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Global region of attraction of a periodic solution to a singularly perturbed parabolic problem in case of exchange of stability

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Abstract

We consider the singularly perturbed parabolic differential equation $\varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\Big) = f(u,x,t,\varepsilon)$ under the assumption that f is T-periodic in t and that the degenerate equation $f(u,x,t,0)=0$ has two intersecting roots. In a previous paper [1] we presented conditions under which there exists an asymptotically stable T-periodic solution $u_p(x,t,\varepsilon)$ satisfying no-flux boundary conditions. In this note we characterize a set of initial functions belonging to the global region of attraction of $u_p(x,t,\varepsilon)$.

1 Formulation of the problem

1.1 Introduction

We consider the singularly perturbed parabolic differential equation

$$L_\varepsilon u := \varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\Big) - f(u,x,t,\varepsilon) = 0 \quad \text{for} \quad (x,t)\in\mathcal{D} \tag{1}$$

with

$$\mathcal{D} := \{(x,t)\in\mathbb{R}^2 : -1<x<1, t\in\mathbb{R}\}$$

and

$$\varepsilon \in I_{\varepsilon_0} := \{\varepsilon\in\mathbb{R} : 0<\varepsilon<\varepsilon_0\}, \quad 0<\varepsilon_0 \ll 1.$$

Under the assumption f to be T-periodic in t, that is

$$f(u,x,t+T,\varepsilon) = f(u,x,t,\varepsilon) \quad \forall (u,x,t,\varepsilon)\in\mathcal{G}\times I_{\varepsilon_0},$$

where $\mathcal{G}$ is some region which is defined in assumption (A_1) below we considered in [1] the periodic boundary value problem

$$\frac{\partial u}{\partial x}(\pm 1,t,\varepsilon) = 0 \quad \forall (t,\varepsilon)\in\mathbb{R}\times I_{\varepsilon_0}, \tag{2}$$

$$u(x,t+T,\varepsilon) = u(x,t,\varepsilon) \quad \forall (x,t,\varepsilon)\in\overline{\mathcal{D}}\times I_{\varepsilon_0} \tag{3}$$

in the case that the degenerate equation

$$f(u,x,t,0) = 0 \tag{4}$$

which we get from (1) by setting $\varepsilon = 0$, has two roots

$$u = \varphi_1(x,t) \quad \text{and} \quad u = \varphi_2(x,t) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}$$

which are T-periodic in t and intersect along some curve whose projection into the (x,t)-plane is located in $\overline{\mathcal{D}}$. This situation is called as case of exchange of stability (see [2]).
In [1] have derived conditions implying the existence of an asymptotically stable periodic solution of (1)-(3). In order to recall the main result of our paper [1] we introduce the following assumptions on the function f.

(A_1). $f \in C^2(\mathcal{G} \times I_{\varepsilon_0}, \mathbb{R})$, and f is T-periodic in the third variable. The region $\mathcal{G}$ is defined by

$$\mathcal{G} := \{(u,x,t) \in \mathbb{R}^3 : \ \underline{u}(x,t) \le u \le \overline{u}(x,t), \ (x,t) \in \overline{\mathcal{D}}\},$$

where $\underline{u}, \overline{u} \in C^2(\overline{\mathcal{D}}, R)$ are certain given functions T-periodic in t.

For the sequel we represent f in the form

$$f(u,x,t,\varepsilon) = f(u,x,t,0) - \varepsilon f_1(u,x,t) + \varepsilon^2 f_2(u,x,t,\varepsilon). \tag{5}$$

Concerning the function $f(u,x,t,0)$ we suppose

(A_2). The function $f(u,x,t,0)$ can be represented in the form

$$f(u,x,t,0) = h(u,x,t)\big(u - \varphi_1(x,t)\big)\big(u - \varphi_2(x,t)\big) \tag{6}$$

with $h \in C^2(\mathcal{G}, \mathbb{R}), \varphi_1, \varphi_2 \in C^2(\overline{\mathcal{D}}, \mathbb{R})$, where all functions are T-periodic in t and satisfy:
There is a positive number m such that

$$h(u,x,t) \ge m > 0 \quad \text{for} \quad (u,x,t) \in \mathcal{G}, \tag{7}$$

$$\underline{u}(x,t) < \varphi_i(x,t) < \overline{u}(x,t) \quad \text{for} \quad i = 1,2, \ (x,t) \in \overline{\mathcal{D}}. \tag{8}$$

Condition (A_2) implies that the degenerate equation (4) has exactly two roots in $\mathcal{G}$.
From the hypothesis (A_2) it follows that there is a positive number M such that

$$|h_u(u,x,t)| \le M \quad \text{for} \quad (u,x,t) \in \mathcal{G}. \tag{9}$$

The next condition describes the intersection of the surfaces $u = \varphi_1(x,t)$ and $u = \varphi_2(x,t)$.

