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Elastoplastic Timoshenko beams

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Abstract

A Timoshenko type elastoplastic beam equation is derived by dimensional reduction from a general 3D system with von Mises plasticity law. It consists of two second-order hyperbolic equations with an anisotropic vectorial Prandtl–Ishlinskii hysteresis operator. Existence and uniqueness of a strong solution for an initial-boundary value problem is proven via standard energy and monotonicity methods.

1 Introduction

We continue in this paper the study of dimensional reduction in oscillating thin elastoplastic structures that we have begun in [4, 7, 8]. In these papers, elastoplastic counterparts of the Euler–Bernoulli beam equation and the Kirchhoff plate equation have been derived using the scaling technique of [1, 3], where the thickness plays the role of the smallness parameter, provided that only terms up to second order are kept and terms of order three and higher are neglected. We show here that the same idea leads to the Timoshenko model if additionally third-order terms are taken into account.

As in the above cases, we consider the standard von Mises single-yield plasticity model, and show that after reduction of the space dimension, the constitutive relation between the projected strain and projected stress can be written in terms of a multi-yield Prandtl–Ishlinskii operator. It is no longer isotropic as in the former cases, but its properties still enable us to prove the existence and uniqueness of solutions to the resulting system with appropriate boundary and initial conditions, using a space discretization and a monotonicity argument.

2 Derivation of the Model

We restrict ourselves to *rectangular beams*, that is, to sets $\Omega \subset \mathbb{R}^3$ of the form $\Omega = (0, L) \times \omega$, where $L > 0$ is the *length* of the beam, and where, with some $h > 0$ and $b > 0$, the set $\omega = (-b, b) \times (-h, h)$ represents its (rectangular) *cross section*. We denote by $x \in (0, L)$ the longitudinal coordinate, by $(y, z) \in \omega$ the transversal coordinates, and by $t \in [0, T]$ the time, where $T > 0$ is given.

In order to compare the resulting equations, we start with the linear elastic isotropic case (Subsection 2.1), and then pass to the elastoplastic model under further simplifying assumptions (Subsection 2.2). We follow the scaling technique of [1, Part A] and [3, Section 5.4] in terms of a small parameter $\alpha > 0$ with the intention to keep only the necessary lowest-order terms in α in the resulting equations. In particular, we assume that

$$h, b = \mathcal{O}(\alpha), \quad L = \mathcal{O}(1).$$

Let us consider smooth displacements $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^3$, decomposed into

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1^L \\ u_2^L \\ u_3^L \end{pmatrix} + \begin{pmatrix} u_1^H \\ u_2^H \\ u_3^H \end{pmatrix} = \mathbf{u}^L + \mathbf{u}^H\,,$$

where the superscripts L and H stand for low-order (second order at most) and high-order components with respect to α, respectively. We neglect longitudinal displacements, and make the following assumptions.

(A1) The low-order displacement of the midsurface $\mathcal{C} = \{(x,y) \in \mathbb{R}^2;\ (x,y,0) \in \Omega\}$ is independent of y, that is,

$$\mathbf{u}^L(x,y,0,t) = \begin{pmatrix} 0 \\ 0 \\ w(x,t) \end{pmatrix}, \quad \forall\,(x,y) \in \mathcal{C}\,, \quad \forall\, t \in (0,T)\,,$$

with $w : (0,L) \times (0,T) \to \mathbb{R}$.

(A2) The low-order deformation

$$\mathbf{F}^L(x,y,z,t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathbf{u}^L(x,y,z,t),$$

leaves the cross sections $\{x\} \times \omega$ perpendicular to the midsurface, and their deformation is proportional to their distance to it. Namely,

$$\mathbf{F}^L(x,y,z,t) = \mathbf{F}^L(x,y,0,t) + z\,\mathbf{n}(x,y,t) \quad \forall\,(x,y,z,t) \in \Omega \times (0,T)\,,$$

where $\mathbf{n}(x,y,t)$ is the unit "upward" normal to the deformed midsurface $\mathcal{C}(t) = \mathcal{C} + \mathbf{F}^L(\mathcal{C},0,t)$ at time t.

(A3) $w_{xx} = \mathcal{O}(\alpha)$.

Under the hypothesis **(A3)**, we can linearize the problem by replacing

$$\mathbf{n}(x,y,t) = \frac{1}{\sqrt{1+w_x^2(x,t)}} \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1 \end{pmatrix}$$

with its approximation

$$\tilde{\mathbf{n}}(x,y,t) := \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

This is justified, since an elementary computation yields that (cf. [4, 7])

$$|\,\tilde{\mathbf{n}}(x,y,t) - \mathbf{n}(x,y,t)| \, < \, \frac{1}{2}\, |w_x(x,t)|^2 \, .$$

This enables us to write for every $(x,y,z,t) \in \Omega \times (0,T)$ the low-order displacement $\mathbf{u}^L(x,y,z,t)$ as

$$\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} -z\, w_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}. \tag{2.2}$$

The smallness assumptions ensure in particular that the deformation

$$\mathbf{F}(x,y,z,t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathbf{u}(x,y,z,t)$$

is a local homeomorphism. We further compute

$$\nabla \mathbf{u}^L(x,y,z,t) = \begin{pmatrix} -z\, w_{xx}(x,t) & 0 & -w_x(x,t) \\ 0 & 0 & 0 \\ w_x(x,t) & 0 & 0 \end{pmatrix},$$

and the low-order strain tensor $\boldsymbol{\varepsilon}^L = (\nabla \mathbf{u}^L + (\nabla \mathbf{u}^L)^T)/2$ becomes

$$\boldsymbol{\varepsilon}^L(x,y,z,t) = \begin{pmatrix} -z\, w_{xx}(x,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

2.1 Small elastic deformations

We denote by " $:$ " the canonical scalar product in the space $\mathbb{T}^{3\times 3}_{\rm sym}$ of symmetric (3×3)-tensors, i. e.,

$$\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{i,j=1}^{3} \xi_{ij}\, \eta_{ij}\,, \quad \forall\, \boldsymbol{\xi} = (\xi_{ij})\,,\ \boldsymbol{\eta} = (\eta_{ij})\ , \quad i,j = 1,2,3\,.$$

Moreover, we define for any given $\boldsymbol{\xi} \in \mathbb{T}^{3\times 3}_{\rm sym}$ its (trace-free) *deviator* $\mathbf{d}(\boldsymbol{\xi})$ by

$$\mathbf{d}(\boldsymbol{\xi}) = \boldsymbol{\xi} - \frac{1}{3}\,(\boldsymbol{\xi} : \boldsymbol{\delta})\,\boldsymbol{\delta}\,, \tag{2.4}$$

where $\boldsymbol{\delta} = (\delta_{ij})$ denotes the Kronecker tensor.

To motivate the elastoplastic case treated below, we first study the case of linear isotropic elasticity, in which the strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ are related to each other through the formula

$$\boldsymbol{\sigma} \ = \ 2\mu\,\boldsymbol{\varepsilon} + \lambda\,(\boldsymbol{\varepsilon} : \boldsymbol{\delta})\,\boldsymbol{\delta}\,, \tag{2.5}$$

where μ, λ are the Lamé constants.

The main issue is to choose a proper scaling of $\boldsymbol{\sigma}$. The component σ_{11} is of the lowest order, which is $\mathcal{O}(\alpha^2)$ due to (2.3) and (2.5). Assuming that the motion is "sufficiently slow" and no volume forces act on the body, we may for scaling purposes refer to the elastostatic equilibrium conditions

$$\mathrm{div}\boldsymbol{\sigma} = \mathbf{0},$$

which according to the natural scaling of the variables $y, z = \mathcal{O}(\alpha)$, $x = \mathcal{O}(1)$, and due to the symmetry of $\boldsymbol{\sigma}$, justify the scaling hypothesis

(A4) $\sigma_{12}, \sigma_{13} = \mathcal{O}(\alpha^3)\,,\ \sigma_{22}, \sigma_{33}, \sigma_{23} = \mathcal{O}(\alpha^4)\,.$

From (2.5) we obtain

$$\boldsymbol{\sigma} : \boldsymbol{\delta} \ = \ (2\mu + 3\lambda)\,(\boldsymbol{\varepsilon} : \boldsymbol{\delta})\,, \tag{2.6}$$

hence

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu+3\lambda)}(\boldsymbol{\sigma} : \boldsymbol{\delta})\,\boldsymbol{\delta}\,. \tag{2.7}$$

Let $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\varepsilon}}$ denote the stress and strain components of the order $\mathcal{O}(\alpha^3)$ at most. We assume in addition to **(A4)** that the shear stresses in the xy-plane are negligible in terms of the α-scaling, that is,

(A5) $\bar{\sigma}_{12} = 0\,.$

Then we have

$$\bar{\boldsymbol{\sigma}} : \boldsymbol{\delta} = \sigma_{11}.$$

Thus, (2.7) yields

$$\begin{cases} \bar{\varepsilon}_{11} = \frac{1}{E}\sigma_{11} \,, \\ \bar{\varepsilon}_{22} = \bar{\varepsilon}_{33} = -\frac{\nu}{E}\sigma_{11} \,, \\ \bar{\varepsilon}_{13} = \frac{1}{2\mu}\sigma_{13} \,. \end{cases} \tag{2.8}$$

where $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$ is the Young modulus and $\nu = \lambda/(2(\mu + \lambda))$ is the Poisson ratio. Namely,

$$\bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} \frac{1}{E}\sigma_{11} & 0 & \frac{1}{2\mu}\sigma_{13} \\ 0 & -\frac{\nu}{E}\sigma_{11} & 0 \\ \frac{1}{2\mu}\sigma_{13} & 0 & -\frac{\nu}{E}\sigma_{11} \end{pmatrix}.$$

Comparing (2.8) with (2.3), we see that the $\mathcal{O}(\alpha^2)$ components of $\bar{\varepsilon}_{22}, \bar{\varepsilon}_{33}$ necessarily originate from the high-order component of the displacement $\mathbf{u}^H$. Taking into account the relations $\bar{\varepsilon}_{12} = \bar{\varepsilon}_{23} = 0$, we have

$$(\bar{u}_1^H)_y + (\bar{u}_2^H)_x = 0 \,, \quad (\bar{u}_3^H)_y + (\bar{u}_2^H)_z = 0.$$

As a consequence of Hypotheses **(A4)**, **(A5)**, we conclude that there exists a function Ψ such that

$$\bar{\mathbf{u}}^H = \begin{pmatrix} -\Psi_x \\ \Psi_y \\ -\Psi_z \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} -z\, w_{xx} - \Psi_{xx} & 0 & -\Psi_{xz} \\ 0 & \Psi_{yy} & 0 \\ -\Psi_{xz} & 0 & -\Psi_{zz} \end{pmatrix}. \tag{2.9}$$

