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On the construction of Dulac-Cherkas functions for generalized Liénard systems

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Abstract

Dulac-Cherkas functions can be used to derive an upper bound for the number of limit cycles of planar autonomous differential systems, at the same time they provide information about their stability. In this paper we present a method to construct such functions for generalized Liénard systems $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \sum_{j=0}^{l} h_j(x)y^j$ with $l \geq 1$ by means of linear differential equations. In case $1 \leq l \leq 3$, the described algorithm works generically. By means of an example we show that this approach can be applied also to systems with $l \geq 4$.

1 Introduction

The problem of estimating the number of limit cycles for two-dimensional systems of autonomous differential equations

$$\frac{dx}{dt} = P(x,y), \ \frac{dy}{dt} = Q(x,y) \tag{1}$$

in some open region $\mathcal{G} \subset R^2$ represents one of the famous problems formulated by D.Hilbert [6] which is still open. There are several approaches to attack this problem, including intentions to weaken it [7]. One known method to estimate the number of limit cycles of (1) from above is the method of Dulac function [2]. Here, the upper bound on the number of limit cycles also depends essentially on the connectivity of the region $\mathcal{G}$. Frequently, this method is used to establish that system (1) has in some simply connected region no limit cycle or in a doubly connected region at most one limit cycle.
The method of Dulac function has been generalized into different directions. One promising generalization is due to the first author who introduced in 1997 a function which we call now Dulac-Cherkas function that not only permits to get an upper bound for the number of limit cycles but also provides an information about their stability (see [1]). The problem of construction of such a function has been investigated by the first and the second author in [4] with respect to the Liénard system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -g(x) - f(x)y \tag{2}$$

with $g(0) = 0$. In that paper, it has been shown that linear differential equations combined with the method of linear programming can be used to determine Dulac-Cherkas functions. Recently, Gasull and Giacomini used in [3] principally the same method to estimate the number of limit cycles for the Kukles system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = h_0(x) + h_1(x)y + h_2(x)y^2 + y^3. \tag{3}$$

Our paper is devoted to the problem of construction of a class of Dulac-Cherkas functions for the generalized Liénard system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \sum_{j=0}^{l} h_j(x)y^j \tag{4}$$

with $l \geq 1$ and

$$h_l(x) \not\equiv 0. \tag{5}$$

It is organized as follows: In section 2 we recall some definitions and known results. In section 3 we present an algorithm to construct a class of functions which represent Dulac-Cherkas functions under some additional conditions. We prove that this algorithm works for $1 \leq l \leq 3$ generically. By considering a class of systems (4), we show in section 4 how this algorithm can be applied also in case $l \geq 4$ in order to derive conditions on the functions h_i implying that the corresponding system (4) has at most one limit cycle.

2 Preliminaries

First we recall the definition of a Dulac function.

Definition 2.1 *Let $P, Q \in C^1(\mathcal{G}, R)$, let X be the vector field defined by (1). A function $B \in C^1(\mathcal{G}, R)$ is called a Dulac-function of (1) in $\mathcal{G}$ if*

$$div(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y}$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}$ of measure zero, where no simply closed curve (oval) in $\mathcal{N}$ is a limit cycle.

The existence of a Dulac function implies the following estimate on the number of limit cycles of system (1) in $\mathcal{G}$.

Proposition 2.2 *Let $\mathcal{G}$ be a p-connected ($p \geq 1$) region in R^2, let $P, Q \in C^1(\mathcal{G}, R)$. If there is a Dulac function B of (1) in $\mathcal{G}$, then (1) has not more than $p-1$ limit cycles in $\mathcal{G}$.*

The method of Dulac function has been generalized in different ways. One possibility is to admit that B is not necessarily C^1 at any equilibrium provided the number of equilibria is finite in $\mathcal{G}$. This generalization has been proposed by the third author in 1968 (see [8]). Another generalization is due to the first author (see [1]). The corresponding generalized Dulac function, which we call Dulac-Cherkas function, is defined as follows.