(A_3). There exists a smooth T-periodic function $x_0 : \mathbb{R} \to \mathbb{R}$ with

$$-1 < x_0(t) < 1 \quad \text{for} \quad t \in \mathbb{R} \tag{10}$$

such that

$$\varphi_1(x_0(t), t) \equiv \varphi_2(x_0(t), t) \quad \text{for} \quad t \in \mathbb{R},$$
$$\varphi_1(x,t) > \varphi_2(x,t) \quad \text{for} \quad -1 \le x < x_0(t), t \in \mathbb{R},$$
$$\varphi_1(x,t) < \varphi_2(x,t) \quad \text{for} \quad x_0(t) < x \le 1, t \in \mathbb{R}.$$

Assumption (A_3) says that the roots $u = \varphi_1(x,t)$ and $u = \varphi_2(x,t)$ of the degenerate equation intersect in a curve whose projection into the (x,t)-plane is located in the region $\mathcal{D}$. We denote this projected curve by Γ_0,

$$\Gamma_0 := \{(x,t) \in \mathcal{D} : x = x_0(t),\ t \in \mathbb{R}\}.$$

By means of the roots φ_1 and φ_2 we construct the following composed roots of equation (4):

$$\check{u}(x,t) = \begin{cases} \varphi_1(x,t) & \text{for} \quad -1 \le x \le x_0(t), \quad t \in \mathbb{R}, \\ \varphi_2(x,t) & \text{for} \quad x_0(t) \le x \le 1, \quad t \in \mathbb{R}, \end{cases} \tag{11}$$

$$\hat{u}(x,t) = \begin{cases} \varphi_2(x,t) & \text{for} \quad -1 \le x \le x_0(t), \quad t \in \mathbb{R}, \\ \varphi_1(x,t) & \text{for} \quad x_0(t) \le x \le 1, \quad t \in \mathbb{R}. \end{cases} \tag{12}$$

It is obvious that the functions $\check{u}$ and $\hat{u}$ are continuous but in general not smooth on the curve Γ_0.
From the hypotheses (A_2) and (A_3) we get

$$\check{u}(x,t) > \hat{u}(x,t) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_0,$$
$$\check{u}(x,t) \equiv \hat{u}(x,t) \quad \text{for} \quad (x,t) \in \Gamma_0,$$

$$\left.\begin{array}{l} f_u(\check{u}(x,t), x, t, 0) > 0 \\ f_u(\hat{u}(x,t), x, t, 0) < 0 \end{array}\right\} \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_0, \tag{13}$$

$$\left.\begin{array}{l} f_u(\check{u}(x,t), x, t, 0) = 0 \\ f_u(\hat{u}(x,t), x, t, 0) = 0 \end{array}\right\} \quad \text{for} \quad (x,t) \in \Gamma_0. \tag{14}$$

The inequalities in (13) yield a justification to call the root $\check{u}$ stable (and to call the root $\hat{u}$ unstable, see [1]). However, the fact that inequality

$$f_u(\check{u}(x,t), x, t, 0) > 0$$

does not hold on the curve Γ_0 is some obstacle to give a unique answer to the question whether there exists a solution $u_p(x,t,\varepsilon)$ to the problem (1)–(3) converging to the composed stable root $\check{u}(x,t)$ in $\overline{\mathcal{D}}$ as ε tends to zero. As we have shown in [1], the sign of the function $f_1(\check{u}(x,t),x,t)$ (see (5)) on the curve Γ_0 plays a crucial role in answering the posed question. Therefore, we require

(A_4). $\quad f_1(\check{u}(x,t),x,t) > 0 \quad \text{for} \quad (x,t) \in \Gamma_0.$

The main result of the paper [1] is the following one:

Theorem 1.1 *Suppose the hypotheses $(A_1) - (A_4)$ hold. Then, for sufficiently small ε, the periodic boundary value problem (1)–(3) has a solution u_p satisfying*

$$\lim_{\varepsilon \to 0} u_p(x,t,\varepsilon)) = \check{u}(x,t) \quad \textit{for} \quad (x,t) \in \overline{\mathcal{D}}, \tag{15}$$

and this solution is asymptotically stable in the sense of Lyapunov.

Since the solution u_p is asymptotically stable, there arises the question for a global region of attraction of this solution. We formulate this problem more precisely in the following subsection including the corresponding main result.

1.2 Global region of attraction of u_p

We consider equation (1) for $\varepsilon \in I_{\varepsilon_1}$ in the region

$$\mathcal{D}_0 := \{(x,t) \in \mathbb{R}^2 : -1 < x < 1, \ t > t_0\}, \tag{16}$$

where t_0 is any number, with the boundary condition (2) and the initial condition

$$u(x,t_0,\varepsilon) = u^0(x) \quad \text{for} \quad -1 \le x \le 1. \tag{17}$$

According to Theorem 1.1, the solution $u_p(x,t,\varepsilon)$ is asymptotically stable for sufficiently small ε. That means that if the initial function $u^0(x)$ in (17) is sufficiently near to $u_p(x,t_0,\varepsilon)$, then the solution $u(x,t,\varepsilon)$ of the initial-boundary value problem (1), (2), (17) exists for $t > t_0$ and satisfies for sufficiently small ε the relation

$$\lim_{t \to \infty} \left[u(x,t,\varepsilon) - u_p(x,t,\varepsilon) \right] = 0 \quad \text{for} \quad x \in [-1,1]. \tag{18}$$

We denote the set of all initial functions $u^0 \in C^1([-1,1],\mathbb{R})$ for which the initial-boundary value problem (1), (2), (17) for sufficiently small ε has a solution satisfying (18) as global region of attraction. The following assumption plays a crucial role in determining such a region.