We have by (2.8) that $\bar{\varepsilon}_{22} = \bar{\varepsilon}_{33}$, hence $\Psi_{yy} + \Psi_{zz} = 0$. The scaling $\bar{u}_2^H, \bar{u}_3^H = \mathcal{O}(\alpha^3)$ suggests to consider in the Taylor expansion of Ψ with respect to y and z only the terms up to order three. Besides, assuming the symmetry condition

(A6) $\Psi(x, -y, z, t) = \Psi(x, y, z, t) = -\Psi(x, y, -z, t)$

also for higher-order displacements, we finally consider Ψ in the form

$$\Psi(x, y, z, t) = (3zy^2 - z^3)\,\xi(x, t) + z\,\eta(x, t) \,, \tag{2.10}$$

with functions ξ, η that are to be identified. From (2.8), it follows for the terms up to the order three that

$$6\xi(x, t) = \nu\big[w(x, t) + \eta(x, t) + (3y^2 - z^2)\xi(x, t)\big]_{xx} \,, \tag{2.11}$$

which can only be consistent if multiples of ξ by powers in y and z are negligible with respect to the other terms independent of y and z, since the left-hand side of (2.11) is independent of y, z. This leads to the following representation formula:

$$\bar{\mathbf{u}} = \begin{pmatrix} -z(w+\eta)_x \\ 0 \\ w-\eta \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} -z\,(w+\eta)_{xx} & 0 & -\eta_x \\ 0 & 6z\xi & 0 \\ -\eta_x & 0 & 6z\xi \end{pmatrix}, \quad \bar{\boldsymbol{\sigma}} = \begin{pmatrix} E\bar{\varepsilon}_{11} & 0 & 2\mu\bar{\varepsilon}_{13} \\ 0 & 0 & 0 \\ 2\mu\bar{\varepsilon}_{13} & 0 & 0 \end{pmatrix}, \tag{2.12}$$

where, by (2.8),

$$\xi = \frac{\nu}{6}(w+\eta)_{xx}.$$

We now introduce the new variables

$$v = w - \eta, \quad \varphi = (w+\eta)_x.$$

Then (2.12) can be rewritten as

$$\bar{\mathbf{u}} = \begin{pmatrix} -z\varphi \\ 0 \\ v \end{pmatrix}, \ \bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} -z\,\varphi_x & 0 & \frac{1}{2}(v_x-\varphi) \\ 0 & z\,\nu\,\varphi_x & 0 \\ \frac{1}{2}(v_x-\varphi) & 0 & z\,\nu\,\varphi_x \end{pmatrix}, \ \bar{\boldsymbol{\sigma}} = \begin{pmatrix} -Ez\,\varphi_x & 0 & \mu(v_x-\varphi) \\ 0 & 0 & 0 \\ \mu(v_x-\varphi) & 0 & 0 \end{pmatrix}. \tag{2.13}$$

On the upper boundary, we prescribe the boundary condition

$$\bar{\boldsymbol{\sigma}}(x,y,h,t)\cdot\boldsymbol{\nu}_3 = \mathbf{f}(x,t), \quad t\in[0,T],$$

where $\boldsymbol{\nu}_3 = (0,0,1)^T$ is the upward normal vector, and $\mathbf{f} = (f_1, 0, f_3)^T$ is a given external surface load. In component form, this boundary condition reads $\bar{\sigma}_{13} = f_1$, $\bar{\sigma}_{23} = 0$, $\bar{\sigma}_{33} = f_3$. In agreement with the scaling hypothesis **(A4)**, we require that $f_1 = \mathcal{O}(\alpha^3), f_3 = \mathcal{O}(\alpha^4)$. On the left boundary $\{0\}\times\omega$, we assume the *clamped boundary condition*

$$v(0,t) = \varphi(0,t) = 0, \quad t\in[0,T].$$

On the right boundary $\{L\}\times\omega$, we assume the *vanishing normal stress* boundary conditions $\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_1 = 0$, where $\boldsymbol{\nu}_1 = (1,0,0)^T$ is the unit rightward normal vector. This means, in particular, that

$$\varphi_x(L,t) = 0\,, \quad (v_x-\varphi)(L,t) = 0 \quad t\in[0,T].$$

Finally, we suppose that the initial conditions

$$\varphi(x,0) = \varphi^0(x)\,,\ \ \varphi_t(x,0) = \varphi^1(x), \quad v(x,0) = v^0(x)\,,\ \ v_t(x,0) = v^1(x),$$

are given.

As in [9], we write the *momentum balance equation* in the variational form

$$\int_\Omega \rho\, \bar{\mathbf{u}}_{tt} \cdot \hat{\mathbf{u}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_\Omega \bar{\boldsymbol{\sigma}} \boldsymbol{:} \hat{\boldsymbol{\varepsilon}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \;=\; \int_{\partial\Omega} (\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}) \cdot \hat{\mathbf{u}}\;\mathrm{d}s\,, \tag{2.14}$$

with the unknown vector $\bar{\mathbf{u}}$ and tensor $\bar{\boldsymbol{\sigma}}$, for all admissible displacements $\hat{\mathbf{u}}$ and strains $\hat{\boldsymbol{\varepsilon}}$ of the form (2.13), i. e., we have

$$\hat{\mathbf{u}} = \begin{pmatrix} -z\hat{\varphi} \\ 0 \\ \hat{v} \end{pmatrix}, \quad \hat{\boldsymbol{\varepsilon}} = \begin{pmatrix} -z\,\hat{\varphi}_x & 0 & \frac{1}{2}(\hat{v}_x - \hat{\varphi}) \\ 0 & z\,\nu\,\hat{\varphi}_x & 0 \\ \frac{1}{2}(\hat{v}_x - \hat{\varphi}) & 0 & z\,\nu\,\hat{\varphi}_x \end{pmatrix}, \tag{2.15}$$

where $(\hat{\varphi}, \hat{v})$ varies over the space

$$V = \big\{(\hat{\varphi}, \hat{v}) \in H^1(0,L) \times H^1(0,L);\; \hat{\varphi}(0) = \hat{v}(0) = 0\big\}\,.$$

It follows from the choice of the boundary conditions that

$$\int_{\partial\Omega} (\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}) \cdot \hat{\mathbf{u}}\; ds = 2b \int_0^L (-hf_1\,\hat{\varphi} + f_3\,\hat{v})\;\mathrm{d}x$$

The left-hand side of (2.14) reads

$$\int_\Omega \big[\rho\,(z^2\varphi_{tt}\,\hat{\varphi} + v_{tt}\,\hat{v}) + Ez^2\,\varphi_x\,\hat{\varphi}_x + \mu(v_x - \varphi)(\hat{v}_x - \hat{\varphi})\big]\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{2.16}$$

The test functions $\hat{\varphi}, \hat{v}$ are independent of each other, and a straightforward calculation shows that (2.14) decouples into the system

$$\int_0^L \rho\, v_{tt}(x,t)\,\hat{v}(x)\,\mathrm{d}x + \int_0^L \mu(v_x - \varphi)(x,t)\,\hat{v}_x(x)\,\mathrm{d}x \;=\; \frac{1}{2h}\int_0^L f_3(x,t)\,\hat{v}(x)\,\mathrm{d}x\,, \tag{2.17}$$

$$\int_0^L \left(\frac{\rho h^2}{3}\varphi_{tt}(x,t) - \mu(v_x - \varphi)\right)\,\hat{\varphi}(x)\;\mathrm{d}x \;+\; \frac{Eh^2}{3}\int_0^L \varphi_x(x,t)\,\hat{\varphi}_x(x)\,\mathrm{d}x$$

$$= -\frac{1}{2}\int_0^L f_1(x,t)\,\hat{\varphi}(x)\,\mathrm{d}x\,. \tag{2.18}$$

The variational system (2.17)–(2.18) leads formally to the partial differential equations

$$\rho\, v_{tt} \;-\; \mu\,(v_x - \varphi)_x \;=\; \frac{1}{2h}f_3\,, \tag{2.19}$$

$$\frac{\rho h^2}{3}\,\varphi_{tt} \;-\; \frac{Eh^2}{3}\,\varphi_{xx} \;-\; \mu(v_x - \varphi) \;=\; -\frac{1}{2}f_1\,, \tag{2.20}$$

subject to the boundary conditions

$$\varphi(0,t) = v(0,t) = 0, \tag{2.21}$$

$$\varphi_x(L,t) = (v_x - \varphi)(L,t) = 0\,. \tag{2.22}$$

System (2.19)–(2.22) represents the classical Timoshenko beam equation; see, for instance, [11].

2.2 Elastoplastic oscillations

In this subsection we turn our interest to elastoplasticity. We still consider $\bar{\mathbf{u}}$ as in (2.13). We make the following hypotheses.

(B1) The strain tensor

$$\bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} -z\,\varphi_x & 0 & \frac{1}{2}(v_x-\varphi) \\ 0 & \bar{\varepsilon}_{22} & 0 \\ \frac{1}{2}(v_x-\varphi) & 0 & \bar{\varepsilon}_{33} \end{pmatrix} \tag{2.23}$$

is decomposed into elastic and plastic components $\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$.
The stress tensor

$$\bar{\boldsymbol{\sigma}} \;=\; \begin{pmatrix} \bar{\sigma}_{11} & 0 & \bar{\sigma}_{13} \\ 0 & 0 & 0 \\ \bar{\sigma}_{13} & 0 & 0 \end{pmatrix} \tag{2.24}$$

is decomposed into elastoplastic and kinematic hardening components $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{ep} + \boldsymbol{\sigma}^{hr}$.

(B2) The *elastic constitutive law* is as in (2.5), i.e.,

$$\boldsymbol{\sigma}^{ep} \;=\; 2\mu\,\boldsymbol{\varepsilon}^e + \lambda\,(\boldsymbol{\varepsilon}^e : \boldsymbol{\delta})\,\boldsymbol{\delta}. \tag{2.25}$$

(B3) The *hardening law* is assumed in the form

$$\boldsymbol{\sigma}^{hr} = \begin{pmatrix} \sigma^{hr}_{11} & 0 & \sigma^{hr}_{13} \\ 0 & 0 & 0 \\ \sigma^{hr}_{13} & 0 & 0 \end{pmatrix} = \begin{pmatrix} -zH_1\,\varphi_x & 0 & \frac{H_2}{2}(v_x-\varphi) \\ 0 & 0 & 0 \\ \frac{H_2}{2}(v_x-\varphi) & 0 & 0 \end{pmatrix} \tag{2.26}$$

with positive constants H_1, H_2.

