stability condition is an immediate consequence of $(\mathrm{DN}_{0,\mathrm{BV}})$, which yields $-\mathrm{D}\mathcal{E}_t(u(t)) \in K^*$ for $\mathscr{L}^1$-a.e. $t \in [0,T]$ and therefore, by continuity, at every point of $[0,T] \setminus \mathrm{J}_u$.

In order to get $(\mathrm{E}'_{\mathrm{ineq}})$, we again apply the chain rule for the composition $\mathcal{E}$ and u, obtaining

$$\begin{aligned}\mathcal{E}_{t_1}(u(t_1)) + \int_{t_0}^{t_1}\Big\langle -\mathrm{D}\mathcal{E}_t(u(t)), \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}\Big\rangle\,\mathrm{d}\mu(t) - \mathrm{Jmp}(\mathcal{E};[t_0,t_1])&\\ = \mathcal{E}_{t_0}(u(t_0)) + \int_{t_0}^{t_1}\partial_t\mathcal{E}_t(u(t))\,\mathrm{d}t&,\end{aligned} \tag{2.32}$$

where

$$\mathrm{Jmp}(\mathcal{E};[t_0,t_1]) = E_+(t_0) + E_-(t_1) + \sum_{t \in \mathrm{J}_u \cap (t_0,t_1)}\Big(E_-(t) + E_+(t)\Big),$$

and

$$E_-(t) := \mathcal{E}_t(u(t)) - \mathcal{E}_t(u(t_-)), \quad E_+(t) := \mathcal{E}_t(u(t_+)) - \mathcal{E}_t(u(t)).$$

By $(\mathrm{DN}_{0,\mathrm{BV}})$ we have

$$\int_{t_0}^{t_1}\Big\langle -\mathrm{D}\mathcal{E}_t(u(t)), \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}\Big\rangle\,\mathrm{d}\mu(t) = \int_{t_0}^{t_1}\Psi_0\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}(t)\Big)\,\mathrm{d}\mu(t) = \int_{t_0}^{t_1}\mathrm{d}\Psi_0(u'_{\mathrm{co}})\,, \tag{2.33}$$

whereas $(\mathrm{J}_{\mathrm{local}})$ yields for every $t \in \mathrm{J}_u$

$$E_-(t) \le -\Delta_{\Psi_0}(u(t_-),u(t)), \quad E_+(t) \le -\Delta_{\Psi_0}(u(t),u(t_+)), \tag{2.34}$$

so that $-\mathrm{Jmp}(\mathcal{E};[t_0,t_1]) \ge \mathrm{Jmp}_{\Psi_0}(u;[t_0,t_1])$ and therefore $(\mathrm{E}'_{\mathrm{ineq}})$ follows from (2.32). □

Remark 2.8. Unlike the case of energetic solutions (cf. Proposition 2.2), a precise description of the behavior of local solutions at jumps in missing here. In fact, the jump inequalities ($\mathrm{J}_{\mathrm{local}}$) are not sufficient to get an energy balance and do not completely capture the jump dynamics, see the discussion of [35, Sec. 5.2].

In order to get more precise insight into the jump properties and to understand the correct energy balance along them, we have to introduce a finer description of the dissipation. It is related to an extra contribution to the jump part of $\mathrm{Var}_{\Psi_0}(u;[\cdot,\cdot])$, which can be better described by using the *vanishing viscosity contact potential* induced by the coupling Ψ, Ψ^*. We describe this notion in the next section.

3. Vanishing viscosity contact potentials and Finsler dissipation costs

3.1. **Heuristics for the concept of vanishing viscosity contact potential.** Suppose for the moment being that, in a given time interval $[r_0, r_1]$, the energy $\mathcal{E}_t(\cdot) = \mathcal{E}(\cdot)$ does not change w.r.t. time. If $\vartheta \in \mathrm{AC}([r_0,r_1];X)$ is a solution of (DN_ε) connecting $u_0 = \vartheta(r_0)$ to $u_1 = \vartheta(r_1)$, then the energy release between the initial and the final state is, by the energy identity (2.27),

$$\begin{aligned} &\mathcal{E}(u_0) - \mathcal{E}(u_1) = \int_{r_0}^{r_1} \Big(\Psi_\varepsilon(v) + \Psi_\varepsilon^*(w) \Big)\,\mathrm{d}t, \\ &\qquad \text{with } \ v(t) = \dot\vartheta(t) \quad \text{and} \quad w(t) = -\mathrm{D}\mathcal{E}(\vartheta(t)) \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{3.1}$$

If one looks for a lower bound of the right-hand side in the above energy identity *which is independent of* $\varepsilon > 0$, it is natural to recur to the functional $\mathfrak{p} : X \times X^* \to [0,+\infty)$ defined by

$$\mathfrak{p}(v,w) := \inf_{\varepsilon>0} \left(\Psi_\varepsilon(v) + \Psi_\varepsilon^*(w)\right) = \inf_{\varepsilon>0} \left(\varepsilon^{-1}\Psi(\varepsilon v) + \varepsilon^{-1}\Psi^*(w)\right) \quad \text{for } v \in X,\ w \in X^*.$$

We obtain

$$\mathcal{E}(u_0) - \mathcal{E}(u_1) \geq \int_{r_0}^{r_1} \mathfrak{p}(v,w)\,\mathrm{d}t \quad \text{with} \quad v(t) = \dot\vartheta(t) \ \text{ and } \ w(t) = -\mathrm{D}\mathcal{E}(\vartheta(t)). \tag{3.2}$$

Since $\mathfrak{p}(\cdot,\cdot)$ is positively 1-homogeneous with respect to its first variable, the right-hand side expression in (3.2) is in fact independent of (monotone) time rescalings. On the other hand, the *vanishing viscosity contact potential* $\mathfrak{p}(\cdot,\cdot)$ has the remarkable properties

$$\mathfrak{p}(v,w) \geq \langle w, v\rangle, \qquad \mathfrak{p}(v,w) \geq \Psi_0(v) \quad \text{for every } v \in X,\ w \in X^*. \tag{3.3}$$

Therefore, if $\tilde\vartheta \in \mathrm{AC}([r_0,r_1];X)$ is another arbitrary curve connecting u_0 to u_1, the chain rule (2.26) for $\mathcal{E}$ yields

$$\mathcal{E}(u_0) - \mathcal{E}(u_1) = \int_{r_0}^{r_1} \langle \tilde w(t), \tilde v(t)\rangle\,\mathrm{d}t \leq \int_{r_0}^{r_1} \left(\Psi_\varepsilon(v(t)) + \Psi_\varepsilon^*(\tilde w(t))\right)\mathrm{d}t$$

(where $\tilde v$ denotes the time derivative of $\tilde\vartheta$ and $\tilde w = -\mathrm{D}\mathcal{E}(\tilde\vartheta)$), whence

$$\mathcal{E}(u_0) - \mathcal{E}(u_1) \leq \int_{r_0}^{r_1} \mathfrak{p}(\tilde v(t), \tilde w(t))\,\mathrm{d}t\,. \tag{3.4}$$

It follows that, in a time regime in which the energy functional $\mathcal{E}$ does not change with respect to time, for every $\varepsilon > 0$ any viscous solution of (DN_ε) (and, therefore, any suitable limit of viscous solutions) should attain the minimum dissipation, measured in terms of the vanishing viscosity contact potential $\mathfrak{p}$. Moreover, this dissipation always provides an upper bound for the energy release, reached exactly along viscous curves and their limits.

Remark 3.1. In some of the cases discussed in Example 2.4, the vanishing viscosity contact potential $\mathfrak{p}$ admits a more explicit representation.

(1) We first consider the Ψ_0-*viscosity* case (2.18), where $\Psi(v) := F(\Psi_0(v))$, $F : [0,+\infty) \to [0,+\infty)$ being a real convex superlinear function with $F(0) = 0, F'(0) = 1$. We introduce the 1-homogeneous support function Ψ_{0*} of the set

$$K := \big\{v \in X : \Psi_0(v) \le 1\big\}, \quad \Psi_{0*}(w) := \sup_{v\in K}\langle w, v\rangle.$$

It is not difficult to show that $\Psi^*(w) = F^*(\Psi_{0*}(w))$ and that for all $(v,w) \in X \times X^*$

$$\mathfrak{p}(v,w) = \Psi_0(v)\,\max(1,\Psi_{0*}(w)) = \begin{cases}\Psi_0(v) & \text{if } w \in K^*,\\ \Psi_0(v)\,\Psi_{0*}(w) & \text{if } w \notin K^*.\end{cases} \tag{3.5}$$

(2) In the *additive viscosity* case of (2.21) one has for all $(v,w) \in X \times X^*$

$$\mathfrak{p}(v,w) = \Psi_0(v) + \mathfrak{p}_V(v,w), \text{ where } \mathfrak{p}_V(v,w) = \inf_{\varepsilon>0}\Big(\varepsilon^{-1}\Psi_V(\varepsilon v) + \varepsilon^{-1}\inf_{z\in K^*}\Psi_V^*(w-z)\Big). \tag{3.6}$$

In particular, when $\Psi_V(v) = F_V(\|v\|)$ for some norm $\|\cdot\|$ of X and a real convex and superlinear function $F_V : [0,+\infty) \to [0,+\infty)$ with $F_V(0) = F_V'(0) = 0$, we have for all $(v,w) \in X \times X^*$

$$\mathfrak{p}(v,w) = \Psi_0(v) + \mathfrak{p}_V(v,w), \quad \text{with } \mathfrak{p}_V(v,w) = \|v\|\,\min_{z\in K^*}\|w-z\|_*. \tag{3.7}$$

Notice that in (3.5) and (3.7) the form of the vanishing viscosity contact potential $\mathfrak{p}$ *does not depend* on the choice of F and F_V, respectively, but only on the chosen viscosity norm.

By the 1-homogeneity of $\mathfrak{p}(\cdot,w)$ and these variational properties, it is then natural to introduce the following Finsler dissipation.

Definition 3.2 (Finsler dissipation)**.** For a fixed $t \in [0,T]$, the Finsler cost induced by $\mathfrak{p}$ and (the differential of) $\mathcal{E}$ at the time t is given by

$$\begin{aligned}\Delta_{\mathfrak{p},\mathcal{E}}(t;u_0,u_1) := \inf\Big\{ &\int_{r_0}^{r_1}\mathfrak{p}(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r)))\,\mathrm{d}r :\\ &\vartheta \in \mathrm{AC}([r_0,r_1];X),\ \vartheta(r_0) = u_0,\ \vartheta(r_1) = u_1\Big\}\end{aligned} \tag{3.8}$$

for every $u_0, u_1 \in X$. We also consider the induced “triple” cost

$$\Delta_{\mathfrak{p},\mathcal{E}}(t;u_-,u,u_+) := \Delta_{\mathfrak{p},\mathcal{E}}(t;u_-,u) + \Delta_{\mathfrak{p},\mathcal{E}}(t;u,u_+).$$

Remark 3.3. Since $\mathfrak{p}(v,w) \ge \Psi_0(v)$ by (3.3), a simple time rescaling argument shows that the infimum in (3.8) is always attained by a *Lipschitz curve* $\vartheta \in \mathrm{AC}([r_0,r_1];X)$ with constant $\mathfrak{p}$-speed, in particular such that

$$\mathfrak{p}(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r)) \equiv 1 \qquad \text{for a.a. } r \in (r_0,r_1)\,.$$

By the heuristical discussion developed throughout (3.1)–(3.4), the cost $\Delta_{\mathfrak{p},\mathcal{E}}$ is the natural candidate to substituting the potential Ψ_0 and the related cost Δ_{Ψ_0} of (2.13) in the jump contributions (2.14) and in the jump conditions ($\mathrm{J}_{\mathrm{ener}}$). Notice that the second relation of (3.3) implies

$$\Delta_{\mathfrak{p},\mathcal{E}}(t;u_0,u_1) \geq \Delta_{\Psi_0}(u_0,u_1) \quad \text{for every } u_0,u_1 \in X. \tag{3.9}$$

The notion of jump variation arising from such replacements is precisely stated as follows.

Definition 3.4 (The total variation induced by $\Delta_{\mathfrak{p},\mathcal{E}}$)**.** Let $u \in \mathrm{BV}([0,T];X)$ a given curve, let u'_{co} be the diffuse part of its distributional derivative u', and let J_u be its pointwise jump set (2.9). For every subinterval $[a,b] \subset [0,T]$ the Jump variation of u induced by $(\mathfrak{p},\mathcal{E})$ on $[a,b]$ is

$$\begin{aligned} \mathrm{Jmp}_{\mathfrak{p},\mathcal{E}}(u;[a,b]) :=& \Delta_{\mathfrak{p},\mathcal{E}}(a;u(a),u(a_+)) + \Delta_{\mathfrak{p},\mathcal{E}}(b;u(b_-),u(b)) + \\ &+ \sum_{t \in \mathrm{J}_u \cap (a,b)} \Delta_{\mathfrak{p},\mathcal{E}}(t;u(t_-),u(t),u(t_+)), \end{aligned} \tag{3.10}$$

and the (pseudo-)total variation induced by $(\mathfrak{p},\mathcal{E})$ is

$$\mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[a,b]) := \int_a^b \mathrm{d}\Psi_0(u'_{\mathrm{co}}) + \mathrm{Jmp}_{\mathfrak{p},\mathcal{E}}(u;[a,b]). \tag{3.11}$$

Remark 3.5 (The (pseudo-)total variation $\mathit{Var}_{\mathfrak{p},\mathcal{E}}$). Let us mention that $\mathit{Var}_{\mathfrak{p},\mathcal{E}}$ enjoys some of the properties of the usual total variation functionals, but it is not lower semicontinuous w.r.t. pointwise convergence. In fact, it is not difficult to see that its lower semicontinuous envelope is simply Var_{Ψ_0}. Furthermore, $\mathit{Var}_{\mathfrak{p},\mathcal{E}}$ is not induced by any distance on X. Indeed, we have used *slanted* fonts in the notation *Var* to stress this fact. In order to recover a more standard total variation in a metric setting, one has to work in the extended space $\mathscr{X} := [0,T] \times X$ and add the local stability constraint $-\mathrm{D}\mathcal{E}_t \in K^*$ on the "continuous" part of the trajectories. We shall discuss this point of view in Section 6.

In view of inequality (3.9) between the Finsler dissipation $\Delta_{\mathfrak{p},\mathcal{E}}$ and Δ_{Ψ_0}, the notion of total variation associated with $\Delta_{\mathfrak{p},\mathcal{E}}$ provides an upper bound for Var_{Ψ_0}, namely

$$\forall\, u \in \mathrm{BV}([0,T];X),\ [a,b] \subset [0,T]: \qquad \mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[a,b]) \geq \mathrm{Var}_{\Psi_0}(u;[a,b]). \tag{3.12}$$

3.2. **Vanishing viscosity contact potentials.** While postponing the definition of BV solutions related to $\mathit{Var}_{\mathfrak{p},\mathcal{E}}$ to the next section, let us add a few remarks about the vanishing viscosity contact potential $\mathfrak{p}$

$$\mathfrak{p}(v,w) := \inf_{\varepsilon>0} \left(\Psi_\varepsilon(v) + \Psi_\varepsilon^*(w)\right) = \inf_{\varepsilon>0} \left(\varepsilon^{-1}\Psi(\varepsilon v) + \varepsilon^{-1}\Psi^*(w)\right) \quad \text{for } v \in X,\ w \in X^*. \tag{3.13}$$

which partly matches the definition introduced by [7]. We first list a set of *intrinsic* properties of $\mathfrak{p}$, which we shall prove at the end of this section.

Theorem 3.6 (Intrinsic properties of $\mathfrak{p}$)**.** *The continuous functional* $\mathfrak{p} : X \times X^* \to [0,+\infty)$ *defined by* (3.13) *satisfies the following properties:*

(I1) *For every* $v \in X, w \in X^*$ *the maps* $\mathfrak{p}(v,\cdot)$ *and* $\mathfrak{p}(\cdot,w)$ *have convex sublevels.*
(I2) $\mathfrak{p}(v,w) \geq \langle w,v\rangle$ *for every* $v \in X, w \in X^*$.

(I3) *For every* $w \in X^*$ *the map* $v \mapsto \mathfrak{p}(v,w)$ *is* 1*-homogeneous and thus convex in* X*, with* $\mathfrak{p}(v,w) > 0$ *if* $v \neq 0$*.*
(I4) *For every* $v \in X, w \in X^*$ *the map* $\lambda \mapsto \mathfrak{p}(v,\lambda w)$ *is nondecreasing in* $[0,+\infty)$*.*
(I5) *If for some* $v_0 \in X$ *and* $\bar w, w \in X^*$ *we have* $\mathfrak{p}(v_0,\bar w) < \mathfrak{p}(v_0,w)$*, then the inequality* $\mathfrak{p}(v,\bar w) \le \mathfrak{p}(v,w)$ *holds for every* $v \in X$*, and there exists* $v_1 \in X$ *such that* $\mathfrak{p}(v_1,\bar w) < \langle w, v_1\rangle$*.*

Remark 3.7 (A dual family of convex sets). Property (I5) has a dual geometric counterpart: let us first observe that for every $w \in X^*$ the map $v \mapsto \mathfrak{p}(v,w)$ is a gauge function and therefore it is the support function of the convex set

$$K^*_w := \Big\{ z \in X^* : \langle z, v\rangle \le \mathfrak{p}(v,w) \text{ for every } v \in X \Big\}, \quad \text{i.e. } \mathfrak{p}(v,w) = \sup \Big\{ \langle z, v\rangle : z \in K^*_w \Big\}.$$

Assertion (I5) then says that for every couple $w, \bar w \in X$

$$\text{we always have} \quad \bar w \in K^*_w \text{ or } w \in K^*_{\bar w} \quad \text{and, moreover,} \quad \bar w \in K^*_w \quad \Leftrightarrow \quad \mathfrak{p}(\cdot,\bar w) \le \mathfrak{p}(\cdot, w). \tag{3.14}$$

Suppose in fact that $w \notin K^*_{\bar w}$: this means that an element $v_0 \in X$ exists such that $\langle w, v_0\rangle > \mathfrak{p}(v_0,\bar w)$; by (I2) we get $\mathfrak{p}(v_0,w) > \mathfrak{p}(v_0,\bar w)$, and therefore by (I5) $\mathfrak{p}(v,w) \ge \mathfrak{p}(v,\bar w) \ge \langle \bar w, v\rangle$ for every $v \in X$, so that $\bar w \in K^*_w$. The second statement of (3.14) is an immediate consequence of the second part of (I5).

Property (I2) suggests that the set where equality holds in plays a crucial role:

Definition 3.8 (Contact set)**.** The *contact set* $\Sigma_{\mathfrak{p}} \subset X \times X^*$ is defined as

$$\Sigma_{\mathfrak{p}} := \Big\{ (v,w) \in X \times X^* : \mathfrak{p}(v,w) = \langle w, v\rangle \Big\}. \tag{3.15}$$

Here are some other useful consequences of (I1–I5)

Lemma 3.9. *If* $\mathfrak{p} : X \times X^* \to [0,+\infty)$ *satisfies* (I1–I5)*, then*

(I6) *for every* $v \in X, w \in X^*$ *we have*

$$\mathfrak{p}(v,0) + I_{K^*_0}(w) \ge \mathfrak{p}(v,w) \ge \mathfrak{p}(v,0). \tag{3.16}$$

(I7) *The contact set can be characterized by*

$$(v,w) \in \Sigma_{\mathfrak{p}} \quad \Leftrightarrow \quad w \in \partial\mathfrak{p}(\cdot,w)(v) \quad \Leftrightarrow \quad v \in \partial I_{K^*_w}(w). \tag{3.17}$$

More generally, if $\bar w \in \partial\mathfrak{p}(\cdot,w)(v)$ *then* $(v,\bar w) \in \Sigma_{\mathfrak{p}}$*,* $\bar w \in K^*_w$*, and* $\mathfrak{p}(v,w) = \mathfrak{p}(v,\bar w)$*. In particular, if* $\bar w \in \partial K^*_w$ *then* $w \in K^*_{\bar w}$*.*

Proof. The chain of inequalities in (3.16) is an immediate consequence of (I4) and of (3.14). (3.17) is a direct consequence of the fact that $v \mapsto \mathfrak{p}(v,w)$ is a gauge function and $I_{K^*_w}$ is its Legendre transform.

