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High-frequency averaging in semi-classical Hartree-type equations

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Abstract. We investigate the asymptotic behavior of solutions to semi-classical Schrödinger equations with nonlinearities of Hartree type. For a weakly nonlinear scaling, we show the validity of an asymptotic superposition principle for slowly modulated highly oscillatory pulses. The result is based on a high-frequency averaging effect due to the nonlocal nature of the Hartree potential, which inhibits the creation of new resonant waves. In the proof we make use of the framework of Wiener algebras.

1. Introduction and main result

In this work we are interested in the asymptotic behavior for $0 < \varepsilon \ll 1$ of the following nonlinear Schrödinger equation

$$i\varepsilon \partial_t \mathrm{u}^\varepsilon = -\frac{\varepsilon^2}{2}\Delta \mathrm{u}^\varepsilon + \left(K * |\mathrm{u}^\varepsilon|^2\right)\mathrm{u}^\varepsilon, \quad \mathrm{u}^\varepsilon\big|_{t=0} = \mathrm{u}_0^\varepsilon, \tag{1.1}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $d \in \mathbb{N}$. This model describes the time-evolution of a complex-valued field $\mathrm{u}^\varepsilon(t,x)$, under the influence of a *Hartree-type nonlinearity*, as it appears for example in the description of superfluids, cf. [2]. Above, $K = K(x)$ denotes a given real-valued interaction kernel, to be specified in detail below, and "$*$" denotes the convolution w.r.t. $x \in \mathbb{R}^d$. The scaling of (1.1) for small $\varepsilon > 0$ corresponds to the *semi-classical* regime, i.e. the regime of *high-frequency* solutions $\mathrm{u}^\varepsilon(t,x)$, which can be approximated via geometric optics.

The asymptotic behavior of (1.1) as $\varepsilon \to 0_+$ has been studied by several authors, mainly for the case $K(x) = \pm\frac{1}{|x|}$: For example in [14, 16, 18], Wigner measure techniques are invoked, which however require mixed states and thus can not be applied to our situation. In the one-dimensional case, this constraint can be overcome [19], but uniqueness of the limiting Wigner measure for $t > 0$ is still open. Turning to multi-scale WKB expansions, which are typically valid for short times only, a variety of asymptotic results can be found in [1, 7, 10, 15], provided that u_0^ε is given as a *single-phase* WKB initial data, i.e

$$\mathrm{u}_0^\varepsilon(x) = a^\varepsilon(x)e^{i\varphi(x)/\varepsilon},$$

with given (ε-independent) phase $\varphi(x) \in \mathbb{C}$ and amplitude $a^\varepsilon(x) \in \mathbb{R}$, such that, asymptotically, $a^\varepsilon \sim a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$.

In the present work we are interested in generalizing these studies to the case of (a superposition of) *several WKB waves.* Due to the expected nonlinear interaction between high-frequency waves (i.e. the appearance of resonances), this problem is notoriously difficult, even on a formal level. We shall therefore simplify the situation considerably by turning our attention to the case of (asymptotically) *small initial data* corresponding to *modulated plane-waves.* More precisely, we consider $\mathrm{u}_0^\varepsilon(x) = \varepsilon^{\alpha/2}u_0^\varepsilon(x)$, where $\alpha \geqslant 1$ and $u_0^\varepsilon(x)$ is given by a superposition of ε-oscillatory plane-waves, i.e.

$$u_0^\varepsilon(x) = \sum_{j=1}^{J} a_j(x)e^{ik_j\cdot x/\varepsilon}, \quad k_j \in \mathbb{R}^d, \tag{1.2}$$

with amplitudes $a_j(x) \in \mathbb{C}$, $j \in \{1, \dots, J\} \subseteq \mathbb{N}$. Here, and in the following, we shall assume for the sake of simplicity that the initial amplitudes a_j do not depend on ε, since we shall only be interested in the leading order asymptotics. A generalization to amplitudes admitting an asymptotic expansion in ε is then straightforward. Without restriction of generality we shall also assume that

$$k_j \neq k_\ell \text{ for all } j \neq \ell \in \Gamma := \{1, \dots, J\} \subseteq \mathbb{N},$$

where $J \in \mathbb{N} \cup \{\infty\}$ and $\{1, \dots, \infty\}$ means simply $\mathbb{N}$.

By rescaling $u^\varepsilon = \varepsilon^{-\alpha/2} \mathrm{u}^\varepsilon$, we can rewrite the considered model into

$$i\varepsilon \partial_t u^\varepsilon = -\frac{\varepsilon^2}{2} \Delta u^\varepsilon + \varepsilon^\alpha \left(K * |u^\varepsilon|^2 \right) u^\varepsilon, \quad u^\varepsilon \big|_{t=0} = u_0^\varepsilon, \tag{1.3}$$

where the initial data is now of order one but the equation displays an (asymptotically) small nonlinearity. We shall from now on take on this point of view since it looks more natural from the physical point of view. In addition, it is well known (see e.g. [3, 4]) that the choice $\alpha = 1$ for (1.3) is *critical* as far as semi-classical behavior is concerned (see Section 2 for more details). We shall therefore pay particular attention to this case. We remark that the same asymptotic scaling has been used in [5, 6] for Schrödinger equations with (gauge invariant) power-law nonlinearities $\propto |u|^{2\sigma} u$, $\sigma \in \mathbb{N}$. In particular, in [5] the problem of high-frequency wave interaction is exhaustively studied in the case $\sigma = 1$ for which a geometric description of all possible nonlinear resonances is given.