(A_5). Let $u^0 \in C([-1,1],\mathbb{R})$ be a function satisfying the inequality

$$\hat{u}(x,t_0) < u^0(x) < \overline{u}(x,t_0) \quad \text{for} \quad -1 \le x \le 1,$$

where $\hat{u}$ is defined in (12) and $\overline{u}$ is the function from assumptions (A_1) and (A_2).

The main result of this paper is the following one.

Theorem 1.2 *Suppose the hypotheses* $(A_1)-(A_5)$ *hold. Then, for sufficiently small* ε*, the initial-boundary value problem* (1), (3), (17) *has a solution* $u(x,t,\varepsilon)$ *satisfying relation* (18)*.*

The proof of this theorem will be given in section 3. As preparation we introduce in section 2 the so-called regularized degenerate equation and estimate its corresponding roots.

2 Regularization of the degenerate equation

As in [1] we consider the equation

$$f(u,x,t,0) - \varepsilon f_1(u,x,t) = 0, \tag{1}$$

which is distinguished from the degenerate equation (4) by taking into account also first order terms in ε and where f_1 is defined in (5). Using the representation (6) and exploiting the relation (7), we rewrite equation (1) in the form

$$\big(u-\varphi_1(x,t)\big)\big(u-\varphi_2(x,t)\big) - \varepsilon a(u,x,t) = 0, \tag{2}$$

where $a(u,x,t) \equiv f_1(u,x,t)/h(u,x,t)$. According to assumption (A_4) and (7) we have

$$a(\check{u}(x,t),x,t) > 0 \quad \text{for} \quad (x,t) \in \Gamma_0. \tag{3}$$

We denote by $\mathcal{C}_\delta$ a δ-neighborhood of the curve

$$\mathcal{C} := \{(u,x,t) \in \mathbb{R}\times[0,1]\times\mathbb{R} : u = \check{u}(x_0(t)), x = x_0(t), t \in \mathbb{R}\}.$$

It follows from (3) that there is a positive number a_0 such that for sufficiently small δ

$$a(u,x,t) \ge a_0^2 \quad \text{for} \quad (u,x,t) \in \mathcal{C}_\delta. \tag{4}$$

Relation (4) implies that for sufficiently small $\varepsilon > 0$ equation (2) has two smooth roots in u. We denote these roots by $u = \varphi(x,t,\varepsilon)$ and $u = \psi(x,t,\varepsilon)$. From (2) we get

$$\begin{aligned}\varphi(x,t,\varepsilon) = \frac{1}{2}\Big\{&\varphi_1(x,t) + \varphi_2(x,t) + \big[(\varphi_1(x,t) - \varphi_2(x,t))^2 + \\ &+ 4\varepsilon a(\varphi(x,t,\varepsilon),x,t)\big]^{1/2}\Big\}, \\ \psi(x,t,\varepsilon) = \frac{1}{2}\Big\{&\varphi_1(x,t) + \varphi_2(x,t) - \big[(\varphi_1(x,t) - \varphi_2(x,t))^2 + \\ &+ 4\varepsilon a(\psi(x,t,\varepsilon),x,t)\big]^{1/2}\Big\},\end{aligned} \tag{5}$$

which imply the asymptotic expressions

$$\begin{aligned}\varphi(x,t,\varepsilon) =& \check{u}(x,t) + O(\sqrt{\varepsilon}) \quad \text{for} \quad (x,t) \in \Gamma_\delta, \\ \psi(x,t,\varepsilon) =& \hat{u}(x,t) + O(\sqrt{\varepsilon}) \quad \text{for} \quad (x,t) \in \Gamma_\delta,\end{aligned} \tag{6}$$

$$\begin{aligned}\varphi(x,t,\varepsilon) =& \check{u}(x,t) + O(\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_\delta, \\ \psi(x,t,\varepsilon) =& \hat{u}(x,t) + O(\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_\delta,\end{aligned} \tag{7}$$

where Γ_δ is any small δ-neighborhood of Γ_0 which does not depend on ε.

As we mentioned in [1], the procedure to replace the degenerate equation (4) by equation (1) represents a regularization in the sense that we approximate the non-smooth functions $\check{u}(x,t)$ and $\hat{u}(x,t)$ by the smooth functions $\varphi(x,t,\varepsilon)$ and $\psi(x,t,\varepsilon)$ for sufficiently small ε.
In [1] we have shown that the solution $u_p(x,t,\varepsilon)$ of the periodic boundary value problem (1)-(3) satisfies

$$u_p(x,t,\varepsilon) = \varphi(x,t,\varepsilon) + O(\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}. \tag{8}$$

For the proof of Theorem 1.2 we need some relations concerning the roots φ and ψ of the regularized equation (2).
From (8) and (5) we get for $(x,t) \in \overline{\mathcal{D}}$

$$\begin{aligned}&2u_p(x,t,\varepsilon) - \varphi_1(x,t) - \varphi_2(x,t) = \\ &= 2\varphi(x,t,\varepsilon) - \varphi_1(x,t) - \varphi_2(x,t) + O(\varepsilon) = \\ &= \big[(\varphi_1(x,t) - \varphi_2(x,t))^2 + 4\varepsilon a(\varphi,x,t)\big]^{1/2} + O(\varepsilon).\end{aligned} \tag{9}$$