(B4) The plastic deformations are *volume preserving* in the sense that

$$\boldsymbol{\varepsilon}^p : \boldsymbol{\delta} = 0.$$

The *von Mises* plastic yield condition is stated in terms of the stress deviator

$$\mathbf{d}(\boldsymbol{\sigma}^{ep}) = \boldsymbol{\sigma}^{ep} - \frac{1}{3}\,(\boldsymbol{\sigma}^{ep} : \boldsymbol{\delta})\,\boldsymbol{\delta}\,,$$

(B5) $\mathbf{d}(\boldsymbol{\sigma}^{ep}) : \mathbf{d}(\boldsymbol{\sigma}^{ep}) \le \frac{2}{3}R^2$, or equivalently

$$(\sigma^{ep}_{11})^2 + 3(\sigma^{ep}_{13})^2 \le R^2, \tag{2.27}$$

where $R > 0$ is a given *yield limit.*

(B6) For the plastic strain, we prescribe the *normality flow rule*

$$\boldsymbol{\varepsilon}^p_t : (\boldsymbol{\sigma}^{ep} - \boldsymbol{\theta}) \geq 0, \quad \forall \boldsymbol{\theta} \in \mathbb{T}^{3\times 3}_{\rm sym} : \quad \mathbf{d}\boldsymbol{\theta} : \mathbf{d}\boldsymbol{\theta} \leq \frac{2}{3} R^2,$$

where the subscript $_t$ denotes the time derivative.

Remark 2.1. Introducing the set

$$K = \left\{ \boldsymbol{\theta} \in \mathbb{T}^{3\times 3}_{\rm sym};\ \mathbf{d}\boldsymbol{\theta} : \mathbf{d}\boldsymbol{\theta} \leq \frac{2}{3} R^2 \right\}$$

of admissible stresses, and using the convex analysis formalism, we can write the assumptions **(B5)**+**(B6)** in subdifferential form as

$$\boldsymbol{\varepsilon}^p_t \in \partial I_K(\boldsymbol{\sigma}^{ep}), \tag{2.28}$$

where I_K is the indicator function of K and ∂I_K its subdifferential.

Similar to the statements in [4,7], we recall other equivalent formulations of the von Mises criterion (cf. [10]):

Proposition 2.2. *Each of the following two conditions is equivalent to* ***(B5)****+****(B6)****.*

(i) (Multiplier formulation) Condition ***(B5)*** *holds, and there exists a multiplier* $l_t \geq 0$ *such that* $l_t = 0$ *if* $\mathbf{d}(\boldsymbol{\sigma}^{ep}) : \mathbf{d}(\boldsymbol{\sigma}^{ep}) < \frac{2}{3} R^2$*, and*

$$\boldsymbol{\varepsilon}^p_t = l_t \mathbf{d}(\boldsymbol{\sigma}^{ep}).$$

(ii) (Dissipation formulation) Let

$$\Psi(\boldsymbol{\xi}) = \begin{cases} \sqrt{\frac{2}{3}} R \sqrt{\boldsymbol{\xi} : \boldsymbol{\xi}} & \text{if } \boldsymbol{\xi} : \boldsymbol{\delta} = 0, \\ +\infty & \text{if } \boldsymbol{\xi} : \boldsymbol{\delta} \neq 0, \end{cases}$$

be the pseudopotential of dissipation. Then we have

$$\boldsymbol{\sigma}^{ep} \ \in \ \partial \Psi(\boldsymbol{\varepsilon}^p_t), \tag{2.29}$$

that is,

$$\boldsymbol{\sigma}^{ep} : (\boldsymbol{\varepsilon}^p_t - \boldsymbol{\xi}) \geq \Psi(\boldsymbol{\varepsilon}^p_t) - \Psi(\boldsymbol{\xi}), \quad \forall\ \boldsymbol{\xi} \in \mathbb{T}^{3\times 3}_{\rm sym}. \tag{2.30}$$

Remark 2.3. We may refer to [4, Section 2.2] for a sketch of the proof of Proposition 2.2. Note that both (2.28) and (2.29) can be viewed as a maximal dissipation principle. On the one hand, in (2.28), for a given stress $\boldsymbol{\sigma}^{ep}$, the strain rate $\boldsymbol{\varepsilon}^p_t$ is required to maximize the dissipation rate $\boldsymbol{\sigma}^{ep} : \boldsymbol{\varepsilon}^p_t$ among all stress values $\boldsymbol{\theta} \in K$. On the other hand, in (2.30) (or (2.29)), for a given strain rate $\boldsymbol{\varepsilon}^p_t$, the stress $\boldsymbol{\sigma}^{ep}$ is chosen to maximize the reduced dissipation rate $\boldsymbol{\sigma}^{ep} : \boldsymbol{\varepsilon}^p_t - \Psi(\boldsymbol{\varepsilon}^p_t)$ over the set of all possible values $\boldsymbol{\xi}$ of the strain rate (cf. [4]).

Now we have

$$\boldsymbol{\varepsilon}^e = \begin{pmatrix} \varepsilon^e_{11} & 0 & \varepsilon^e_{13} \\ 0 & -\nu\varepsilon^e_{11} & 0 \\ \varepsilon^e_{13} & 0 & -\nu\varepsilon^e_{11} \end{pmatrix}, \quad \boldsymbol{\sigma}^{ep} = \begin{pmatrix} E\varepsilon^e_{11} & 0 & 2\mu\varepsilon^e_{13} \\ 0 & 0 & 0 \\ 2\mu\varepsilon^e_{13} & 0 & 0 \end{pmatrix}. \tag{2.31}$$

Assume that $\varepsilon^p_{13} = \varepsilon^p_{23} = 0$ at time $t = 0$. It follows from **(B4)** and Proposition 2.2 that

$$\boldsymbol{\varepsilon}^p = \begin{pmatrix} \varepsilon^p_{11} & 0 & \varepsilon^p_{13} \\ 0 & -\frac{1}{2}\varepsilon^p_{11} & 0 \\ \varepsilon^p_{13} & 0 & -\frac{1}{2}\varepsilon^p_{11} \end{pmatrix}. \tag{2.32}$$

We notice that there are two scalar parameters in each of the tensors $\boldsymbol{\sigma}^{ep}, \boldsymbol{\sigma}^{hr}, \boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^p$. It would be convenient to consider them as vectors with two components (cf. [4]). For this purpose, we introduce the following notations:

$$\boldsymbol{\sigma}^{ep}_* = \begin{pmatrix} \sigma^{ep}_{11} \\ \sigma^{ep}_{13} \end{pmatrix}, \ \boldsymbol{\sigma}^{hr}_* = \begin{pmatrix} \sigma^{hr}_{11} \\ \sigma^{hr}_{13} \end{pmatrix}, \ \boldsymbol{\varepsilon}^e_* = \begin{pmatrix} \varepsilon^e_{11} \\ \varepsilon^e_{13} \end{pmatrix}, \ \boldsymbol{\varepsilon}^p_* = \begin{pmatrix} \varepsilon^p_{11} \\ \varepsilon^p_{13} \end{pmatrix}, \ \bar{\boldsymbol{\varepsilon}}_* = \begin{pmatrix} -z\varphi_x \\ \frac{1}{2}(v_x - \varphi) \end{pmatrix}. \tag{2.33}$$

Hypothesis **(B1)** implies that

$$\bar{\boldsymbol{\varepsilon}}_* = \boldsymbol{\varepsilon}^e_* + \boldsymbol{\varepsilon}^p_*,$$

and $\bar{\varepsilon}_{22}, \bar{\varepsilon}_{33}$ in $\bar{\boldsymbol{\varepsilon}}$ can be determined by $\bar{\boldsymbol{\varepsilon}}_*$. Moreover, let $\mathbf{C}$ be the following positive definite matrix

$$\mathbf{C} = \begin{pmatrix} E & 0 \\ 0 & 2\mu \end{pmatrix}.$$

Then we have (cf., e.g., (2.12))

$$\boldsymbol{\sigma}^{ep}_* = \mathbf{C}\boldsymbol{\varepsilon}^e_*. \tag{2.34}$$

Next, we restate the assumptions **(B5)** and **(B6)**. Let

$$\mathfrak{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then, in view of (2.27), **(B5)** can be written as

$$\boldsymbol{\sigma}^{ep}_* \cdot \mathfrak{D}\boldsymbol{\sigma}^{ep}_* \le R^2,$$

while condition **(B6)** reads

$$\mathbf{J}(\boldsymbol{\varepsilon}^p_*)_t \cdot (\boldsymbol{\sigma}^{ep}_* - \boldsymbol{\theta}_*) \ge 0, \quad \forall \boldsymbol{\theta}_* \in K_*,$$

where the set K_* is defined as

$$K_* = \{\boldsymbol{\theta}_* \in \mathbb{R}^2; \boldsymbol{\theta}_* \cdot \mathfrak{D}\boldsymbol{\theta}_* \leq R^2\}.$$

We can also write the variational inequality in terms of $\boldsymbol{\varepsilon}^e_*$. Namely, we have $\boldsymbol{\varepsilon}^e_* \in \mathbf{C}^{-1}(K_*)$, and

$$\mathbf{JC}(\bar{\boldsymbol{\varepsilon}}_* - \boldsymbol{\varepsilon}^e_*)_t \cdot (\boldsymbol{\varepsilon}^e_* - \boldsymbol{\eta}_*) \geq 0, \quad \forall \boldsymbol{\eta}_* \in \mathbf{C}^{-1}(K_*). \tag{2.35}$$

Since $\mathbf{JC} = \mathbf{CJ}$ is a symmetric positive definite matrix, we can choose in $\mathbb{R}^2$ the scalar product

$$\langle \boldsymbol{\xi}_*, \boldsymbol{\eta}_* \rangle = \mathbf{JC}\boldsymbol{\xi}_* \cdot \boldsymbol{\eta}_*. \tag{2.36}$$

Then we can prescribe the canonical initial condition

$$\boldsymbol{\varepsilon}^e_*(0) = P_{\mathbf{C}^{-1}(K_*)}(\bar{\boldsymbol{\varepsilon}}(0)), \tag{2.37}$$

where $P_{\mathbf{C}^{-1}(K_*)}$ is the orthogonal projection onto the set $\mathbf{C}^{-1}(K_*)$ with respect to the scalar product defined above.