However, for the considered case of Hartree type nonlinearities, the situation is very different, due to the *nonlocal nature* of $K * |u^\varepsilon|^2$. Since we expect the solution $u^\varepsilon(t, x)$ to be given asymptotically as a superposition of highly oscillatory waves, we clearly can not regard the Hartree term to be slowly varying (as in the case of a single wave). The notion of a resonance though, seems to be not clearly defined in this case. A more sophisticated analysis of the oscillatory structure of $K * |u^\varepsilon|^2$ is needed therefore. As we shall see, the nonlocal nature of the Hartree potential yields a kind of *averaging effect.* In particular *no new* highly oscillatory phases are created in leading order (via resonances), in sharp contrast to the situation for local nonlinearities.

In order to be more precise, we need to introduce some notation: The Fourier transform of $f \in L^1 \cap L^2(\mathbb{R}^d)$ will be denoted as

$$(\mathcal{F} f)(\xi) \equiv \widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.$$

Our analytical approach will be based on the use of the *Wiener Algebra* (see Section 3 for more details). Within this framework it turns out that the natural space for the amplitudes $a = (a_j)_{j\in\Gamma}$ is given as follows.

Definition 1.1.

$$\mathcal{A}(\mathbb{R}^d) := \{a = (a_j)_{j\in\Gamma} : (\widehat{a}_j)_{j\in\Gamma} \in \ell^1(\Gamma; L^1(\mathbb{R}^d))\},$$

$\Gamma = \{1, \dots, J\} \subseteq \mathbb{N}$, equipped with the norm

$$\|a\|_{\mathcal{A}(\mathbb{R}^d)} = \sum_{j=1}^{J} \|\widehat{a}_j\|_{L^1(\mathbb{R}^d)}.$$

We are now in the position to state the main theorem of this work.

Theorem 1.2. *For $d \geqslant 1$, consider the Cauchy problem* (1.3) *with $\alpha \geqslant 1$, subject to initial data u_0^ε of the form* (1.2)*, where the initial amplitudes $a_j \in L^2(\mathbb{R}^d) \cap \mathcal{A}(\mathbb{R}^d)$ are such that $(\partial_x^p a_j)_{j\in\Gamma} \in \mathcal{A}(\mathbb{R}^d)$ for all $|p| \leqslant 2$. In addition assume*

$$|\Lambda|_\infty^{-1} := \inf\{|k_\ell - k_m| : \ \ell, m \in \Gamma, \ \ell \neq m\} > 0, \tag{1.4}$$

and let the interaction kernel K satisfy $(1 + |\xi|)\widehat{K}(\xi) \in L^\infty(\mathbb{R}^d)$.

Then, for all $T > 0$ there exists $C, \varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, the exact solution to (1.3) *can be approximated by*

$$\left\|u^\varepsilon - u^\varepsilon_{\mathrm{app}}\right\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant C\varepsilon^\beta \quad \text{with } \beta = \begin{cases} 1 & \text{if } \alpha \in \{1\} \cup [2, \infty), \\ \alpha - 1 & \text{if } \alpha \in (1, 2). \end{cases} \tag{1.5}$$

Here the approximate solution $u^\varepsilon_{\mathrm{app}}$ is given by

$$u^\varepsilon_{\mathrm{app}}(t, x) = \sum_{j=1}^{J} a_j(x - tk_j) e^{iS_j(t,x)} e^{ik_j\cdot x/\varepsilon - it|k_j|^2/(2\varepsilon)}, \tag{1.6}$$

where $S_j(t, x) \in \mathbb{R}$ is defined in (2.10) *if $\alpha = 1$ and $S_j(t, x) = 0$ if $\alpha > 1$, respectively.*

In (1.6), the total number of (highly oscillatory) phases $J \in \mathbb{N} \cup \infty$, is the same as for the initial data (1.2). Hence, no new phases are created by the nonlinearity. Nonlinear effects in leading order only show up via *self-modulation of the amplitudes* (provided $\alpha = 1$). The condition (1.4) can be seen as a *small divisor assumption* required in the case of infinitely many phases (though, not of the same type as the one used in [5]). Obviously, (1.4) is satisfied if $J < +\infty$.

Under the general assumptions of this work, we can not infer global well posedness of equation (1.3) in the Wiener space. The usual arguments for proving global existence (see e.g. [17]) involve the conservation of the L^2 norm, which also holds in our case. However, this is not sufficient to control the nonlinearity $(K * |u|^2)u$ in the Wiener space. Nevertheless, the above theorem shows that for initial data u_0^ε in the form (1.2), the solutions cannot blow up too fast: If $T^\varepsilon > 0$ is the blow-up time, then $T^\varepsilon \to +\infty$ for $\varepsilon \to 0$.

Remark 1.3. A particular example for K, satisfying the assumptions is given by the one considered in [2, equation (15)] and more importantly by the so-called *Yukawa potential*

$$K(x) = \pm \frac{e^{-\lambda|x|}}{|x|}, \quad x \in \mathbb{R}^3, \lambda > 0,$$

for which the corresponding Fourier transform is found to be

$$\widehat{K}(\xi) = \pm \frac{1}{\lambda^2 + |\xi|^2}.$$

Clearly, $(1+|\xi|)\widehat{K} \in L^\infty(\mathbb{R}^3)$ in this case. Note that in the limit $\lambda \to 0$ we obtain the Fourier transform of the Coulomb potential $K(x) = \pm\frac{1}{|x|}$, which, however, is too singular, to directly apply our theorem. It remains an interesting open problem to establish the same result for the Coulomb case in $d = 3$.