From the estimate (4) we get for sufficiently small ε

$$a(\varphi(x,t),x,t) \geq a_0^2, \; a(\psi(x,t),x,t) \geq a_0^2 \text{ for } (x,t) \in \Gamma_\delta \tag{10}$$

and the obvious inequality

$$|\varphi_1(x,t) - \varphi_2(x,t)| \geq 2c_\delta > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_\delta, \tag{11}$$

where the constant c_δ depends on δ but not on ε. Using these relations we obtain from (9) for sufficiently small δ and ε

$$2u_p(x,t,\varepsilon) - \varphi_1(x,t) - \varphi_2(x,t) \geq \left\{ \begin{array}{lll} a_0\sqrt{\varepsilon} & \text{for} & (x,t) \in \Gamma_\delta, \\ c_\delta & \text{for} & (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_\delta. \end{array} \right. \tag{12}$$

Furthermore, from (5) we get

$$\begin{aligned} &\varphi(x,t,\varepsilon) - \psi(x,t,\varepsilon) = \\ &= \frac{1}{2}\Big[(\varphi_1(x,t) - \varphi_2(x,t))^2 + 4\varepsilon a(\varphi(x,t,\varepsilon),x,t)\Big]^{1/2} + \\ &+ \frac{1}{2}\Big[(\varphi_1(x,t) - \varphi_2(x,t))^2 + 4\varepsilon a(\psi(x,t,\varepsilon),x,t)\Big]^{1/2}. \end{aligned} \tag{13}$$

Using (10) and (11) we get from (13) for sufficiently small ε

$$\varphi(x,t,\varepsilon) - \psi(x,t,\varepsilon) \geq 2a_0\sqrt{\varepsilon} \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}. \tag{14}$$

From (5) we also obtain

$$\begin{aligned} &\varphi(x,t,\varepsilon) + \psi(x,t,\varepsilon) = \varphi_1 + \varphi_2 \\ &+ \frac{1}{2}\Big\{\Big[(\varphi_1 - \varphi_2)^2 + 4\varepsilon a(\varphi,x,t)\Big]^{1/2} - \Big[(\varphi_1 - \varphi_2)^2 + 4\varepsilon a(\psi,x,t)\Big]^{1/2}\Big\}. \end{aligned} \tag{15}$$

In what follows we will show that the expression in the curly brackets on the right hand side of (15) has the order $O(\varepsilon)$ in $\overline{\mathcal{D}}$.
In order to get this we use the identity

$$\begin{aligned} &\Big[(\varphi_1 - \varphi_2)^2 + 4\varepsilon a(\varphi,x,t)\Big]^{1/2} - \Big[(\varphi_1 - \varphi_2)^2 + 4\varepsilon a(\psi,x,t)\Big]^{1/2} = \\ &= \frac{4\varepsilon[a(\varphi,x,t) - a(\psi,x,t)]}{\Big[(\varphi_1 - \varphi_2)^2 + 4\varepsilon a(\varphi,x,t)\Big]^{1/2} + \Big[(\varphi_1 - \varphi_2)^2 + 4\varepsilon a(\psi,x,t)\Big]^{1/2}} \end{aligned}$$

According to the mean value theorem we have

$$a(\varphi,x,t) - a(\psi,x,t) = a_u(\varphi + \theta(\varphi - \psi),x,t)(\varphi - \psi).$$

Taking into account (13) we get that the expression in the curly brackets on the right hand side of (15) is equal to $2\varepsilon a_u(\varphi + \theta(\varphi - \psi),x,t) = O(\varepsilon)$. Thus, we obtain from (15)

$$\varphi(x,t,\varepsilon) + \psi(x,t,\varepsilon) = \varphi_1(x,t) + \varphi_2(x,t) + O(\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}. \tag{16}$$

For the proof of Theorem 1.2 we also need the following estimates of some derivatives of φ and ψ derived in [1] for sufficiently small ε:

$$\varphi_x(x,t,\varepsilon) = O(1),\ \varphi_t(x,t,\varepsilon) = O(1) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}, \tag{17}$$

$$\varphi_{xx}(x,t,\varepsilon) \leq \left\{ \begin{array}{lll} \frac{c}{\sqrt{\varepsilon}} & \text{for} & (x,t) \in \Gamma_\delta, \\ c & \text{for} & (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_\delta, \end{array} \right. \tag{18}$$

where the constant c does not depend on δ and ε for sufficiently small ε.
The same estimates hold for the function ψ.

3 Proof of Theorem 1.2

3.1 Definition of lower and upper solutions

The proof of Theorem 1.2 is based on the method of differential inequalities. For this reason we will construct for the problem (1), (2), (17) upper and lower solutions in the subsections 3.2 and 3.3. First we recall their definitions.