Definition 2.3 *Let $P, Q \in C^1(\mathcal{G}, R)$. A function $\Psi \in C^1(\mathcal{G}, R)$ is called a Dulac-Cherkas function of system (1) in $\mathcal{G}$ if there exists a real number $k \neq 0$ such that*

$$\Phi := (grad\ \Psi, X) + k\Psi\ div\ X > 0 \quad (< 0) \quad in \quad \mathcal{G}. \tag{1}$$

Remark 2.4 *Condition (1) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set of measure zero, and that no closed curve of this set is a limit cycle of (1).*

For the sequel we introduce the subset $\mathcal{W}$ of $\mathcal{G}$ by

$$\mathcal{W} := \{(x, y) \in \mathcal{G} : \Psi(x, y) = 0\}.$$

The following two theorems can be found in [1].

Theorem 2.5 *Let $\mathcal{G}$ be a p-connected region, let Ψ be a Dulac-Cherkas function of* (1) *in $\mathcal{G}$. If we additionally assume that $\mathcal{W}$ has no oval in $\mathcal{G}$, then system* (1) *has at most $p-1$ limit cycles in $\mathcal{G}$.*

Theorem 2.6 *Let Ψ be a Dulac-Cherkas function of* (1) *in the region $\mathcal{G}$. Then any limit cycle Γ of* (1) *in $\mathcal{G}$ is hyperbolic and its stability is determined by the sign of the expression $k\Phi\Psi$ on Γ.*

Theorem 2.5 has been generalized in [5] by the second and the third authors as follows.

Theorem 2.7 *Let $\mathcal{G}$ be a p-connected region, let Ψ be a Dulac-Cherkas function of* (1) *in $\mathcal{G}$ such that $\mathcal{W}$ has s ovals in $\mathcal{G}$. Then system* (1) *has at most $p-1+s$ limit cycles in $\mathcal{G}$.*

Remark 2.8 *In [5] it has been also shown that the differentiability conditions of Ψ in Theorem 2.7 can be weakened in the same manner as in case of a Dulac function.*

The problem to construct a Dulac-Cherkas function has been solved by the first author for the Liénard system (2). He uses as Ψ the function

$$\Psi(x,y) \equiv y^2 + G(x) - \alpha, \tag{2}$$

where α is an appropriate constant and G is defined by $G(x) := \int_0^x g(\sigma)d\sigma$. According this choice of Ψ, the curve $\Psi(x,y) = 0$ has at most one oval. Moreover, we get from (1) and (2)

$$\Phi(x,y) \equiv -(k+2)f(x)y^2 - k(G(x) - \alpha)f(x).$$

Setting $k = -2$ we obtain

$$\Phi(x,y) \equiv 2(G(x) - \alpha)f(x).$$

Thus, Φ does not depend on y, and applying Theorem 2.7 we get the result:

Theorem 2.9 *Suppose $f, g : R \to R$ to be continuous. Additionally, we assume that there is a constant α^* such that the function Φ_1 defined by*

$$\Phi_1(x) := \Big(G(x) - \alpha^*\Big)f(x) \tag{3}$$

does not change sign in R and vanishes only at finitely many points. Then system (2) has at most one limit cycle.

In the case

$$g(x) \equiv x, \quad f(x) \equiv \mu(x^2 - 1)$$

system (2) represents the van der Pol equation, and we get

$$\Psi(x,y) \equiv y^2 + \frac{x^2}{2} - \alpha, \quad \Phi_1(x) \equiv \mu\Big(\frac{x^2}{2} - \alpha\Big)(x^2 - 1).$$

Setting $\alpha = 1/2$ we have

$$\Phi_1(x) \equiv \frac{\mu}{2}(x^2 - 1)^2,$$

that is, condition (ii) in Theorem 2.9 is fulfilled for $\mu \neq 0$, and the curve $\Psi(x,y) = 0$ consists in R^2 of exactly one oval. Thus, we get the well-known result that for $\mu \neq 0$ the van der Pol equation has at most one limit cycle.

We note that in case of Liénard system (2), to the Dulac-Cherkas function Ψ in (2) there belongs a function Φ defined in (1) that does not depend on y for a special value of k. This was the reason for the first and second author to look in [4] for an algorithmic way to construct a Dulac-Cherkas function Ψ for the Liénard system (2) in the form

$$\Psi(x,y) = \sum_{j=0}^{n} \Psi_j(x)y^j \tag{4}$$

with

$$\Psi_n(x) \not\equiv 0, \tag{5}$$

where the coefficient functions Ψ_j can be determined by means of linear differential equations such that the corresponding function Φ in (1) does not depend on y. Additionally, the problem to derive conditions such that Φ is either positive or negative in the considered region was formulated as a problem of linear programming.
In [3] Gasull and Giacomini consider the class of planar autonomous systems (3), where the functions $h_i : R \to R, 0 \leq i \leq 2$, are continuous. This system represents a generalized Liénard system. They also look for a Dulac-Cherkas function in the form (4) and prove that to any given positive integer n there is a function Ψ as in (4) and a special value k such that the corresponding function Φ does not depend on y, and that the functions Ψ_j can be determined by solving linear differential equations. They did not mention that this approach in case of the Liénard system (2) has been introduced by the first and second author in [4], probably, they were not aware of that paper.