The paper is organized as follows: In Section 2, we formally derive the approximate solution and draw some further conclusions from it. In Section 3 we set up the Wiener framework for the exact and the approximate solution. Finally, we prove the required estimates on the remainder of the approximation and consequently state the proof of Theorem 1.2 in Section 4.

2. Derivation of the approximate solution

We seek an approximation of solutions to (1.3) in the form

$$u^\varepsilon_{\rm app}(t,x) = \sum_{j=1}^J A_j(t,x) e^{i\phi_j(t,x)/\varepsilon}. \tag{2.1}$$

Assuming sufficient smoothness, we can plug this ansatz into (1.3), which yields

$$i\varepsilon\partial_t u^\varepsilon_{\rm app} + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_{\rm app} - \varepsilon^\alpha \left(K * |u^\varepsilon_{\rm app}|^2\right) u^\varepsilon_{\rm app} = \sum_{n=0}^2 \varepsilon^n X_n + \varepsilon^\alpha (Y + Y_R) \tag{2.2}$$

with

$$X_0 = -\sum_{j=1}^J e^{i\phi_j/\varepsilon}\Big(\partial_t \phi_j + \frac{1}{2}|\nabla\phi_j|^2\Big) A_j,$$

$$X_1 = i\sum_{j=1}^J e^{i\phi_j/\varepsilon}\Big(\partial_t A_j + \nabla A_j \cdot \nabla\phi_j + \frac{1}{2} A_j \Delta\phi_j\Big),$$

and

$$X_2 = \frac{1}{2}\sum_{j=1}^J e^{i\phi_j/\varepsilon}\Delta A_j. \tag{2.3}$$

These terms are the same as in the linear case $K \equiv 0$. Due to the presence of the Hartree type nonlinearity, we also obtain

$$Y = -\sum_{j=1}^J e^{i\phi_j/\varepsilon}\Big(K * \sum_{\ell=1}^J |A_\ell|^2\Big) A_j, \tag{2.4}$$

$$Y_R = -\sum_{j=1}^J e^{i\phi_j/\varepsilon}\Big(K * \sum_{\substack{\ell,m=1\\ \ell\neq m}}^J \left(A_\ell \overline{A}_m\, e^{i(\phi_\ell-\phi_m)/\varepsilon}\right)\Big) A_j. \tag{2.5}$$

Obviously, Y_R carries high-frequency oscillations, which are not captured by our ansatz (2.1). Thus we need to develop a more careful analysis in the following, which shows that Y_R is of higher order.

Ignoring this problem for the moment, we consequently aim to eliminate equal powers of ε. Hence, in leading order, we set $X_0 = 0$, which is equivalent to the *Hamilton-Jacobi equation*

$$\partial_t \phi_j + \frac{1}{2}|\nabla \phi_j|^2 = 0, \quad \phi_j\big|_{t=0} = k_j \cdot x. \tag{2.6}$$

Solutions to (2.6) determine the characteristic high-frequency oscillations present in $u^\varepsilon_{\rm app}$. In our case, they are easily found to be

$$\phi_j(t,x) = k_j \cdot x - \frac{t}{2}|k_j|^2. \tag{2.7}$$

These phases obviously solve (2.6) for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, i.e. *no caustics* appear in our study.

In the next step we set $X_1 = 0$ if $\alpha > 1$ and $X_1 + Y = 0$ if $\alpha = 1$ (note that we do not include Y_R here). Comparing the prefactors of the terms multiplied by $e^{i\phi_j/\varepsilon}$, yields the following system of transport equations for the amplitudes:

$$\partial_t A_j + k_j \cdot \nabla A_j = \begin{cases} 0 & \text{if } \alpha > 1, \\ -iV_{\rm eff}(A)A_j & \text{if } \alpha = 1, \end{cases} \tag{2.8}$$

where we have used the fact that $\Delta \phi_j \equiv 0$, in view of (2.7). For $\alpha = 1$, the effective (nonlinear) potential $V_{\rm eff}(A)$ is given by

$$V_{\rm eff}(A) := K * \Big(\sum_{\ell=1}^{J} |A_\ell|^2 \Big).$$

We see that for $\alpha > 1$ no nonlinear effects are present in transport equations for the leading order amplitudes. The case $\alpha = 1$ is therefore seen to be critical as far as semi-classical asymptotics is concerned.