Definition 3.1 *Let $\underline{U}(x,t,\varepsilon)$ and $\overline{U}(x,t,\varepsilon)$ be functions mapping $\overline{\mathcal{D}}_0 \times I_{\varepsilon_0}$ into $\mathbb{R}$, twice continuously differentiable in x and continuously differentiable in t, where $\mathcal{D}_0$ is defined in (16). The functions $\underline{U}$ and $\overline{U}$ are called lower and upper solutions of the initial boundary value problem* (1), (2), (17), respectively, if they satisfy for sufficiently small ε the inequalities

$$L_\varepsilon \overline{U} - f(\overline{U}, x, t, \varepsilon) \le 0 \ \le L_\varepsilon \underline{U} - f(\underline{U}, x, t, \varepsilon) \quad \text{for} \quad (x,t) \in \mathcal{D}_0, \tag{1}$$

$$\begin{aligned} &\frac{\partial \underline{U}}{\partial x}(1,t,\varepsilon) \le 0 \le \frac{\partial \underline{U}}{\partial x}(-1,t,\varepsilon) \quad \text{for} \quad t \ge t_0, \\ &\frac{\partial \overline{U}}{\partial x}(-1,t,\varepsilon) \le 0 \le \frac{\partial \overline{U}}{\partial x}(1,t,\varepsilon) \quad \text{for} \quad t \ge t_0. \end{aligned} \tag{2}$$

$$\underline{U}(x,t_0,\varepsilon) \le u^0(x) \le \overline{U}(x,t_0,\varepsilon) \quad \text{for} \quad -1 \le x \le 1, \tag{3}$$

It is well-known [3] that the lower and upper solutions defined above are ordered and that their existence implies the existence of a solution $u(x,t,\varepsilon)$ to problem (1), (2), (17) satisfying

$$\underline{U}(x,t,\varepsilon) \le u(x,t,\varepsilon) \le \overline{U}(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0. \tag{4}$$

3.2 Construction of an upper solution

We construct an upper solution $\overline{U}$ to (1), (2), (17) in the form

$$\overline{U}(x,t,\varepsilon) := u_p(x,t,\varepsilon) + \alpha(x,t,\varepsilon)E(t,\varepsilon), \tag{5}$$

where u_p is the solution of the periodic boundary value problem (1)–(3) (see Theorem 1.1),

$$\alpha(x,t,\varepsilon) := \overline{u}(x,t) - \varphi(x,t,\varepsilon) + \varepsilon z(x,\varepsilon), \tag{6}$$

where $\overline{u}$ is introduced in assumption (A_1) and satisfies (8), φ is defined in (5),

$$z(x,\varepsilon) := \exp\Big\{-k\,\frac{x+1}{\varepsilon}\Big\} + \exp\Big\{k\,\frac{x-1}{\varepsilon}\Big\}, \quad k > 0, \tag{7}$$

$$E(t,\varepsilon) := \exp\Big\{ -\varrho\,\frac{t-t_0}{\varepsilon^{3/2}}\Big\}, \quad \varrho > 0, \tag{8}$$

the numbers k and ϱ will be chosen suitably later.
We note that for sufficiently small ε the following relations hold:

(i). There is a positive constant α_0 such that

$$\alpha(x,t,\varepsilon) \geq \alpha_0 > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0.$$

(ii).

$$\alpha_x(x,t,\varepsilon) = O(1),\ \alpha_t(x,t,\varepsilon) = O(1) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0.$$

(iii). α_{xx} satisfies outside some small δ - neighborhoods of the straight lines $x = -1$ and $x = 1$ the same estimates as φ_{xx} (see (18), while inside this δ-neighborhoods we have $\varepsilon^2\alpha_{xx} = O(\varepsilon)$ since in these neighborhoods it holds

$$\varepsilon^2 z_{xx} = O\Big(\frac{1}{\varepsilon}\Big).$$

Taking into account (i) we obtain

$$\Big|\varepsilon^2\alpha_{xx}(x,t,\varepsilon) - \alpha_t(x,t,\varepsilon)\Big| \leq \varepsilon c_0 \alpha \qquad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0. \tag{9}$$

where in the sequel we denote by $c_0, c_1, ...$ suitable positive constants not depending of ε and δ.

Next we will show that for sufficiently large k and ϱ the function $\overline{U}$ defined in (5) satisfies the inequalities (2) in Definition 3.1.
Indeed, by taking into account (5) we have

$$\frac{\partial \overline{U}}{\partial x}(-1,t,\varepsilon) = \frac{\partial u_p}{\partial x}(-1,t,\varepsilon) + \frac{\partial \alpha}{\partial x}(-1,t,\varepsilon)E(t,\varepsilon) =$$

$$= \Big[\frac{\partial \overline{u}}{\partial x}(-1,t) - \frac{\partial \varphi}{\partial x}(-1,t,\varepsilon) - k + k\exp\Big(\frac{-2}{\varepsilon}\Big)\Big]E(t,\varepsilon),$$

where we used the relation $\frac{\partial u_p}{\partial x}(-1,t,\varepsilon) = 0$.
From the relations

$$\Big|\frac{\partial \overline{u}}{\partial x}(-1,t)\Big| = O(1), \quad \frac{\partial \varphi}{\partial x}(-1,t,\varepsilon) = O(1) \quad \text{for} \quad t \geq t_0$$

(see (17)) we have for sufficiently large k

$$\frac{\partial \overline{U}}{\partial x}(-1,t,\varepsilon) \leq 0 \quad \text{for} \quad t \geq t_0.$$