In order to check the last statement, given $v \in X, w \in X^*$ let us take $\bar w \in \partial\mathfrak{p}(\cdot,w)(v)$ so that $\bar w \in K^*_w$ and $\mathfrak{p}(v,w) = \langle \bar w, v\rangle$. Combining (I2) with (3.14) we get $\mathfrak{p}(v,w) = \mathfrak{p}(v,\bar w)$, so that $(v,\bar w) \in \Sigma_{\mathfrak{p}}$. □

Remark 3.10. Properties (I1,I2,I5) suggest a strong analogy between $\mathfrak{p}$ and the notion of bipotential introduced by [7]: according to [7], a bipotential is a functional $\mathfrak{b} : X \times X^* \to (-\infty, +\infty]$ which is *convex* and lower semicontinuous in each argument, satisfies (I2), and whose contact set fulfils a condition similar to (3.17)

$$(v, w) \in \Sigma_{\mathfrak{b}} \quad \Leftrightarrow \quad w \in \partial\mathfrak{b}(\cdot, w)(v) \quad \Leftrightarrow \quad v \in \partial\mathfrak{b}(v, \cdot)(w).$$

In our situation, (3.17) is a direct consequence of the homogeneity of $\mathfrak{p}$, but the convexity condition with respect to w looks too restrictive, as shown by this simple example. Consider the case $X = X^* = \mathbb{R}^2$, with $\Psi(v) := \|v\|_1 + \Psi_V(v)$, $\|v\|_1 := |v_1| + |v_2|$, and

$$\Psi_V(v) := \frac{1}{2}v_1^2 + \frac{1}{4}v_2^4, \quad v = (v_1, v_2) \in \mathbb{R}^2; \qquad \Psi_V^*(w) = \frac{1}{2}w_1^2 + \frac{3}{4}w_2^{4/3} \quad w = (w_1, w_2) \in \mathbb{R}^2.$$

By (3.6) we have $\mathfrak{p}(v, w) = \|v\|_1 + \mathfrak{p}_V(v, w)$ with $\mathfrak{p}_V(v, w) = \inf_{\varepsilon>0} \frac{1}{\varepsilon}\Big(\Psi_V(\varepsilon v) + \Psi^*(w)\Big)$ and find

$$\Psi^*(w) = \frac{1}{2}(|w_1| - 1)_+^2 + \frac{3}{4}(|w_2| - 1)_+^{4/3}.$$

Considering the special case $v = (v_1, 0),\ w = (0, w_2)$, we obtain

$$\mathfrak{p}_V((v_1, 0), (0, w_2)) = \sqrt{3/2}\,|v_1|\Big((|w_2| - 1)_+\Big)^{2/3}.$$

The map $w_2 \mapsto \mathfrak{p}((v_1, 0), (0, w_2))$ is therefore not convex.

Let us now consider some properties of $\mathfrak{p}$ and its contact set $\Sigma_{\mathfrak{p}}$ involving explicitly the functional Ψ. Since the vanishing viscosity contact potential $\mathfrak{p}$ is defined through the minimum procedure (3.13), the contact set is strictly related to the set of optimal $\varepsilon > 0$ attaining the minimum in (3.13).

Definition 3.11 (Lagrange multipliers)**.** For every $(v, w) \in X \times X^*$ we introduce the multivalued function Λ (with possibly empty values)

$$\Lambda(v, w) := \Big\{\varepsilon \geq 0 : \mathfrak{p}(v, w) = \Psi_\varepsilon(v) + \Psi_\varepsilon^*(w)\Big\} \subset [0, +\infty). \tag{3.18}$$

Notice that for every $(v, w) \in X \times X^*$ the function $\varepsilon \mapsto \varepsilon^{-1}\Psi(\varepsilon v) + \varepsilon^{-1}\Psi^*(w)$ is convex on $(0, +\infty)$. Since Ψ has superlinear growth at infinity, it goes to $+\infty$ as $\varepsilon \uparrow +\infty$ if $v \neq 0$, so that

(3.19) the set $\Lambda(v, w)$ is always a bounded closed interval if $v \neq 0$.

Theorem 3.12 (Properties of $\mathfrak{p}, \Psi$ and $\Sigma_{\mathfrak{p}}$)**.**

(P1) *The vanishing viscosity contact potential* $\mathfrak{p}$ *satisfies* $\mathfrak{p}(v, 0) = \Psi_0(v)$, $K_0^* = K^*$, *and in particular*

$$\mathfrak{p}(v, w) \geq \langle w, v\rangle, \quad \Psi_0(v) + \mathrm{I}_{K^*}(w) \geq \mathfrak{p}(v, w) \geq \Psi_0(v) \geq 0 \quad \textit{for every } v \in X,\ w \in X^*, \tag{3.20}$$

$$\mathfrak{p}(v, w) = \Psi_0(v) \quad \Leftrightarrow \quad w \in K^*. \tag{3.21}$$

(P2) *For every $w \in X^*$, the convex sets K^*_w are the sublevels of Ψ^**

$$K^*_w = \Big\{ z \in X^* : \Psi^*(z) \le \Psi^*(w) \Big\}, \tag{3.22}$$

and $\mathfrak{p}$ admits the dual representation

$$\mathfrak{p}(v,w) = \sup \Big\{ \langle z, v\rangle \,:\, z \in X^*,\ \Psi^*(z) \le \Psi^*(w) \Big\}. \tag{3.23}$$

In particular, $\Psi^*(w_1) \le \Psi^*(w_2)$ for some $w_1, w_2 \in X^*$ if and only if $\mathfrak{p}(v,w_1) \le \mathfrak{p}(v,w_2)$ for every $v \in X$.

(P3) *The multivalued function Λ defined in* (3.18) *is upper semicontinuous, i.e.*

$$\text{if } (v_n,w_n) \to (v,w) \in X \times X^* \text{ and } \varepsilon_n \in \Lambda(v_n,w_n) \to \varepsilon, \quad \text{then} \quad \varepsilon \in \Lambda(v,w). \tag{3.24}$$

(P4) *The contact set $\Sigma_{\mathfrak{p}}$* (3.15) *can be characterized by*

$$w \in \partial\Psi_0(v) \subset K^* \quad \text{or,} \quad \text{if } w \notin K^*, \quad \exists\, \varepsilon > 0 : w \in \partial\Psi(\varepsilon v), \tag{3.25}$$

and the last inclusion holds exactly for $\varepsilon \in \Lambda(v,w)$. Equivalently,

$$(v,w) \in \Sigma_{\mathfrak{p}} \quad \Leftrightarrow \quad w \in \partial\Psi_\varepsilon(v) \quad \text{for every } \varepsilon \in \Lambda(v,w).$$

In particular, in the case of additive viscosity, *with $\Psi(v) = \Psi_0(v) + \Psi_V(v)$ and Ψ_V satisfying* (2.20), *we simply have*

$$(v,w) \in \Sigma_{\mathfrak{p}} \quad \Longleftrightarrow \quad \exists\, \lambda \ge 0 : \quad w \in \partial\Psi_0(v) + \partial\Psi_V(\lambda v). \tag{3.26}$$

Proofs of Theorems 3.12 and 3.6.
Ad (P1). Inequalities (3.20) are immediate consequences of the definition of $\mathfrak{p}$. The equality $\Psi_0(v) = \mathfrak{p}(v,w)$ is equivalent to the existence of a sequence $\varepsilon_k > 0$ such that (recall that $\varepsilon^{-1}\Psi_\varepsilon(\varepsilon v) \ge \Psi_0(v)$)

$$\lim_{k\to\infty} \varepsilon_k^{-1}\Psi(\varepsilon_k v) = \Psi_0(v), \quad \lim_{k\to\infty} \varepsilon_k^{-1}\Psi^*(w) = 0.$$

Since the first inequality prevents ε_k from diverging to $+\infty$ (being Ψ superlinear), from the second limit we get $\Psi^*(w) = 0$, i.e.

$$\langle w, z\rangle \le \Psi(z) \quad \forall\, z \in X.$$

Replacing z with εz, multiplying the previous inequality by ε^{-1}, and passing to the limit as $\varepsilon \downarrow 0$, in view of (Ψ.3) we conclude

$$\langle w, z\rangle \le \Psi_0(z) \quad \forall\, z \in X, \quad \text{so that} \quad w \in K^*.$$

The converse implication in (3.21) is immediate.
Ad (P2). Since the sublevels of Ψ^* are closed and convex, a duality argument shows that (3.22) is equivalent to (3.23). In order to prove the latter formula, let us observe that, if $\Psi^*(z) \le \Psi^*(w)$, then $\langle z, v\rangle \le \mathfrak{p}(v,w)$, because the Fenchel inequality yields

$$\langle z, v\rangle = \varepsilon^{-1}\langle z, \varepsilon v\rangle \le \varepsilon^{-1}\Psi(\varepsilon v) + \varepsilon^{-1}\Psi(z) = \Psi_\varepsilon(v) + \Psi^*_\varepsilon(z) \le \Psi_\varepsilon(v) + \Psi^*_\varepsilon(w) \quad \text{for every } \varepsilon > 0.$$

We show that there exists $z \in X^*$ such that $\Psi^*(z) \le \Psi^*(w)$ and $\mathfrak{p}(v,w) = \langle z, v\rangle$. Due to (3.21), if $w \in K^*$, then $\mathfrak{p}(v,w) = \Psi_0(v)$ and the thesis follows from (2.5) Hence, let us suppose that $w \notin K^*$ and $v \neq 0$; then we can choose $\varepsilon_0 \in \Lambda(v,w)$, $\varepsilon_0 > 0$, such that

$$\mathfrak{p}(v,w) = \varepsilon_0^{-1}\Psi(\varepsilon_0 v) + \varepsilon_0^{-1}\Psi^*(w) \le \varepsilon^{-1}\Psi(\varepsilon v) + \varepsilon^{-1}\Psi^*(w) \quad \text{for every } \varepsilon > 0. \tag{3.27}$$

Choosing $z_\varepsilon \in \partial\Psi(\varepsilon v)$ we have

$$\Psi(\varepsilon v) - \Psi(\varepsilon_0 v) \le \langle z_\varepsilon, (\varepsilon - \varepsilon_0)v\rangle \quad \text{for every } \varepsilon > 0$$

so that, in view of inequality (3.27),

$$\big(\varepsilon^{-1} - \varepsilon_0^{-1}\big)\Big(\Psi(\varepsilon_0 v) + \Psi^*(w)\Big) + \varepsilon^{-1}\langle z_\varepsilon, (\varepsilon - \varepsilon_0)v\rangle \ge 0 \quad \text{for every } \varepsilon > 0.$$

Dividing by $\varepsilon - \varepsilon_0$ and passing to the limit first as $\varepsilon \downarrow \varepsilon_0$ and then as $\varepsilon \uparrow \varepsilon_0$, we thus find $z_\pm \in \partial\Psi(\varepsilon_0 v)$ (accumulation points of the sequences $(z_\varepsilon : \varepsilon > \varepsilon_0)$ and $(z_\varepsilon : \varepsilon < \varepsilon_0)$, respectively), such that

$$\langle z_-, v\rangle \le \mathfrak{p}(v, w) = \varepsilon_0^{-1}\Big(\Psi(\varepsilon_0 v) + \Psi^*(w)\Big) \le \langle z_+, v\rangle. \tag{3.28}$$

On the other hand, the Fenchel identity of convex analysis yields

$$\varepsilon_0^{-1}\Psi^*(z) = \langle z, v\rangle - \varepsilon_0^{-1}\Psi(\varepsilon_0 v) \quad \text{for every } z \in \partial\Psi(\varepsilon_0 v) \tag{3.29}$$

so that the map $z \mapsto \Psi^*(z)$ is affine on $\partial\Psi(\varepsilon_0 v)$ and a comparison between (3.28) and (3.29) yields

$$\Psi^*(z_-) \le \Psi^*(w) \le \Psi^*(z_+).$$

Using formula (3.29) we can thus find $\theta \in [0, 1]$ and $z_\theta := (1 - \theta)z_- + \theta z_+ \in \partial\Psi(\varepsilon_0 v)$ such that

$$\Psi^*(z_\theta) = \Psi^*(w), \quad \langle z_\theta, v\rangle = \mathfrak{p}(v, w) = \varepsilon_0^{-1}\Big(\Psi(\varepsilon_0 v) + \Psi^*(w)\Big).$$

The last statement of (P2) follows easily. One implication is immediate. On the other hand, if $\Psi^*(w_1) > \Psi^*(w_2)$, then by the Hahn-Banach separation theorem we can find $\bar v \in X$ and $\delta > 0$ such that

$$\langle w_1, \bar v\rangle \ge \delta + \langle z, \bar v\rangle \quad \text{for every } z \in X^* \text{ such that } \Psi^*(z) \le \Psi^*(w_2),$$

and, therefore, by (3.23) we conclude $\mathfrak{p}(\bar v, w_1) \ge \langle w_1, \bar v\rangle \ge \delta + \mathfrak{p}(\bar v, w_2)$.

Ad (I1,2,3,4,5) These properties directly follow from (P2).

Ad (P3) and continuity of $\mathfrak{p}$. Notice that $\mathfrak{p}$ is upper semicontinuous, being defined as the infimum of a family of continuous functions. Take now converging sequences (v_n), (w_n), (ε_n) as in (3.24): we have that

$$\liminf_{n\to\infty} \big(\varepsilon_n^{-1}\Psi(\varepsilon_n v_n) + \varepsilon_n^{-1}\Psi^*(w_n)\big) \ge \Psi_\varepsilon(v) + \Psi_\varepsilon^*(w) = \begin{cases} \varepsilon^{-1}\Psi(\varepsilon v) + \varepsilon^{-1}\Psi^*(w) & \text{if } \varepsilon > 0, \\ \Psi_0(v) + \mathrm{I}_K^*(w) & \text{if } \varepsilon = 0. \end{cases}$$

Since

$$\begin{aligned} \mathfrak{p}(v, w) \ge \liminf_{n\to\infty} \mathfrak{p}(v_n, w_n) &\ge \liminf_{n\to\infty} \big(\varepsilon_n^{-1}\Psi(\varepsilon_n v_n) + \varepsilon_n^{-1}\Psi^*(w_n)\big) \\ &\ge \Psi_\varepsilon(v) + \Psi_\varepsilon^*(w) \ge \mathfrak{p}(v, w), \end{aligned} \tag{3.30}$$

we obtain $\varepsilon \in \Lambda(v, w)$.
Inequality (3.30) shows that $\mathfrak{p}$ is also lower semicontinuous, since, if $v \ne 0$, any sequence $\varepsilon_n \in \Lambda(v_n, w_n)$ admits a converging subsequence, in view of (3.19).
Ad (P4). Concerning the characterization (3.25) of $\Sigma_{\mathfrak{p}}$, it is easy to check that, if (v, w) satisfies (3.25), then by the Fenchel identity and formula (2.2) we have, when $w \in K^*$,

$$\mathfrak{p}(v, w) \ge \langle w, v\rangle = \Psi_0(v) = \mathfrak{p}(v, w),$$

and, when $w \notin K^*$,

$$\mathfrak{p}(v,w) \geq \langle w, v\rangle = \varepsilon^{-1}\langle w, \varepsilon v\rangle = \varepsilon^{-1}\Psi(\varepsilon v) + \varepsilon^{-1}\Psi^*(w) \geq \mathfrak{p}(v,w)$$

so that $(v,w) \in \Sigma_{\mathfrak{p}}$ and $\varepsilon \in \Lambda(v,w)$. Conversely, if $\mathfrak{p}(v,w) = \langle w, v\rangle$ and $w \in K^*$, then by (3.20) $\Psi_0(v) = \langle w, v\rangle$ and therefore $w \in \partial\Psi_0(v)$. If $w \notin K^*$, then, choosing $\varepsilon \in \Lambda(v,w)$, we have

$$\Psi(\varepsilon v) + \Psi^*(w) = \varepsilon\mathfrak{p}(v,w) = \langle w, \varepsilon v\rangle, \quad \text{so that} \quad w \in \partial\Psi(\varepsilon v).$$

In the particular case of (2.21), (3.26) follows now from (3.25) by the sum rule of the subdifferentials and the 0-homogeneity of $\partial\Psi_0$. □

4. BV solutions and energy-driven dissipation

4.1. BV **solutions.** We can now give our precise definition of BV solution of the rate-independent system $(X, \mathcal{E}, \mathfrak{p})$, driven by the vanishing viscosity contact potential $\mathfrak{p}$ (3.13) and the energy $\mathcal{E}$. From a formal point of view, the definition simply replaces the global stability condition (S) by the local one ($\mathrm{S}_{\mathrm{loc}}$), and the Ψ_0-total variation in the energy balance (E) by the "Finsler" total variation (3.11), induced by $\mathfrak{p}$ and $\mathcal{E}$.

Definition 4.1 (BV solutions, variational characterization)**.** A curve $u \in \mathrm{BV}([0,T];X)$ is a BV solution of the rate independent system $(X, \mathcal{E}, \mathfrak{p})$ the *local stability* ($\mathrm{S}_{\mathrm{loc}}$) and the $(\mathfrak{p}, \mathcal{E})$-*energy balance* hold:

$$(\mathrm{S}_{\mathrm{loc}}) \qquad -\mathrm{D}\mathcal{E}_t(u(t)) \in K^* \quad \text{for a.a. } t \in [0,T] \setminus J_u$$

$$(\mathrm{E}_{\mathfrak{p},\mathcal{E}}) \qquad \mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[0,t]) + \mathcal{E}_t(u(t)) = \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T].$$

We shall see in the next Section 4.3 that any pointwise limit, as $\varepsilon \downarrow 0$, of the solutions (u_ε) of the viscous equation (DN_ε) or, as τ, $\varepsilon \downarrow 0$, of the discrete solutions $(\overline{\mathrm{U}}_{\tau,\varepsilon})$ of the viscous incremental problems (IP_ε), is a BV solutions induced by the vanishing viscosity contact potential $\mathfrak{p}$. Let us first get more insight into Definition 4.1.

Properties of BV **solutions.** As in the case of energetic solutions, it is not difficult to see that the energy balance ($\mathrm{E}_{\mathfrak{p},\mathcal{E}}$) holds on any subinterval $[t_0, t_1] \subset [0,T]$; moreover, if the local stability condition ($\mathrm{S}_{\mathrm{loc}}$) holds, to check ($\mathrm{E}_{\mathfrak{p},\mathcal{E}}$) it is sufficient to prove the corresponding inequality.