Lemma 2.1. *The transport equation* (2.8) *with initial data* $(a_j)_{j\in\Gamma} \in L^2 \cap \mathcal{A}(\mathbb{R}^d)$ *admits global-in-time solutions* $A \in C([0,\infty); L^2 \cap \mathcal{A}(\mathbb{R}^d))$, *which can be written in the form*

$$A_j(t,x) = a_j(x - tk_j)e^{iS_j(t,x)}, \tag{2.9}$$

where $S_j \equiv 0$ *for* $\alpha > 1$ *and* $S_j \in C([0,\infty) \times \mathbb{R}^d)$ *for* $\alpha = 1$ *is given by*

$$S_j(t,\cdot) = -\int_0^t \Big(K * \sum_{\ell=1}^{J} \big| a_\ell\big(\,\cdot + (\tau - t)k_j - \tau k_\ell\big)\big|^2 \Big)\, d\tau \tag{2.10}$$

In particular we have mass conservation for each individual mode

$$\|A_j(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|a_j\|_{L^2(\mathbb{R}^d)}, \quad \forall\, t \in \mathbb{R}. \tag{2.11}$$

In contrast to ϕ_j the phases S_j are only *slowly varying*, i.e. they oscillate with frequencies larger than $1/\varepsilon$. They describe the nonlinear self-modulation of the amplitudes but do not show up in quadratic quantities, like the mass density $|A_j|^2$ etc.

Proof. By multiplying (2.8) with $\bar{A}_j$ and taking the real part, we see

$$(\partial_t + k_j \cdot \nabla) |A_j|^2 = 0,$$

which yields $|A_j(t,x)|^2 = |a_j(x - tk_j)|^2$ and thus (2.9) and (2.11). Finally, inserting (2.9) into (2.8) and integrating along characteristics, we obtain the expression (2.10) for S_j. □

Having obtained the characteristic phases ϕ_j and the leading order amplitudes A_j we shall now turn our attention towards the remainder, i.e.

$$R(u^\varepsilon_{\text{app}}) = \begin{cases} \varepsilon^2 X_2 + \varepsilon Y_R & \text{if } \alpha = 1, \\ \varepsilon^2 X_2 + \varepsilon^\alpha (Y + Y_R) & \text{if } \alpha > 1. \end{cases} \tag{2.12}$$

For $\alpha = 1$ the term Y_R appearing within $R(u^\varepsilon_{\text{app}})$ is formally of order $\mathcal{O}(\varepsilon)$. Thus, at first glance, Y_R seems to be too large to be considered a part of the remainder. It will be our main task to show that Y_R is indeed sufficiently small as $\varepsilon \to 0_+$. To this end, we shall rely on the framework of Wiener algebras.

3. The Wiener algebra framework

We now present the analytical framework of Wiener algebras which already proved its use in similar circumstances, cf. [5, 8, 12]. We start with the following definition.

Definition 3.1 (Wiener Algebra)**.** We define

$$W(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d;\mathbb{C}), \quad \|f\|_W := \|\widehat{f}\|_{L^1(\mathbb{R}^d)} < \infty \right\}.$$

The following properties of $W(\mathbb{R}^d)$ have been proved in [8, 12].

Lemma 3.2.

- **i.** *$W(\mathbb{R}^d)$ is a Banach space, continuously embedded into $L^\infty(\mathbb{R}^d)$.*
- **ii.** *$W(\mathbb{R}^d)$ is an algebra, in the sense that the mapping $(f,g) \mapsto fg$ is continuous from $W(\mathbb{R}^d)^2$ to $W(\mathbb{R}^d)$, and moreover*

$$\forall f, g \in W(\mathbb{R}^d), \quad \|fg\|_W \leqslant \|f\|_W \|g\|_W.$$

- **iii.** *Let $U^\varepsilon(t) = e^{i\varepsilon \frac{t}{2}\Delta}$ denote the free Schrödinger group. Then, for all $t \in \mathbb{R}$, $U^\varepsilon(t)$ is unitary on $W(\mathbb{R}^d)$.*

Assertion iii. follows from the fact that $\widehat{U}^\varepsilon(t) = e^{it|\xi|^2/2}$, acting as a multiplication operator in Fourier space. From now on, we shall consider the Cauchy problem (1.3) to be posed in $W(\mathbb{R}^d)$. To this end we need the following well-posedness result (which is an adaptation of the one given in [5] to the case of Hartree nonlinearities).

Lemma 3.3. *Consider the initial value problem*

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon^\alpha (K * |u^\varepsilon|^2) u^\varepsilon, \quad u^\varepsilon\big|_{t=0} = u_0^\varepsilon, \tag{1.3}$$

with $\alpha \geqslant 1$. If $\widehat{K} \in L^\infty(\mathbb{R}^d)$ and $u_0^\varepsilon \in W(\mathbb{R}^d)$ with $\|u_0^\varepsilon\|_W \leqslant D_0$, then there exists a $T_0 > 0$, which depends on D_0 but not on ε, and a unique solution $u^\varepsilon \in C([0,T_0]; W(\mathbb{R}^d))$ of (1.3).

Proof. Duhamel's formulation of (1.3) reads

$$u^\varepsilon(t) = U^\varepsilon(t)u_0^\varepsilon - i\varepsilon^{\alpha-1}\int_0^t U^\varepsilon(t-\tau)\left((K*|u^\varepsilon|^2)u^\varepsilon)\right)(\tau)\,d\tau.$$

Denote, for fixed u_0^ε, the right hand side of this formula by $\Phi^\varepsilon(u^\varepsilon)(t)$. From Lemma 3.2 iii. we have

$$\|\Phi^\varepsilon(u^\varepsilon)(t)\|_W \leqslant D_0 + \varepsilon^{\alpha-1}\int_0^t \|\left((K*|u^\varepsilon|^2)u^\varepsilon)\right)(\tau)\|_W\,d\tau. \tag{3.1}$$