In the next section we consider the generalized Liénard system (4) and describe an algorithm to find a function Ψ and a number k such that the corresponding function Φ in (1) does not depend on y.

3 Algorithm to construct a function Ψ such that Φ does not depend on y

We consider the vector field $X_l(x, y)$ defined by the differential system (4) in some region $\mathcal{G} \subset R^2$. For the Dulac-Cherkas function $\Psi(x, y)$ of (4) in $\mathcal{G}$ we make the ansatz (4) with $n \geq 2$. In what follows we describe an algorithm to determine the functions $\Psi_j(x)$ in (4) and the constant k such that the corresponding function $\Phi(x, y)$ determined by

$$\Phi(x, y) := (grad\, \Psi(x, y), X_l(x, y)) + k\Psi(x, y)\, div\, X_l(x, y) \tag{1}$$

does not depend on y.
If we put (4) into the right hand side of (1) and take into account that the vector field X_l is determined by (4) we get

$$\begin{aligned}\Phi(x, y) \equiv & \Big(\Psi_0'(x) + \Psi_1'(x)y + ... + \Psi_n'(x)y^n\Big)y \\ & + \Big(\Psi_1(x) + 2\Psi_2(x)y + ... + n\Psi_n(x)y^{n-1}\Big) \\ & \times \Big(h_0(x) + h_1(x)y + ... + h_l(x)y^l\Big) \\ & + k\Big(\Psi_0(x) + \Psi_1(x)y + ... + \Psi_n(x)y^n\Big) \\ & \times \Big(h_1(x) + 2h_2(x)y + ... + lh_l(x)y^{l-1}\Big).\end{aligned} \tag{2}$$

For the sequel we represent $\Phi(x, y)$ in the form

$$\Phi(x,y) \equiv \sum_{i=0}^{m} \Phi_i(x) y^i, \tag{3}$$

where $\Phi_i(x)$ is a function of the known coefficient functions $h_0(x), ..., h_l(x)$, of the unknown coefficient functions $\Psi_0(x), ..., \Psi_n(x)$, of their first derivatives $\Psi_0'(x), ..., \Psi_n'(x)$, and of k.
Concerning the highest power m of y in (3) we get from (2)

$$m = max\{n+1, n+1+l-2\}. \tag{4}$$

Our goal is to determine the functions $\Psi_j(x), j = 0, ..., n$, and the real number k in such a way that we have

$$\Phi_i(x) \equiv 0 \quad \text{for} \quad i = 1, ..., m. \tag{5}$$

Then it holds

$$\Phi(x,y) \equiv \Phi_0(x) \equiv \Psi_1(x)h_0(x) + k\Psi_0(x)h_1(x). \tag{6}$$

If we additionally require

$$\Phi_0(x) \geq 0 \quad (\leq 0) \quad \text{in } \mathcal{G} \tag{7}$$

and if $\Phi_0(x)$ vanishes only at finitely many values of x, then Ψ is a Dulac-Cherkas function of (4) in $\mathcal{G}$.

From (2)–(4) we get that for $l = 1$ and $l = 2$ the relations (5) represent a system of $n+1$ linear differential equations to determine the $n+1$ functions $\Psi_j, j = 0, ..., n$. In case $l = 1$ this system reads

$$\begin{aligned}
&0 \equiv \Psi_n'(x), \\
&0 = \Psi_{n-1}'(x) + (k+n)h_1(x)\Psi_n(x), \\
&0 \equiv \Psi_{n-2}'(x) + (k+n-1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x), \\
&\;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&0 \equiv \Psi_1'(x) + (k+2)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x), \\
&0 \equiv \Psi_0'(x) + (k+1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).
\end{aligned} \tag{8}$$

It is easy to see that this system can be solved successively by simple quadratures, starting with Ψ_n. The general solution depends on $n+1$ integration constants and on the constant k. An appropriate choice of these constants leads to conditions on the functions h_i such that Ψ is a Dulac-Cherkas function for (4) in $\mathcal{G}$.