Proposition 4.2. *If $u \in \mathrm{BV}([0,T];X)$ satisfies ($\mathrm{E}_{\mathfrak{p},\mathcal{E}}$), then for every subinterval $[t_0,t_1]$ there holds*

$$(\mathrm{E}'_{\mathfrak{p},\mathcal{E}}) \qquad \mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[t_0,t_1]) + \mathcal{E}_{t_1}(u(t_1)) = \mathcal{E}_{t_0}(u(t_0)) + \int_{t_0}^{t_1} \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s.$$

Moreover, if u satisfies ($\mathrm{S}_{\mathrm{loc}}$)*, then* ($\mathrm{E}_{\mathfrak{p},\mathcal{E}}$) *is equivalent to the energy inequality*

$$(\mathrm{E}_{\mathfrak{p},\mathcal{E};\mathrm{ineq}}) \qquad \mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[0,T]) + \mathcal{E}_T(u(T)) \leq \mathcal{E}_0(u(0)) + \int_0^T \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s.$$

Proof. ($\mathrm{E}'_{\mathfrak{p},\mathcal{E}}$) easily follows from the additivity property

$$\forall\, 0 \le t_0 < t_1 < t_2 \le T: \quad \mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[t_0,t_1]) + \mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[t_1,t_2]) = \mathit{Var}_{\mathfrak{p},\mathcal{E}}(u;[t_0,t_2]). \tag{4.1}$$

In order to prove the second inequality we argue as in [35, Prop. 4], taking ($\mathrm{S}_{\mathrm{loc}}$) into account. □

Notice that, by (3.12), any BV solution is also a local solution according to Definition 2.6, i.e. it satisfies the local stability condition and energy inequality ($\mathrm{E}'_{\mathrm{ineq}}$). In fact, one has a more accurate description of the jump conditions, as the following Theorem shows (cf. with Propositions 2.2 and 2.7).

Theorem 4.3 (Differential characterization of BV solutions)**.** *A curve* $u \in \mathrm{BV}([0,T];X)$ *is a* BV *solution of the rate-independent system* $(X,\mathcal{E},\mathfrak{p})$ *if and only if it satisfies the doubly nonlinear differential inclusion in the* BV *sense*

$$\partial\Psi_0\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{d\mu}(t)\Big) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \quad \text{for } \mu\text{-a.e. } t \in [0,T], \qquad \mu := \mathscr{L}^1 + |u'_{\mathrm{C}}|, \tag{$\mathrm{DN}_{0,\mathrm{BV}}$}$$

and the following jump conditions at each point $t \in \mathrm{J}_u$ *of the jump set* (2.9)

$$\begin{aligned} \mathcal{E}_t(u(t)) - \mathcal{E}_t(u(t_-)) &= -\Delta_{\mathfrak{p},\mathcal{E}}(t;u(t_-),u(t)),\\ \mathcal{E}_t(u(t_+)) - \mathcal{E}_t(u(t)) &= -\Delta_{\mathfrak{p},\mathcal{E}}(t;u(t),u(t_+)),\\ \mathcal{E}_t(u(t_+)) - \mathcal{E}_t(u(t_-)) &= -\Delta_{\mathfrak{p},\mathcal{E}}(t;u(t_-),u(t_+)). \end{aligned} \tag{J_{BV}}$$

Proof. We have already seen (see Lemma 2.7) that local solutions satisfy ($\mathrm{DN}_{0,\mathrm{BV}}$). The jump conditions ($\mathrm{J}_{\mathrm{BV}}$) can be obtained by localizing ($\mathrm{E}'_{\mathfrak{p},\mathcal{E}}$) around any jump time $t \in \mathrm{J}_u$.

Conversely, to prove ($\mathrm{E}_{\mathfrak{p},\mathcal{E};\mathrm{ineq}}$) (as seen in the proof of Lemma 2.7, ($\mathrm{S}_{\mathrm{loc}}$) ensues from ($\mathrm{DN}_{0,\mathrm{BV}}$)), we argue as in the second part of the proof of Lemma 2.7, still applying (2.32) and (2.33), but replacing inequalities (2.34) with the following identities,

$$E_-(t) = -\Delta_{\mathfrak{p},\mathcal{E}}(t;u(t_-),u(t)), \quad E_+(t) = -\Delta_{\mathfrak{p},\mathcal{E}}(t;u(t),u(t_+)) \quad \text{for all } t \in \mathrm{J}_u,$$

which are due to (J_{BV}). Hence, $-\mathrm{Jmp}(\mathcal{E};[0,T]) = \mathrm{Jmp}_{\mathfrak{p},\mathcal{E}}(u;[0,T])$. Then, ($\mathrm{E}_{\mathfrak{p},\mathcal{E};\mathrm{ineq}}$) follows from (2.32). □

The next section is devoted to a refined description of the behavior of a BV solution along the jumps.

4.2. **Jumps and optimal transitions.**

Let us first introduce the notion of *optimal transition*.

Definition 4.4. Let $t \in [0,T]$, $u_-, u_+ \in X$ with $-\mathrm{D}\mathcal{E}_t(u_-)$, $-\mathrm{D}\mathcal{E}_t(u_+) \in K^*$, and $-\infty \le r_0 < r_1 \le +\infty$. An absolutely continuous curve $\vartheta : [r_0,r_1] \to X$ connecting $u_- = \vartheta(r_0)$ and $u_+ = \vartheta(r_1)$ is an *optimal* $(\mathfrak{p},\mathcal{E}_t)$*-transition* between u_- and u_+ if

$$\dot\vartheta(r) \neq 0 \quad \text{for a.a. } r \in (r_0,r_1); \quad \Psi_{0*}(-\mathrm{D}\mathcal{E}_t(\vartheta(r))) \ge 1 \quad \forall\, r \in [r_0,r_1], \tag{O.1}$$

$$\mathcal{E}_t(u_-) - \mathcal{E}_t(u_+) = \Delta_{\mathfrak{p},\mathcal{E}}(t;u_-,u_+) = \int_{r_0}^{r_1} \mathfrak{p}(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r)))\,\mathrm{d}r. \tag{O.2}$$

We also say that an optimal transition ϑ is of

$$
\begin{array}{lrll}
(\mathrm{O}_{\mathrm{sliding}}) & \textit{sliding} \text{ type if} & -\mathrm{D}\mathcal{E}_t(\vartheta(r)) \in K^* & \text{for every } r \in [r_0, r_1], \\
(\mathrm{O}_{\mathrm{viscous}}) & \textit{viscous} \text{ type if} & -\mathrm{D}\mathcal{E}_t(\vartheta(r)) \notin K^* & \text{for every } r \in (r_0, r_1), \\
(\mathrm{O}_{\mathrm{ener}}) & \textit{energetic} \text{ type if} & \mathcal{E}_t(u_+) - \mathcal{E}_t(u_-) = -\Psi_0(u_+ - u_-). &
\end{array}
$$

We denote by $\Theta(t; u_-, u_+)$ the (possibly empty) collection of such optimal transitions, with normalized domain $[0,1]$ and *constant Finsler velocity*

$$
\mathfrak{p}(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r))) \equiv \mathcal{E}_t(u_-) - \mathcal{E}_t(u_+) \qquad \text{for a.a.}\, r \in (0,1)\,. \tag{4.2}
$$

Remark 4.5. Notice that the notion of optimal transition is invariant by absolutely continuous (monotone) time rescalings with absolutely continuous inverse; moreover, any optimal transition ϑ has finite length, it admits a reparametrization with constant Finsler velocity $\mathfrak{p}(\dot\vartheta(\cdot), -\mathrm{D}\mathcal{E}_t(\vartheta(\cdot)))$, and is a minimizer of (3.8), so that it is not restrictive to assume $\vartheta \in \Theta(t, u_-, u_+)$.

Theorem 4.6. *A local solution $u \in \mathrm{BV}([0,T]; X)$ is a* BV *solution according to Definition 4.1 if and only if at every jump time $t \in \mathrm{J}_u$ the initial and final values $u(t_-)$ and $u(t_+)$ can be connected by an optimal transition curve $\vartheta^t \in \Theta(t; u(t_-), u(t_+))$, and there exists $r \in [0,1]$ such that $u(t) = \vartheta^t(r)$. Any optimal transition curve ϑ satisfies the* contact condition

$$
(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r))) \in \Sigma_{\mathfrak{p}} \quad \textit{for a.a. } r \in (0,1). \tag{4.3}
$$

Proof. Taking into account Theorem 4.3, the proof of the first part of the statement is immediate. To prove (4.3), let t be a jump point of u and let us first suppose that $u(t_-) = u(t) \neq u(t_+)$. By Remark 3.3, we can find a Lipschitz curve $\vartheta_{01} \in \mathrm{AC}([r_0, r_1]; X)$ with normalized speed $\mathfrak{p}(\dot\vartheta, -\mathrm{D}\mathcal{E}_t(\vartheta)) \equiv 1$, connecting $u(t_-)$ to $u(t_+)$, so that the jump condition $(\mathrm{J}_{\mathrm{BV}})$ yields

$$
\int_{r_0}^{r_1} \langle -\mathrm{D}\mathcal{E}_t(\vartheta(r)), \dot\vartheta(r)\rangle \,\mathrm{d}r = \mathcal{E}_t(u(t_-)) - \mathcal{E}_t(u(t_+)) = \int_{r_0}^{r_1} \mathfrak{p}(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r)))\,\mathrm{d}r.
$$

This shows that ϑ is an optimal transition curve and satisfies

$$
\int_{r_0}^{r_1} \Big(\mathfrak{p}(\dot\vartheta, -\mathrm{D}\mathcal{E}_t(\vartheta(r)))\,\mathrm{d}r - \langle -\mathrm{D}\mathcal{E}_t(\vartheta(r)), \dot\vartheta(r)\rangle \Big)\,\mathrm{d}r = 0.
$$

Since the integrand is always nonnegative, it follows that (4.3) holds.

In the general case, when u is not left or right continuous at t, we join two (suitably rescaled) optimal transition curves $\vartheta_{01} \in \Theta(t; u(t_-), u(t))$ and $\vartheta_{12} \in \Theta(t; u(t), u(t_+))$. $\square$

The next result provides a careful description of $(\mathfrak{p}, \mathcal{E}_t)$-optimal transitions.

Theorem 4.7. *Let $t \in [0,T]$, u_-, $u_+ \in X$, and $\vartheta : [0,1] \to X$ be an optimal transition curve in $\Theta(t; u_-, u_+)$. Then,*

(1) ϑ *is a constant-speed minimal geodesic for the (possibly asymmetric) Finsler cost* $\Delta_{\mathfrak{p},\mathcal{E}}(t;u_-,u_+)$*, and for every* $0 \le \rho_0 < \rho_1 \le 1$ *it satisfies*

$$\begin{aligned}\mathcal{E}_t(\vartheta(\rho_0)) - \mathcal{E}_t(\vartheta(\rho_1)) &= \Delta_{\mathfrak{p},\mathcal{E}}(t;\vartheta(\rho_0),\vartheta(\rho_1)) \\ &= (\rho_1-\rho_0)\Delta_{\mathfrak{p},\mathcal{E}}(t;u_-,u_+) = (\rho_1-\rho_0)\Big(\mathcal{E}_t(u_-)-\mathcal{E}_t(u_+)\Big);\end{aligned} \tag{4.4}$$

In particular, the map $\rho \mapsto \mathcal{E}_t(\vartheta(\rho))$ *is affine.*

(2) *An optimal transition* ϑ *is of* sliding *type* $(\mathrm{O}_{\mathrm{sliding}})$ *if and only if it satisfies*

$$\partial\Psi_0(\dot\vartheta(r)) + \mathrm{D}\mathcal{E}_t(\vartheta(r)) \ni 0 \quad \textit{for a.a. } r \in (0,1), \tag{4.5}$$

$$\Psi_{0*}(-\mathrm{D}\mathcal{E}_t(\vartheta(r))) = 1 \quad \textit{for every } r \in [0,1]. \tag{4.6}$$

(3) *An optimal transition* ϑ *is of* viscous *type* $(\mathrm{O}_{\mathrm{viscous}})$ *if and only if there holds for every selection* $(0,1) \ni r \mapsto \varepsilon(r)$ *in* $\Lambda(\dot\vartheta(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r))$

$$\partial\Psi(\varepsilon(r)\dot\vartheta(r)) + \mathrm{D}\mathcal{E}_t(\vartheta(r)) \ni 0 \quad \textit{for a.a.}\, r \in (0,1). \tag{4.7}$$

Equivalently, there exists an absolutely continuous, surjective time rescaling $\mathsf{r} : (\rho_0,\rho_1) \to (0,1)$*, with* $-\infty \le \rho_0 < \rho_1 \le \infty$ *and* $\dot{\mathsf{r}}(s) > 0$ *for* $\mathscr{L}^1$ *a.e.* $s \in (\rho_0,\rho_1)$*, such that the rescaled transition* $\theta(s) := \vartheta(\mathsf{r}(s))$ *satisfies the viscous differential inclusion*

$$\partial\Psi(\dot\theta(s)) + \mathrm{D}\mathcal{E}_t(\theta(s)) \ni 0 \quad \textit{for a.a. } s \in (\rho_0,\rho_1)\,, \quad \textit{with} \ \lim_{s\downarrow\rho_0}\theta(s) = u_-, \quad \lim_{s\uparrow\rho_1}\theta(s) = u_+\,. \tag{4.8}$$

(4) *Any optimal transition* ϑ *can be decomposed in a canonical way into an (at most) countable collection of optimal* sliding and viscous *transitions. In other words, there exists (uniquely determined) disjoint open intervals* $(S_j)_{j\in\sigma}$ *and* $(V_k)_{k\in\upsilon}$ *of* $(0,1)$*, with* $\sigma,\upsilon \subset \mathbb{N}$*, such that* $(0,1) \subset \big(\cup_{j\in\sigma}\overline{S_j}\big) \cup \big(\cup_{k\in\upsilon} V_k\big)$ *and*

$$\vartheta|_{\overline{S_j}} \quad \textit{is of sliding type,} \quad \vartheta|_{V_k} \quad \textit{is of viscous type.}$$

(5) *An optimal transition* ϑ *is of* energetic *type* $(\mathrm{O}_{\mathrm{ener}})$ *if and only if* ϑ *is of sliding type and it is a* Ψ_0*-minimal geodesic, i.e.*

$$\Psi_0(\vartheta(r_1)-\vartheta(r_0)) = (r_1-r_0)\Psi_0(u_1-u_0) \quad \textit{for every } 0 \le r_0 < r_1 \le 1. \tag{4.9}$$

If Ψ_0 *has strictly convex sublevels, then* ϑ *is linear and* $r \mapsto (\vartheta(r),\mathcal{E}_t(\vartheta(r)))$ *is a linear segment contained in the graph of* $\mathcal{E}_t$*.*
If Ψ_0 *is Gâteaux-differentiable at* $X \setminus \{0\}$ *then*

$$-\mathrm{D}\mathcal{E}_t(\vartheta(r)) = \mathrm{D}\Psi_0(u_+-u_-) \quad \textit{for every } r \in [0,1].$$

In particular, the map $r \mapsto -\mathrm{D}\mathcal{E}_t(\vartheta(r))$ *is constant.*

Remark 4.8. It follows from the characterization in **(2)** of Theorem 4.7 (cf. with (4.5)–(4.6)) that sliding optimal transitions are independent of the form of the vanishing viscosity contact potential $\mathfrak{p}$, and thus on the particular viscosity potential Ψ.

Instead, as one may expect, Ψ occurs in the doubly nonlinear equation (4.7) (equivalently, in (4.8)), which in fact describes the viscous transient regime. Hence, different choices of the viscous dissipation Ψ shall give raise to a different behavior in the viscous jumping regime, see also the example in [51, Sec. 2.2]. The latter paper sets forth a different characterization of rate-independent evolution, still oriented towards local stability, but derived from a global-in-time variational principle and not a vanishing viscosity approach.

Proof. **Ad (1).** The geodesic property follows from the minimality of ϑ (cf. with (O.2) in Definition 4.4). Then, there holds

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{E}_t(\vartheta(r)) = -\mathfrak{p}(\dot{\vartheta}(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r))) \equiv \mathcal{E}_t(u_+) - \mathcal{E}_t(u_-) \quad \text{for a.a. } r \in (0,1), \tag{4.10}$$

where the first identity ensues from the chain rule (2.26) for $\mathcal{E}$ and the contact condition (4.3), and the second one from (4.2). Clearly, (4.10) implies (4.4).

Ad (2). If ϑ is of sliding type, then the contact condition (4.3), with (3.25), yields (4.5); (4.6) follows since $\dot{\vartheta} \neq 0$ a.e. in $(0,1)$.

Ad (3). Equation (4.7) still follows from (3.25). Choosing $r_0 \in (0,1)$ and a Borel selection $\varepsilon(r) \in \Lambda(\dot{\vartheta}(r), -\mathrm{D}\mathcal{E}_t(\vartheta(r)))$ (which is therefore locally bounded away from 0), we set

$$\mathsf{s}(r) := \int_{r_0}^{r} \varepsilon^{-1}(\rho)\,\mathrm{d}\rho, \quad \mathsf{r} := \mathsf{s}^{-1}, \tag{4.11}$$

so that r is defined in a suitable interval of $\mathbb{R}$ and satisfies

$$\dot{\mathsf{r}}(s) = \varepsilon(\mathsf{r}(s)), \quad \dot{\theta}(s) = \varepsilon(\mathsf{r}(s))\vartheta(\mathsf{r}(s)).$$

Ad (4). We simply introduce the disjoint open sets

$$V := \Big\{ r \in (0,1) : -\mathrm{D}\mathcal{E}_t(\vartheta(r)) \notin K^* \Big\}, \quad S := (0,1) \setminus \overline{V}$$

and we consider their canonical decomposition in connected components.

Ad (5). If ϑ is energetic, then by ($\mathrm{O}_{\mathrm{ener}}$) and (4.4) there holds $\Delta_{\mathfrak{p},\mathcal{E}}(t; u_-, u_+) = \Psi_0(u_+ - u_-)$. Thus, taking into account (4.2) and (3.3) as well, we find $\mathfrak{p}(\dot{\vartheta}, -\mathrm{D}\mathcal{E}_t(\vartheta(r))) = \Psi_0(\dot{\vartheta}(r))$ for a.a. $r \in (0,1)$. Since its Ψ_0-velocity is constant and the total length is $\Psi_0(u_+ - u_-)$, we deduce that ϑ is a constant speed minimal geodesic for Ψ_0. Conversely, the constraint $-\mathrm{D}\mathcal{E}_t(\vartheta(r)) \in K^*$ satisfied by sliding transitions yields, in view of (3.21), that $\mathfrak{p}(\dot{\vartheta}, -\mathrm{D}\mathcal{E}_t(\vartheta(r))) = \Psi_0(\dot{\vartheta}(r))$ for a.a. $r \in (0,1)$. Therefore,

$$\Delta_{\mathfrak{p},\mathcal{E}}(t; u_-, u_+) = \int_0^1 \Psi_0(\dot{\vartheta}(r))\,\mathrm{d}r = \Psi_0(u_+ - u_-)$$

by the geodesic property (4.9).

It is well known that, if Ψ_0 has strictly convex sublevels, the related geodesics are linear segments. In order to prove the last statement, let us observe that for every $\xi \in \partial\Psi_0(u_+ - u_-) \subset K^*$ there holds

$$\int_0^1 \langle \xi, \dot{\vartheta}(r) \rangle\,\mathrm{d}r = \langle \xi, u_+ - u_- \rangle = \Psi_0(u_+ - u_-) = \int_0^1 \Psi_0(\dot{\vartheta}(r))\,\mathrm{d}r\,,$$

where the second equality follows from the characterization (2.2) of $\partial\Psi_0(u_+ - u_-)$. Hence,

$$\int_0^1 \Big(\Psi_0(\dot{\vartheta}(r)) - \langle \xi, \dot{\vartheta}(r) \rangle \Big)\,\mathrm{d}r = 0.$$

Since the above integrand is nonnegative (being $\xi \in K^*$), again by (2.2) we deduce that $\xi \in \partial\Psi_0(\dot{\vartheta}(r))$ for a.a. $r \in (0,1)$. On the other hand, if Ψ_0 is Gâteaux-differentiable outside 0, its subdifferential contains just one point. Ultimately, (4.5) (recall that ϑ is of sliding type) shows that $-\mathrm{D}\mathcal{E}_t(\vartheta(r)) = \xi$ for every $r \in [0,1]$. $\square$

The next result clarifies the relationships between energetic and BV solutions.