As in [4], for every $\bar{\boldsymbol{\varepsilon}}_* \in W^{1,1}(0,T;\mathbb{R}^2)$, problem (2.35)–(2.37) admits a unique solution $\boldsymbol{\varepsilon}^e_*$ in the metric space

$$W^{1,1}(0,T;\mathbf{C}^{-1}(K_*)) := \{\boldsymbol{\xi}_* \in W^{1,1}(0,T;\mathbb{R}^2);\ \boldsymbol{\xi}_*(t) \in \mathbf{C}^{-1}(K_*),\ \forall t \in [0,T]\}.$$

The solution mapping

$$\mathcal{S}_{\mathbf{C}^{-1}(K_*)}:\ W^{1,1}(0,T;\mathbb{R}^2) \ \rightarrow\ W^{1,1}(0,T;\mathbf{C}^{-1}(K_*)); \quad \bar{\boldsymbol{\varepsilon}}_* \mapsto \boldsymbol{\varepsilon}^e_*, \tag{2.38}$$

is called *the stop with characteristic* $\mathbf{C}^{-1}(K_*)$ (cf. [6]), whose properties are listed in Section 3. For the sake of simplicity, we denote in the following

$$\mathcal{K} = \mathbf{C}^{-1}(K_*).$$

Then we can write

$$\boldsymbol{\varepsilon}^e_* = \mathcal{S}_{\mathcal{K}}[\bar{\boldsymbol{\varepsilon}}_*],$$

and, by (2.34),

$$\boldsymbol{\sigma}^{ep}_* = \mathbf{C}\mathcal{S}_{\mathcal{K}}[\bar{\boldsymbol{\varepsilon}}_*]. \tag{2.39}$$

Remark 2.4. Obviously, $\mathcal{K} \subset \mathbb{R}^2$ is the ellipsoid $\{\boldsymbol{\eta}_* \in \mathbb{R}^2;\ \boldsymbol{\eta}_* \cdot \mathbf{C}\mathfrak{D}\mathbf{C}\boldsymbol{\eta}_* \leq R^2\}$, which is a uniformly strictly convex bounded closed set with nonempty interior and smooth boundary. As a consequence, all the properties listed in Proposition 3.1 and Proposition 3.3 below are valid.

For $c \in \mathbb{R}$, we put

$$\mathbf{I}_c = \begin{pmatrix} \operatorname{sgn} c & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B}_c = \begin{pmatrix} c & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

where " sgn " denotes the standard sign function

$$\operatorname{sgn} x = \begin{cases} 1 & : \ x > 0, \\ 0 & : \ x = 0, \\ -1 & : \ x < 0. \end{cases}$$

By definition of $\mathcal{S}_{\mathcal{K}}$ and the symmetry of $\mathcal{K}$, we easily verify that the operator $\mathcal{S}_{\mathcal{K}}$ commutes with $\mathbf{I}_c$ for $c \neq 0$, i.e.,

$$\mathcal{S}_{\mathcal{K}}[\mathbf{I}_c \boldsymbol{\xi}_*] = \mathbf{I}_c \mathcal{S}_{\mathcal{K}}[\boldsymbol{\xi}_*] \quad \text{for all } \ \boldsymbol{\xi}_* \in W^{1,1}(0,T;\mathbb{R}^2) \quad \text{and} \ \ c \neq 0\,.$$

Using the simple fact that

$$\begin{pmatrix} -z\varphi_x \\ \frac{1}{2}(v_x - \varphi) \end{pmatrix} = \mathbf{I}_{-z}\mathbf{B}_{|z|} \begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix}, \tag{2.40}$$

we deduce from **(B3)**, (2.33), (2.39)–(2.40), that

$$\boldsymbol{\sigma}_*^{ep} = \mathbf{C}\mathcal{S}_{\mathcal{K}}\left[\begin{pmatrix} -z\varphi_x \\ \frac{1}{2}(v_x - \varphi) \end{pmatrix}\right] = \mathbf{C}\mathbf{I}_{-z}\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_{|z|}\begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix}\right] \tag{2.41}$$

for all admissible arguments, and

$$\boldsymbol{\sigma}_*^{hr} = \mathbf{H}_* \begin{pmatrix} -z\varphi_x \\ \frac{1}{2}(v_x - \varphi) \end{pmatrix} = \mathbf{H}_*\mathbf{I}_{-z}\mathbf{B}_{|z|}\begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix}, \tag{2.42}$$

where $\mathbf{H}_*$ is a constant positive definite diagonal matrix.

Next, we proceed to derive an elastoplastic counterpart of the system (2.19)–(2.20) from the momentum balance equation (2.14). We take the same test functions as in (2.15), which means, in the 2D representation, that

$$\hat{\boldsymbol{\varepsilon}}_* = \begin{pmatrix} -z\hat{\varphi}_x \\ \frac{1}{2}(\hat{v}_x - \hat{\varphi}) \end{pmatrix} = \mathbf{I}_{-z}\mathbf{B}_{|z|}\begin{pmatrix} \hat{\varphi}_x \\ \hat{v}_x - \hat{\varphi} \end{pmatrix}. \tag{2.43}$$

For the sake of simplicity, we put

$$\mathbf{u} = \begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix}, \quad \hat{\mathbf{u}} = \begin{pmatrix} \hat{\varphi}_x \\ \hat{v}_x - \hat{\varphi} \end{pmatrix}. \tag{2.44}$$

Now we take a look at the second term in (2.14). From (2.40)–(2.44), we obtain that

$$\begin{aligned}\int_\Omega \bar{\boldsymbol{\sigma}} : \hat{\boldsymbol{\varepsilon}}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \;&=\; \int_\Omega \mathbf{J}(\boldsymbol{\sigma}^{ep}_* + \boldsymbol{\sigma}^{hr}_*) \cdot \hat{\boldsymbol{\varepsilon}}_*\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ =\;& 2b \int_0^h \int_0^L \mathbf{J}\left(\mathbf{C}\mathbf{I}_{-1}\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_z\mathbf{u}\right] + \mathbf{H}_*\left(\mathbf{I}_{-1}\mathbf{B}_z\mathbf{u}\right)\right) \cdot \left(\mathbf{I}_{-1}\mathbf{B}_z\hat{\mathbf{u}}\right) \mathrm{d}x\, \mathrm{d}z \\ &+2b \int_{-h}^0 \int_0^L \mathbf{J}\left(\mathbf{C}\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_{|z|}\mathbf{u}\right] + \mathbf{H}_*\left(\mathbf{B}_{|z|}\mathbf{u}\right)\right) \cdot (\mathbf{B}_{|z|}\hat{\mathbf{u}}) \mathrm{d}x \mathrm{d}z \\ =\;& 4b \int_0^L \left[\int_0^h \left(\mathbf{J}\mathbf{C}\mathbf{B}_q\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\mathbf{u}\right] + \mathbf{H}_q\mathbf{u}\right)\, \mathrm{d}q\right] \cdot \hat{\mathbf{u}} \mathrm{d}x, \end{aligned} \tag{2.45}$$

where $\mathbf{H}_q = \mathbf{J}\mathbf{B}_q\mathbf{H}_*\mathbf{B}_q$ is a diagonal matrix, which is positive definite for $q > 0$. We now set

$$\mathbf{F}\left[\mathbf{u}\right] := \int_0^h \mathbf{B}_q\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\mathbf{u}\right]\, \mathrm{d}q\,, \quad \mathbf{H}^* := \int_0^h \mathbf{H}_q\, \mathrm{d}q\,. \tag{2.46}$$

Then $\mathbf{H}^*$ is a positive definite diagonal matrix, and $\mathbf{F}$ is an anisotropic version of the vectorial Prandtl–Ishlinskii operator; see Section 3.

Finally, we can write the whole equation for the elastoplastic Timoshenko beam in the following variational form

$$\begin{aligned}&\int_0^L \rho\left(\frac{2}{3}h^3\varphi_{tt}\,\hat{\varphi} + 2hv_{tt}\,\hat{v}\right)\, \mathrm{d}x \\ &\quad + 2\int_0^L \left(\mathbf{J}\mathbf{C}\mathbf{F}\left[\begin{pmatrix}\varphi_x \\ v_x - \varphi\end{pmatrix}\right] + \mathbf{H}^*\begin{pmatrix}\varphi_x \\ v_x - \varphi\end{pmatrix}\right) \cdot \begin{pmatrix}\hat{\varphi}_x \\ \hat{v}_x - \hat{\varphi}\end{pmatrix} \mathrm{d}x \\ =\;& \int_0^L \left(-hf_1\,\hat{\varphi} + f_3\,\hat{v}\right)\, \mathrm{d}x\,. \end{aligned} \tag{2.47}$$

3 Prandtl–Ishlinskii Operators

We consider a real separable Hilbert space X endowed with a scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. In our present case, we consider $\mathrm{X} = \mathbb{R}^2$ with inner product (2.36).

Assume that a convex closed set $\mathrm{Z} \subset \mathrm{X}$ containing the origin is given. For any $u \in W^{1,1}(0,T;\mathrm{X})$, we define $\chi \in W^{1,1}(0,T;X)$ as the unique solution to the variational inequality

$$\chi(t) \in \mathrm{Z}, \quad \forall\, t \in [0,T], \tag{3.1}$$

$$\chi(0) = P_{\mathrm{Z}}(v(0)), \tag{3.2}$$

$$\langle v_t(t) - \chi_t(t),\; \chi(t) - y(t)\rangle \geq 0, \quad \text{a.e. in } (0,T), \quad \forall\, y \in \mathrm{Z}, \tag{3.3}$$

where $P_{\mathrm{Z}} : \mathrm{X} \to \mathrm{Z}$ is the orthogonal projection onto Z. The solution mapping

$$\mathcal{S}_{\mathrm{Z}} : \ W^{1,1}(0,T;\mathrm{X}) \to W^{1,1}(0,T;\mathrm{X}); \quad v \mapsto \chi,$$

is called *the stop with characteristic* Z. We recall some analytic properties of $\mathcal{S}_{\mathrm{Z}}$ (cf. [2, Chapter 2]).

Proposition 3.1. *The mapping $\mathcal{S}_{\mathrm{Z}}$ defined by (3.1)–(3.3) has the following properties.*

(1) $\mathcal{S}_{\mathrm{Z}}$ is continuous in the strong topology of $W^{1,1}(0,T;X)$, and depends continuously on Z in the sense of Hausdorff distance;

(2) If the boundary of Z is of class $W^{2,\infty}$ (that is, if the outward normal mapping is Lipschitz continuous), then $\mathcal{S}_{\mathrm{Z}}$ is locally Lipschitz continuous in $W^{1,1}(0,T;\mathrm{X})$;

(3) If Z has a nonempty interior, then $\mathcal{S}_{\mathrm{Z}}$ can be extended to a continuous mapping $C([0,T];\mathrm{X}) \to C([0,T];\mathrm{X})$;

(4) If Z is uniformly strictly convex, then $\mathcal{S}_{\mathrm{Z}} : C([0,T];\mathrm{X}) \to C([0,T];\mathrm{X})$ is Hölder continuous with exponent $\frac{1}{2}$;

(5) The mapping is monotone in the sense that

$$\langle \mathcal{S}_{\mathrm{Z}}[u_1](t) - \mathcal{S}_{\mathrm{Z}}[u_2](t), \ u_{1t}(t) - u_{2t}(t) \rangle \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |\mathcal{S}_{\mathrm{Z}}[u_1](t) - \mathcal{S}_{\mathrm{Z}}[u_2](t)|^2 \quad a.e. \ in \ (0,T),$$

for every $u_1, u_2 \in W^{1,1}(0,T;\mathrm{X})$;

(6) The mapping $\mathcal{S}_{\mathrm{Z}}$ is locally monotone, i.e.,

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_{\mathrm{Z}}[u](t), u_t(t) \right\rangle = \left| \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_{\mathrm{Z}}[u](t) \right|^2 \leq |u_t(t)|^2 \quad a.e. \ in \ (0,T),$$

for every $u \in W^{1,1}(0,T;\mathrm{X})$;

(7) The "second-order energy inequality"

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_{\mathrm{Z}}[u](t), u_{tt}(t) \right\rangle \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_{\mathrm{Z}}[u](t), u_t(t) \right\rangle$$

holds in the sense of distributions for every $u \in W^{1,1}(0,T;\mathrm{X})$.