In order to control the nonlinear term, we need to estimate expressions of the form $(K*(uv))w$ in $W(\mathbb{R}^d)$. To this end, we first use Lemma 3.2 ii, to obtain

$$\|(K*(uv))w\|_W \leqslant \|K*(uv)\|_W\|w\|_W.$$

By Hölder's inequality we also get

$$\|K*(uv)\|_W = \|\widehat{K*(uv)}\|_{L^1} \leqslant \|\widehat{K}\|_{L^\infty}\|uv\|_W, \tag{3.2}$$

and applying again Lemma 3.2 ii, we arrive at

$$\|(K*(uv))w\|_W \leqslant \|\widehat{K}\|_{L^\infty}\|u\|_W\|v\|_W\|w\|_W \quad \text{for } u,v,w\in W(\mathbb{R}^d). \tag{3.3}$$

Thus, from (3.1) we obtain

$$\|\Phi^\varepsilon(u^\varepsilon)(t)\|_W \leqslant D_0 + \varepsilon^{\alpha-1}\|\widehat{K}\|_{L^\infty}\int_0^t \|u^\varepsilon(\tau)\|_W^3\,d\tau.$$

Moreover, $u^\varepsilon\mapsto\Phi^\varepsilon(u^\varepsilon)$ is locally Lipschitz in $U:=C([0,T],W(\mathbb{R}^d))$: If $\|u^\varepsilon\|_U\leqslant D$, $\|v^\varepsilon\|_U\leqslant D$, then there exists $C=C(D)$ such that

$$\|\Phi^\varepsilon(u^\varepsilon)(t)-\Phi^\varepsilon(v^\varepsilon)(t)\|_W \leqslant C(D)\int_0^t \|u^\varepsilon(\tau)-v^\varepsilon(\tau)\|_W\,d\tau, \quad \forall t\in[0,T].$$

A fixed point argument in $\left\{u\in U:\ \sup_{t\in[0,T]}\|u(t)\|_W\leqslant D\right\}$, with $D>D_0$, for $T=T_0$ sufficiently small, then yields Lemma 3.3. ☐

Having set up an existence result for the exact solution u^ε in $W(\mathbb{R}^d)$, we now turn to the approximate solution $u^\varepsilon_{\rm app}$ given by (1.6). To this end, we shall need the following lemma, which shows that the Wiener space is perfectly adapted to our problem.

Lemma 3.4. *Let $k_j\in\mathbb{R}^d$, $c_j\in\mathbb{R}$, and $b\in\ell^1(\Gamma,W(\mathbb{R}^d))$, and set*

$$f(x,y)=\sum_{j=1}^J b_j(x)e^{i(k_j\cdot y+c_j)}.$$

Then, for all $\varepsilon>0$ the function $f(\cdot,\cdot/\varepsilon): x\mapsto f(x,x/\varepsilon)$ lies in $W(\mathbb{R}^d)$ with

$$\|f(\cdot,\cdot/\varepsilon)\|_W \leqslant \|b\|_{\mathcal{A}} = \sum_{j\in\Gamma}\|b_j\|_W.$$

Proof. We write

$$\|f(\cdot,\cdot/\varepsilon)\|_W = \Big\|\sum_{j=1}^J e^{ic_j}\widehat{b_j}(\cdot - k_j/\varepsilon)\Big\|_{L^1} \leqslant \sum_{j=1}^J \|\widehat{b_j}(\cdot - k_j/\varepsilon)\|_{L^1} = \sum_{j=1}^J \|\widehat{b_j}\|_{L^1}.$$

The last term is, by definition, $\|b\|_{\mathcal{A}}$. $\square$

Remark 3.5. This lemma in general does *not* hold for functions of the form $f(x,y) = \sum_{j=1}^J b_j(x)e^{i\varphi_j(y)}$, with $\varphi_j(y) \neq k_j \cdot y + c_j$. A generalization of our study to non-plane wave WKB phases therefore seems to be a delicate issue (at least within the Wiener framework) and by no means straightforward.

Since the phases ϕ_j considered in this work are of plane-wave form (2.7), applying Lemma 3.4 with $c_j = -\frac{t}{2\varepsilon}|k_j|^2$ to (1.6), we immediately obtain

$$\|u^\varepsilon_{\text{app}}(t,\cdot)\|_W \leqslant \|A(t,\cdot)\|_{\mathcal{A}}. \tag{3.4}$$

Similarly, we can estimate the expression (2.3) for X_2 by

$$\|X_2(t,\cdot)\|_W \leqslant \frac{1}{2}\|\Delta A(t,\cdot)\|_{\mathcal{A}} = \frac{1}{2}\sum_{j=1}^J \|\Delta A_j(t,\cdot)\|_W. \tag{3.5}$$

In addition, we obtain the following estimate for Y, defined in (2.4):

$$\|Y(t,\cdot)\|_W \leqslant \Big\|K * \sum_{\ell=1}^J |A_\ell(t,\cdot)|^2\Big\|_W \|A(t,\cdot)\|_{\mathcal{A}},$$

where we have used Lemma 3.4 and Lemma 3.2 ii. This can be estimated further similarly to (3.2) by using Lemma 3.4 (with $k_j = 0$, $c_j = 0$), as well as Assertion ii. of Lemma 3.2 and the fact that $\ell^\infty \subset \ell^1$, to obtain

$$\|Y(t,\cdot)\|_W \leqslant \|\widehat{K}\|_{L^\infty}\|A(t,\cdot)\|^3_{\mathcal{A}}. \tag{3.6}$$

In order to close the argument, we consequently require appropriate bounds in $\mathcal{A}(\mathbb{R}^d)$ on the amplitudes $A_j(t,\cdot)$, together with their spatial derivatives.