As an example, we consider system (4) with $l = 1$, i.e.

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = h_0(x) + h_1(x)y. \tag{9}$$

We look for a Dulac-Cherkas function in the form

$$\Psi(x,y) = \Psi_0(x) + \Psi_1(x)y + \Psi_2(x)y^2 \tag{10}$$

with $\Psi_2(x) \not\equiv 0$. Putting $n = 2$ in (8) we obtain the following system of differential equations

$$\begin{aligned} &\Psi_2' = 0, \\ &\Psi_1' = -(k+2)h_1(x)\Psi_2, \\ &\Psi_0' = -(k+1)h_1(x)\Psi_1 - 2h_0(x)\Psi_2. \end{aligned} \tag{11}$$

Setting $k = -2$ we get from the first two equations

$$\Psi_2(x) \equiv c_2 \neq 0, \Psi_1(x) \equiv c_1, \tag{12}$$

where c_2 and c_1 are real constants. Putting $c_1 = 0$ we obtain from the last differential equation in (11)

$$\Psi_0(x) \ \ \equiv -2c_2 \textstyle\int_0^x h_0(\tau)d\tau + c_0, \tag{13}$$

where c_0 is any real constant. Thus, we have

$$\Psi(x,y) = -2c_2 \int_0^x h_0(\tau)d\tau + c_0 + c_2y^2,$$

$$\Phi_0(x) = -2\Big(- 2c_2 \int_0^x h_0(\tau)d\tau + c_2\Big)h_1(x) =$$

$$4c_2\Big(\int_0^x h_0(\tau)d\tau + c_0^*\Big)h_1(x).$$

To guarantee the validity of one of the inequalities $\Phi_0(x) \leq 0, \Phi_0(x) \geq 0$, we impose on h_0 and h_1 the following assumption.
(H). $h_0, h_1 : R \to R$ are continuous and such that there is a constant c_0^* ensuring that the function

$$\tilde{\Phi}_0(x) := \Big(\int_0^x h_0(\tau)d\tau + c_0^*\Big)h_1(x)$$

satisfies one of the inequalities $\tilde{\Phi}_0(x) \leq 0,\ \tilde{\Phi}_0(x) \geq 0$, where $\tilde{\Phi}_0(x)$ vanishes only in finite many points x_k.

Proposition 3.1 *Suppose hypothesis (H) to be valid. Then system (4) has at most one limit cycle in the finite part of the phase plane.*

We note that Proposition 3.1 coincides with Theorem 2.9.

In case $l = 2$ we get the system

$$\begin{aligned}
&0 \equiv \Psi_n'(x) + (2k+n)h_2(x)\Psi_n(x),\\
&0 \equiv \Psi_{n-1}'(x) + (2k+n-1)h_2(x)\Psi_{n-1}(x)\\
&\quad + (k+n)h_1(x)\Psi_n(x),\\
&0 \equiv \Psi_{n-2}'(x) + (2k+n-2)h_2(x)\Psi_{n-2}(x)\\
&\quad + (k+n-1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x),\\
&\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&0 \equiv \Psi_1'(x) + (2k+1)h_2(x)\Psi_1(x)\\
&\quad + (k+2)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x),\\
&0 = \Psi_0'(x) + 2kh_2(x)\Psi_0(x)\\
&\quad + (k+1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).
\end{aligned} \tag{14}$$

This system can also be integrated successively by solving inhomogeneous linear differential equations, starting with Ψ_n. We note that the functions Ψ_j depend on the parameter k, but we get no restriction on k in the process of solving this system. Of course, in order to be able to fulfill the inequalities (7) we have to choose k and the integration constants appropriately.