Corollary 4.9 (Energy balance and comparison with energetic solutions)**.**

(1) *A* BV *solution* u *of the rate-independent system* $(X, \mathcal{E}, \mathfrak{p})$ *satisfies the energy balance* (E) *if and only if every optimal transition associated with its jump set is of energetic type* ($\mathrm{O}_{\mathrm{ener}}$)*.*
(2) *A* BV *solution* u *is an energetic solution if and only if it satisfies the global stability condition* (S)*. In that case, all of its optimal transition curves are of energetic type.*
(3) *Conversely, an energetic solution* u *is a* BV *solution if and only if, for every* $t \in \mathrm{J}_u$*, any jump couple* $(u(t_-), u(t_+))$ *can be connected by a sliding optimal transition.*

Proof. **Ad (1).** Let u be a BV solution such that every optimal transition is of energetic type ($\mathrm{O}_{\mathrm{ener}}$). Now, taking into account (J_{BV}), one sees that ($\mathrm{O}_{\mathrm{ener}}$) is equivalent to the jump conditions ($\mathrm{J}_{\mathrm{ener}}$). Then, equation ($\mathrm{DN}_{0,\mathrm{BV}}$) (which holds by Theorem 4.3) and ($\mathrm{J}_{\mathrm{ener}}$) yield the energy balance (E) (cf. the proofs of Propositions 2.2 and 2.7). The converse implication ensues by analogous arguments.
Ad (2). The necessity is obvious; for the sufficiency we observe that, for every jump point $t \in \mathrm{J}_u$, the global stability condition (S) (written first for $u(t_-)$ with test functions $v = u(t)$ and $v = u(t_+)$, and then for $u(t)$ with $v = u(t_+)$), yields

$$\begin{aligned}
\Psi_0(u(t) - u(t_-)) \geq \;& \mathcal{E}_t(u(t_-)) - \mathcal{E}_t(u(t)) = \Delta_{\mathfrak{p},\mathcal{E}}(t; u(t_-), u(t)) \geq \Psi_0(u(t) - u(t_-)),\\
\Psi_0(u(t_+) - u(t_-)) \geq \;& \mathcal{E}_t(u(t_-)) - \mathcal{E}_t(u(t_+)) = \Delta_{\mathfrak{p},\mathcal{E}}(t; u(t_-), u(t_+)) \geq \Psi_0(u(t_+) - u(t_-)),\\
\Psi_0(u(t_+) - u(t)) \geq \;& \mathcal{E}_t(u(t)) - \mathcal{E}_t(u(t_+)) = \Delta_{\mathfrak{p},\mathcal{E}}(t; u(t), u(t_+)) \geq \Psi_0(u(t_+) - u(t)),
\end{aligned}$$

where the intermediate equalities are due to (O.2) and the subsequent inequalities to (3.9). The resulting identities ultimately show that the transition is energetic, by the very definition ($\mathrm{O}_{\mathrm{ener}}$).
Ad (3). The condition is clearly sufficient. It is also necessary by the previous point, since energetic transitions are in particular of sliding type. □

4.3. **Viscous limit.** We conclude this section by our main asymptotic results:

Theorem 4.10 (Convergence of viscous approximations to BV solutions)**.** *Consider a sequence*
$(u_\varepsilon) \subset \mathrm{AC}([0,T]; X)$ *of solutions of the viscous equation* (DN_ε)*, with* $u_\varepsilon(0) \to u_0$ *as* $\varepsilon \downarrow 0$*.*

Then, every vanishing sequence $\varepsilon_k \downarrow 0$ *admits a further subsequence (still denoted by* (ε_k)*), and a limit function* $u \in \mathrm{BV}([0,T]; X)$ *such that*

$$u_{\varepsilon_k}(t) \to u(t) \quad \textit{for every } t \in [0,T] \textit{ as } k \uparrow +\infty, \tag{4.12}$$

and u *is a* BV *solution of* (DN_0)*, induced by the vanishing viscosity contact potential* $\mathfrak{p}$ *according to Definition 4.1.*

Proof. It follows from the discussion developed in Section 2.5 that for every sequence $\varepsilon_k \downarrow 0$ there exists a not relabeled subsequence (u_{ε_k}) such that (4.12) holds, and u complies with the local stability condition ($\mathrm{S}_{\mathrm{loc}}$). In view of Proposition 4.2, it is then sufficient to check that ($\mathrm{E}_{\mathfrak{p},\mathcal{E};\mathrm{ineq}}$) holds. The latter energy inequality is a direct consequence of the ε-energy identity (2.27) and the lower semicontinuity property stated in Lemma 6.15 later on. □

Our next result concerns the convergence of the discrete solutions to the *viscous time-incremental* problem $(\mathrm{IP}_\varepsilon)$, as both the viscosity parameter ε and the time-step τ tend to 0.

Theorem 4.11 (Convergence of discrete solutions of the viscous incremental problems)**.** *Let* $\overline{\mathrm{U}}_{\tau,\varepsilon} : [0,T] \to X$ *be the left-continuous piecewise constant interpolants of the discrete solutions of the viscous incremental problem* $(\mathrm{IP}_\varepsilon)$*, with* $\mathrm{U}^0_{\tau,\varepsilon} \to u_0$ *as* $\varepsilon, \tau \downarrow 0$.

Then, all vanishing sequences $\tau_k, \varepsilon_k \downarrow 0$ *satisfying*

$$\lim_{k\downarrow 0} \frac{\varepsilon_k}{\tau_k} = +\infty \tag{4.13}$$

admit further subsequences (still denoted by (τ_k) *and* (ε_k)*) and a limit function* $u \in \mathrm{BV}([0,T];X)$ *such that*

$$\overline{\mathrm{U}}_{\tau_k,\varepsilon_k}(t) \to u(t) \quad \text{for every } t \in [0,T] \text{ as } k \uparrow +\infty,$$

and u *is a* BV *solution of* (DN_0) *induced by the vanishing viscosity contact potential* $\mathfrak{p}$ *according to Definition 4.1.*

The reader may compare this result to [16, 21, 22, 49], where the same double passage to the limit was performed for specific applied problems and conditions analogous to (4.13) were imposed.

Proof. The standard energy estimate associated with the variational problem $(\mathrm{IP}_\varepsilon)$ yields

$$\frac{\tau}{\varepsilon}\Psi\Big(\frac{\varepsilon}{\tau}(U^n_{\tau,\varepsilon} - U^{n-1}_{\tau,\varepsilon})\Big) + \mathcal{E}_{t_n}(U^n_{\tau,\varepsilon}) \le \mathcal{E}_{t_n}(U^{n-1}_{\tau,\varepsilon}) = \mathcal{E}_{t_{n-1}}(U^{n-1}_{\tau,\varepsilon}) + \int_{t_{n-1}}^{t_n} \partial_t \mathcal{E}_t(U^{n-1}_{\tau,\varepsilon})\,\mathrm{d}t\,. \tag{4.14}$$

Thanks to (2.7), we easily get from (4.14) the following uniform bounds for every $1 \le n \le N$ (here C is a constant independent of n, τ, ε)

$$\mathcal{E}_{t_n}(U^n_{\tau,\varepsilon}) \le C, \quad \sum_{n=1}^N \frac{\tau}{\varepsilon}\Psi\Big(\frac{\varepsilon}{\tau}(U^n_{\tau,\varepsilon} - U^{n-1}_{\tau,\varepsilon})\Big) \le C, \quad \sum_{n=1}^N \Psi_0(U^n_{\tau,\varepsilon} - U^{n-1}_{\tau,\varepsilon}) \le C\,,$$

the latter estimate thanks to $(\Psi.2)$.

Denoting by $\underline{\mathrm{U}}_{\tau,\varepsilon}$ (resp. $\mathrm{U}_{\tau,\varepsilon}$) the *right-continuous* piecewise constant interpolants (resp. piecewise linear interpolant) of the discrete values $(U^n_{\tau,\varepsilon})$ which take the value $U^n_{\tau,\varepsilon}$ at $t = t_n$, we have

$$\mathcal{E}_t(\overline{\mathrm{U}}_{\tau,\varepsilon}(t)) \le C, \quad \mathrm{Var}_{\Psi_0}(\mathrm{U}_{\tau,\varepsilon};[0,T]) \le C \tag{4.15a}$$

(4.15b)

$$\|\mathrm{U}_{\tau,\varepsilon} - \overline{\mathrm{U}}_{\tau,\varepsilon}\|_{L^\infty(0,T;X)}, \ \|\mathrm{U}_{\tau,\varepsilon} - \underline{\mathrm{U}}_{\tau,\varepsilon}\|_{L^\infty(0,T;X)}, \le \sup_n \|U^n_{\tau,\varepsilon} - U^{n-1}_{\tau,\varepsilon}\|_X \le C\omega(\tau/C\varepsilon),$$

where

$$\omega(r) := \sup_{x\in X}\big\{\|x\|_X : r\Psi(r^{-1}x) \le 1\big\}$$

satisfies $\lim_{r\downarrow 0}\omega(r) = 0$ thanks to $(\Psi.1)$. By Helly's theorem, these bounds show that (up to the extraction of suitable subsequences (τ_k) and (ε_k) satisfying (4.13)), the sequences $(\mathrm{U}_{\tau_k,\varepsilon_k})$, $(\overline{\mathrm{U}}_{\tau_k,\varepsilon_k})$ and $(\underline{\mathrm{U}}_{\tau_k,\varepsilon_k})$ pointwise converge to the same limit u.

By differentiating the variational characterization of $U^n_{\tau,\varepsilon}$ given by $(\mathrm{IP}_\varepsilon)$ we obtain

$$\partial\Psi_\varepsilon\Big(\frac{U^n_{\tau,\varepsilon}-U^{n-1}_{\tau,\varepsilon}}{\tau}\Big)+W^n_{\tau,\varepsilon}\ni 0,\quad W^n_{\tau,\varepsilon}:=-\mathrm{D}\mathcal{E}_{t_n}(U^n_{\tau,\varepsilon}),$$

which yields in each interval $(t_{n-1},t_n]$ (here, $\overline{\mathrm{W}}_{\tau,\varepsilon}$ denotes the left-continuous piecewise constant interpolant of the values $(W^n_{\tau,\varepsilon})_{n=1}^N$)

$$\begin{aligned}&\tau\Psi_\varepsilon\big(\dot{\mathrm{U}}_{\tau,\varepsilon}\big)+\tau\Psi^*_\varepsilon(\overline{\mathrm{W}}_{\tau,\varepsilon})=-\left\langle\mathrm{D}\mathcal{E}_{t_n}(\mathrm{U}_{\tau,\varepsilon}(t_n)),\mathrm{U}_{\tau,\varepsilon}(t_n)-\mathrm{U}_{\tau,\varepsilon}(t_{n-1})\right\rangle\\&=\mathcal{E}_{t_{n-1}}(\mathrm{U}_{\tau,\varepsilon}(t_{n-1}))-\mathcal{E}_{t_n}(\mathrm{U}_{\tau,\varepsilon}(t_n))+\int_{t_{n-1}}^{t_n}\partial_t\mathcal{E}_t(\underline{\mathrm{U}}_{\tau,\varepsilon}(t))\,\mathrm{d}t-R(t_n;\mathrm{U}_{\tau,\varepsilon}(t_{n-1}),\mathrm{U}_{\tau,\varepsilon}(t_n))\end{aligned}$$

where

$$R(t;x,y):=\mathcal{E}_t(y)-\mathcal{E}_t(x)-\langle\mathrm{D}\mathcal{E}_t(y),y-x\rangle.$$

Since $\mathcal{E}$ is of class C^1, for every convex and bounded set $B\subset X$ there exists a concave modulus of continuity $\sigma_B:[0,+\infty)\to[0,+\infty)$ such that $\lim_{r\downarrow 0}\sigma_B(r)=\sigma_B(0)=0$ and

$$R(t;x,y)\le\sigma_B(\|y-x\|_X)\|y-x\|_X\quad\text{for every }t\in[0,T],\ x,y\in B.$$

We thus obtain

(4.16)
$$\begin{aligned}&\int_0^T\Big(\Psi_\varepsilon(\dot{\mathrm{U}}_{\tau,\varepsilon}(t))+\Psi^*_\varepsilon(\overline{\mathrm{W}}_{\tau,\varepsilon}(t))\Big)\,\mathrm{d}t+\mathcal{E}_{t_N}(\mathrm{U}_{\tau,\varepsilon}(t_N))\le\mathcal{E}_0(u_0)+\int_0^{t_N}\partial_t\mathcal{E}_t(\underline{\mathrm{U}}_{\tau,\varepsilon}(t))\,\mathrm{d}t\\&+\sup_{1\le n\le N}\sigma_B(\|U^n_{\tau,\varepsilon}-U^{n-1}_{\tau,\varepsilon}\|)\sum_{n=1}^N\|U^n_{\tau,\varepsilon}-U^{n-1}_{\tau,\varepsilon}\|,\quad\overline{\mathrm{W}}_{\tau,\varepsilon}(t))=-\mathrm{D}\mathcal{E}_{\bar{\mathrm{t}}_\tau(t)}(\overline{\mathrm{U}}_{\tau,\varepsilon}(t)).\end{aligned}$$

We pass to the limit along suitable subsequences (τ_k) and (ε_k) such that $\mathrm{U}_{\tau_k,\varepsilon_k}$, $\overline{\mathrm{U}}_{\tau_k,\varepsilon_k}\to u$ pointwise; since $\mathrm{U}_{\tau,\varepsilon}$ and $\overline{\mathrm{U}}_{\tau,\varepsilon}$ are uniformly bounded, (4.15b) and (4.13) yield the convergence to 0 of the third term on the right-hand side of (4.16), which thus tends to $\mathcal{E}_0(u_0)+\int_0^T\partial_t\mathcal{E}_t(u(t))\,\mathrm{d}t$. Since $\overline{\mathrm{W}}_{\tau_k,\varepsilon_k}(t)\to w(t)=-\mathrm{D}\mathcal{E}_t(u(t))$, applying the lower semicontinuity result of Lemma 6.15 we obtain that u satisfies $(\mathrm{E}_{\mathfrak{p},\mathcal{E};\mathrm{ineq}})$ and the local stability condition. In view of Proposition 4.2, this concludes the proof. □

5. Parametrized solutions

In this section, we restart from the discussions in Sections 2.4 and 2.5, and adopt a different point of view, which relies on the rate-independent structure of the limit problem. The main idea, which was introduced by [18], is to rescale time in order to gain a uniform Lipschitz bound on the (rescaled) viscous approximations. Keeping track of the asymptotic behavior of time rescalings, one can retrieve the BV limit we analyzed in Section 4. In particular, we shall recover that the limiting jump pathes reflect the viscous approximation.

5.1. **Vanishing viscosity analysis: a rescaling argument.**

Let us recall that for every $\varepsilon>0$ u_ε are the solutions of the viscous differential inclusion

$$(\mathrm{DN}_\varepsilon)\qquad\partial\Psi_\varepsilon(\dot u_\varepsilon(t))+\mathrm{D}\mathcal{E}_t(u_\varepsilon(t))\ni 0\quad\text{in }X^*\quad\text{for a.a. }t\in(0,T),$$

which we split into the system

$$\begin{aligned}&\partial\Psi_\varepsilon(\dot u_\varepsilon(t)) \ni w_\varepsilon,\\ &\mathrm{D}\mathcal{E}_t(u_\varepsilon(t)) = -w_\varepsilon, \qquad \partial_t\mathcal{E}_t(u_\varepsilon(t)) = -p_\varepsilon.\end{aligned}$$

We follow the ideas of [18, 35] to capture the aforementioned limiting viscous jump pathes,. However, owing to the dissipation bound (2.28), we use a different time rescaling $\mathsf{s}_\varepsilon : [0,T] \to [0,\mathsf{S}_\varepsilon]$

$$\mathsf{s}_\varepsilon(t) := t + \int_0^t \Big(\Psi_\varepsilon(\dot u_\varepsilon(r)) + \Psi_\varepsilon^*(w_\varepsilon(r))\Big)\,\mathrm{d}r \quad \text{and} \quad \mathsf{S}_\varepsilon := \mathsf{s}_\varepsilon(T). \tag{5.1}$$

Thus, s_ε may be interpreted as some sort of "energy arclength" of the curve u_ε. Notice that, thanks to (2.28), the sequence (S_ε) is uniformly bounded with respect to the parameter ε. Let us consider the rescaled functions $(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon) : [0,\mathsf{S}_\varepsilon] \to [0,T]\times X$ and $(\mathsf{p}_\varepsilon, \mathsf{w}_\varepsilon) : [0,\mathsf{S}_\varepsilon] \to \mathbb{R}\times X^*$ defined by

$$\begin{aligned}&\mathsf{t}_\varepsilon(s) := \mathsf{s}_\varepsilon^{-1}(s)\,, && \mathsf{u}_\varepsilon(s) := u_\varepsilon(\mathsf{t}_\varepsilon(s)),\\ &\mathsf{p}_\varepsilon(s) := p_\varepsilon(\mathsf{t}_\varepsilon(s)) = -\partial_t\mathcal{E}_{\mathsf{t}_\varepsilon(s)}(\mathsf{u}_\varepsilon(s))\,, && \mathsf{w}_\varepsilon(s) := w_\varepsilon(\mathsf{t}_\varepsilon(s)) = -\mathrm{D}\mathcal{E}_{\mathsf{t}_\varepsilon(s)}(\mathsf{u}_\varepsilon(s)).\end{aligned} \tag{5.2}$$

We now study the limiting behavior as $\varepsilon \downarrow 0$ of the *reparametrized trajectories*

$$\begin{aligned}\big\{\big(\mathsf{t}_\varepsilon(s), \mathsf{u}_\varepsilon(s)\big) \ : \ s \in [0,\mathsf{S}_\varepsilon]\big\} \subset \mathscr{X} = [0,T]\times X,\\ \big\{\big(\dot{\mathsf{t}}_\varepsilon(s), \dot{\mathsf{u}}_\varepsilon(s); \mathsf{p}_\varepsilon(s), \mathsf{w}_\varepsilon(s)\big) \ : \ s \in [0,\mathsf{S}_\varepsilon]\big\} \subset \mathscr{B},\end{aligned}$$

where we use the notation

$$\mathscr{B} := [0,+\infty)\times X\times\mathbb{R}\times X^*. \tag{5.3}$$

In order to rewrite the "rescaled energy identity" fulfilled by the triple $(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon, \mathsf{w}_\varepsilon)$, we define the viscous space-time vanishing viscosity contact potential $\mathfrak{P}_\varepsilon : (0,+\infty)\times X\times\mathbb{R}\times X^* \to [0,+\infty)$ by setting

$$\mathfrak{P}_\varepsilon(\alpha, \mathsf{v}; \mathsf{p}, \mathsf{w}) := \alpha\Psi_\varepsilon(\mathsf{v}/\alpha) + \alpha\Psi_\varepsilon^*(\mathsf{w}) + \alpha\mathsf{p} = \frac{\alpha}{\varepsilon}\Psi(\frac{\varepsilon}{\alpha}\mathsf{v}) + \frac{\alpha}{\varepsilon}\Psi^*(\mathsf{w}) + \alpha\mathsf{p} \tag{5.4}$$