Lemma 3.6. *Let $\alpha \geqslant 1$ and $\widehat{K} \in L^\infty(\mathbb{R}^d)$. For all $a = (a_j)_{j\in\Gamma} \in \mathcal{A}(\mathbb{R}^d)$, there exists a unique solution $A \in C([0,\infty);\mathcal{A}(\mathbb{R}^d))$ to the system (2.8). Moreover, if $(\partial_x^p a_j)_{j\in\Gamma} \in \mathcal{A}(\mathbb{R}^d)$, for $|p| \leqslant 2$, then $(\partial_x^p A_j)_{j\in\Gamma} \in C([0,\infty);\mathcal{A}(\mathbb{R}^d))$.*

Proof. For $\alpha > 1$ the statements of the lemma are immediately clear, since in this case $A_j(t,x) = a_j(x - tk_j)$, cf. (2.8). For $\alpha = 1$ we rewrite (2.8) in its integral form

$$A_j(t,x) = a_j(x - tk_j) + \int_0^t \mathcal{N}(A)_j(\tau, x + (\tau - t)k_j)d\tau, \tag{3.7}$$

where the nonlinearity $\mathcal{N}(A)_j$ is given by

$$\mathcal{N}(A)_j = -i\Big(K * \big(\sum_{\ell=1}^J |A_\ell|^2\big)\Big)A_j$$

Invoking the same arguments as for the derivation of (3.6), we obtain

$$\|\mathcal{N}(A)\|_{\mathcal{A}} = \sum_{j=1}^{J} \|\mathcal{N}(A)_j\|_W \leqslant \|\widehat{K}\|_{L^\infty} \|A\|_{\mathcal{A}}^3.$$

This shows that $\mathcal{N}(A)$ defines a continuous mapping from $\mathcal{A}^3$ to $\mathcal{A}$ and a local-in-time existence result immediately follows from the standard Cauchy-Lipschitz theorem for ordinary differential equations. That the solutions A_j indeed exist for all $t \geqslant 0$ then follows from the explicit representation (2.9). From the latter we additionally obtain the propagation of regularity, by explicit calculation of $\partial_x^p A$. $\square$

Lemma 3.6 consequently establishes the estimates in $W(\mathbb{R}^d)$ for $u^\varepsilon_{\text{app}}$, X_2 and Y in a rigorous way. Note however, that the above given estimates do not yield an estimate for the remainder $R(u^\varepsilon_{\text{app}})$, given by (2.12), since it also includes Y_R, which we completely ignored so far. We will make up for it in the following section.

4. Estimates on the remainder and proof of the main theorem

It remains to estimate in $W(\mathbb{R}^d)$ the term Y_R given in (2.5). To this end we shall prove the following key technical result.

Proposition 4.1. *Let Y_R be defined by* (2.5) *with plane-wave phases ϕ_j given by* (2.7) *and assume $|\Lambda|_\infty^{-1} := \inf\{|k_\ell - k_m| : \ \ell, m \in \Gamma, \ \ell \neq m\} > 0$. Moreover, let K be such that $\widehat{K} \in L^\infty(\mathbb{R}^d)$ and $\widehat{\nabla K} \in L^\infty(\mathbb{R}^d)$. Then we have the following bound:*

$$\|Y_R(t,\cdot)\|_W \leqslant \varepsilon\, C_K \|A(t,\cdot)\|_{\mathcal{A}}^2 \big(\|A(t,\cdot)\|_{\mathcal{A}} + \|\nabla A(t,\cdot)\|_{\mathcal{A}}\big), \tag{4.1}$$

where $C_K > 0$ is independent of ε.

Proof. Recalling the definition of Y_R given in (2.5) and taking into account the particular plane-wave form (2.7) of the phases ϕ_j, we obtain from Lemma 3.4 and Assertion ii. of Lemma 3.2 that

$$\|Y_R\|_W \leqslant \|A\|_{\mathcal{A}} \Big\| \sum_{\substack{\ell,m=1\\ \ell\neq m}}^{J} K * \big(A_\ell \overline{A}_m e^{i(\phi_\ell - \phi_m)/\varepsilon}\big) \Big\|_W. \tag{4.2}$$

Using,

$$e^{iy\cdot k/\varepsilon + z} = -i\varepsilon \frac{k}{|k|^2} \cdot \nabla_y e^{iy\cdot k/\varepsilon + z} \quad (z \in \mathbb{C}),$$

we can perform a partial integration w.r.t y, and rewrite

$$\begin{aligned} K * \big(A_\ell \bar{A}_m e^{i(\phi_\ell - \phi_m)/\varepsilon}\big) &= \int_{\mathbb{R}^d} K(x-y) A_\ell(y) \bar{A}_m(y) e^{i(\phi_\ell(y) - \phi_m(y))/\varepsilon}\, dy \\ &= i\varepsilon \int_{\mathbb{R}^d} \frac{k_\ell - k_m}{|k_\ell - k_m|^2} \cdot \nabla_y \big(K(x-y) A_\ell(y) \bar{A}_m(y)\big) e^{i(\phi_\ell(y) - \phi_m(y))/\varepsilon}\, dy. \end{aligned}$$

To show that the boundary terms vanish, assume first that $A_j \in \mathcal{S}(\mathbb{R}^d)$, the set of Schwartz functions (for which the boundary terms clearly vanish). Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$, Fourier transformation implies that $\mathcal{S}(\mathbb{R}^d)$ is also dense in

$W(\mathbb{R}^d)$. Consequently, the fact that the expressions on both sides are norm-bounded sesquilinear forms establishes the above formula.