Next we consider the case $l = 3$. From (2) and (3) we obtain

$$\begin{aligned}
&0 \equiv (n+3k)h_3(x)\Psi_n(x),\\
&0 \equiv \Psi_n'(x) + (2k+n)h_2(x)\Psi_n(x)\\
&\quad + (n-1+3k)h_3(x)\Psi_{n-1}(x),\\
&0 \equiv \Psi_{n-1}'(x) + (n-1+2k)h_2(x)\Psi_{n-1}(x)\\
&\quad + (n+k)h_1(x)\Psi_n(x) + (n-2+3k)h_3(x)\Psi_{n-2},\\
&0 \equiv \Psi_{n-2}'(x) + (2k+n-2)h_2(x)\Psi_{n-2}(x)\\
&\quad + (k+n-1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x)\\
&\quad + (n-3+3k)h_3(x)\Psi_{n-3}(x),\\
&\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&0 \equiv \Psi_1'(x) + (1+2k)h_2(x)\Psi_1(x) + 3kh_3(x)\Psi_0(x)\\
&\quad + (2+k)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x),\\
&0 \equiv \Psi_0'(x) + 2kh_2(x)\Psi_0(x)\\
&\quad + (k+1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).
\end{aligned} \tag{15}$$

The first equation is an algebraic equation which determines according to (5) and (5) the constant k uniquely as $k = -\frac{n}{3}$. The remaining equations represent a system of $n+1$ linear differential equations. Its general solution depends on $n+1$ integration constants which can be used to try to fulfill the relations (7). If this is not possible we have to look for corresponding conditions on the functions h_i.

In case $l \geq 4$ system (5) consist of $n+1$ linear differential equations and $l-2$ algebraic equations to determine k and the functions $\Psi_0, ..., \Psi_n$. Thus, this system has generically no solution. In what follows we show that under additional conditions on the functions h_i system (5) has a nontrivial solution which satisfies the inequalities (7).

4 Construction of Cherkas-Dulac functions in case $l = 4$

In what follows we consider the case $l = 4$ in (4). Our aim is to construct a Dulac-Cherkas function in the form (4) with $n = 2$ under the assumption $h_i(x) \equiv c_i$ for $i = 3, 4$, where c_3 and c_4 are real parameters satisfying

$$c_3 c_4 \neq 0. \tag{1}$$

Thus, the system under consideration reads

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = h_0(x) + h_1(x)y + h_2(x)y^2 + c_3 y^3 + c_4 y^4. \tag{2}$$

It is easy to see that the equilibria of system (2) are determined by the roots of $h_0(x) = 0$. In order to exclude the trivial situation that (2) has no equilibrium, that means, system (2) has no limit cycle in any finite part of the phase plane, we assume

$$h_0(0) = 0. \tag{3}$$

We look for a Dulac-Cherkas function in the form

$$\Psi(x, y) = \sum_{j=0}^{2} \Psi_j(x) y^j \tag{4}$$

with

$$\Psi_2(x) \not\equiv 0. \tag{5}$$

By (4) it holds $m = 5$, that is, the corresponding function Φ in (3) has the form

$$\Phi(x, y) = \sum_{j=0}^{5} \Phi_j(x) y^j.$$

Taking into account (2) we get from (5) and (2)

$$2(1+2k) c_4 \Psi_2(x) \equiv 0, \tag{6}$$

$$(1+4k)c_4\Psi_1(x) + (2+3k)c_3\Psi_2(x) \equiv 0, \tag{7}$$

$$\Psi_2' + 2(1+k)h_2(x)\Psi_2 + (1+3k)c_3\Psi_1 + 4kc_4\Psi_0 \equiv 0, \tag{8}$$

$$\Psi_1' + (1+2k)h_2(x)\Psi_1 + (2+k)h_1(x)\Psi_2 + 3kc_3\Psi_0 \equiv 0, \tag{9}$$

$$\Psi_0' + 2kh_2(x)\Psi_0 + (1+k)h_1(x)\Psi_1 + 2h_0(x)\Psi_2 \equiv 0. \tag{10}$$

By (1) and (5) we get from (6)

$$k = -\frac{1}{2}. \tag{11}$$

Taking into account (1) and (11) we obtain from (7)

$$\Psi_1(x) = \frac{c_3}{2c_4}\Psi_2(x). \tag{12}$$

Substituting (12) and (11) into (8) and (9) we get

$$\Psi_2' + \Big(h_2(x) - \frac{c_3^2}{4c_4}\Big)\Psi_2 - 2c_4\Psi_0 = 0, \tag{13}$$

$$\Psi_2' + \frac{3c_4}{c_3}h_1(x)\Psi_2 - 3c_4\Psi_0 = 0. \tag{14}$$

A function Ψ_2 satisfying the differential equations (13) and (14) has also to obey the homogeneous equation