Hence, (2.27) becomes for all $0 \le s_1 \le s_2 \le \mathsf{S}_\varepsilon$

$$\int_{s_1}^{s_2} \mathfrak{P}_\varepsilon\big(\dot{\mathsf{t}}_\varepsilon(s), \dot{\mathsf{u}}_\varepsilon(s); \mathsf{p}_\varepsilon(s), \mathsf{w}_\varepsilon(s)\big)\,\mathrm{d}s + \mathcal{E}_{\mathsf{t}_\varepsilon(s_2)}(\mathsf{u}_\varepsilon(s_2)) = \mathcal{E}_{\mathsf{t}_\varepsilon(s_1)}(\mathsf{u}_\varepsilon(s_1)), \tag{5.5}$$

and (5.1) yields

$$\mathfrak{P}_\varepsilon\big(\dot{\mathsf{t}}_\varepsilon(s), \dot{\mathsf{u}}_\varepsilon(s); 1, \mathsf{w}_\varepsilon(s)\big) = 1 \quad \text{for a.a. } s \in (0,\mathsf{S}_\varepsilon)\,.$$

A priori estimates and passage to the limit. Due to estimate (2.28), there exists $\mathsf{S} > 0$ such that, along a (not relabeled) subsequence, we have $\mathsf{s}_\varepsilon(T) \to \mathsf{S}$ as $\varepsilon \downarrow 0$. Exploiting again (2.28), the Arzelà-Ascoli compactness theorem, and the fact that X is finite-dimensional (see also the proof of [35, Thm. 3.3]), we find two curves $\mathsf{t} \in W^{1,\infty}(0,\mathsf{S})$ and $\mathsf{u} \in W^{1,\infty}([0,\mathsf{S}];X)$ such that, along the same subsequence,

$$\mathsf{t}_\varepsilon \to \mathsf{t} \text{ in } \mathrm{C}^0([0,\mathsf{S}]), \qquad \dot{\mathsf{t}}_\varepsilon \rightharpoonup^* \dot{\mathsf{t}} \text{ in } L^\infty(0,\mathsf{S}), \tag{5.6a}$$

$$\mathsf{u}_\varepsilon \to \mathsf{u} \text{ in } \mathrm{C}^0([0,\mathsf{S}];X), \qquad \dot{\mathsf{u}}_\varepsilon \rightharpoonup^* \dot{\mathsf{u}} \text{ in } L^\infty(0,\mathsf{S};X), \tag{5.6b}$$

$$\mathsf{p}_\varepsilon \to \mathsf{p} \text{ in } \mathrm{C}^0([0,\mathsf{S}]), \qquad \mathsf{w}_\varepsilon \to \mathsf{w} \text{ in } \mathrm{C}^0([0,\mathsf{S}];X^*), \tag{5.6c}$$

with

$$\mathcal{E}_{\mathsf{t}_\varepsilon}(\mathsf{u}_\varepsilon) \to \mathcal{E}_{\mathsf{t}}(\mathsf{u}), \quad \mathsf{p}(s) = -\partial_t \mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)), \quad \mathsf{w}(s) = -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \tag{5.6d}$$

for all $s \in [0, \mathsf{S}]$. Then, to pass to the limit in (5.5) we exploit a lower semicontinuity result (see Proposition 6.2), based on the fact that the sequence of functionals $(\mathfrak{P}_\varepsilon)$ Γ-converges to the *augmented* vanishing viscosity contact potential $\mathfrak{P} : [0, +\infty) \times X \times \mathbb{R} \times X^* \to [0, +\infty]$ (see Lemma 6.1) defined by

$$\mathfrak{P}(\alpha, v; p, w) := \begin{cases} \Psi_0(v) + \mathrm{I}_{K^*}(w) + \alpha\, p & \text{if } \alpha > 0, \\ \mathfrak{p}(v, w) & \text{if } \alpha = 0. \end{cases} \tag{5.7}$$

By (5.6) and Proposition 6.2, we take the lim inf as $\varepsilon \downarrow 0$ of (5.5) and conclude that the pair (t, u) fulfils, for all $0 \le s_1 \le s_2 \le \mathsf{S}$, the estimate

$$\int_{s_1}^{s_2} \mathfrak{P}\big(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s); \mathsf{p}(s), \mathsf{w}(s)\big)\, \mathrm{d}s + \mathcal{E}_{\mathsf{t}(s_2)}(\mathsf{u}(s_2)) \le \mathcal{E}_{\mathsf{t}(s_1)}(\mathsf{u}(s_1))\,. \tag{5.8}$$

5.2. **Vanishing viscosity contact potentials and rate-independent evolution.** The augmented space-time contact potential $\mathfrak{P}$ is closely related to $\mathfrak{p}$ introduced by (3.13). The following result fixes some properties of $\mathfrak{P}$. Its proof, which we choose to omit, can be easily developed starting from Theorems 3.6 and 3.12 for the vanishing viscosity contact potential $\mathfrak{p}$.

Lemma 5.1 (General properties of $\mathfrak{P}$)**.**

(1) $\mathfrak{P}$ *is lower semicontinuous,* 1*-homogeneous and convex in the pair* (α, v)*; for every* $(\alpha, v) \in [0, +\infty) \times X$ *the function* $\mathfrak{P}(\alpha, v; \cdot, \cdot)$ *has convex sublevels.*

(2) *For all* $(\alpha, v, p, w) \in \mathscr{B}$ *(cf.* (5.3)*) it satisfies*

$$\mathfrak{P}(\alpha, v; p, w) \ge \langle w, v\rangle + \alpha p, \quad \mathfrak{P}(0, v; p, w) \ge \mathfrak{p}(v, w) \ge \Psi_0(v), \tag{5.9}$$

$$\mathfrak{P}(0, v; p, w) = \Psi_0(v) \quad \Leftrightarrow \quad w \in K^*. \tag{5.10}$$

(3) *The* contact set *of* $\mathfrak{P}$

$$\Sigma_{\mathfrak{P}} := \Big\{(\alpha, v; p, w) \in \mathscr{B} : \mathfrak{P}(\alpha, v; p, w) = \langle w, v\rangle + \alpha p\Big\} \tag{5.11}$$

does not impose any constraint on p*. It can be characterized by*

$$(\alpha, v; p, w) \in \Sigma_{\mathfrak{P}} \quad \Leftrightarrow \quad w \in \partial\, \mathfrak{P}(\alpha, \cdot\,; p, w)(v). \tag{5.12}$$

We also have

$$\text{for } \alpha > 0, \quad (\alpha, v; p, w) \in \Sigma_{\mathfrak{P}} \quad \text{if and only if} \quad w \in \partial\Psi_0(v), \tag{5.13}$$

$$\text{for } \alpha = 0, \quad (\alpha, v; p, w) \in \Sigma_{\mathfrak{P}} \quad \text{if and only if} \quad (v, w) \in \Sigma_{\mathfrak{p}}. \tag{5.14}$$

Equivalently, $(\alpha, v; p, w) \in \Sigma_{\mathfrak{P}}$ *if and only if*

$$w \in \partial\Psi_0(v) \subset K^* \qquad \text{or} \qquad \Big(w \notin K^*, \quad \alpha = 0, \quad \exists\, \varepsilon \in \Lambda(v, w) : \quad w \in \partial\Psi(\varepsilon v)\Big), \tag{5.15}$$

where $\Lambda(v, w)$ *is defined in* (3.18)*. In particular, in the additive viscosity case* (2.21)*, we simply have*

$$(\alpha, v; p, w) \in \Sigma_{\mathfrak{P}} \quad \Longleftrightarrow \quad \exists\, \lambda \ge 0 : \quad w \in \partial\Psi_0(v) + \partial\Psi_V(\lambda v) \quad \text{and} \quad \alpha\lambda = 0. \tag{5.16}$$

Conclusion of the vanishing viscosity analysis. We are now going to show that (5.8) is in fact an equality. This can be easily checked relying on the chain rule (2.26), which yields for a.a. $s \in (0, \mathsf{S})$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) &= -\partial_t \mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))\,\dot{\mathsf{t}}(s) - \langle -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)), \dot{\mathsf{u}}(s)\rangle \\ &\stackrel{(5.2)}{=} -\mathsf{p}(s)\dot{\mathsf{t}}(s) - \langle \mathsf{w}(s), \dot{\mathsf{u}}(s)\rangle \geq -\mathfrak{P}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s), \mathsf{p}(s), \mathsf{w}(s))\,.\end{aligned} \tag{5.17}$$

Collecting (5.17) and (5.8), we conclude that the latter holds with an equality sign and, with an elementary argument, that such equality also holds in the differential form, namely for a.a. $s \in (0, \mathsf{S})$

$$\begin{aligned} &\mathsf{p}(s) = -\partial_t \mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)), \qquad \mathsf{w}(s) = -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \\ &\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) = -\mathsf{p}(s)\dot{\mathsf{t}}(s) - \langle \mathsf{w}(s), \dot{\mathsf{u}}(s)\rangle = -\mathfrak{P}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s), \mathsf{p}(s), \mathsf{w}(s))\end{aligned} \tag{5.18}$$

which yields

$$\Big(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s); -\partial_t \mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)), -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))\Big) \in \Sigma_{\mathfrak{P}} \quad \text{for a.a. } s \in (0, \mathsf{S}). \tag{5.19}$$

Finally, we take the lim sup as $\varepsilon \downarrow 0$ of (5.5), using (5.6) and (5.18), whence

$$\limsup_{\varepsilon \downarrow 0} \int_0^{\mathsf{S}} \mathfrak{P}_\varepsilon(\dot{\mathsf{t}}_\varepsilon(s), \dot{\mathsf{u}}_\varepsilon(s), \mathsf{p}_\varepsilon(s), \mathsf{w}_\varepsilon(s))\,\mathrm{d}s \leq \int_0^{\mathsf{S}} \mathfrak{P}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s), \mathsf{p}(s), \mathsf{w}(s))\,\mathrm{d}s.$$

In particular, we find that for a.a. $s \in (0, \mathsf{S})$

$$\mathfrak{P}\big(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s); 1, \mathsf{w}(s)\big) = 1. \tag{5.20}$$

5.3. **Parametrized solutions of rate-independent systems.** Motivated by the discussion of the previous section, we now give the notion of parametrized rate-independent evolution, driven by a general vanishing viscosity contact potential $\mathfrak{P}$, satisfying conditions (1), (2) of Lemma 5.1.

Definition 5.2 (Parametrized solutions of rate-independent systems)**.** Let $\mathfrak{P} : \mathscr{B} \to (-\infty, +\infty]$ be the vanishing viscosity contact potential (5.7). We say that a Lipschitz continuous curve $(\mathsf{t}, \mathsf{u}) : [a, b] \to [0, T] \times X$ is a *parametrized rate-independent solution for the system* $(X, \mathcal{E}, \mathfrak{P})$ if t is nondecreasing and, setting $\mathsf{p}(s) = -\partial_t \mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))$, $\mathsf{w}(s) = -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))$ for all $s \in [a, b]$, we have

$$\int_{s_1}^{s_2} \mathfrak{P}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s); \mathsf{p}(s), \mathsf{w}(s))\,\mathrm{d}s + \mathcal{E}_{\mathsf{t}(s_2)}(\mathsf{u}(s_2)) \leq \mathcal{E}_{\mathsf{t}(s_1)}(\mathsf{u}(s_1)) \quad \forall\, a \leq s_1 \leq s_2 \leq b. \tag{5.21}$$

Furthermore,

(1) if $\dot{\mathsf{t}}(s) + \Psi_0(\dot{\mathsf{u}}(s)) > 0$ for a.a. $s \in (a, b)$ we say that (t, u) is *nondegenerate*;
(2) if $\mathsf{t}(a) = 0, \mathsf{t}(b) = T$ we say that (t, u) is *surjective*;
(3) if (t, u) satisfies (5.20), we say that it is *normalized*.

Definition 5.2 generalizes to the present setting the notion which we first introduced in [35].

Remark 5.3. The nice feature of the previous definition is its invariance with respect to (nondecreasing, Lipschitz) time rescalings. Namely, if $(\mathsf{t},\mathsf{u}) : [a,b] \to [0,T]\times X$ is a parametrized solution and $\mathsf{s} : [\alpha,\beta] \to [a,b]$ is a Lipschitz nondecreasing map, then $(\mathsf{t}\circ\mathsf{s},\mathsf{u}\circ\mathsf{s})$ is a parametrized solution in $[\alpha,\beta]$.

The next result provides equivalent characterizations of parametrized solutions.

Proposition 5.4. *A Lipschitz continuous curve* $(\mathsf{t},\mathsf{u}) : [a,b] \to [0,T]\times X$, *with* t *nondecreasing, is a parametrized solution of* $(X,\mathcal{E},\mathfrak{P})$ *if and only if one of the following (equivalent) conditions (involving as usual* $\mathsf{p} = -\partial_t\mathcal{E}_{\mathsf{t}}(\mathsf{u}), \mathsf{w} = -\mathrm{D}\mathcal{E}_{\mathsf{t}}(\mathsf{u})$*) is satisfied:*

(1) *The energy inequality* (5.21) *holds just for* $s_1 = a$ *and* $s_2 = b$, *i.e.*

$$\int_a^b \mathfrak{P}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s);\mathsf{p}(s),\mathsf{w}(s))\,\mathrm{d}s + \mathcal{E}_{\mathsf{t}(b)}(\mathsf{u}(b)) \le \mathcal{E}_{\mathsf{t}(a)}(\mathsf{u}(a)). \tag{5.22}$$

(2) *The energy inequality* (5.21) *holds in the differential form*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) + \mathfrak{P}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s);\mathsf{p}(s),\mathsf{w}(s)) \le 0 \quad \textit{for a.a. } s\in(a,b). \tag{5.23}$$

(3) *The energy identity holds, in the differential form*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) + \mathfrak{P}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s);\mathsf{p}(s),\mathsf{w}(s)) = 0 \quad \textit{for a.a. } s\in(a,b), \tag{5.24}$$

or in the integrated form

$$\int_{s_1}^{s_2} \mathfrak{P}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s);\mathsf{p}(s),\mathsf{w}(s))\,\mathrm{d}s + \mathcal{E}_{\mathsf{t}(s_2)}(\mathsf{u}(s_2)) = \mathcal{E}_{\mathsf{t}(s_1)}(\mathsf{u}(s_1)) \quad \textit{for } a\le s_1\le s_2\le b. \tag{5.25}$$

(4) *There holds*

$$\big(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s);-\partial_t\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)),-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))\big) \in \Sigma_{\mathfrak{P}} \quad \textit{for a.a. } s\in(a,b)\,.$$

(5) *The pair* (t,u) *satisfy the differential inclusion*

$$\partial\mathfrak{P}\big(\dot{\mathsf{t}}(s),\cdot\,;-\partial_t\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)),-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))\big)(\dot{\mathsf{u}}(s)) + \mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \ni 0 \qquad \textit{a.e. in } (a,b)\,. \tag{5.26}$$

In particular, for a.a. $s\in(a,b)$ *we have the implications*

$$\begin{aligned} \dot{\mathsf{t}}(s) > 0 \ &\Rightarrow\ -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \in K^*, \\ -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \in K^* \ &\Rightarrow\ -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \in \partial\Psi_0(\dot{\mathsf{u}}(s)), \end{aligned} \tag{5.27}$$

and for every Borel map λ *defined in the open set* J *by*

$$\begin{gathered} \mathsf{J} := \big\{s\in(a,b) : -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \notin K^*\big\}, \\ \textit{with}\ \ \lambda(s)\in\Lambda(\dot{\mathsf{u}}(s),-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))) \ \ \textit{for a.a. } s\in\mathsf{J}, \end{gathered} \tag{5.28}$$

we have

$$-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \in \partial\Psi(\lambda(s)\dot{\mathsf{u}}(s)), \quad \dot{\mathsf{t}}(s) = 0 \quad \textit{for a.a. } s\in\mathsf{J}. \tag{5.29}$$

The *proof* follows from the chain rule (2.26) (arguing as for (5.17), (5.18), (5.19)), and from the characterization of the contact set $\Sigma_{\mathfrak{P}}$ of Lemma 5.1 (see also [35, Prop. 2]).

Corollary 5.5 (Differential characterization in the additive viscosity case)**.** *Let* $\mathfrak{P} : \mathscr{B} \to (-\infty, +\infty]$ *be a vanishing viscosity contact potential satisfying conditions* (1), (2) *of Lemma 5.1, and suppose also that the contact set of* $\mathfrak{P}$ *satisfies the characterization* (5.16) *of Lemma 5.1 in the additive viscosity case* (2.21) $\Psi = \Psi_0 + \Psi_V$.

Then, a Lipschitz continuous curve $(\mathsf{t}, \mathsf{u}) : [a, b] \to [0, T] \times X$ *is a parametrized solution of* $(X, \mathcal{E}, \mathfrak{P})$ *if and only if there exists a Borel function* $\lambda : (a, b) \to [0, +\infty)$ *such that for a.a.* $s \in (a, b)$

(5.30) $$\partial\Psi_0(\dot{\mathsf{u}}(s)) + \partial\Psi_V(\lambda(s)\dot{\mathsf{u}}(s)) + \mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)) \ni 0, \quad \lambda(s)\dot{\mathsf{t}}(s) = 0 \quad \textit{for a.a. } s \in (a, b).$$

The vanishing viscosity analysis developed in Sections 5.1 and 5.2 provides the following convergence result.

Theorem 5.6 (Convergence to parametrized solutions)**.** *Let* (u_n) *be viscous solutions of* $(\mathrm{DN}_\varepsilon)$ *corresponding to a vanishing sequence* (ε_n)*, let* $\mathsf{t}_n : [0, \mathsf{S}] \to [0, T]$ *be uniformly Lipschitz and surjective time rescalings and let* $\mathsf{u}_n : [0, \mathsf{S}] \to X$ *be defined as* $\mathsf{u}_n(s) := u_n(\mathsf{t}_n(s))$ *for all* $s \in [0, \mathsf{S}]$*. Suppose that*

$$\exists\, \alpha > 0 \quad \forall\, n \in \mathbb{N} : \quad \mathsf{m}_n(s) := \mathfrak{P}_{\varepsilon_n}(\dot{\mathsf{t}}_n(s), \dot{\mathsf{u}}_n(s); 1, -\mathrm{D}\mathcal{E}_{\mathsf{t}_n(s)}(\mathsf{u}_n(s))) \in [\alpha, \alpha^{-1}]$$

for a.a. $s \in (0, \mathsf{S})$*. If the functions* $(\mathsf{t}_n, \mathsf{u}_n, \mathsf{m}_n)$ *pointwise converge to* $(\mathsf{t}, \mathsf{u}, \mathsf{m})$ *as* $n \to \infty$*, then* (t, u) *is a (nondegenerate, surjective) parametrized rate-independent solution according to Definition 5.2, and*

$$\mathfrak{P}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s); 1, -\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))) = \mathsf{m}(s) \qquad \textit{for a.a. } s \in (0, \mathsf{S}).$$

The following remark, to be compared with Remark 4.8, highlights the different mechanical regimes encompassed in the notion of parametrized rate-independent solution.

Remark 5.7 (Mechanical interpretation)*.* The evolution described by (5.26) in Proposition 5.4 bears the following mechanical interpretation (cf. with [18] and [35]):

- the regime ($\dot{\mathsf{t}} > 0$, $\dot{\mathsf{u}} = 0$) corresponds to *sticking*,
- the regime ($\dot{\mathsf{t}} > 0, \dot{\mathsf{u}} \neq 0$) corresponds to *rate-independent sliding*. In both these two regimes $-\mathrm{D}\mathcal{E}_{\mathsf{t}}(\mathsf{u}) \in K^*$.
- when $-\mathrm{D}\mathcal{E}_{\mathsf{t}}(\mathsf{u})$ cannot obey the constraint K^*, then the system switches to a *viscous regime*. The time is frozen (i.e., $\dot{\mathsf{t}} = 0$), and the solution follows a viscous path. In the additive viscosity case (2.21) it is governed by the rescaled viscous equation (5.30) with $\lambda > 0$. These viscous motions can be seen as a jump in the (slow) external time scale.

We conclude this section with the main equivalence result between parametrized and BV solutions of rate-independent systems (compare with the analogous [35, Prop. 6]). We postpone its proof at the end of the next section.