Using $k_\ell \neq k_m \in \mathbb{R}^d$, we know that $\Lambda_{\ell,m} := \frac{k_\ell - k_m}{|k_\ell - k_m|^2} \in \mathbb{R}^d$ is well defined and consequently

$$K * \left(A_\ell \bar{A}_m e^{i(\phi_\ell - \phi_m)/\varepsilon}\right) = - i\varepsilon(\Lambda_{\ell,m} \cdot \nabla K) * \left(A_\ell \bar{A}_m e^{i(\phi_\ell - \phi_m)/\varepsilon}\right) + i\varepsilon K * \left(\left(\Lambda_{\ell,m} \cdot \nabla (A_\ell \bar{A}_m)\right) e^{i(\phi_\ell - \phi_m)/\varepsilon}\right).$$

Using the estimate (3.2) and Lemma 3.4 (with $J = 1$) we get

$$\|K * \left(A_\ell \bar{A}_m e^{i(\phi_\ell - \phi_m)/\varepsilon}\right)\|_W \leqslant \varepsilon \|\Lambda_{\ell,m} \cdot \widehat{\nabla K}\|_{L^\infty} \|A_\ell\|_W \|A_m\|_W + \varepsilon \|\widehat{K}\|_{L^\infty} \|\Lambda_{\ell,m} \cdot \nabla (A_\ell \bar{A}_m)\|_W.$$

Invoking again Lemma 3.4 (with $k_j = 0$, $c_j = 0$) and Lemma 3.2 ii, we conclude

$$\begin{aligned} &\Big\| \sum_{\substack{\ell,m=1 \\ \ell \neq m}}^{J} K * \left(A_\ell \bar{A}_m e^{i(\phi_\ell - \phi_m)/\varepsilon}\right) \Big\|_W \\ &\leqslant \varepsilon \sum_{\substack{\ell,m=1 \\ \ell \neq m}}^{J} \Big(d |\Lambda_{\ell,m}| \|\widehat{\nabla K}\|_{L^\infty} \|A_\ell\|_W \|A_m\|_W + 2 \|\widehat{K}\|_{L^\infty} |\Lambda_{\ell,m}| \|\nabla A_\ell\|_{\mathcal{A}} \|A_m\|_W \Big) \\ &\leqslant \varepsilon |\Lambda|_\infty \|A\|_{\mathcal{A}} \Big(d \|\widehat{\nabla K}\|_{L^\infty} \|A\|_{\mathcal{A}} + 2 \|\widehat{K}\|_{L^\infty} \|\nabla A\|_{\mathcal{A}} \Big) \end{aligned}$$

where $\|\widehat{\nabla K}\|_{L^\infty} = \max\limits_{n=1,\dots,d} \|\widehat{\partial_{x_n} K}\|_{L^\infty}$, and

$$\|\nabla A\|_{\mathcal{A}} = \sum_{\ell=1}^{J} \|\nabla A_\ell\|_{\mathcal{A}} = \sum_{\ell=1}^{J} \sum_{n=1}^{d} \|\partial_{x_n} A_\ell\|_W.$$

This, together with (4.2) yields the estimate (4.1). □

Remark 4.2. Proposition 4.1 shows that $\|Y_R\|_W = \mathcal{O}(\varepsilon)$ and thus can indeed be considered a part of the remainder. Note that in the proof it is essential to invoke a stationary-phase type argument first, before starting to take estimates. In fact we would not succeed to show that $\|Y_R\|_W = \mathcal{O}(\varepsilon)$, if we would estimate Y_R in its original form.

By combining the results of Lemma 3.6 and Proposition 4.1, we obtain the following result.

Corollary 4.3. *Under the assumptions of Lemma 3.6 and Proposition 4.1, there exists, for every $T > 0$, a constant $C_R(T) > 0$, independent of $\varepsilon > 0$, such that the remainder $R(u^\varepsilon_{\text{app}})$ given by (2.12) satisfies*

$$\|R(u^\varepsilon_{\text{app}})\|_W \leqslant \varepsilon^\gamma C_R \qquad \forall\, t \in [0, T],\ \forall\, \varepsilon > 0, \tag{4.3}$$

where

$$\gamma = \begin{cases} 2 & \text{if } \alpha \in \{1\} \cup [2, \infty), \\ \alpha & \text{if } \alpha \in (1, 2). \end{cases}$$

Proof. Lemma 3.6 guarantees the existence of the norms $\|A(t,\cdot)\|_{\mathcal{A}}$, $\|\nabla A(t,\cdot)\|_{\mathcal{A}}$, $\|\Delta A(t,\cdot)\|_{\mathcal{A}} < \infty$ for all $t \in [0,\infty)$, which are independent of $\varepsilon > 0$ and continuous in t. Hence, taking their maximum over $t \in [0,T]$, we obtain (4.3) from (3.5), (3.6), (4.1) and the definition (2.12) of $R(u^\varepsilon_{\text{app}})$. □

With the estimate of Corollary 4.3 on $R(u^\varepsilon_{\text{app}})$ at hand, we can finally state the proof of our main theorem, which follows the basic ideas established in [13] for justifying the nonlinear Schrödinger equation as a modulation equation for dispersive waves.