$$\Psi_2' + h(x)\Psi_2 = 0 \tag{15}$$

with

$$h(x) := 3h_2(x) - \frac{3c_3^2}{4c_4} - \frac{6c_4}{c_3}h_1(x). \tag{16}$$

Thus, we have

$$\Psi_2(x) = ce^{-\int_0^x h(\sigma)d\sigma}, \tag{17}$$

where $c \neq 0$ by (5). From (1), (5), (12) and (17) we get that the functions Ψ_1 and Ψ_2 never take the value zero.
A solution of (15) satisfies the differential equation (13) only if the relation

$$\Big(-h_2(x) + \frac{c_3^2}{4c_4} + \frac{3c_4}{c_3}h_1(x)\Big)\Psi_2(x) = c_4\Psi_0(x) \tag{18}$$

is valid. We get the same relation if we consider equation (14).

Substituting (17) together with (11) and (12) into (10) we obtain

$$\Psi_0' - h_2(x)\Psi_0 + \Big(\frac{c_3}{4c_4}h_1(x) + 2h_0(x)\Big)ce^{-\int_0^x h(\sigma)d\sigma} = 0. \tag{19}$$

Introducing the function

$$\hat{h}(x) := \frac{c_3}{4c_4}h_1(x) + 2h_0(x), \tag{20}$$

the differential equation (19) takes the form

$$\Psi_0' - h_2(x)\Psi_0 + c\hat{h}(x)e^{-\int_0^x h(\sigma)d\sigma} = 0. \tag{21}$$

Its general solution reads

$$\Psi_0(x) = e^{\int_0^x h_2(\sigma)d\sigma}\Big(d - c\int_0^x \hat{h}(\sigma)e^{-\int_0^\sigma \tilde{h}(\tau)d\tau}d\sigma\Big), \tag{22}$$

where

$$\tilde{h}(x) \equiv h(x) + h_2(x) \tag{23}$$

and d is any real constant. If we substitute the function $\Psi_2(x)$ defined in (17) and the function $\Psi_0(x)$ defined in (22) into (18) we get the relation

$$\begin{aligned} &c\Big(-h_2(x) + \frac{c_3^2}{4c_4} + \frac{3c_4}{c_3}h_1(x)\Big)e^{-\int_0^x \tilde{h}(\sigma)d\sigma} \\ &\equiv c_4\Big(d - c\int_0^x \hat{h}(\sigma)e^{-\int_0^\sigma \tilde{h}(\tau)d\tau}d\sigma\Big). \end{aligned} \tag{24}$$

That means, the functions Ψ_0, Ψ_2, and Ψ_1 defined in (22), (17) and (12), respectively, satisfy the equations (7)–(10) with $k = -1/2$ only if the relation (24) is fulfilled. This relation represents a restriction for the coefficient functions h_0, h_1 and h_2.

In order to guarantee that Ψ defined in (4) is a Dulac-Cherkas function we have to require that the function Φ_0 defined in (6) satisfies (7). Thus, we have the following Theorem.

Theorem 4.1 *Consider system (2) under the following assumptions:*

(A_1). *The coefficients c_3, c_4 satisfy $c_3c_4 \neq 0$.*

(A_2). *The functions $h_i : R \to R, i = 0, 1, 2,$ are continuous, and such that*

- *(i).* $h_0(0) = 0$.
- *(ii). There are constants c and d such that the relation (24) is valid for all $x \in R$, where $\hat{h}$ and $\tilde{h}$ are defined in (20) and (23), respectively, and that the function*

$$\Phi_0(x) := \Psi_1(x)h_0(x) - \frac{1}{2}\Psi_0(x)h_1(x) \tag{25}$$

satisfies

$$\Phi_0(x) \geq 0 \quad (\Phi_0(x) \leq 0) \quad \forall x \tag{26}$$

and vanishes only in finitely many points. Here, Ψ_0 and Ψ_1 are defined by (22), (12), (17), (16), respectively.

Then system (2) has at most one limit cycle in R^2.