Theorem 5.8 (Equivalence between BV and parametrized solutions)**.** *Let* $(\mathsf{t}, \mathsf{u}) : [0, \mathsf{S}] \to [0, T] \times X$ *be a (nondegenerate, surjective) parametrized solution of the rate independent system* $(X, \mathcal{E}, \mathfrak{P})$*. For every* $t \in [0, T]$ *set*

(5.31) $$\mathsf{s}(t) := \big\{ s \in [0, \mathsf{S}] : \mathsf{t}(s) = t \big\}$$

Then, any curve $u : [0,T] \to X$ *such that*

$$u(t) \in \{\mathsf{u}(s) : s \in \mathsf{s}(t)\} \tag{5.32}$$

is a BV *solution of the rate-independent system* $(X, \mathcal{E}, \mathfrak{p})$.
Conversely, if $u : [0,T] \to X$ *is a* BV *solution, then there exists a parametrized solution* (t, u) *such that* (5.32) *holds for a time-rescaling function* s *defined as in* (5.31).

6. Auxiliary results

After proving some lower semicontinuity results for vanishing viscosity contact potentials, in Section 6.2 we develop some auxiliary results concerning the total variation induced by time-dependent (and possibly asymmetric) Finsler norms.

6.1. Lower semicontinuity for vanishing viscosity contact potentials.

Let us start with a lemma which shows that $\mathfrak{P}_\varepsilon$, which is defined in (5.4), Γ-converges to $\mathfrak{P}$ as $\varepsilon \downarrow 0$ (compare with [35, Lemma 3.1]), where $\mathfrak{P}$ is defined in (5.7).

Lemma 6.1 (Γ-convergence of $\mathfrak{P}_\varepsilon$)**.**

Γ-liminf estimate: *For every choice of sequences* $\varepsilon_n \downarrow 0$ *and* $(\alpha_n, v_n, p_n, w_n) \to (\alpha, v, p, w)$ *in* $\mathscr{B}$, *we have*

$$\liminf_{n\to\infty} \mathfrak{P}_{\varepsilon_n}(\alpha_n, v_n; p_n, w_n) \geq \mathfrak{P}(\alpha, v; p, w). \tag{6.1}$$

Γ-limsup estimate: *For every* $(\alpha, v; p, w) \in \mathscr{B}$ *there exists* $(\alpha_\varepsilon, v_\varepsilon, p_\varepsilon, w_\varepsilon)_{\varepsilon>0}$ *such that*

$$\limsup_{\varepsilon\downarrow 0} \mathfrak{P}_\varepsilon(\alpha_\varepsilon, v_\varepsilon; p_\varepsilon, w_\varepsilon) \leq \mathfrak{P}(\alpha, v; p, w). \tag{6.2}$$

Proof. The Γ-liminf estimate is easy: if $\alpha > 0$ then, also recalling (2.29), one verifies that

$$\begin{aligned} \liminf_{n\to\infty} \mathfrak{P}_{\varepsilon_n}(\alpha_n, v_n; p_n, w_n) &\geq \liminf_{n\to\infty} \left(\Psi_0(v_n) + \alpha_n \varepsilon_n^{-1} \Psi_0^*(w_n) + \alpha_n p_n\right) \\ &\geq \mathfrak{P}_0(\alpha, v; p, w), \end{aligned} \tag{6.3}$$

where we have used the notation

$$\mathfrak{P}_0(\alpha, v; p, w) := \Psi_0(v) + \mathrm{I}_K^*(w) + \alpha p. \tag{6.4}$$

The first inequality in (6.3) is also due to (2.16). If $\alpha = 0$, we use the obvious lower bound

$$\mathfrak{P}_{\varepsilon_n}(\alpha_n, v_n; p_n, w_n) \geq \mathfrak{p}(v_n, w_n) + \alpha_n p_n$$

and the continuity of $\mathfrak{p}$ (cf. Theorem 3.12).

To show the limsup estimate (6.2) for $w \in K^*$, we simply choose $\alpha_\varepsilon := \alpha + \varepsilon, v_\varepsilon := v, p_\varepsilon := p, w_\varepsilon := w$, observing that in this case

$$\mathfrak{P}_\varepsilon(\alpha_\varepsilon, v_\varepsilon; p_\varepsilon, w_\varepsilon) \leq \varepsilon(\alpha+\varepsilon)\Psi(v/(\varepsilon(\alpha+\varepsilon))) + (\alpha+\varepsilon)p \overset{\varepsilon\downarrow 0}{\to} \Psi_0(v) + \alpha p = \mathfrak{P}(\alpha, v; p, w)\,,$$

the first passage due to (2.24). If $w \notin K^*$, we choose a coefficient $\lambda \in \Lambda(v, w)$ as in (3.18), and we set $\alpha_\varepsilon := \lambda\varepsilon$, $v_\varepsilon := v, p_\varepsilon := p, w_\varepsilon := w$, obtaining

$$\mathfrak{P}_\varepsilon(\alpha_\varepsilon, v_\varepsilon; p_\varepsilon, w_\varepsilon) = \mathfrak{p}(v, w) + \lambda\varepsilon p \overset{\varepsilon\downarrow 0}{\to} \mathfrak{p}(v, w) = \mathfrak{P}(\alpha, v; p, w). \quad \square$$

An important consequence of the previous Lemma is provided by the following lower-semicontinuity result for the integral functional associated with $\mathfrak{P}_\varepsilon$.

Proposition 6.2 (Lower-semicontinuity of the ε-energy)**.** *Let us fix an interval* (s_0, s_1)*. For every choice of a vanishing sequence* $\varepsilon_n > 0$ *and of functions* $\alpha_n \in L^\infty(s_0, s_1)$*,* $p_n \in L^1(s_0, s_1)$*,* $v_n \in L^1(0,T;X)$*,* $w_n \in L^1(0,T;X^*)$ *such that*

$$\alpha_n \rightharpoonup^* \alpha \quad \text{in } L^\infty(s_0,s_1), \qquad p_n \to p \quad \text{in } L^1(0,T),$$
$$v_n \rightharpoonup v \quad \text{in } L^1(0,T;X), \qquad w_n \to w \quad \text{in } L^1(s_0,s_1),$$

we have the liminf estimates

$$\liminf_{n\to\infty} \int_{s_0}^{s_1} \mathfrak{P}_{\varepsilon_n}(\alpha_n(s), v_n(s); p_n(s), w_n(s))\,\mathrm{d}s \geq \int_{s_0}^{s_1} \mathfrak{P}(\alpha(s), v(s); p(s), w(s))\,\mathrm{d}s, \tag{6.5}$$

$$\liminf_{n\to\infty} \int_{s_0}^{s_1} \mathfrak{P}_0(\alpha_n(s), v_n(s); p_n(s), w_n(s))\,\mathrm{d}s \geq \int_{s_0}^{s_1} \mathfrak{P}(\alpha(s), v(s); p(s), w(s))\,\mathrm{d}s, \tag{6.6}$$

where $\mathfrak{P}_0$ *is defined in* (6.4).

Proof. It is sufficient to prove this result in the case $p_n \equiv p = 0$. Then we notice that, by Lemma 6.1, the integrand

$$\tilde{\mathfrak{P}}(\varepsilon,\alpha,v,w) := \mathfrak{P}_\varepsilon(\alpha,v;0,w) \quad \text{for } (\varepsilon,\alpha,v,w) \in [0,+\infty)\times[0,+\infty)\times X\times X^*$$

is lower semicontinuous and convex in the pair (α, v). Then, inequality (6.5) follows from Ioffe's Theorem (see e.g. [3, Thm. 5.8]). A similar argument yields (6.6). □

6.2. Asymmetric dissipations, pseudo-total variation, and extended space-time curves.

Notation. Hereafter, $\mathscr{X}$ shall stand for the extended space-time domain $[0,T]\times X$, with elements $\boldsymbol{x} = (t,u)$ denoted by bold letters. We shall denote by $\mathscr{V}$ the tangent cone $[0,+\infty)\times X$ to $\mathscr{X}$ and by $\boldsymbol{v} = (\alpha, v)$ the elements in $\mathscr{V}$.

We shall consider lower semicontinuous dissipation functionals $\mathcal{R} : \mathscr{X}\times\mathscr{V} \to [0,+\infty)$ satisfying the following properties:

$$\forall\, \boldsymbol{x}\in\mathscr{X} :\ \mathcal{R}(\boldsymbol{x};\cdot) \text{ is convex and positively 1-homogeneous;} \tag{6.7a}$$

$$\exists\, C>0\ \forall\, \boldsymbol{x}\in\mathscr{X},\ \boldsymbol{v}=(\alpha,v)\in\mathscr{V} :\quad \mathcal{R}(\boldsymbol{x};\boldsymbol{v}) \geq C\|v\|_X \tag{6.7b}$$

$$\mathcal{R} \text{ is lower semicontinuous on } \mathscr{X}\times\mathscr{V}. \tag{6.7c}$$

In order to keep track of the time-component of $\boldsymbol{v}$ we also set, for all $\beta \geq 0$,

$$\mathcal{R}_\beta(\boldsymbol{x};\boldsymbol{v}) = \alpha\beta + \mathcal{R}(\boldsymbol{x};\boldsymbol{v}) \quad \text{for all } \boldsymbol{x}\in\mathscr{X},\ \boldsymbol{v}=(\alpha,v)\in\mathscr{V}.$$

Notice that, for any dissipation $\mathcal{R}$ complying with properties (6.7), the corresponding functional $\mathcal{R}_\beta$ satisfies the subadditivity property for all $\boldsymbol{x}\in\mathscr{X}$ and $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \mathscr{V}$

$$\mathcal{R}_\beta(\boldsymbol{x};\boldsymbol{v}_1+\boldsymbol{v}_2) \leq \mathcal{R}_\beta(\boldsymbol{x};\boldsymbol{v}_1) + \mathcal{R}_\beta(\boldsymbol{x};\boldsymbol{v}_2).$$

Example 6.3 (Dissipations induced by Ψ_0 and $\mathfrak{P}$)**.**

(1) Our first trivial example of a dissipation fulfilling properties (6.7) is given by

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{v}) := \Psi_0(v) \quad \text{for } \boldsymbol{x}\in\mathscr{X},\ \boldsymbol{v}=(\alpha,v)\in\mathscr{V}. \tag{6.8}$$

(2) Our **main example** will be provided by the dissipation induced by the vanishing viscosity contact potential $\mathfrak{P}$, namely

$$\mathcal{B}(\boldsymbol{x};\boldsymbol{v}) := \mathcal{P}(\alpha, v, 0, -\mathrm{D}\mathcal{E}_t(u)) \quad \text{for } \boldsymbol{x}=(t,u)\in\mathscr{X},\ \boldsymbol{v}=(\alpha,v)\in\mathscr{V}\,. \tag{6.9}$$

It is not difficult to check that $\mathcal{B}$ satisfies all of assumptions (6.7). Hence, for all $\beta \geq 0$ we set

$$\mathcal{B}_\beta(\boldsymbol{x};\boldsymbol{v}) := \mathfrak{P}(\alpha, v; \beta, -\mathrm{D}\mathcal{E}_t(u)) \quad \text{for } \boldsymbol{x}=(t,u)\in\mathscr{X},\ \boldsymbol{v}=(\alpha,v)\in\mathscr{V}. \tag{6.10}$$

Definition 6.4 (Pseudo-Finsler distance induced by $\mathcal{R}$)**.** Given a dissipation function $\mathcal{R}:\mathscr{X}\times\mathscr{V}\to[0,+\infty)$ complying with (6.7), for every $\boldsymbol{x}_i=(t_i,u_i)\in\mathscr{X},\ i=0,1$, with $0\leq t_0\leq t_1\leq T$, we set

$$\begin{aligned}\Delta_{\mathcal{R}_\beta}(\boldsymbol{x}_0,\boldsymbol{x}_1) := \inf\Big\{ &\int_{r_0}^{r_1} \mathcal{R}_\beta(\mathsf{x}(r);\dot{\mathsf{x}}(r))\,\mathrm{d}r \ : \\ &\mathsf{x}=(\mathsf{t},\mathsf{u})\in \mathrm{Lip}(r_0,r_1;\mathscr{X}),\ \boldsymbol{x}(r_i)=\boldsymbol{x}_i,\ i=0,1,\quad \dot{\mathsf{t}}\geq 0\Big\}.\end{aligned} \tag{6.11}$$

If $t_0>t_1$ we set $\Delta_{\mathcal{R}_\beta}(\boldsymbol{x}_0,\boldsymbol{x}_1):=+\infty$. We also define

$$\Delta_{\mathcal{R}_0}(t;u_0,u_1,u_2) := \Delta_{\mathcal{R}_0}((t,u_0),(t,u_1)) + \Delta_{\mathcal{R}_0}((t,u_1),(t,u_2)).$$

(notice that this quantity is independent of β).

Remark 6.5. The link with the Finsler cost $\Delta_{\mathfrak{p},\mathcal{E}}$ (3.8) induced by $(\mathfrak{p},\mathcal{E})$ is clear. For $\mathcal{R}=\mathcal{B}$ given by (6.9), using $\mathfrak{P}(0,v;0,w)=\mathfrak{p}(v,w)$ we have, for $t_0=t_1=t$,

$$\Delta_{\mathcal{B}_0}((t,u_0),(t,u_1)) = \Delta_{\mathfrak{p},\mathcal{E}}(t;u_0,u_1) \quad \text{for every } u_0,u_1\in X. \tag{6.12}$$

When $\mathcal{R}=\mathcal{P}$ is given by (6.8), we simply have

$$\Delta_{\mathcal{P}_\beta}((t_0,u_0),(t_1,u_1)) = \beta(t_1-t_0)+\Psi_0(u_1-u_0) \quad \text{for } u_0,u_1\in X,\ 0\leq t_0<t_1\leq T.$$

General properties of $\Delta_{\mathcal{R}_\beta}(\cdot,\cdot)$. It is not difficult to check that the infimum in (6.11) is attained and, by the usual rescaling argument (cf. Remark 4.5), one can always choose an optimal Lipschitz curve $\mathsf{x}=(\mathsf{t},\mathsf{u})$ defined in $[0,1]$ such that

$$\mathcal{R}_1(\mathsf{x};\dot{\mathsf{x}}) \quad \text{is essentially constant and equal to } \Delta_{\mathcal{R}_1}(\boldsymbol{x}_0,\boldsymbol{x}_1) = (t_1-t_0)+\Delta_{\mathcal{R}_0}(\boldsymbol{x}_0,\boldsymbol{x}_1). \tag{6.13}$$

Properties (6.7b)–(6.7c) yield, for every $u_0,u_1\in X$ and $0\leq t_0\leq t_1\leq T$, the estimate

$$\beta(t_1-t_0)+C\|u_1-u_0\|_X \leq \Delta_{\mathcal{R}_\beta}((t_0,u_0),(t_1,u_1)). \tag{6.14}$$

Notice that $\Delta_{\mathcal{R}_\beta}(\cdot,\cdot)$ is not symmetric but still satisfies the triangle inequality: for $\boldsymbol{x}_i=(t_i,u_i)\in\mathscr{X}$ with $t_0\leq t_1\leq t_2$, there holds

$$\Delta_{\mathcal{R}_\beta}(\boldsymbol{x}_0,\boldsymbol{x}_2) \leq \Delta_{\mathcal{R}_\beta}(\boldsymbol{x}_0,\boldsymbol{x}_1)+\Delta_{\mathcal{R}_\beta}(\boldsymbol{x}_1,\boldsymbol{x}_2).$$

Another useful property, direct consequence of (6.7c), is the lower semicontinuity with respect to convergence in $\mathscr{X}$: if $\boldsymbol{x}_{i,n}=(t_{i,n},u_{i,n})\to\boldsymbol{x}_i=(t_i,u_i)$ in $\mathscr{X}$ as $n\uparrow+\infty$, $i=0,1$, then

$$\liminf_{n\uparrow+\infty}\Delta_{\mathcal{R}_\beta}(\boldsymbol{x}_{0,n},\boldsymbol{x}_{1,n}) \geq \Delta_{\mathcal{R}_\beta}(\boldsymbol{x}_0,\boldsymbol{x}_1). \tag{6.15}$$

Indeed, assuming that the lim inf in (6.15) is finite and that, up to the extraction of a suitable subsequence, that it is a limit, it is sufficient to choose an optimal sequence

$\mathsf{x}_n = (\mathsf{t}_n, \mathsf{u}_n)$ of Lipschitz curves as in (6.13), which therefore satisfies a uniform Lipschitz bound and, up to the extraction of a further subsequence, converges to some Lipschitz curve $\mathsf{x} = (\mathsf{t}, \mathsf{u})$. Then, (6.15) can be proved in the same way as (6.6).

In the case of the cost induced by $\mathcal{B}$ induced by the vanishing viscosity contact potential $\mathfrak{P}$, we have a refined lower-semicontinuity result:

Lemma 6.6. *Let $u_n, w_n : [t_0, t_1] \to X$ be Borel maps and (ε_n) be a vanishing sequence. Suppose that u_n is absolutely continuous for every $n \in \mathbb{N}$ and that the following convergences hold as $n \to \infty$*

$$u_n(t) \to u(t) \quad \textit{and} \quad w_n(t) \to w(t) \quad \textit{for all } t \in [t_0, t_1]; \quad \sup_{t \in [t_0, t_1]} \|w_n(t) + \mathrm{D}\mathcal{E}_t(u_n(t))\|_{X^*} \to 0.$$

Then,

$$\liminf_{n \uparrow +\infty} \int_{t_0}^{t_1} \Big(\Psi_{\varepsilon_n}(\dot{u}_n(t)) + \Psi^*_{\varepsilon_n}(w_n(t)) \Big) \, \mathrm{d}t \geq \Delta_{\mathcal{B}_0}((t_0, u(t_0)), (t_1, u(t_1))). \tag{6.16}$$

Proof. Up to extracting a further subsequence, it is not restrictive to assume that the lim inf in (6.16) is in fact a limit. We set as in (5.1), (5.2)

$$\begin{gathered} \mathsf{s}_n(t) := t - t_0 + \int_{t_0}^{t} \Big(\Psi_{\varepsilon_n}(\dot{u}_n(r)) + \Psi^*_{\varepsilon_n}(w_n(r)) \Big) \, \mathrm{d}r, \quad \mathsf{S}_n := \mathsf{s}_n(t_1), \\ \mathsf{t}_n(s) := \mathsf{s}_n^{-1}(s), \quad \mathsf{u}_n(s) := u_n(\mathsf{t}_n(s)), \quad \mathsf{w}_n(s) := w_n(\mathsf{t}_n(s)) \quad \text{for all } s \in [0, \mathsf{S}_n] \end{gathered}$$

so that

$$\int_{t_0}^{t_1} \Big(\Psi_{\varepsilon_n}(\dot{u}_n(t)) + \Psi^*_{\varepsilon_n}(w_n(t)) \Big) \, \mathrm{d}t = \int_0^{\mathsf{S}_n} \mathfrak{P}_{\varepsilon_n}(\dot{\mathsf{t}}_n(s), \dot{\mathsf{u}}_n(s); 0, \mathsf{w}_n(s)) \, \mathrm{d}s. \tag{6.17}$$

Since the sequences (t_n) and (u_n) are uniformly Lipschitz, applying the Ascoli-Arzelà Theorem we can extract a (not relabeled) subsequence such that $\mathsf{S}_n \to \mathsf{S}$, and find functions $\mathsf{t} : [0, \mathsf{S}] \to [t_0, t_1]$, $\mathsf{u} : [0, \mathsf{S}] \to X$, and $\mathsf{w} : [0, \mathsf{S}] \to X^*$, such that

$$\mathsf{t}_n \to \mathsf{t}, \quad \mathsf{u}_n \to \mathsf{u}, \quad \mathsf{w}_n \to \mathsf{w} = -\mathrm{D}\mathcal{E}_{\mathsf{t}}(\mathsf{u}) \quad \text{uniformly in } [0, \mathsf{S}].$$

By construction, we have $\mathsf{t}(0) = t_0$, $\mathsf{u}(0) = u(t_0)$, $\mathsf{t}(\mathsf{S}) = t_1$, and $\mathsf{u}(\mathsf{S}) = u(t_1)$. Applying Proposition 6.2, we have

$$\begin{aligned} \liminf_{n \uparrow +\infty} \int_0^{\mathsf{S}_n} \mathfrak{P}_{\varepsilon_n}(\dot{\mathsf{t}}_n(s), \dot{\mathsf{u}}_n(s); 0, \mathsf{w}_n(s)) \, \mathrm{d}s &\geq \int_0^{\mathsf{S}} \mathfrak{P}_0(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s); 0, \mathsf{w}(s)) \, \mathrm{d}s \\ &\geq \Delta_{\mathcal{B}_0}((\mathsf{t}(0), \mathsf{u}(0)), (\mathsf{t}(\mathsf{S}), \mathsf{u}(\mathsf{S}))). \end{aligned} \tag{6.18}$$

Combining (6.17) and (6.18), we conclude (6.16). □

The total variation associated with $\Delta_{\mathcal{R}_\beta}$. In the same way as in Definition 3.4 we introduced the total variation $\mathit{Var}_{\mathfrak{p},\mathcal{E}}$ associated with the Finsler cost $\Delta_{\mathfrak{p},\mathcal{E}}$, it is now natural to define the total variation associated with $\Delta_{\mathcal{R}_\beta}$.