Analogously we can show that for sufficiently large k

$$\frac{\partial \overline{U}}{\partial x}(1, t, \varepsilon) \geq 0 \quad \text{for} \quad t \geq t_0.$$

Thus, the function $\overline{U}$ satisfies the inequalities (2) in Definition 3.1. Next we prove that $\overline{U}$ obeys the inequality (3) in Definition 3.1.
By (8), (5) - (8) we have

$$\overline{U}(x, t_0, \varepsilon) = u_p(x, t_0, \varepsilon) + \alpha(x, t_0, \varepsilon) = \varphi(x, t_0, \varepsilon) + O(\varepsilon) +$$

$$+ \overline{u}(x, t_0) - \varphi(x, t_0, \varepsilon) + \varepsilon z(x, \varepsilon) = \overline{u}(x, t_0) + O(\varepsilon).$$

According to assumption (A_5) it holds $\overline{u}(x, t_0) > u^0(x)$. Hence, for sufficiently small ε it holds

$$\overline{U}(x, t_0, \varepsilon) \geq u^0(x) \quad \text{for} \quad -1 \leq x \leq 1,$$

that is, $\overline{U}$ satisfies the inequalities (3).
Now we show that $\overline{U}$ also obeys the inequality (1). By (1), (5), (6) we have

$$\begin{aligned}
& L_\varepsilon \overline{U} \equiv \varepsilon^2 \Big(\frac{\partial^2 \overline{U}}{\partial x^2} - \frac{\partial \overline{U}}{\partial t} \Big) - f(\overline{U}, x, t, \varepsilon) = \\
& = \varepsilon^2 \Big(\frac{\partial^2 u_p}{\partial x^2} - \frac{\partial u_p}{\partial t} \Big) + \varepsilon^2 \Big(\frac{\partial^2 \alpha}{\partial x^2} - \frac{\partial \alpha}{\partial t} \Big) E + \sqrt{\varepsilon} \varrho \alpha E - \\
& - h(u_p + \alpha E, x, t)(u_p + \alpha E - \varphi_1)(u_p + \alpha E - \varphi_2) + \\
& + \varepsilon f_1(u_p + \alpha E, x, t) - \varepsilon^2 f_2(u_p + \alpha E, x, t, \varepsilon) = \\
& = \Big[\varepsilon^2 \Big(\frac{\partial^2 u_p}{\partial x^2} - \frac{\partial u_p}{\partial t} \Big) - h(u_p, x, t)(u_p - \varphi_1)(u_p - \varphi_2) + \\
& + \varepsilon f_1(u_p, x, t) - \varepsilon^2 f_2(u_p, x, t) \Big] + \varepsilon^2 \Big(\frac{\partial^2 \alpha}{\partial x^2} - \frac{\partial \alpha}{\partial t} \Big) E + \\
& + \sqrt{\varepsilon} \varrho \alpha E - \\
& - \Big(h(u_p + \alpha E, x, t) - h(u_p, x, t) \Big) (u_p - \varphi_1)(u_p - \varphi_2) - \\
& - h(u_p + \alpha E, x, t)(2u_p - \varphi_1 - \varphi_2 + \alpha E) \alpha E + \\
& + \varepsilon \Big(f_1(u_p + \alpha E, x, t) - f_1(u_p, x, t) \Big) - \\
& - \varepsilon^2 \Big(f_2(u_p + \alpha E, x, t) - f_2(u_p, x, t) \Big).
\end{aligned} \tag{10}$$

Taking into account that u_p solves (1) and that f has the representation (5) and (6) we can conclude that the expression in the square bracket vanishes. Using the inequalities (7), (9) we get the obvious estimates

$$h(u_p + \alpha E, x, t) \geq m > 0,$$

$$|h(u_p + \alpha E, x, t) - h_p(u_p, x, t)| \leq M \alpha E,$$

$$|f_1(u_p + \alpha E, x, t) - f_1(u_p, x, t)| \le c_1 \alpha E,$$
$$|f_2(u_p + \alpha E, x, t) - f_2(u_p, x, t)| \le c_2 \alpha E.$$

Using these estimates we get from (10)

$$\begin{aligned} L_\varepsilon \overline{U} &\le \varepsilon c_0 \alpha E + \sqrt{\varepsilon} \varrho \alpha E + M|u_p - \varphi_1||u_p - \varphi_2|\alpha E - \\ &- m(2u_p - \varphi_1 - \varphi_2)\alpha E + \varepsilon c_1 \alpha E + \varepsilon^2 c_2 \alpha E = \\ &= \alpha E \Big[\sqrt{\varepsilon}\varrho + M|u_p - \varphi_1||u_p - \varphi_2| - \\ &- m(2u_p - \varphi_1 - \varphi_2) + O(\varepsilon)\Big]. \end{aligned} \tag{11}$$

According to (8) we have

$$u_p - \varphi_1 = \varphi - \varphi_1 + O(\varepsilon), \quad u_p - \varphi_2 = \varphi - \varphi_2 + O(\varepsilon).$$