In what follows we consider (2) under the assumption

$$\tilde{h}(x) := h(x) + h_2(x) \equiv 0. \tag{27}$$

In that case we have by (12), (17), and (22)

$$\Psi_1(x) := \frac{cc_3}{2c_4} e^{\int_0^x h_2(\sigma)d\sigma}, \tag{28}$$

$$\Psi_0(x) := e^{\int_0^x h_2(\sigma)d\sigma}\Big(d - c\int_0^x \hat{h}(\sigma)d\sigma\Big). \tag{29}$$

Substituting these relations into (25) we get

$$\Phi_0(x) = \frac{c}{2} e^{\int_0^x h_2(\sigma)d\sigma}\Big(\frac{c_3}{c_4}h_0(x) - h_1(x)\Big[\frac{d}{c} - \int_0^x \hat{h}(\sigma)d\sigma\Big]\Big). \tag{30}$$

Thus, introducing the function

$$\tilde{\Phi}_0(x) := \frac{c_3}{c_4}h_0(x) - h_1(x)\Big[\frac{d}{c} - \int_0^x \hat{h}(\sigma)d\sigma\Big] \tag{31}$$

we have

$$\Phi_0(x) = \frac{c}{2} e^{\int_0^x h_2(\sigma)d\sigma}\tilde{\Phi}_0(x), \tag{32}$$

and the inequalities $\Phi_0(x) \geq 0 \quad (\Phi_0(x) \leq 0)$ are fulfilled if it holds

$$\tilde{\Phi}_0(x) \geq 0 \quad (\tilde{\Phi}_0(x) \leq 0). \tag{33}$$

We note that the assumption (27) implies

$$h_2(x) \equiv \frac{3}{16}\frac{c_3^2}{c_4} + \frac{3c_4}{2c_3}h_1(x). \tag{34}$$

Taking into account (27) and (34), relation (24) takes the form

$$\frac{c_3^2}{16c_4} - \frac{dc_4}{c} + \frac{3c_4}{2c_3}h_1(x) \equiv -\int_0^x \hat{h}(\sigma)d\sigma. \tag{35}$$

Using this relation we obtain from (31)

$$\tilde{\Phi}_0(x) \equiv \frac{c_3}{c_4}h_0(x) - h_1(x)\Big[\frac{c_3^2}{16c_4} + \frac{d}{c}(1-c_4)\Big] - \frac{3c_4}{2c_3}h_1^2(x). \tag{36}$$

To determine the function h_1 we substitute (20) into (35) and get the integral equation

$$\frac{c_3^2}{16c_4} - \frac{dc_4}{c} + \frac{3c_4}{2c_3}h_1(x) \equiv -\int_0^x \Big[\frac{c_3}{c_4}h_1(\sigma) + 2h_0(\sigma)\Big]d\sigma \tag{37}$$

which is equivalent to the initial value problem

$$\begin{aligned} &h_1' + \frac{c_3^2}{6c_4^2}h_1 + \frac{4c_3}{3c_4}h_0(x) = 0, \\ &h_1(0) = \frac{2}{3}c_3\Big(\frac{d}{c} - \frac{c_3^2}{16c_4^2}\Big). \end{aligned} \tag{38}$$

Its explicit solution reads

$$h_1(x) = e^{-\frac{c_3^2 x}{6c_4^2}}\Big(h_1(0) - \frac{4c_3}{3c_4}\int_0^x e^{\frac{c_3^2\sigma}{6c_4^2}}h_0(\sigma)d\sigma\Big). \tag{39}$$

Theorem 4.2 *Consider system (2) under the following assumptions:*

(A_1). *The coefficients c_3, c_4 satisfy $c_3c_4 \neq 0$.*

(A_2). *The function $h_0 : R \to R$ is continuous and satisfies $h_0(0) = 0$.*

(A_3).

$$h_2(x) := \frac{3}{16}\frac{c_3^2}{c_4} + \frac{3c_4}{2c_3}h_1(x).$$

(A_4).

$$h_1(x) := e^{-\frac{c_3^2 x}{6c_4^2}}\Big(\alpha - \frac{4c_3}{3c_4}\int_0^x e^{\frac{c_3^2\sigma}{6c_4^2}}h_0(\sigma)d\sigma\Big)$$

with

$$\alpha = \frac{2c_3}{3c_4}\Big(\frac{dc_4}{c} - \frac{c_3^2}{16c_4}\Big) \neq 0,$$

where c and d are real constants different from zero.

(A_5). *There are constants $c \neq 0$ and $d \neq 0$ such that the function $\tilde{\Phi}_0$ defined in (36) satisfies one of the inequalities in (33) for all $x \in R$.*

Then system (2) has at most one limit cycle in R^2.

Taking into account (3) and using (31) we get from (32)

$$\Phi_0(0) = -\frac{d}{2}\, h_1(0).$$

Thus, we have the following corollary.