Definition 6.7 (Total variation for the pseudo-Finsler distance $\Delta_{\mathcal{R}_\beta}$)**.** For every curve $\mathsf{x} = (\mathsf{t}, \mathsf{u}) : [0, \mathsf{S}] \to \mathscr{X}$ such that t is *nondecreasing* and every interval $[\mathsf{a}, \mathsf{b}] \subset [0, \mathsf{S}]$ we set

$$\begin{aligned} \mathrm{Var}_{\mathcal{R}_\beta}(\mathsf{x}; [\mathsf{a}, \mathsf{b}]) := \sup \Big\{ \sum_{m=1}^{M} \Delta_{\mathcal{R}_\beta}(\mathsf{x}(s_m), \mathsf{x}(s_{m-1})) : \\ \mathsf{a} = s_0 < s_1 < \cdots < s_{M-1} < s_M = \mathsf{b} \Big\}. \end{aligned} \tag{6.19}$$

For a *non-parametrized* curve $u : [0, T] \to X$ and $[a, b] \subset [0, T]$, we simply set

$$\mathrm{Var}_{\mathcal{R}_\beta}(u; [a, b]) := \mathrm{Var}_{\mathcal{R}_\beta}(\boldsymbol{u}; [a, b]), \quad \text{with } \ \boldsymbol{u}(t) := (t, u(t)) \in \mathscr{X}, \quad t \in [0, T].$$

In view of (6.7b), it is immediate to check that a curve u with $\mathrm{Var}_{\mathcal{R}_0}(u; [0, T]) < +\infty$ belongs to $\mathrm{BV}([0, T]; X)$.

In contrast to the (pseudo)-total variation defined in (3.11), the above notion of total variation is lower semicontinuous with respect to pointwise convergence (compare with Remark 3.5).

Proposition 6.8 (Lower semicontinuity of $\mathrm{Var}_{\mathcal{R}_\beta}(\cdot; [\mathsf{a}, \mathsf{b}])$)**.** *If $\mathsf{x}_n = (\mathsf{t}_n, \mathsf{u}_n) : [0, \mathsf{S}] \to \mathscr{X}$ is a sequence of curves pointwise converging to $\mathsf{x} = (\mathsf{t}, \mathsf{u})$ as $n \uparrow \infty$, we have*

$$\liminf_{n \uparrow \infty} \mathrm{Var}_{\mathcal{R}_\beta}(\mathsf{x}_n; [\mathsf{a}, \mathsf{b}]) \geq \mathrm{Var}_{\mathcal{R}_\beta}(\mathsf{x}; [\mathsf{a}, \mathsf{b}]). \tag{6.20}$$

Proof. The argument is standard: for an arbitrary subdivision $\mathsf{a} = s_0 < s_1 < \cdots < s_{M-1} < s_M = \mathsf{b}$, (6.15) yields

$$\sum_{m=1}^{M} \Delta_{\mathcal{R}_\beta}(\mathsf{x}(s_m), \mathsf{x}(s_{m-1})) \leq \liminf_{n \uparrow +\infty} \sum_{m=1}^{M} \Delta_{\mathcal{R}_\beta}(\mathsf{x}_n(s_m), \mathsf{x}_n(s_{m-1})) \leq \liminf_{n \uparrow \infty} \mathrm{Var}_{\mathcal{R}_\beta}(\mathsf{x}_n; [\mathsf{a}, \mathsf{b}]).$$

Taking the supremum with respect to all subdivisions of $[\mathsf{a}, \mathsf{b}]$ we obtain (6.20). □

Lipschitz curves. The next result shows that, for Lipschitz curves, the total variation can be calculated by integrating the corresponding dissipation potential.

Proposition 6.9 (The total variation for Lipschitz curves)**.** *Given β, $L > 0$, a bounded curve $\mathsf{x} := (\mathsf{t}, \mathsf{u}) : [0, \mathsf{S}] \to \mathscr{X}$ satisfies the $\Delta_{\mathcal{R}_\beta}$-Lipschitz condition with Lipschitz constant L*

$$\Delta_{\mathcal{R}_\beta}(\mathsf{x}(s_1), \mathsf{x}(s_2)) \leq L(s_2 - s_1) \quad \textit{for every } 0 \leq s_1 \leq s_2 \leq \mathsf{S}, \tag{6.21}$$

if and only if it is Lipschitz continuous (with respect to the usual distance in $\mathscr{X}$), t is nondecreasing, and

$$\mathcal{R}_\beta\left(\mathsf{x}(s); \dot{\mathsf{x}}(s)\right) \leq L \quad \textit{for a.a. } s \in (0, \mathsf{S}). \tag{6.22}$$

In this case, for every $\gamma \geq 0$

$$\mathrm{Var}_{\mathcal{R}_\gamma}(\mathsf{x}; [\mathsf{a}, \mathsf{b}]) = \gamma(\mathsf{t}(\mathsf{b}) - \mathsf{t}(\mathsf{a})) + \int_{\mathsf{a}}^{\mathsf{b}} \mathcal{R}_0\left(\mathsf{x}(s); \dot{\mathsf{x}}(s)\right) \mathrm{d}s. \tag{6.23}$$

Proof. The sufficiency of condition (6.22) is clear. Let us now consider a curve $\boldsymbol{x}$ satisfying (6.21): by the coercivity (6.14), $\boldsymbol{x}$ is a Lipschitz curve in the usual sense and [47, Prop. 2.2] yields

$$\Delta_{\mathcal{R}_\beta}(\mathsf{x}(s_0),\mathsf{x}(s_1)) \le \int_{s_0}^{s_1} \mathsf{m}(s)\,\mathrm{d}s, \quad \text{where} \quad \mathsf{m}(s) := \lim_{h\downarrow 0} \frac{\Delta_{\mathcal{R}_\beta}(\mathsf{x}(s),\mathsf{x}(s+h))}{h}$$

is the so-called *metric derivative* of $\boldsymbol{x}$ (see [1, 4]). The minimality of m ensures that

$$\mathsf{m}(s) \le \mathcal{R}_\beta\left(\mathsf{x}(s);\dot{\mathsf{x}}(s)\right) \quad \text{for a.a. } s\in[0,\mathsf{S}]. \tag{6.24}$$

On the other hand, since $\mathcal{R}_\beta$ is lower semicontinuous and 1-homogeneous in $\boldsymbol{v}$, for every $0<\sigma<1$ and $s\in[0,\mathsf{S}]$ we find a constant $\delta>0$ such that

$$\mathcal{R}_\beta\left(\mathsf{x}(r);\boldsymbol{v}\right) \ge \sigma\mathcal{R}_\beta\left(\mathsf{x}(s);\boldsymbol{v}\right) \quad \text{for every } \boldsymbol{v}\in\mathscr{V} \quad \text{if } |r-s|\le\delta,$$

so that a comparison with the linear segment joining $\mathsf{x}(s)$ with $\mathsf{x}(s+h)$ yields

$$\Delta_{\mathcal{R}_\beta}(\mathsf{x}(s),\mathsf{x}(s+h)) \le \sigma^{-1}\mathcal{R}_\beta\left(\mathsf{x}(s);\mathsf{x}(s+h)-\mathsf{x}(s)\right)$$

Dividing by h and passing to the limit first as $h\downarrow 0$ and eventually as $\sigma\uparrow 1$, we obtain the opposite inequality of (6.24). Combining (6.24) (which holds as an equality) with (6.21), we infer (6.22), and (6.23) ensues. □

Proposition 6.10 (Reparametrization)**.** *Let $u:[0,T]\to X$ be a curve with finite total variation $V:=\mathrm{Var}_{\mathcal{R}_0}(u;[0,T])<+\infty$, and let us set*

$$\mathsf{s}(t) := t + \mathrm{Var}_{\mathcal{R}_0}(u;[0,t]) = \mathrm{Var}_{\mathcal{R}_1}(u;[0,t]) \quad \textit{for every } t\in[0,T]. \tag{6.25}$$

Then, there exists a Lipschitz parametrization $\mathsf{x}=(\mathsf{t},\mathsf{u}):[0,\mathsf{S}]\to\mathscr{X}$, with $\mathsf{S}=V+T$, such that

$$\mathcal{R}_1\left(\mathsf{x}(s);\dot{\mathsf{x}}(s)\right) = 1 \quad \textit{for a.a. } s\in(0,\mathsf{S}), \tag{6.26}$$

$$\mathsf{t}(\mathsf{s}(t)) = t, \quad \mathsf{u}(\mathsf{s}(t)) = u(t) \quad \textit{for every } t\in[0,T]. \tag{6.27}$$

In particular,

$$b-a+\mathrm{Var}_{\mathcal{R}_0}(u;[a,b]) = \mathsf{s}(b)-\mathsf{s}(a) = \int_{\mathsf{s}(a)}^{\mathsf{s}(b)} \mathcal{R}_1\left(\mathsf{x}(s);\dot{\mathsf{x}}(s)\right)\mathrm{d}s. \tag{6.28}$$

Proof. The proof is classical, at least when the dissipation $\mathcal{R}$ is continuous and even in its second argument: we briefly sketch the main ideas and refer to [35, Lemma 4.1].

Notice that the jump set J_s of the curve s given by (6.25) coincides with the jump set J_u of u, and s is injective in $\mathrm{C}_u := (0,T)\setminus\mathrm{J}_u$. We denote by t its inverse, defined on $\mathsf{C}_u := \mathsf{s}(\mathrm{C}_u)$ and extended to $\overline{\mathsf{C}}_u$ by its (Lipschitz) continuity; we also set $\mathsf{u}(s) := u(t)$ if $s=\mathsf{s}(t)\in\mathsf{C}_u$. Suppose now that $(\mathsf{s}_-,\mathsf{s}_+)$ is a connected component of $[0,\mathsf{S}]\setminus\overline{\mathsf{C}}_u$, corresponding to some time $\bar t\in[0,T]$ with $\mathsf{s}_\pm=\mathsf{s}(\bar t_\pm)$ and $\bar{\mathsf{s}}=\mathsf{s}(\bar t)\in[\mathsf{s}_-,\mathsf{s}_+]$. We have

$$\mathsf{u}(\mathsf{s}_-) = \lim_{s\uparrow\mathsf{s}_-}\mathsf{u}(s) = u(\bar t_-), \qquad \mathsf{u}(\mathsf{s}_+) = \lim_{s\downarrow\mathsf{s}_+}\mathsf{u}(s) = u(\bar t_+),$$
$$\bar{\mathsf{s}}-\mathsf{s}_- = \Delta_{\mathcal{R}_0}((\bar t,u(\bar t_-)),(\bar t,u(\bar t))), \quad \mathsf{s}_+-\bar{\mathsf{s}} = \Delta_{\mathcal{R}_0}((\bar t,u(\bar t)),(\bar t,u(\bar t_+))).$$

By Definition 6.4, we can join $(\bar t,\mathsf{u}(\mathsf{s}_-))$ to $(\bar t,\mathsf{u}(\mathsf{s}_+))$ by a $\Delta_{\mathcal{R}_0}$-Lipschitz curve (still denoted by (t,u)) defined in $[\mathsf{s}_-,\mathsf{s}_+]$ with constant first component $\mathsf{t}(s)=\bar t$, and satisfying (6.13) as well as $\mathsf{u}(\bar{\mathsf{s}})=u(\bar t)$.

It is then easy to check that the final curve $\mathsf{x} = (\mathsf{t}, \mathsf{u})$ obtained by "filling" in this way all the (at most countable) holes in $[0, \mathsf{S}] \setminus \mathsf{C}_u$ satisfies (6.27) and the Lipschitz condition (6.21) with $L \leq 1$. Applying (6.22) and (6.23) we get

$$\int_{\mathsf{s}(a)}^{\mathsf{s}(b)} \mathcal{R}_1 \left(\mathsf{x}(s); \dot{\mathsf{x}}(s)\right) \,\mathrm{d}s \leq \mathsf{s}(b) - \mathsf{s}(a) = \mathrm{Var}_{\mathcal{R}_1}(u; [a, b]) \leq \mathrm{Var}_{\mathcal{R}_1}(\mathsf{x}; [\mathsf{s}(a), \mathsf{s}(b)])$$
$$= \int_{\mathsf{s}(a)}^{\mathsf{s}(b)} \mathcal{R}_1 \left(\mathsf{x}(s); \dot{\mathsf{x}}(s)\right) \,\mathrm{d}s$$

where the first inequality follows from the 1-Lipschitz condition, the subsequent identity from the definition of s, and the last one from (6.28). □

The reparametrization of Proposition 6.10 is also useful to express the distributional derivative of u. If $\mathrm{Var}_{\mathcal{R}_0}(u; [0, T]) < +\infty$, we can introduce the distributional derivative $\mu_{\mathcal{R}_1,u} := \mathsf{s}'$ of s, which is a finite positive measure satisfying

$$\mu_{\mathcal{R}_1,u}([a, b]) = \mathsf{s}(b) - \mathsf{s}(a), \quad \int_0^T \zeta(t) \,\mathrm{d}\mu_{\mathcal{R}_1,u}(t) = -\int_0^T \dot{\zeta}(t)\mathsf{s}(t) \,\mathrm{d}t \quad \text{for every } \zeta \in \mathrm{C}^0_0(0, T).$$

Notice that a singleton $\{t\}$ has strictly positive measure if and only if $t \in \mathrm{J}_u$; more precisely

$$\mu_{\mathcal{R}_1,u}(\{t\}) = \Delta_{\mathcal{R}_0}(t; u(t_-), u(t), u(t_+)) \quad \text{if } t \in \mathrm{J}_u;$$
$$\mu_{\mathcal{R}_1,u}(\{t\}) = 0 \quad \text{if } t \in \mathrm{C}_u = (0, T) \setminus \mathrm{J}_u\,,$$

with obvious modification for $t = 0, T$. As a general fact we have the representation formula (recall that t is the inverse of s)

$$\mathsf{t}_\# \left(\mathscr{L}^1|_{(0,\mathsf{S})}\right) = \mu_{\mathcal{R}_1,u}, \quad \text{i.e.} \int_0^T \zeta(t) \,\mathrm{d}\mu_{\mathcal{R}_1,u}(t) = \int_0^{\mathsf{S}} \zeta(\mathsf{t}(s)) \,\mathrm{d}s, \tag{6.29}$$

for every bounded Borel function $\zeta : [0, T] \to \mathbb{R}$. Since t is injective in $\mathsf{C}_u := \mathsf{t}^{-1}(\mathrm{C}_u) \subset (0, \mathsf{S})$, a Borel subset A of C_u is $\mathscr{L}^1$-negligible if and only if $\mathsf{t}(A)$ has $\mu_{\mathcal{R}_1,u}$-measure 0. Therefore, as the derivatives $\dot{\mathsf{t}}, \dot{\mathsf{u}}$ are Borel functions defined up to a $\mathscr{L}^1$-negligible subset of $(0, \mathsf{S})$, the compositions $\dot{\mathsf{t}} \circ \mathsf{s}, \dot{\mathsf{u}} \circ \mathsf{s}$ are well defined in C_u. The next lemma shows that they play an important role.

Proposition 6.11. *The Lebesgue measure $\mathscr{L}^1|_{(0,T)}$ and the vector measure $u'_{\mathrm{co}} = u'_{\mathscr{L}} + u'_{\mathrm{C}}$ are absolutely continuous w.r.t. $\mu_{\mathcal{R}_1,u}$, and we have*

$$\frac{\mathrm{d}\mathscr{L}^1}{\mathrm{d}\mu_{\mathcal{R}_1,u}} = \dot{\mathsf{t}} \circ \mathsf{s} \quad \text{and} \quad \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu_{\mathcal{R}_1,u}} = \dot{\mathsf{u}} \circ \mathsf{s} \qquad \mu_{\mathcal{R}_1,u}\text{-a.e. in } \mathrm{C}_u. \tag{6.30}$$

Proof. The absolute continuity of both measures is easy, since $\mathscr{L}^1 \leq \mu_{\mathcal{R}_1,u}$ by (6.29) and the total variation $\|u'\|_X$ is absolutely continuous w.r.t. $\mu_{\mathcal{R}_1,u}$ thanks to (6.14). The first identity of (6.30) can be proved as in [35, Lemma 4.1]. Concerning the second one, let us set for every smooth function $\zeta \in \mathrm{C}^\infty_0(0, T)$

$$J_u(\zeta) := \sum_{t \in \mathrm{J}_u} \zeta(t)\big(u(t_+) - u(t_-)\big),$$

and let us observe that we have

$$-\int_{\mathsf{C}_u} (\zeta\circ\mathsf{t})'(s)\,\mathsf{u}(s)\,\mathrm{d}s = \int_{\mathsf{C}_u} \zeta(\mathsf{t}(s))\,\dot{\mathsf{u}}(s)\,\mathrm{d}s + J_u(\zeta). \tag{6.31}$$

Indeed, denoting by $A_t = (\mathsf{a}_t,\mathsf{b}_t) = \mathsf{t}^{-1}(t)$, $t\in \mathrm{J}_u$, the connected components of $[0,\mathsf{S}]\setminus \mathsf{C}_u$, and recalling that $\mathsf{u}(\mathsf{a}_t) = u(t_-), \mathsf{u}(\mathsf{b}_t) = u(t_+)$, we have

$$\begin{aligned}
-\int_{\mathsf{C}_u} (\zeta\circ\mathsf{t})'(s)\,\mathsf{u}(s)\,\mathrm{d}s &= -\int_0^{\mathsf{S}} (\zeta\circ\mathsf{t})'(s)\,\mathsf{u}(s)\,\mathrm{d}s + \sum_{t\in\mathrm{J}_u}\int_{\mathsf{a}_t}^{\mathsf{b}_t} (\zeta\circ\mathsf{t})'(s)\,\mathsf{u}(s)\,\mathrm{d}s \\
= \int_0^{\mathsf{S}} \zeta(\mathsf{t}(s))\,\dot{\mathsf{u}}(s)\,\mathrm{d}s - \sum_{t\in\mathrm{J}_u}\int_{\mathsf{a}_t}^{\mathsf{b}_t} &(\zeta\circ\mathsf{t})(s)\,\dot{\mathsf{u}}(s)\,\mathrm{d}s + J_u(\zeta) = \int_{\mathsf{C}_u} \zeta(\mathsf{t}(s))\,\dot{\mathsf{u}}(s)\,\mathrm{d}s + J_u(\zeta).
\end{aligned}$$