Proof of Theorem 1.2. We consider a fixed $T > 0$ and introduce the following scaled error r^ε between the original solution u^ε to (1.3) subject to the initial data (1.2) and the approximation (1.6):

$$\varepsilon^\beta r^\varepsilon := u^\varepsilon - u^\varepsilon_{\text{app}},$$

with a parameter $\beta > 0$ to be specified below. Hence, $r^\varepsilon(0) = 0$. From (3.4) and Lemma 3.6 we know that there exists a constant $C_A > 0$, independent of ε, such that $\|u^\varepsilon_{\text{app}}(t,\cdot)\|_W \leqslant C_A$, for all $t \in [0,T]$. Since $r^\varepsilon(0) = 0$, it follows $\|u^\varepsilon_0\|_W \leqslant C_A$. Consequently, for any $D > C_A$, Lemma 3.3 yields the existence of a unique solution $u^\varepsilon \in C([0,T_0], W(\mathbb{R}^d))$ for some $T_0 > 0$ with $\|u^\varepsilon(t)\|_W \leqslant D$ for $t \in [0,T_0]$.

Moreover, from (1.3) and (2.2) it follows that r^ε satisfies

$$i\varepsilon\partial_t r^\varepsilon + \frac{\varepsilon^2}{2}\Delta r^\varepsilon = \varepsilon^{\alpha-\beta}\big(\mathcal{M}(u^\varepsilon_{\text{app}} + \varepsilon^\beta r^\varepsilon) - \mathcal{M}(u^\varepsilon_{\text{app}})\big) - \varepsilon^{-\beta} R(u^\varepsilon_{\text{app}})$$

with $\mathcal{M}(u) = (K * |u|^2)u$ for $t \leqslant \tau := \min\{T_0, T\}$, and by Duhamel's formula and Lemma 3.2 iii. we obtain

$$\begin{aligned}\|r^\varepsilon(t)\|_W \leqslant \varepsilon^{\alpha-\beta-1}\int_0^t \|\mathcal{M}(u^\varepsilon_{\text{app}}(\tau) + \varepsilon^\beta r^\varepsilon(\tau)) - \mathcal{M}(u^\varepsilon_{\text{app}}(\tau))\|_W \, d\tau \\ + \varepsilon^{-\beta-1}\int_0^t \|R(u^\varepsilon_{\text{app}}(\tau))\|_W \, d\tau,\end{aligned} \tag{4.4}$$

for all $t \leqslant \tau$. Writing

$$\mathcal{M}(u+r) - \mathcal{M}(u) = \big(K * (u\bar r + \bar u r + |r|^2)\big)(u+r) + (K * |u|^2)r,$$

the estimate (3.3) gives

$$\|\mathcal{M}(u+r) - \mathcal{M}(u)\|_W \leqslant \|\widehat{K}\|_{L^\infty}(3\|u\|_W^2 + 3\|u\|_W\|r\|_W + \|r\|_W^2)\,\|r\|_W. \tag{4.5}$$

Hence, replacing $u = u^\varepsilon_{\text{app}}$ and $r = \varepsilon^\beta r^\varepsilon$, and recalling $\beta > 0$, we obtain for any $C > 0$ and $\varepsilon_0 \in (0,1]$, such that $3\varepsilon_0^\beta C_A C + \varepsilon_0^{2\beta} C^2 = C_A^2$, that

$$\|\mathcal{M}(u^\varepsilon_{\text{app}} + \varepsilon^\beta r^\varepsilon) - \mathcal{M}(u^\varepsilon_{\text{app}})\|_W \leqslant \varepsilon^\beta C_{\mathcal{M}}\,\|r^\varepsilon\|_W \quad \forall\, \varepsilon \leqslant \varepsilon_0,\ t \leqslant \tau \tag{4.6}$$

where $C_{\mathcal{M}} := 4\|\widehat{K}\|_{L^\infty} C_A^2$, as long as $\|r^\varepsilon\|_W \leqslant C$.

Inserting the bounds (4.6) and (4.3) into (4.4), and recalling that $\varepsilon_0 \leqslant 1$, $\alpha \geqslant 1$, $\tau \leqslant T$, we consequently obtain for $\beta \in (0, \gamma - 1]$

$$\|r^\varepsilon(t)\|_W \leqslant C_R T + C_{\mathcal{M}}\int_0^t \|r^\varepsilon(\tau)\|_W \, d\tau \quad \forall\, \varepsilon \leqslant \varepsilon_0,\ t \leqslant \tau.$$

By Gronwall's lemma this yields

$$\|r^\varepsilon(t)\|_W \leqslant C_R T e^{C_{\mathcal{M}} t} \quad \forall\, \varepsilon \leqslant \varepsilon_0,\ t \leqslant \tau.$$

Hence, setting above $C := C_R T e^{C_{\mathcal{M}} T}$ and $D := C + C_A$ this estimate guarantees that the solution u^ε exists on the whole time interval $[0,T]$, cf. the proof of Lemma 3.3. Moreover, recalling Lemma 3.2 i, we finally obtain the error estimate (1.5) which finishes the proof. □

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