Corollary 4.3 *Under the assumptions of Theorem 4.2 there exist an interval I containing the origin such that system (2) has in the region $I \times R$ at most one limit cycle.*

The following example shows that the interval I can coincide with the real axis. We consider system (2) under the assumptions $(A_1) - (A_4)$ of Theorem 4.2. As function h_0 we choose

$$h_0(x) \equiv -x.$$

Then, the function h_1 reads

$$h_1(x) = e^{-\frac{c_3^2 x}{6c_4^2}}\left[h_1(0) + \frac{48c_4^3}{c_3^3}\right] + \frac{8c_4}{c_3}\left(x - \frac{6c_4^2}{c_3^2}\right). \tag{40}$$

Setting

$$c_3 = c_4 = c = d = 1 \tag{41}$$

we have

$$h_1(x) = \frac{389}{8} e^{-\frac{x}{6}} + 8(x-6). \tag{42}$$

The following relations cab be easyly verified

$$\lim_{x\to\pm\infty} h_1(x) = +\infty, \tag{43}$$

$$h_1'(x) = -\frac{389}{48} e^{-\frac{x}{6}} + 8, \quad h_1''(x) = \frac{389}{288} e^{-\frac{x}{6}}. \tag{44}$$

Hence, we have

$$h_1''(x) > 0 \quad \forall x \in R, \tag{45}$$

and we can conclude that $h_1(x)$ has a unique minimum at $x = x_m$. From

$$h_1'(x_m) = -\frac{389}{48} e^{-\frac{x_m}{6}} + 8 = 0$$

and from (42) we get

$$x_m = -6 \ln \frac{384}{389} > 0, \quad h_1(x_m) = 8x_m > 0.$$

Thus, we have

$$h_1(x) > 0 \quad \forall x \in R. \tag{46}$$

Especially, we obtain from (42) and (43)

$$h_1(0) = 5/8, \quad h_1'(0) = -5/48, \tag{47}$$

$$h_1\Big(\frac{-1}{2}\Big) = \frac{1}{8}\Big(389e^{\frac{1}{12}} - 416\Big) > 0.85, \tag{48}$$

$$h_1'\Big(\frac{-1}{2}\Big) = \frac{1}{48}\Big(-389e^{\frac{1}{12}} + 384\Big) < -0.80. \tag{49}$$

From (36) and (41) we obtain

$$\tilde{\Phi}_0(x) = -x - h_1(x)\Big(\frac{1}{16} + \frac{3}{2}h_1(x)\Big), \tag{50}$$

$$\tilde{\Phi}_0'(x) = -1 - \frac{1}{16}h_1'(x) - 3h_1(x)h_1'(x), \tag{51}$$

$$\tilde{\Phi}_0''(x) = -\frac{1}{16}h_1''(x) - 3h_1(x)h_1''(x) - 3(h_1'(x))^2. \tag{52}$$

From (42), (43), and (50) we get

$$\lim_{x\to\pm\infty} \tilde{\Phi}_0(x) = -\infty.$$

From (45), (46), and (52) it follows $\tilde{\Phi}_0''(x) < 0$, that is $\tilde{\Phi}_0(x)$ has a unique maximum at $x = x_M$. By (51) and (47) we have $\tilde{\Phi}_0'(0) = -613/768 < 0$, that is $x_M < 0$. By (48), (49) we have

$$-3h_1\Big(-\frac{1}{2}\Big)h_1'\Big(-\frac{1}{2}\Big) > 2.04$$

such that by (51) it holds $\tilde{\Phi}_0'(-\frac{1}{2}) > 0$, that is $x_M > -1/2$. By (50) and our results about $h_1(x)$ we have

$$\tilde{\Phi}_0(x) < 0.5 - \frac{3}{2}\Big(\frac{5}{8}\Big)^2 < 0 \quad \text{for} \quad -1/2 \le x \le 0$$

such that all conditions of Theorem 4.2 are fulfilled and we have the result

Corollary 4.4 *The system*

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + h_1(x)y + h_2(x)y^2 + y^3 + y^4 \tag{53}$$

with

$$h_1(x) \equiv \frac{389}{8}e^{-\frac{x}{6}} + 8(x-6), \; h_2(x) \equiv \frac{3}{16} + \frac{3}{2}h_1(x)$$

has at most one limit cycle in R^2.

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