Therefore, there holds

$$\begin{aligned}
\int_0^T \zeta(t)\,\mathrm{d}u'(t) &= -\int_0^T \dot{\zeta}(t)\,u(t)\,\mathrm{d}t = -\int_0^{\mathsf{S}} \dot{\zeta}(\mathsf{t}(s))\,u(\mathsf{t}(s))\,\dot{\mathsf{t}}(s)\,\mathrm{d}s \\
&= -\int_{\mathsf{C}_u} \dot{\zeta}(\mathsf{t}(s))\,u(\mathsf{t}(s))\,\dot{\mathsf{t}}(s)\,\mathrm{d}s \\
&= -\int_{\mathsf{C}_u} (\zeta\circ\mathsf{t})'(s)\,\mathsf{u}(s)\,\mathrm{d}s = \int_{\mathsf{C}_u} \zeta(\mathsf{t}(s))\,\dot{\mathsf{u}}(s)\,\mathrm{d}s + J_u(\zeta) \\
&= \int_{\mathrm{C}_u} \zeta(t)\,\dot{\mathsf{u}}(\mathsf{s}(t))\,\mathrm{d}\mu_{\mathcal{R}_1,u}(t) + J_u(\zeta).
\end{aligned}$$

where the fifth identity ensues from (6.31) and the last one from (6.29). Since

$$\int_0^T \zeta(t)\,\mathrm{d}u'_{\mathrm{co}}(t) = \int_0^T \zeta(t)\,\mathrm{d}u'(t) - J_u(\zeta)$$

we conclude the second of (6.30). □

Corollary 6.12 (Integral expression for $\mathrm{Var}_{\mathcal{R}}$)**.** *Let $u:[0,T]\to X$ fulfil $\mathrm{Var}_{\mathcal{R}_0}(u;[0,T]) < +\infty$, let μ be a positive finite measure such that $\mathscr{L}^1 \ll \mu$ and $u'_{\mathrm{co}} \ll \mu$, and let us set*

$$\begin{aligned}
\mathrm{Jmp}_{\mathcal{R}_0}(u;[a,b]) := \Delta_{\mathcal{R}_0}(a;u(a),u(a_+)) &+ \Delta_{\mathcal{R}_0}(b;u(b_-),u(b)) \\
&+ \sum_{t\in\mathrm{J}_u\cap(a,b)} \Delta_{\mathcal{R}_0}(t;u(t_-),u(t),u(t_+)).
\end{aligned} \tag{6.32}$$

Then,

$$\mathrm{Var}_{\mathcal{R}_0}(u;[a,b]) = \int_a^b \mathcal{R}_0\left(\left(t,u(t)\right);\left(\frac{\mathrm{d}\mathscr{L}^1}{\mathrm{d}\mu}(t),\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}(t)\right)\right)\,\mathrm{d}\mu(t) + \mathrm{Jmp}_{\mathcal{R}_0}(u;[a,b]). \tag{6.33}$$

Proof. Since the expression on the right-hand side is independent of the measure μ, it is not restrictive to choose $\mu = \mu_{\mathcal{R}_1,u}$; by (6.28) we have

$$\begin{aligned} b - a + \mathrm{Var}_{\mathcal{R}_0}(u;[a,b]) &= \int_{(\mathsf{s}(a),\mathsf{s}(b))\cap \mathsf{C}_u} \mathcal{R}_1\left(\mathsf{x}(s);\dot{\mathsf{x}}(s)\right)\,\mathrm{d}s + \mathscr{L}^1((\mathsf{s}(a),\mathsf{s}(b))\setminus \mathsf{C}_u) \\ &= \int_{(a,b)\cap \mathrm{C}_u} \mathcal{R}_1\left(\mathsf{x}(\mathsf{s}(t));\dot{\mathsf{x}}(\mathsf{s}(t))\right)\,\mathrm{d}\mu + \mu([a,b]\cap \mathrm{J}_u) \\ &= \int_{(a,b)\cap \mathrm{C}_u} \mathcal{R}_1\Big((t,u(t));(\dot{\mathsf{t}}(\mathsf{s}(t)),\dot{\mathsf{u}}(\mathsf{s}(t)))\Big) + \mathrm{Jmp}_{\mathcal{R}_0}(u;[a,b]), \end{aligned}$$

and we conclude by (6.30). $\square$

6.3. **Total variation for** BV **solutions.** We focus now on the particular case (6.10) of Example 6.3, when the dissipation $\mathcal{R}$ is associated with the vanishing viscosity contact potential $\mathfrak{P}$.

Theorem 6.13 (Comparison between $\mathrm{Var}_{\mathcal{B}_0}(u;[\cdot,\cdot])$ and $Var_{\mathfrak{p},\mathcal{E}}(u;[\cdot,\cdot])$)**.** *For every curve $u\in \mathrm{BV}([0,T];X)$ and every interval $[a,b]\subset[0,T]$ we have*

$$\mathrm{Var}_{\mathfrak{p},\mathcal{E}}(u;[a,b]) \le \mathrm{Var}_{\mathcal{B}_0}(u;[a,b]), \tag{6.34}$$

and equality holds in (6.34) *if and only if u satisfies the local stability condition* ($\mathrm{S}_{\mathrm{loc}}$) *on* (a,b)*. Furthermore, if* $\mathrm{Var}_{\mathcal{B}_0}(u;[a,b])<+\infty$*, then u satisfies* ($\mathrm{S}_{\mathrm{loc}}$) *on* (a,b)*.*

Proof. Let us first notice that the jump contributions to the total variations $Var_{\mathfrak{p},\mathcal{E}}$ and $\mathrm{Var}_{\mathcal{B}_0}$ are the same by (6.12). Inequality (6.34) then follows by applying (6.33) and observing that for μ-a.a. $t\in[0,T]$

$$\mathcal{B}_0\Big((t,u(t)),\Big(\frac{\mathrm{d}\mathscr{L}^1}{\mathrm{d}\mu}(t),\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}(t)\Big)\Big) = \mathfrak{P}\Big(\frac{\mathrm{d}\mathscr{L}^1}{\mathrm{d}\mu}(t),\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}(t);0,w(t)\Big) \ge \Psi_0\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\mu}(t)\Big) \tag{6.35}$$

(where we have used the notation $w(t) = -\mathrm{D}\mathcal{E}_t(u(t))$), the latter inequality ensuing from (5.9). On the other hand, in view of (5.10), (6.35) is an identity if and only if $w(t)\in K^*$ for μ-a.a $t\in(0,T)$, i.e. if the local stability property ($\mathrm{S}_{\mathrm{loc}}$) holds.

Finally, since $\mathscr{L}^1 \ll \mu$, $\frac{\mathrm{d}\mathscr{L}^1}{\mathrm{d}\mu}(t)>0$ for $\mathscr{L}^1$-a.a $t\in(0,T)$. Therefore, on account of (5.7) we conclude the last part of the statement. $\square$

Corollary 6.14. *A curve $u:[0,T]\to X$ is a* BV *solution if and only if it satisfies one of the following (equivalent) two conditions:*

$$\mathrm{Var}_{\mathcal{B}_0}(u;[t_0,t_1]) + \mathcal{E}_{t_1}(u(t_1)) = \mathcal{E}_{t_0}(u(t_0)) + \int_{t_0}^{t_1} \partial_t\mathcal{E}_t(u(t))\,\mathrm{d}t \quad \textit{for every } 0\le t_0\le t_1\le T, \tag{6.36}$$

$$\mathrm{Var}_{\mathcal{B}_0}(u;[0,T]) + \mathcal{E}_T(u(T)) \le \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s. \tag{6.37}$$

Lemma 6.15. *Suppose that $u_\varepsilon \in \mathrm{AC}([0,T];X)$, $\varepsilon > 0$, is a family pointwise converging to u as $\varepsilon \downarrow 0$, and $w_\varepsilon : [0,T] \to X^*$ satisfies $\|w_\varepsilon(t) + \mathrm{D}\mathcal{E}_t(u_\varepsilon(t))\|_{X^*} \to 0$ uniformly in $[0,T]$. Then,*

$$\liminf_{\varepsilon\downarrow 0} \int_0^T \Big(\Psi_\varepsilon(\dot u_\varepsilon) + \Psi_\varepsilon^*(w_\varepsilon(t))\Big)\,\mathrm{d}t \ge \mathrm{Var}_{\mathfrak{B}_0}(u;[0,T]) \ge \mathrm{Var}_{\mathfrak{p},\mathcal{E}}(u;[0,T]). \tag{6.38}$$

Proof. Choosing a finite partition $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ of the time interval $[0,T]$, Lemma 6.6 yields

$$\liminf_{\varepsilon\downarrow 0} \int_0^T \Big(\Psi_\varepsilon(\dot u_\varepsilon) + \Psi_\varepsilon^*(w_\varepsilon(t))\Big)\,\mathrm{d}t \ge \sum_{j=1}^N \Delta_{\mathfrak{B}_0}((t_{j-1},u(t_{j-1})),(t_j,u(t_j))).$$

Taking the supremum of the right-hand side with respect to all partitions of $[0,T]$, we end up with (6.38). □

We conclude this section with the **proof of Theorem 5.8.**

Proof. Let (t,u) be a parametrized solution as in the statement of the theorem. It is easy to check directly from definitions (6.11) and (6.19) that

$$\begin{aligned}\mathrm{Var}_{\mathfrak{B}_0}(u;[0,T]) &\le \int_0^{\mathsf{S}} \mathfrak{P}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s);0,-\mathrm{D}\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s)))\,\mathrm{d}s \\ &\le \mathcal{E}_0(\mathsf{u}(0)) - \mathcal{E}_{\mathsf{t}(\mathsf{S})}(\mathsf{u}(\mathsf{S})) + \int_0^{\mathsf{S}} \partial_t\mathcal{E}_{\mathsf{t}(s)}(\mathsf{u}(s))\dot{\mathsf{t}}(s)\,\mathrm{d}s \\ &= \mathcal{E}_0(u(0)) - \mathcal{E}_T(u(T)) + \int_0^T \partial_t\mathcal{E}_t(u(t))\,\mathrm{d}t\,,\end{aligned}$$

where the second inequality ensues from (5.22). Thus, (6.37) holds, so that u is a BV solution by Corollary 6.14. The converse implication follows from Proposition 6.10. □

References

[1] L. Ambrosio, *Minimizing movements*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5), 19 (1995), pp. 191–246.

[2] L. Ambrosio and G. Dal Maso, *A general chain rule for distributional derivatives*, Proc. Amer. Math. Soc., 108 (1990), pp. 691–702.

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 2000.

[4] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, second ed., 2008.

[5] F. Auricchio, A. Mielke, and U. Stefanelli, *A rate-independent model for the isothermal quasi-static evolution of shape-memory materials*, $\mathrm{M^3AS}$ Math. Models Meth. Appl. Sci., 18 (2008), pp. 125–164.

[6] G. Bouchitté, A. Mielke, and T. Roubíček, *A complete-damage problem at small strains*, Z. Angew. Math. Phys., 60 (2009), pp. 205–236.

[7] M. Buliga, G. de Saxcé, and C. Vallée, *Existence and construction of bipotentials for graphs of multivalued laws*, J. Convex Anal., 15 (2008), pp. 87–104.

[8] P. Colli, *On some doubly nonlinear evolution equations in Banach spaces*, Japan J. Indust. Appl. Math., 9 (1992), pp. 181–203.

[9] P. Colli and A. Visintin, *On a class of doubly nonlinear evolution equations*, Comm. Partial Differential Equations, 15 (1990), pp. 737–756.

[10] G. Dal Maso, A. DeSimone, and M. G. Mora, *Quasistatic evolution problems for linearly elastic-perfectly plastic materials*, Arch. Ration. Mech. Anal., 180 (2006), pp. 237–291.

[11] G. Dal Maso, A. DeSimone, M. G. Mora, and M. Morini, *Globally stable quasistatic evolution in plasticity with softening*, Netw. Heterog. Media, 3 (2008), pp. 567–614.

[12] G. Dal Maso, A. DeSimone, M. G. Mora, and M. Morini, *A vanishing viscosity approach to quasistatic evolution in plasticity with softening*, Arch. Ration. Mech. Anal., 189 (2008), pp. 469–544.

[13] G. Dal Maso, A. DeSimone, and F. Solombrino, *Quasistatic evolution for cam-clay plasiticity: a weak formulation via viscoplastic regularization and time rescaling*, (2009). Preprint. Available at `http://cvgmt.sns.it`.

[14] G. Dal Maso, G. Francfort, and R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Rational Mech. Anal., 176 (2005), pp. 165–225.

[15] G. Dal Maso and R. Toader, *A model for quasi–static growth of brittle fractures: existence and approximation results*, Arch. Rational Mech. Anal., 162 (2002), pp. 101–135.

[16] G. Dal Maso and R. Toader, *A model for the quasi-static growth of brittle fractures based on local minimization*, Math. Models Methods Appl. Sci., 12 (2002), pp. 1773–1799.

[17] G. Dal Maso and C. Zanini, *Quasi-static crack growth for a cohesive zone model with prescribed crack path.*, Proc. R. Soc. Edinb., Sect. A, Math., 137 (2007), pp. 253–279.

[18] M. Efendiev and A. Mielke, *On the rate–independent limit of systems with dry friction and small viscosity*, J. Convex Analysis, 13 (2006), pp. 151–167.

[19] A. Fiaschi, *A vanishing viscosity approach to a quasistatic evolution problem with nonconvex energy*, Ann. Inst. H. Poincaré Anal. Non Linéaire, (2009). To appear.

[20] G. Francfort and A. Mielke, *Existence results for a class of rate-independent material models with nonconvex elastic energies*, J. reine angew. Math., 595 (2006), pp. 55–91.

[21] D. Knees, A. Mielke, and C. Zanini, *On the inviscid limit of a model for crack propagation*, Math. Models Methods Appl. Sci., 18 (2008), pp. 1529–1569.

[22] ——, *Crack propagation in polyconvex materials*, Special Issue of Physica D: Nonlinear phenomena, (2009). To appear, DOI: 10.1016/j.physd.2009.02.008.

[23] M. Kočvara, A. Mielke, and T. Roubíček, *A rate–independent approach to the delamination problem*, Math. Mech. Solids, 11 (2006), pp. 423–447.

[24] P. Krejčí, *Evolution variational inequalities and multidimensional hysteresis operators*, in Nonlinear differential equations (Chvalatice, 1998), vol. 404 of Chapman & Hall/CRC Res. Notes Math., Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 47–110.

[25] P. Krejčí and M. Liero, *Rate independent Kurzweil processes*, Appl. Math., 54 (2009), pp. 117–145.

[26] C. Larsen, *ϵ-stable quasi-static brittle fracture evolution*, Comm. Pure Appl. Math, (2009). To appear.

[27] A. Mainik and A. Mielke, *Existence results for energetic models for rate–independent systems*, Calc. Var. PDEs, 22 (2005), pp. 73–99.

[28] A. Mainik and A. Mielke, *Global existence for rate-independent gradient plasticity at finite strain*, J. Nonlinear Science, 19 (2009), pp. 221–248.

[29] A. Mielke, *Finite elastoplasticity, Lie groups and geodesics on $SL(d)$*, in Geometry, Dynamics, and Mechanics, P. Newton, A. Weinstein, and P. J. Holmes, eds., Springer–Verlag, New York, 2002, pp. 61–90.

[30] ——, *Energetic formulation of multiplicative elasto–plasticity using dissipation distances*, Cont. Mech. Thermodynamics, 15 (2003), pp. 351–382.

[31] ——, *Existence of minimizers in incremental elasto–plasticity with finite strains*, SIAM J. Math. Analysis, 36 (2004), pp. 384–404.

[32] A. Mielke, *Evolution in rate-independent systems (Ch. 6)*, in Handbook of Differential Equations, Evolutionary Equations, vol. 2, C. Dafermos and E. Feireisl, eds., Elsevier B.V., Amsterdam, 2005, pp. 461–559.

[33] ——, *Complete-damage evolution based on energies and stresses*, Special Issue of Discrete Cont. Dyn. Syst. Ser. S, (2009). Submitted. WIAS preprint 1414.

[34] ——, *Differential, energetic and metric formulations for rate-independent processes*, October 2009. Lecture Notes, Summer School Cetraro 2008.

[35] A. Mielke, R. Rossi, and G. Savaré, *Modeling solutions with jumps for rate-independent systems on metric spaces*, Discrete Contin. Dyn. Syst., 25 (2009), pp. 585–615.

[36] A. Mielke, R. Rossi, and G. Savaré, *Nonsmooth analysis of doubly nonlinear evolution equations*, in preparation, (2009).

[37] A. Mielke and T. Roubíček, *A rate–independent model for inelastic behavior of shape–memory alloys*, Multiscale Model. Simul., 1 (2003), pp. 571–597.

[38] ——, *Rate-independent damage processes in nonlinear elasticity*, M^3AS Math. Models Methods Appl. Sci., 16 (2006), pp. 177–209.

[39] A. Mielke and T. Roubíček, *Rate-Independent Systems: Theory and Application*, In preparation, 2009.

[40] A. Mielke and F. Theil, *A mathematical model for rate-independent phase transformations with hysteresis*, in Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering", H.-D. Alber, R. Balean, and R. Farwig, eds., Aachen, 1999, Shaker-Verlag, pp. 117–129.

[41] ——, *On rate–independent hysteresis models*, Nonl. Diff. Eqns. Appl. (NoDEA), 11 (2004), pp. 151–189. (Accepted July 2001).

[42] A. Mielke, F. Theil, and V. I. Levitas, *A variational formulation of rate-independent phase transformations using an extremum principle*, Arch. Ration. Mech. Anal., 162 (2002), pp. 137–177.

[43] A. Mielke and A. Timofte, *An energetic material model for time-dependent ferroelectric behavior: existence and uniqueness*, Math. Meth. Appl. Sciences, 29 (2006), pp. 1393–1410.

[44] A. Mielke and S. Zelik, *On the vanishing viscosity limit in parabolic systems with rate-independent dissipation terms*, in preparation, (2009).

[45] M. Negri and C. Ortner, *Quasi-static crack propagation by Griffith's criterion*, Math. Models Methods Appl. Sci., 18 (2008), pp. 1895–1925.

[46] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[47] R. Rossi, A. Mielke, and G. Savaré, *A metric approach to a class of doubly nonlinear evolution equations and applications*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 7 (2008), pp. 97–169.

[48] R. Rossi and G. Savaré, *Gradient flows of non convex functionals in Hilbert spaces and applications*, ESAIM Control Optim. Calc. Var., 12 (2006), pp. 564–614 (electronic).

[49] T. Roubíček, *Rate independent processes in viscous solids at small strains*, Math. Methods Appl. Sci., 32 (2009), pp. 825–862.

[50] F. Schmid and A. Mielke, *Vortex pinning in super-conductivity as a rate-independent process*, Europ. J. Appl. Math., 16 (2005), pp. 799–808.

[51] U. Stefanelli, *A variational characterization of rate-independent evolution*, Math. Nachr., (2009). To appear.

[52] M. Thomas and A. Mielke, *Damage of nonlinearly elastic materials at small strains: existence and regularity results*, ZAMM Zeits. Angew. Math. Mech., (2009). Submitted. WIAS preprint 1397.

[53] R. Toader and C. Zanini, *An artificial viscosity approach to quasistatic crack growth*, Boll. Unione Mat. Ital. (9), 2 (2009), pp. 1–35.

[54] A. Visintin, *Differential models of hysteresis*, vol. 111 of Applied Mathematical Sciences, Springer-Verlag, Berlin, 1994.