Let $\mathrm{Lin}(\mathrm{X})$ denote the space of all bounded linear mappings $\mathrm{X} \to \mathrm{X}$. Given a parameter set Q endowed with a measure ν, a mapping $A : Q \to \mathrm{Lin}(\mathrm{X})$, and a family $\{\mathrm{Z}_q; q \in Q\}$ of nonempty convex closed subsets of X, we define the Prandtl–Ishlinskii operator $\tilde{\mathbf{F}} : W^{1,1}(0,T;\mathrm{X}) \to W^{1,1}(0,T;\mathrm{X})$ by the formula

$$\tilde{\mathbf{F}}[u](t) = \int_Q A^*(q)\mathcal{S}_{\mathrm{Z}_q}\left[A(q)u\right](t)\,\mathrm{d}\nu(q)\,, \tag{3.4}$$

where $A^*(q)$ is the dual mapping to $A(q)$. In the case of (2.46), we have

$$Q = [0,h], \quad \mathrm{Z}_q = \mathcal{K} \subset \mathrm{X} = \mathbb{R}^2, \quad A(q) = A^*(q) = \mathbf{B}_q.$$

Now we consider the operator $\mathbf{F}$ defined in (2.46),

$$\mathbf{F}[\mathbf{u}](t) = \int_0^h \mathbf{B}_q \mathcal{S}_{\mathcal{K}} \left[\mathbf{B}_q \mathbf{u}\right](t)\,\mathrm{d}q\,. \tag{3.5}$$

Remark 3.2. One can see that our Prandtl–Ishlinskii operator (3.5) has a different form from those introduced in [4,7,8]. It is no longer isotropic (cf. the definition of $\mathbf{B}_q$).

As a direct consequence of Proposition 3.1, we can easily deduce the following result.

Proposition 3.3. *The mapping* $\mathbf{F}$ *defined by (3.5) has the following properties.*

(i) The mapping $\mathbf{F} : W^{1,1}(0,T;\mathbb{R}^2) \to W^{1,1}(0,T;\mathbb{R}^2)$ *is locally Lipschitz continuous, and* $\mathbf{F} : C([0,T];\mathbb{R}^2) \to C([0,T];\mathbb{R}^2)$ *is bounded and continuous in the respective strong topologies.*

(ii) The mapping $\mathbf{F}$ *is monotone in the sense that*

$$\langle \mathbf{F}[\mathbf{u}_1](t) - \mathbf{F}[\mathbf{u}_2](t), \mathbf{u}_{1t}(t) - \mathbf{u}_{2t}(t)\rangle \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^h \left|\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\mathbf{u}_1\right](t) - \mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\mathbf{u}_2\right](t)\right|^2 \mathrm{d}q,$$

a.e. in $(0,T)$*, for every* $\mathbf{u}_1, \mathbf{u}_2 \in W^{1,1}(0,T;\mathbb{R}^2)$*.*

(iii) The mapping $\mathbf{F}$ *is locally monotone in the sense that*

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}[\mathbf{u}](t), \mathbf{u}_t(t)\right\rangle = \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_{\mathcal{K}}[\mathbf{B}_q\mathbf{u}](t)\right|^2 \mathrm{d}q,$$

$$\min\left\{\frac{3}{h^3}, \frac{4}{h}\right\}\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}[\mathbf{u}](t)\right|^2 \leq \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}[\mathbf{u}](t), \mathbf{u}_t(t)\right\rangle \leq \max\left\{\frac{h^3}{3}, \frac{h}{4}\right\}|\mathbf{u}_t(t)|^2,$$

a.e. in $(0,T)$*, for every* $\mathbf{u} \in W^{1,1}(0,T;\mathbb{R}^2)$*.*

(iv) The "second-order energy inequality"

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}[\mathbf{u}](t), \mathbf{u}_{tt}(t)\right\rangle \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}[\mathbf{u}](t), \mathbf{u}_t(t)\right\rangle \tag{3.6}$$

holds in the sense of distributions for every for every $\mathbf{u} \in W^{2,1}(0,T;\mathbb{R}^2)$*.*

Remark 3.4. As in [4, Section 3], we can evaluate $\mathbf{F}[\mathbf{u}](t)$ at $t = 0$:

$$\mathbf{F}[\mathbf{u}](0) = \int_0^h \mathbf{B}_q P_{\mathcal{K}}[\mathbf{B}_q\mathbf{u}(0)]\mathrm{d}q. \tag{3.7}$$

The initial value mapping

$$\mathbf{A}_{\mathbf{F}}(\boldsymbol{\xi}) : \mathbb{R}^2 \to \mathbb{R}^2;\ \boldsymbol{\xi} \mapsto \int_0^h \mathbf{B}_q P_{\mathcal{K}}[\mathbf{B}_q\boldsymbol{\xi}]\mathrm{d}q,$$

is Lipschitz continuous in $\mathbb{R}^2$, and $\mathbf{A}_{\mathbf{F}}(\mathbf{0}) = \mathbf{0}$.

4 Existence and Uniqueness of Solutions

For the sake of simplicity, we study our problem in $Q_T := (0,1)\times(0,T)$ and set all positive constants that have no influence on the existence and uniqueness result to unity. We also put $\mathbf{JC} = \mathbf{H}^* = \mathbf{I}_1$ (the identity matrix). We now restate the equation of the elastoplastic Timoshenko beam (2.47) as

$$\int_0^1 \begin{pmatrix} \varphi_{tt} \\ v_{tt} \end{pmatrix} \cdot \begin{pmatrix} \hat{\varphi} \\ \hat{v} \end{pmatrix} \mathrm{d}x + \int_0^1 \left(\mathbf{F}\left[\begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix} \right] + \begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix} \right) \cdot \begin{pmatrix} \hat{\varphi}_x \\ \hat{v}_x - \hat{\varphi} \end{pmatrix} \mathrm{d}x$$
$$= \int_0^1 \mathbf{g} \cdot \begin{pmatrix} \hat{\varphi} \\ \hat{v} \end{pmatrix} \mathrm{d}x, \tag{4.1}$$

where $\mathbf{g} = (g^1, g^2)^T$ is a given vector. Equation (4.1) is subject to the boundary conditions

$$\varphi(0,t) = v(0,t) = 0, \quad \left(\mathbf{F}\left[\begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix} \right] + \begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix} \right)(1,t) = 0, \quad t \in [0,T], \tag{4.2}$$

and to the initial conditions

$$\varphi(x,0) = \varphi^0(x)\,, \quad \varphi_t(x,0) = \varphi^1(x), \quad v(x,0) = v^0(x)\,, \quad v_t(x,0) = v^1(x). \tag{4.3}$$

In order to prove the existence and uniqueness of a solution to problem (4.1)–(4.3), we transform the vector equation (4.1) into a first-order system by introducing the new variables

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \varphi_t \\ v_t \end{pmatrix}, \quad \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \varphi_x \\ v_x - \varphi \end{pmatrix}, \quad \begin{pmatrix} r \\ s \end{pmatrix} = \mathbf{F}\left[\begin{pmatrix} w \\ z \end{pmatrix} \right] + \begin{pmatrix} w \\ z \end{pmatrix}. \tag{4.4}$$

As a consequence, equation (4.1) can then be rewritten in the form

$$\left.\begin{aligned} p_t &= r_x + s + g^1, \\ q_t &= s_x + g^2, \\ w_t &= p_x, \\ z_t &= q_x - p, \end{aligned}\right\} \tag{4.5}$$

with the boundary conditions

$$p(0,t) = q(0,t) = r(1,t) = s(1,t) = 0, \quad t \in [0,T], \tag{4.6}$$

and the initial conditions

$$\begin{pmatrix} p(x,0) \\ q(x,0) \end{pmatrix} = \begin{pmatrix} p^0(x) \\ q^0(x) \end{pmatrix} := \begin{pmatrix} \varphi^1(x) \\ v^1(x) \end{pmatrix}, \quad \begin{pmatrix} w(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ z^0(x) \end{pmatrix} := \begin{pmatrix} \varphi^0_x(x) \\ v^0_x(x) - \varphi^0(x) \end{pmatrix} \tag{4.7}$$

for $x \in [0,1]$.

We make the following assumptions:

(H1) $\mathbf{g},\ \mathbf{g}_t \in L^2(Q_T;\mathbb{R}^2)$.

(H2) $v^0,\varphi^0 \in H^2(0,1)$, $v^1,\varphi^1 \in H^1(0,1)$ satisfy the *compatibility conditions*

$$v^i(0) = \varphi^i(0) = 0, \quad \varphi^i_x(1) = 0, \quad \varphi^i(1) = v^i_x(1), \quad i = 0,1. \tag{4.8}$$

(H3) $\mathbf{F}$ has the form as in (3.5) (cf. also (2.44) and (2.46)).

Remark 4.1. It follows from **(H2)** and from the Proposition 3.3 (ii) with $\mathbf{u}_1 = 0$ that the boundary conditions (4.6) can be written equivalently as

$$p(0,t) = q(0,t) = w(1,t) = z(1,t) = 0\,. \tag{4.9}$$

In terms of the new unknowns p,q,w,z, Hypothesis **(H2)** then reads
(H2)' $p^0,q^0,w^0,z^0 \in H^1(0,1)$ satisfy the compatibility conditions

$$p^0(0) = q^0(0) = 0, \quad w^0(1) = z^0(1) = 0\,. \tag{4.10}$$

We now state the main result on existence and uniqueness.