Consider the neighborhood Γ_δ of the curve Γ_0. By (6) it holds

$$\varphi(x, t, \varepsilon) = \check{u}(x, t) + O(\sqrt{\varepsilon}),$$

hence one of the differences $\varphi - \varphi_1$ and $\varphi - \varphi_2$ is $\mathrm{O}(\sqrt{\varepsilon})$, while the other one satisfies $|\varphi_1 - \varphi_2| + O(\sqrt{\varepsilon})$, that is, it is an expression of order $O(\delta) + O(\sqrt{\varepsilon})$. Therefore,

$$|u_p - \varphi_1||u_p - \varphi_2| \le c_3 \sqrt{\varepsilon}(\delta + \sqrt{\varepsilon}) \quad \text{for} \quad (x, t) \in \Gamma_\delta. \tag{12}$$

By using inequality (12) we obtain from (11)

$$L_\varepsilon \overline{U} \le \alpha E \sqrt{\varepsilon}\Big[\varrho + M c_3(\delta + \sqrt{\varepsilon}) - m a_0 + O(\sqrt{\varepsilon})\Big] \quad \text{for} \quad (x, t) \in \Gamma_\delta.$$

Due to the term $-m a_0$, for sufficiently small ϱ, δ, and ε the expression in the square bracket is negative, and we have

$$L_\varepsilon \overline{U} < 0 \quad \text{for} \quad (x, t) \in \Gamma_\delta.$$

Outside the neighborhood Γ_δ one of the differences $u_p - \varphi_1$ and $u_p - \varphi_2$ is a term of order $O(\varepsilon)$ according (8), (7). Thus

$$|u_p - \varphi_1||u_p - \varphi_2| \le c_4 \varepsilon \quad \text{for} \quad (x, t) \in \overline{\mathcal{D}}_0 \setminus \Gamma_\delta. \tag{13}$$

For the expression $2u_p - \varphi_1 - \varphi_2$ the estimate (12) holds such that we get from (11)

$$L_\varepsilon \overline{U} \le \alpha E\Big[\sqrt{\varepsilon}\varrho + M c_4 \varepsilon - m c_\delta + O(\varepsilon)\Big] \quad \text{for} \quad (x, t) \in \overline{\mathcal{D}}_0 \setminus \Gamma_\delta.$$

Since the term $-m c_\delta$ is negative and does not depend on ε, we can conclude that the expression in the square bracket is negative for sufficiently small ε and we have

$$L_\varepsilon \overline{U} < 0 \quad \text{for} \quad (x, t) \in \overline{\mathcal{D}}_0 \setminus \Gamma_\delta.$$

That implies $L_\varepsilon < 0$ in $\overline{\mathcal{D}}_0$, therefore, the function $\overline{U}$ defined in (5) satisfies for sufficiently small ε and δ the inequality (1) in definition 3.1 and is an upper solution of the problem (1), (2), (17).

3.3 Construction of a lower solution

We construct a lower solution to (1), (2), (17) in the form

$$\underline{U}(x,t,\varepsilon) = u_p(x,t,\varepsilon) - \beta(x,t,\varepsilon)E(t,\varepsilon), \tag{14}$$

where u_p and E are the same functions as in (5),

$$\beta(x,t,\varepsilon) = \varphi(x,t,\varepsilon) - \psi(x,t,\varepsilon) + \varepsilon z(x,\varepsilon) - a_0\sqrt{\varepsilon}, \tag{15}$$

φ and ψ are the roots of equation (2), z is defined in (7) and a_0 satisfies (12), (14). We note that due to (14) the following inequality holds

$$\beta(x,t,\varepsilon) \geq a_0\sqrt{\varepsilon} \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0, \tag{16}$$

and for the derivatives of β outside some δ-neighborhoods of the straight lines $x = -1, x = 1$ there hold the same estimates as for the functions φ and ψ in (17) and (18), respectively. Hence, outside the mentioned neighborhoods we have the estimate analogously to (9)

$$\left|\varepsilon^2\Big(\beta_{xx} - \beta_t\Big)\right| \leq \varepsilon c_0\beta. \tag{17}$$

Inside the δ-neighborhood of the straight lines $x = -1$ and $x = 1$ we have $\varepsilon^2\frac{\partial^2\beta}{\partial x^2} = O(\varepsilon)$ such that β can be estimated by

$$\beta(x,t,\varepsilon) \geq c > 0$$

which is stronger than the corresponding inequality (16). Therefore, the estimate (17) holds in the full region $\overline{\mathcal{D}}_0$. As in the case of the function $\overline{U}$ it can be easily checked that for sufficiently large k the function $\underline{U}$ satisfies the inequality (2) in Definition 3.1.

Now we verify inequality (3). Using the relations (8), (6), (7) we have

$$\underline{U}(x,t_0,\varepsilon) = u_p(x,t_0,\varepsilon) - \varphi(x,t_0,\varepsilon) + \psi(x,t_0,\varepsilon) - \varepsilon z(x,\varepsilon) + a_0\sqrt{\varepsilon} =$$

$$= \psi(x,t_0,\varepsilon) + O(\sqrt{\varepsilon}) = \hat{u}(x,t_0) + O(\sqrt{\varepsilon}).$$

Since according to assumption (A_5) it holds $\hat{u}(x,t_0) < u^0(x)$, we can conclude that for sufficiently small ε the inequality

$$\underline{U}(x,t_0,\varepsilon) \leq u^0(x) \quad \text{for} \quad -1 \leq x \leq, 1$$

is valid, that is $\underline{U}$ satisfies inequality (3).

Finally, we have to verify inequality (1). Analogously to (11) we obtain

$$\begin{aligned}
&L_\varepsilon\underline{U} \geq -\varepsilon c_0\beta E - \sqrt{\varepsilon}\varrho\beta E - M|u_p - \varphi_1||u_p - \varphi_2| + \\
&+ h(u_p - \beta E, x, t)(2u_p - \varphi_1 - \varphi_2 - \beta E)\beta E - \varepsilon c_1\beta E - \\
&- \varepsilon^2 c_2\beta E = \beta E\Big[- \sqrt{\varepsilon}\varrho - M|u_p - \varphi_1||u_p - \varphi_2| + \\
&+ h(u_p - \beta E, x, t)(2u_p - \varphi_1 - \varphi_2 - \beta E) + O(\varepsilon)\Big].
\end{aligned} \tag{18}$$

From (12) and (13) it follows that to any $\delta > 0$ for sufficiently small ε there holds the inequality

$$|u_p - \varphi_1||u_p - \varphi_2| \leq c_5\sqrt{\varepsilon}(\delta + \sqrt{\varepsilon}) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0. \tag{19}$$

Furthermore, using (8), the obvious inequality $0 < E(t,\varepsilon) \leq 1$ and the relation $\varphi - \beta = \psi - \varepsilon z + a_0\sqrt{\varepsilon}$ (see (15)) we obtain

$$2u_p - \varphi_1 - \varphi_2 - \beta E = 2\varphi + O(\varepsilon) - \varphi_1 - \varphi_2 - \beta E \geq$$

$$2\varphi - \varphi_1 - \varphi_2 - \beta + O(\varepsilon) = \varphi + (\varphi - \beta) - \varphi_1 - \varphi_2 + O(\varepsilon) =$$
$$= (\varphi + \psi - \varphi_1 - \varphi_2) + a_0\sqrt{\varepsilon} + O(\varepsilon).$$

By (16) we have $\varphi + \psi - \varphi_1 - \varphi_2 = O(\varepsilon)$, thus it holds

$$2u_p - \varphi_1 - \varphi_2 - \beta E \geq a_0\sqrt{\varepsilon} + O(\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0. \tag{20}$$

Taking into account the estimates (19), (20) and the inequality $h(u_p - \beta E, x, t) \geq m > 0$ we get from (18)

$$L_\varepsilon \underline{U} \geq \beta E\sqrt{\varepsilon}\Big[-\varrho - Mc_5(\delta + \sqrt{\varepsilon}) + ma_0 + O(\sqrt{\varepsilon})\Big] \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0.$$

For sufficiently small $\varrho, \delta, \varepsilon$ the expression in the square bracket is positive due to the presence of the term ma_0 such that we have

$$L_\varepsilon \underline{U} > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_0,$$

that is, the function $\underline{U}$ satisfies the inequality (1) in Definition 3.1. Therefore, the function $\underline{U}$ defined in (14) is a lower solution for the problem (1), (2), (17) provided k occurring in the function z is sufficiently large, ϱ arising in the function E is sufficiently small and ε is sufficiently small.

3.4 Completing the proof of Theorem 1.2

As we mentioned in subsection 3.1, the existence of ordered upper and lower solutions for the problem (1), (2), (17) implies the existence of a solution of that problem satisfying the inequalities (4). Taking into account (5) and (14) we obtain from these inequalities

$$-\beta(x,t,\varepsilon)E(t,\varepsilon) \leq u(x,t,\varepsilon) - u_p(x,t,\varepsilon) \leq \alpha(x,t,\varepsilon)E(t,\varepsilon). \tag{21}$$

Since $E(t,\varepsilon) \to 0$ as $t \to \infty$ we have

$$u(x,t,\varepsilon) - u_p(x,t,\varepsilon) \to 0 \quad \text{as} \quad t \to \infty,$$

that is the limit relation (18) is valid, and the proof of Theorem 1.2 is complete.

Remark 3.1 *Theorem 1.2 states that all smooth initial functions u^0 satisfying the condition (A_5) belong to the region of attraction of the periodic solution u_p.*

Remark 3.2 *From the inequalities (21) and the form (8) of the function E we can conclude*

$$|u(x,t,\varepsilon) - u_p(x,t,\varepsilon)| \leq c \exp\Big(- \varrho \frac{t-t_0}{\varepsilon^{3/2}}\Big) \quad \textit{for} \quad t \leq t_0,$$

that is, the solution u tends to the periodic solution u_p exponentially fast. Especially, if $t - t_0$ satisfies $t - t_0 = O(\varepsilon^{3/2-\gamma})$, where γ is any small positive number, we have the estimate for any natural number n

$$u(x,t,\varepsilon) - u_p(x,t,\varepsilon) \leq cexp\Big(-\frac{\varrho}{\varepsilon^\gamma}\Big) = o(\varepsilon^n).$$

Remark 3.3 *As well in our paper [1] as in this paper the assumption (A_4) plays an important role. If the function f does not depend on ε, then assumption (A_4) is not fulfilled. In that case the problem of the existence of a solution to (1)-(3) is more complicated and we try to contribute to that problem in a forthcoming paper.*

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