Theorem 4.2. *Suppose that the hypotheses **(H1)**, **(H2)'**, and **(H3)** are satisfied. Then, for any $T > 0$, problem (4.5)–(4.7) admits a unique solution (p,q,w,z) such that*

$$p,q,r,s \in W^{1,\infty}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1(0,1)), \quad w,z \in W^{1,\infty}(0,T;L^2(0,1)). \tag{4.11}$$

Putting

$$\varphi(x,t) = \varphi^0(x) + \int_0^t p(x,\tau)\,\mathrm{d}\tau, \quad v(x,t) = v^0(x) + \int_0^t q(x,\tau)\,\mathrm{d}\tau,$$

we easily obtain the following consequence:

Corollary 4.3. *Suppose that the hypotheses **(H1)**, **(H2)**, and **(H3)** are satisfied. Then, for any $T > 0$, problem (4.1)–*(4.3) *admits a unique solution (φ,v) such that*

$$\varphi, v \in W^{2,\infty}(0,T;L^2(0,1)) \cap W^{1,\infty}(0,T;H^1(0,1)). \tag{4.12}$$

The rest of the paper is devoted to the proof of Theorem 4.2. It is divided into the subsections 4.1–4.3.

4.1 Space discretization

We fix some $n \in \mathbb{N}$. For a generic vector $\mathbf{u} = (u_0, ..., u_n)^T$, we introduce the notation

$$\mathrm{d}_k \mathbf{u} = n(u_k - u_{k-1}), \quad k = 1, ..., n.$$

A space-discrete counterpart of (4.5)–(4.7) is considered in the following form:

$$\dot{p}_k(t) = \mathrm{d}_{k+1}\mathbf{r}(t) + s_k(t) + g_k^1(t), \tag{4.13}$$

$$\dot{q}_k(t) = \mathrm{d}_{k+1}\mathbf{s}(t) + g_k^2(t), \tag{4.14}$$

$$\dot{w}_k(t) = \mathrm{d}_k \mathbf{p}(t), \tag{4.15}$$

$$\dot{z}_k(t) = \mathrm{d}_k \mathbf{q}(t) - p_k(t), \tag{4.16}$$

for $k = 1, \ldots, n-1$, with the “boundary conditions” (cf. (4.9))

$$p_0(t) = q_0(t) = r_n(t) = s_n(t) = 0, \quad t \in (0, T], \tag{4.17}$$

and the initial conditions

$$p_k(0) = p^0(k/n)\,,\ q_k(0) = q^0(k/n)\,, w_k(0) = w^0(k/n)\,,\ z_k(0) = z^0(k/n)\,. \tag{4.18}$$

It follows from (4.10) that relations (4.17) hold also for $t = 0$. In (4.13) and (4.14), we let

$$g_k^i(t) = n \int_{k/n}^{(k+1)/n} g^i(x,t)\,\mathrm{d}x, \qquad i = 1, 2\,, \quad k = \ 0, 1, \ldots, n-1\,, \tag{4.19}$$

and

$$\begin{pmatrix} r_k \\ s_k \end{pmatrix} = \mathbf{F}\left[\begin{pmatrix} w_k \\ z_k \end{pmatrix}\right] + \begin{pmatrix} w_k \\ z_k \end{pmatrix}. \tag{4.20}$$

Remark 4.4. From (4.13)–(4.20) and Remark 4.1, we compute the missing values on the “discrete boundary” $k = 0$ and $k = n$, that is,

$$\left.\begin{aligned} &w_n(t) = z_n(t) = 0\,, \quad n(q_n(t) - q_{n-1}(t)) = p_n(t) = p_{n-1}(t)\,, \\ &n(r_0(t) - r_1(t)) = s_0(t) + g_0^1(t)\,, \quad n(s_0(t) - s_1(t)) = g_0^2(t)\,, \end{aligned}\right\} \tag{4.21}$$

for all $t \in [0, T]$.

Problem (4.13)–(4.20) is a system of $4(n-1)$ ODEs with a right-hand side that is locally Lipschitz continuous in $W^{1,1}(0, T; \mathbb{R}^{4(n-1)})$. By the contraction mapping principle, it is standard to prove that (4.13)–(4.20) admits a unique local solution. Hence, we omit the details here.

In what follows, we derive some uniform estimates that will enable us to fulfill two purposes: (1) extend the local solution of problem (4.13)–(4.20) to $[0, T]$ for arbitrary

$T > 0$; (2) pass to the limit as $n \to \infty$. In the subsequent proof, we shall denote by C any constant that possibly depends on the data and T, but not on the discretization parameter n. Below, we simply denote by $\|\cdot\|$ the norm in $L^2(0,1)$, and by $H^m(0,1)$ the Sobolev spaces $W^{m,2}(0,1)$ with norm $\|\cdot\|_{H^m}$, $m \in \mathbb{N}$.

First Estimate. Testing (4.13) by $p_k(t)$, (4.14) by $q_k(t)$, (4.15) by $r_k(t)$, (4.16) by $s_k(t)$, and using summation by parts together with (4.17), (4.20), we obtain, for a.e. $t \in (0,T)$,

$$\sum_{k=1}^{n-1}(\dot{p}_k p_k + \dot{q}_k q_k + \dot{w}_k w_k + \dot{z}_k z_k) = -\sum_{k=1}^{n-1} \mathbf{F}\left[\begin{pmatrix} w_k \\ z_k \end{pmatrix}\right] \cdot \begin{pmatrix} \dot{w}_k \\ \dot{z}_k \end{pmatrix} + \sum_{k=1}^{n-1} \begin{pmatrix} g_k^1 \\ g_k^2 \end{pmatrix} \cdot \begin{pmatrix} p_k \\ q_k \end{pmatrix}. \quad (4.22)$$

It follows from Proposition 3.3(ii) (with $\mathbf{u}_2 = 0$) that

$$\sum_{k=1}^{n-1} \mathbf{F}\left[\begin{pmatrix} w_k \\ z_k \end{pmatrix}\right] \cdot \begin{pmatrix} \dot{w}_k \\ \dot{z}_k \end{pmatrix} \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=1}^{n-1} \int_0^h \left| \mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\begin{pmatrix} w_k \\ z_k \end{pmatrix}\right]\right|^2 \mathrm{d}q. \quad (4.23)$$

Define

$$\begin{aligned} V_1(t) \;=\; & \frac{1}{n}\sum_{k=1}^{n-1}(p_k^2(t) + q_k^2(t) + w_k^2(t) + z_k^2(t)) + \frac{1}{n}\sum_{k=1}^{n-1}\int_0^h \left|\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\begin{pmatrix} w_k(t) \\ z_k(t) \end{pmatrix}\right]\right|^2 \mathrm{d}q \\ & -2\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\begin{pmatrix} g_k^1(\tau) \\ g_k^2(\tau) \end{pmatrix} \cdot \begin{pmatrix} p_k(\tau) \\ q_k(\tau) \end{pmatrix} \mathrm{d}\tau. \end{aligned}$$

The function V_1 is absolutely continuous. We infer from (4.22) and (4.23) that it is decreasing in time. Hence,

$$V_1(t) \leq V_1(0) \quad \text{for } t \in [0,T]. \quad (4.24)$$

From (4.18) and Remark 3.4, we have

$$V_1(0) \;\leq\; C(\|p^0\|_{L^\infty}^2 + \|q^0\|_{L^\infty}^2 + \|w^0\|_{L^\infty}^2 + \|z^0\|_{L^\infty}^2). \quad (4.25)$$

Furthermore, (4.19) implies that

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}[(g_k^1(\tau))^2 + (g_k^2(\tau))^2]\mathrm{d}\tau \leq C\int_0^t \|\mathbf{g}(\tau)\|^2 \mathrm{d}\tau. \quad (4.26)$$

From (4.24)–(4.26), **(H1)**, **(H2)'**, and Gronwall's inequality, we deduce the estimate

$$\frac{1}{n}\sum_{k=1}^{n-1}(p_k^2(t) + q_k^2(t) + w_k^2(t) + z_k^2(t)) \leq C, \quad \forall\, t \in [0,T]. \quad (4.27)$$

Besides, we infer from Proposition 3.3, (4.20), and (4.27), that

$$\frac{1}{n}\sum_{k=1}^{n-1}(r_k^2(t) + s_k^2(t)) \leq C, \quad \forall\, t \in [0,T]. \quad (4.28)$$

Second Estimate. We differentiate (4.13)–(4.16) by t, and test by $\dot{p}_k, \dot{q}_k, \dot{r}_k, \dot{s}_k$, respectively. Using summation by parts, and (4.17), (4.20), we obtain that

$$\sum_{k=1}^{n-1}(\ddot{p}_k\dot{p}_k+\ddot{q}_k\dot{q}_k+\ddot{w}_k\dot{w}_k+\ddot{z}_k\dot{z}_k) = -\sum_{k=1}^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}\left[\begin{pmatrix} w_k \\ z_k \end{pmatrix}\right]\cdot\begin{pmatrix} \ddot{w}_k \\ \ddot{z}_k \end{pmatrix}+\sum_{k=1}^{n-1}\begin{pmatrix} \dot{g}_k^1 \\ \dot{g}_k^2 \end{pmatrix}\cdot\begin{pmatrix} \dot{p}_k \\ \dot{q}_k \end{pmatrix}. \tag{4.29}$$

It follows from Proposition 3.3(iv) (namely, the second-order energy inequality (3.6)) that

$$-\sum_{k=1}^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}\left[\begin{pmatrix} w_k \\ z_k \end{pmatrix}\right]\cdot\begin{pmatrix} \ddot{w}_k \\ \ddot{z}_k \end{pmatrix} \le -\frac{1}{2}\sum_{k=1}^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}\left[\begin{pmatrix} w_k \\ z_k \end{pmatrix}\right]\cdot\begin{pmatrix} \dot{w}_k \\ \dot{z}_k \end{pmatrix}\right\} \tag{4.30}$$

in the sense of distributions.

We now define

$$\begin{aligned} V_2(t) &= \frac{1}{n}\sum_{k=1}^{n-1}(\dot{p}_k^2(t)+\dot{q}_k^2(t)+\dot{w}_k^2(t)+\dot{z}_k^2(t))+\frac{1}{n}\sum_{k=1}^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}\left[\begin{pmatrix} w_k(t) \\ z_k(t) \end{pmatrix}\right]\cdot\begin{pmatrix} \dot{w}_k(t) \\ \dot{z}_k(t) \end{pmatrix} \\ &\quad +2\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\begin{pmatrix} \dot{g}_k^1(\tau) \\ \dot{g}_k^2(\tau) \end{pmatrix}\cdot\begin{pmatrix} \dot{p}_k(\tau) \\ \dot{q}_k(\tau) \end{pmatrix}\mathrm{d}\tau. \end{aligned}$$

We can infer from (4.29) and (4.30) that $V_2(t)$ is decreasing in time, similarly to $V_1(t)$. However, $V_2(t)$ is no longer necessarily continuous. We thus introduce the continuous functions

$$\begin{aligned} \underline{V}_2(t) &= \frac{1}{n}\sum_{k=1}^{n-1}(\dot{p}_k^2(t)+\dot{q}_k^2(t)+\dot{w}_k^2(t)+\dot{z}_k^2(t)) \\ &\quad -\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}[(\dot{g}_k^1(\tau))^2+(\dot{g}_k^2(\tau))^2+\dot{p}_k^2(t)+\dot{q}_k^2(t)]\mathrm{d}\tau, \\ \overline{V}_2(t) &= \frac{1}{n}\sum_{k=1}^{n-1}(\dot{p}_k^2(t)+\dot{q}_k^2(t)+C\dot{w}_k^2(t)+C\dot{z}_k^2(t)) \\ &\quad +\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}[(\dot{g}_k^1(\tau))^2+(\dot{g}_k^2(\tau))^2+\dot{p}_k^2(t)+\dot{q}_k^2(t)]\mathrm{d}\tau, \end{aligned}$$

with a suitably chosen constant $C>0$. It follows from the local monotonicity of $\mathbf{F}$ (cf. Proposition 3.3 (iii)) that for $t\in[0,T]$ and C sufficiently large we obtain the inequalities

$$\underline{V}_2(t)\le V_2(t)\le \overline{V}_2(t).$$

Hence, for a.e. $0\le s\le t\le T$, we have

$$\underline{V}_2(t)\le V_2(t)\le V_2(s)\le \overline{V}_2(s).$$

In particular, it holds

$$\underline{V}_2(t) \le \overline{V}_2(0), \quad t \in [0,T]. \tag{4.31}$$

We estimate the initial value $\overline{V}_2(0)$ using the equations (4.13)–(4.16). Note that by Remark 3.4, we can estimate $|\mathrm{d}_{k+1}\mathbf{r}(0)|^2 + |\mathrm{d}_{k+1}\mathbf{s}(0)|^2$ from above by $C(|\mathrm{d}_{k+1}\mathbf{w}(0)|^2 + |\mathrm{d}_{k+1}\mathbf{z}(0)|^2)$.

For a generic function $f \in H^1(0,1)$, we have

$$n\sum_{k=0}^{n-1}\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right|^2 \;=\; n\sum_{k=0}^{n-1}\left|\int_{k/n}^{(k+1)/n} f'(x)\,\mathrm{d}x\right|^2 \;\le\; \|f'\|^2 .$$

Applying this formula successively to $f = p^0, q^0, w^0, z^0$, and using (4.25), we eventually obtain the estimate

$$\overline{V}_2(0) \le C(\|p^0\|_{H^1}^2 + \|q^0\|_{H^1}^2 + \|w^0\|_{H^1}^2 + \|z^0\|_{H^1}^2 + \|\mathbf{g}(x,0)\|^2). \tag{4.32}$$

Since

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}[(\dot{g}_k^1(\tau))^2 + (\dot{g}_k^2(\tau))^2]\mathrm{d}\tau \le C\int_0^t \|\mathbf{g}_t(\tau)\|^2\mathrm{d}\tau, \tag{4.33}$$

it follows from (4.31), (4.32), (4.33), and from Gronwall's inequality, that

$$\frac{1}{n}\sum_{k=1}^{n-1}(\dot{p}_k^2(t) + \dot{q}_k^2(t) + \dot{w}_k^2(t) + \dot{z}_k^2(t)) \le C, \quad \forall\, t \in [0,T]. \tag{4.34}$$

Owing to (4.15), (4.16), and (4.27), the above estimate implies

$$\frac{1}{n}\sum_{k=1}^{n-1}[(\mathrm{d}_k\mathbf{p})^2(t) + (\mathrm{d}_k\mathbf{q})^2(t)] \le C, \quad \forall\, t \in [0,T]. \tag{4.35}$$

Besides, (4.20), (4.34) and Proposition 3.3 yield that

$$\frac{1}{n}\sum_{k=1}^{n-1}(\dot{r}_k^2(t) + \dot{s}_k^2(t)) \le C, \quad \forall\, t \in [0,T]. \tag{4.36}$$

Moreover, (4.13), (4.14), (4.28), (4.34), and **(H1)** imply that

$$\frac{1}{n}\sum_{k=1}^{n-1}[(\mathrm{d}_{k+1}\mathbf{r})^2(t) + (\mathrm{d}_{k+1}\mathbf{s})^2(t)] \le C, \quad \forall\, t \in [0,T]. \tag{4.37}$$

4.2 Passage to the limit as $n \to \infty$

First, we introduce the approximations of p, q, r, s, w, z. For $k = 1, ..., n$, $i = 1, 2$, $t \in [0, T]$, $x \in ((k-1)/n, k/n]$, we define

$$\begin{array}{lll} \bar{p}^{(n)}(x,t) = p_k(t), & \underline{p}^{(n)}(x,t) = p_{k-1}(t), & \underline{q}^{(n)}(x,t) = q_{k-1}(t), \\ \bar{w}^{(n)}(x,t) = w_k(t), & \bar{z}^{(n)}(x,t) = z_k(t), & \underline{g}^{(n)i}(x,t) = g^i_{k-1}(t), \\ \bar{r}^{(n)}(x,t) = r_k(t), & \bar{s}^{(n)}(x,t) = s_k(t), & \underline{s}^{(n)}(x,t) = s_{k-1}(t), \end{array}$$

as well as the interpolates

$$\begin{aligned} p^{(n)}(x,t) &= p_k(t) + \left(x - \frac{k}{n}\right) \mathrm{d}_k \mathbf{p}(t), \quad q^{(n)}(x,t) = q_k(t) + \left(x - \frac{k}{n}\right) \mathrm{d}_k \mathbf{q}(t), \\ r^{(n)}(x,t) &= r_k(t) + \left(x - \frac{k}{n}\right) \mathrm{d}_k \mathbf{r}(t), \quad s^{(n)}(x,t) = s_k(t) + \left(x - \frac{k}{n}\right) \mathrm{d}_k \mathbf{s}(t), \end{aligned}$$

and extend the above functions continuously to $x = 0$. We note that, in view of Remark 4.4, the definitions are meaningful.

For a.e. $(x,t) \in Q_T$, we have that

$$\underline{p}^{(n)}_t(t) = r^{(n)}_x(t) + \underline{s}^{(n)}(t) + \underline{g}^{(n)1}(t), \tag{4.38}$$

$$\underline{q}^{(n)}_t(t) = s^{(n)}_x(t) + \underline{g}^{(n)2}(t), \tag{4.39}$$

$$\bar{w}^{(n)}_t(t) = p^{(n)}_x(t), \tag{4.40}$$

$$\bar{z}^{(n)}_t(t) = q^{(n)}_x(t) - \bar{p}^{(n)}(t), \tag{4.41}$$

and

$$\begin{pmatrix} \bar{r}^{(n)} \\ \bar{s}^{(n)} \end{pmatrix} = \mathbf{F}\left[\begin{pmatrix} \bar{w}^{(n)} \\ \bar{z}^{(n)} \end{pmatrix}\right] + \begin{pmatrix} \bar{w}^{(n)} \\ \bar{z}^{(n)} \end{pmatrix}. \tag{4.42}$$

Using the estimates (4.27)–(4.28) and (4.34)–(4.37), we now derive bounds for $p^{(n)}$, $q^{(n)}$, $r^{(n)}$, $s^{(n)}$, $\bar{w}^{(n)}$, $\bar{z}^{(n)}$ independent of n, which will enable us to pass to the limit as $n \to \infty$. Taking into account the identities (4.21), we have

$$\begin{aligned} \|p^{(n)}\|^2 &= \sum_{k=1}^{n-1} \int_{(k-1)/n}^{k/n} |p_k + (nx - k)(p_k - p_{k-1})|^2 \, \mathrm{d}x + \frac{1}{n} p_{n-1}^2 \ \leq \ \frac{4}{n} \sum_{k=1}^{n-1} p_k^2 \,, \\ \|q^{(n)}\|^2 &= \sum_{k=1}^{n-1} \int_{(k-1)/n}^{k/n} |q_k + (nx - k)(q_k - q_{k-1})|^2 \, \mathrm{d}x \\ &\quad + \int_{(n-1)/n}^{1} \left| q_{n-1} + \left(x - \frac{n-1}{n}\right) p_{n-1} \right|^2 \mathrm{d}x \\ &\leq \frac{4}{n} \sum_{k=1}^{n-1} q_k^2 + \frac{2}{n^3} p_{n-1}^2 \,, \end{aligned}$$

$$
\begin{aligned}
\|s^{(n)}\|^2 &= \sum_{k=2}^{n}\int_{(k-1)/n}^{k/n} |s_k + (nx-k)(s_k - s_{k-1})|^2 \,\mathrm{d}x + \int_0^{1/n} \left| s_1 - \Big(x - \frac{1}{n}\Big) g_0^2 \right|^2 \,\mathrm{d}x \\
&\le \frac{4}{n}\sum_{k=1}^{n-1} s_k^2 + \frac{2}{n^3}(g_0^2)^2 , \\
\|r^{(n)}\|^2 &= \sum_{k=2}^{n}\int_{(k-1)/n}^{k/n} |r_k + (nx-k)(r_k - r_{k-1})|^2 \,\mathrm{d}x \\
&\quad + \int_0^{1/n} \left| r_1 - \Big(\frac{1}{n} - x\Big)\Big(s_1 + \frac{1}{n} g_0^2 + g_0^1\Big) \right|^2 \,\mathrm{d}x \\
&\le \frac{4}{n}\sum_{k=1}^{n-1} r_k^2 + \frac{3}{n^3}(g_0^1)^2 + \frac{3}{n^5}(g_0^2)^2 + \frac{3}{n^3} s_1^2 .
\end{aligned}
$$

Hence,

$$
\max_{0\le t\le T} \left(\|p^{(n)}(t)\|^2 + \|q^{(n)}(t)\|^2 + \|r^{(n)}(t)\|^2 + \|s^{(n)}(t)\|^2 \right) \le C \,. \tag{4.43}
$$

Besides, we easily deduce from (4.27) that

$$
\max_{0\le t\le T} \left(\|\bar{w}^{(n)}(t)\|^2 + \|\bar{z}^{(n)}(t)\|^2 \right) \le C \,. \tag{4.44}
$$

In the same way, we can prove that

$$
\max_{0\le t\le T} \Big(\|p_t^{(n)}(t)\|^2 + \|q_t^{(n)}(t)\|^2 + \|r_t^{(n)}(t)\|^2 + \|s_t^{(n)}(t)\|^2 + \|\bar{w}_t^{(n)}(t)\|^2 + \|\bar{z}_t^{(n)}(t)\|^2 \Big) \le C \,. \tag{4.45}
$$

We easily evaluate the norms

$$
\begin{aligned}
\|p_x^{(n)}\|^2 &= \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} |\mathrm{d}_k \mathbf{p}|^2 \mathrm{d}x \;=\; \frac{1}{n}\sum_{k=1}^{n-1} |\mathrm{d}_k \mathbf{p}|^2 , \\
\|q_x^{(n)}\|^2 &= \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} |\mathrm{d}_k \mathbf{q}|^2 \mathrm{d}x \;=\; \frac{1}{n}\sum_{k=1}^{n-1} |\mathrm{d}_k \mathbf{q}|^2 + \frac{1}{n} p_{n-1}^2 , \\
\|s_x^{(n)}\|^2 &= \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} |\mathrm{d}_k \mathbf{s}|^2 \mathrm{d}x \;=\; \frac{1}{n}\sum_{k=1}^{n-1} |\mathrm{d}_{k+1} \mathbf{s}|^2 + \frac{1}{n}|g_0^2|^2 , \\
\|r_x^{(n)}\|^2 &= \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} |\mathrm{d}_k \mathbf{r}|^2 \mathrm{d}x \;=\; \frac{1}{n}\sum_{k=1}^{n-1} |\mathrm{d}_{k+1} \mathbf{r}|^2 + \frac{1}{n}\left| \frac{1}{n} g_0^2 + g_0^1 + s_1 \right|^2 ,
\end{aligned}
$$

which, together with (4.35), (4.37) and Remark 4.4, yield that

$$
\max_{0\le t\le T} \left(\|p_x^{(n)}(t)\|^2 + \|q_x^{(n)}(t) + \|r_x^{(n)}(t)\|^2 + \|s_x^{(n)}(t)\|^2 \right) \le C. \tag{4.46}
$$

By definition, for $x \in ((k-1)/n, k/n)$, $k = 1, ..., n$, we have

$$|\underline{p}^{(n)}(x,t) - p^{(n)}(x,t)| \le \frac{1}{n} |\mathrm{d}_k \mathbf{p}(t)|.$$

Hence, by the above argument,

$$\sup_{(x,t)\in Q_T} |\underline{p}^{(n)}(x,t) - p^{(n)}(x,t)|^2 \le \frac{1}{n^2} \sum_{k=1}^{n} |\mathrm{d}_k \mathbf{p}(t)|^2 \ \le \ \frac{C}{n}. \tag{4.47}$$

Therefore, $p^{(n)}$ and $\underline{p}^{(n)}$ have the same limit as $n \to \infty$, provided that the limit exists. By the same argument, we have similar results for $\bar{p}^{(n)} - p^{(n)}$, $\underline{q}^{(n)} - q^{(n)}$, $\bar{r}^{(n)} - r^{(n)}$, $\bar{s}^{(n)} - s^{(n)}$, and $\underline{s}^{(n)} - s^{(n)}$.

Combining the above estimates, and selecting a suitable subsequence of $n \to \infty$, we see that there exist functions p, q, w, z, r, s in appropriate Sobolev spaces (cf. (4.11), (4.12)) such that

$$\left.\begin{array}{llll} p_x^{(n)} \to p_x, & q_x^{(n)} \to q_x, & p_t^{(n)} \to p_t, & q_t^{(n)} \to q_t, \\ \bar{w}^{(n)} \to w, & \bar{z}^{(n)} \to z, & \bar{w}_t^{(n)} \to w_t, & \bar{z}_t^{(n)} \to z_t, \\ r_t^{(n)} \to r_t, & s_t^{(n)} \to s_t, & r_x^{(n)} \to r_x, & s_x^{(n)} \to s_x, \\ \underline{p}_t^{(n)} \to p_t, & \underline{q}_t^{(n)} \to q_t, & \bar{r}_t^{(n)} \to r_t, & \bar{s}_t^{(n)} \to s_t, \end{array}\right\} \text{weakly-star in } L^\infty(0,T;L^2(0,1)). \tag{4.48}$$

Then, by compact embedding and (4.47), we have

$$\left.\begin{array}{lllll} p^{(n)} \to p, & q^{(n)} \to q, & \underline{p}^{(n)} \to p, & \bar{p}^{(n)} \to p, & \underline{q}^{(n)} \to q, \\ r^{(n)} \to r, & s^{(n)} \to s, & \bar{r}^{(n)} \to r, & \bar{s}^{(n)} \to s, & \underline{s}^{(n)} \to s, \end{array}\right\} \text{strongly in } L^\infty(Q_T). \tag{4.49}$$

The boundary conditions are preserved in the limit, and the convergence of the initial conditions as $n \to \infty$ easily follows from (4.49). From the definition of $\underline{\mathbf{g}}^{(n)}$ and **(H1)**, it is easy to see that $\underline{\mathbf{g}}^{(n)} \to \mathbf{g}$ strongly in $C([0,T]; L^2(0,1;\mathbb{R}^2))$. Hence, we may pass to the limit in (4.38)–(4.41) to obtain (4.5).

To finish the existence proof, it remains to verify that

$$\begin{pmatrix} r \\ s \end{pmatrix} = \mathbf{F}\left[\begin{pmatrix} w \\ z \end{pmatrix}\right] + \begin{pmatrix} w \\ z \end{pmatrix} := \mathbf{G}\left[\begin{pmatrix} w \\ z \end{pmatrix}\right]. \tag{4.50}$$

The proof follows from Minty's trick as, e.g., in [8], based on the monotonicity of the Prandtl–Ishlinskii operator $\mathbf{F}$ (cf. Proposition 3.3(ii)). To this end, we take an arbitrary vector function $\boldsymbol{\xi} \in C(\overline{Q_T}; \mathbb{R}^2)$ and define $\tilde{\boldsymbol{\xi}}(x,t) = \int_0^t \boldsymbol{\xi}(x,\tau)\,\mathrm{d}\tau$. For all $\delta > 0$ and $n \in \mathbb{N}$, we deduce

$$\int_0^T \int_0^1 \left\{ \mathbf{G}\left[\begin{pmatrix} \bar{w}^{(n)} \\ \bar{z}^{(n)} \end{pmatrix}\right] - \mathbf{G}\left[\begin{pmatrix} w \\ z \end{pmatrix} + \delta\tilde{\boldsymbol{\xi}}\right] \right\} \cdot \left[\begin{pmatrix} \bar{w}^{(n)} \\ \bar{z}^{(n)} \end{pmatrix}_t - \begin{pmatrix} w \\ z \end{pmatrix}_t - \delta\boldsymbol{\xi}\right] \mathrm{d}x\,\mathrm{d}t$$

$$\geq \ -\frac{1}{2}\int_0^1\int_0^h\left|\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\begin{pmatrix}\bar{w}^{(n)}\\ \bar{z}^{(n)}\end{pmatrix}\right]-\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\begin{pmatrix}w\\ z\end{pmatrix}+\delta\mathbf{B}_q\tilde{\boldsymbol{\xi}}\right]\right|^2(x,0)\,\mathrm{d}q\,\mathrm{d}x$$
$$-\frac{1}{2}\left\|\begin{pmatrix}\bar{w}^{(n)}\\ \bar{z}^{(n)}\end{pmatrix}(x,0)-\begin{pmatrix}w\\ z\end{pmatrix}(x,0)\right\|^2$$
$$\geq \ -C\left\|\begin{pmatrix}\bar{w}^{(n)}\\ \bar{z}^{(n)}\end{pmatrix}(x,0)-\begin{pmatrix}w\\ z\end{pmatrix}(x,0)\right\|^2, \tag{4.51}$$

where we have used the fact that the initial value map (3.2) of $\mathcal{S}_{\mathcal{K}}$ is Lipschitz continuous. Passing to the limit as $n\to\infty$ in (4.51), we infer from (4.48) and (4.49) that

$$\int_0^T\int_0^1\left\{\begin{pmatrix}r\\ s\end{pmatrix}-\mathbf{G}\left[\begin{pmatrix}w\\ z\end{pmatrix}+\delta\tilde{\boldsymbol{\xi}}\right]\right\}\cdot\boldsymbol{\xi}\,\mathrm{d}x\,\mathrm{d}t\leq 0,\quad\forall\,\boldsymbol{\xi}\in C(\overline{Q_T};\mathbb{R}^2). \tag{4.52}$$

Besides, owing to Proposition 3.3(i), the mapping $\mathbf{G}$ is continuous in $C(\overline{Q_T};\mathbb{R}^2)$. Hence,

$$\lim_{\delta\to 0}\mathbf{G}\left[\begin{pmatrix}w\\ z\end{pmatrix}+\delta\tilde{\boldsymbol{\xi}}\right]=\mathbf{G}\left[\begin{pmatrix}w\\ z\end{pmatrix}\right]. \tag{4.53}$$

Since $\boldsymbol{\xi}\in C(\overline{Q_T};\mathbb{R}^2)$ in (4.52) is arbitrary, we obtain that (4.50) holds.

The proof for existence is complete.

4.3 Proof of uniqueness

The uniqueness of the solution to our problem is a consequence of the monotonicity of $\mathbf{F}$ (cf. Proposition 3.3(ii)) and of the energy estimate. Consider two solutions $(p_i,q_i,w_i,z_i)^T$, $i=1,2$, to problem (4.5)–(4.7). Then p_1-p_2 and q_1-q_2 satisfy

$$\left.\begin{aligned}(p_1-p_2)_t &= (r_1-r_2)_x+(s_1-s_2),\\ (q_1-q_2)_t &= (s_1-s_2)_x,\\ (w_1-w_2)_t &= (p_1-p_2)_x,\\ (z_1-z_2)_t &= (q_1-q_2)_x-(p_1-p_2),\end{aligned}\right\} \tag{4.54}$$

with the boundary conditions

$$(p_1-p_2)(0,t)=(q_1-q_2)(0,t)=(w_1-w_2)(1,t)=(z_1-z_2)(1,t)=0. \tag{4.55}$$

Testing the first equation by p_1-p_2, and the second equation by q_1-q_2, integrating over $(0,1)$, and using (4.55), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|p_1-p_2\|^2+\|q_1-q_2\|^2+\|w_1-w_2\|^2+\|z_1-z_2\|^2)$$

$$
\begin{aligned}
&= -\int_0^1 \mathbf{F}\left[\begin{pmatrix} w_1 - w_2 \\ z_1 - z_2 \end{pmatrix}\right] \cdot \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} w_1 - w_2 \\ z_1 - z_2 \end{pmatrix} \mathrm{d}x \\
&\geq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^h \left|\mathcal{S}_{\mathcal{K}}\left[\mathbf{B}_q\begin{pmatrix} w_1 - w_2 \\ z_1 - z_2 \end{pmatrix}\right]\right|^2 \mathrm{d}q\mathrm{d}x.
\end{aligned}
$$

If the initial data of $(p_i, q_i, w_i, z_i)^T$, $i = 1, 2$, are the same, then we are able to conclude that $(p_i, q_i, w_i, z_i)^T$, $i = 1, 2$, coincide for $t \in [0, T]$. The uniqueness of solution to problem (4.5)–(4.7) is proved.

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