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On the unitary equivalence of absolutely continuous parts of self-adjoint extensions

Dedicated to the memory of M. S. Birman

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Abstract

The classical Weyl-von Neumann theorem states that for any self-adjoint operator A in a separable Hilbert space $\mathfrak{H}$ there exists a (non-unique) Hilbert-Schmidt operator $C = C^*$ such that the perturbed operator $A + C$ has purely point spectrum. We are interesting whether this result remains valid for non-additive perturbations by considering self-adjoint extensions of a given densely defined symmetric operator A in $\mathfrak{H}$ and fixing an extension $A_0 = A_0^*$. We show that for a wide class of symmetric operators the absolutely continuous parts of extensions $\widetilde{A} = \widetilde{A}^*$ and A_0 are unitarily equivalent provided that their resolvent difference is a compact operator. Namely, we show that this is true whenever the Weyl function $M(\cdot)$ of a pair $\{A, A_0\}$ admits bounded limits $M(t) :=$ w-$\lim_{y\to+0} M(t+iy)$ for a.e. $t \in \mathbb{R}$. This result is applied to direct sums of symmetric operators and Sturm-Liouville operators with operator potentials.

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1 Introduction

Let A_0 be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let $C = C^*$ be a trace class operator in $\mathfrak{H}$, $C \in \mathfrak{S}_1(\mathfrak{H})$. Recall, that according to the Kato-Rosenblum theorem, cf. [19, 29] the absolutely continuous parts A_0^{ac} and $\widetilde{A}^{ac}$, in short the ac-parts, of the operators A_0 and $\widetilde{A} = A_0 + C$ are unitarily equivalent. In other words, the absolutely continuous spectrum, in short ac-spectrum, of A_0 and its spectral multiplicity are stable under *additive* trace class perturbations. At the same time, the Weyl-von Neumann-Kuroda theorem [1, Theorem 94.2], [30], [24] shows that the condition $C \in \mathfrak{S}_1(\mathfrak{H})$ cannot be replaced by $C \in \mathfrak{S}_p(\mathfrak{H})$ with $p \in (1, \infty]$ (where $\mathfrak{S}_p(\mathfrak{H})$ denotes the Neumann-Schatten operator ideals).

Theorem 1.1 ([20, Theorem 10.2.1 and Theorem 10.2.3]) *For any operator $A_0 = A_0^*$ in $\mathfrak{H}$ and any $p \in (1, \infty]$ there exists an operator $C = C^* \in \mathfrak{S}_p(\mathfrak{H})$ such that the perturbed operator $\widetilde{A} = A_0 + C$ has purely point spectrum. In particular, $\sigma_{ac}(A_0 + C) = \emptyset$.*

The Kato-Rosenblum theorem was generalized by Birman [4] and Birman and Krein [6] to the case of *non-additive* perturbations. Namely, it was shown that A_0^{ac} and $\widetilde{A}^{ac}$ still remain unitary equivalent whenever

$$(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_1(\mathfrak{H}).$$

In particular, this is true if $A_0 = A_0^*$ and $\widetilde{A} = \widetilde{A}^*$ are self-adjoint extensions of a symmetric operator A (in short $A_0, \widetilde{A} \in \mathrm{Ext}_A$). This rises the following Weyl-von Neumann problem for extensions: Given $p \in (1, \infty]$ and a self-adjoint extension A_0 of A. Does there exist a self-adjoint extension $\widetilde{A}$ of A such that $\widetilde{A}$ has purely point spectrum and the difference $(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1}$ belongs to $\mathfrak{S}_p(\mathfrak{H})$? To the best of our knowledge this problem was not investigated.

In the present paper we show that the Weyl-von Neumann theorem for extensions becomes false in general. We show that under an additional assumption on the symmetric operator A the ac-part of a certain extension $A_0 = A_0^*$ is unitarily equivalent to the ac-part of any extension $\widetilde{A} = \widetilde{A}^*$ of A provided that their resolvent difference is compact, that is,

$$K_{\widetilde{A}} := (\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}). \tag{1.1}$$

The additional assumption on the pair $\{A, A_0\}$ is formulated in terms of the Weyl function of the pair $\{A, A_0\}$. The latter is the main object in the boundary triplet approach to the extension theory developed in the last three decades, see [12, 13, 17] and references therein.

The core of this approach is the following abstract version of Green's formula

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \mathrm{dom}\,(A^*), \tag{1.2}$$

where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}\,(A^*) \to \mathcal{H}$ are linear mappings. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for the operator A^* if (1.2) holds and the mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : \mathrm{dom}\,(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

With a boundary triplet Π for A^* one associates in a natural way the Weyl function $M(\cdot) = M_\Pi(\cdot)$ (see Definition 2.10), which is the key object of this approach. It is an operator-valued Nevanlinna function with values in $[\mathcal{H}]$ (i.e. $R_{\mathcal{H}}$-function) and its role in the extension theory is similar to that of the classical Weyl function in the spectral theory of Sturm-Liouville operators. In particular, if A is simple, then $M(\cdot)$ determines the pair $\{A, A_0\}$, where $A_0 := A^* \upharpoonright \ker \Gamma_0$, uniquely, up to unitary equivalence. Moreover, $M(\cdot)$ is regular (holomorphic) precisely on the resolvent set $\varrho(A_0)$ of A_0 and the spectral properties of A_0 are described in terms of the limits $M(t + i0)$ at the real line (see [9]).

One of our main results (Theorem 4.3) reads now as follows.

Theorem 1.2 *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that the corresponding Weyl function $M(\cdot)$ has weak limits*

$$M(t + i0) := \text{w-}\lim_{y \downarrow 0} M(t + iy) \quad \textit{for a.e.} \quad t \in \mathbb{R}. \tag{1.3}$$

If a self-adjoint extension $\widetilde{A}$ of A satisfies condition (1.1)*, then the ac-parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A}$ and $A_0\,(= A^* \upharpoonright \ker(\Gamma_0))$ are unitarily equivalent.*

We apply this result to direct sums $A := \oplus_{n=1}^{\infty} S_n$ of symmetric operators S_n with equal and finite deficiency indices $n_{\pm}(S_n)$. Let S_{0n} be a self-adjoint extension of S_n for each $n \in \mathbb{N}$. We show that the ac-part of $A_0 := \oplus_{n=1}^{\infty} S_{0n}$ is unitarily equivalent to the ac-part of any other extension $\widetilde{A} = \widetilde{A}^* \in \operatorname{Ext}_A$ provided that condition (1.1) is satisfied and the symmetric operators S_n are unitarily equivalent to S_1 for any $n \in \mathbb{N}$.

The second part of the paper is concerned with a spectral extremal property of certain self-adjoint extensions of A described by the following definition.

Definition 1.3 (i) Let $T_j = T_j^* \in \mathcal{C}(\mathfrak{H}_j)$, $j = 1, 2$. We say that T_1 is a part of T_2 if there is an isometry V from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ such that $V T_1 V^* \subseteq T_2$.

(ii) Let $A_0 = A_0^*$ be an extension of A. We say that A_0 is ac-minimal if A_0^{ac} is a part of any self-adjoint extension $\widetilde{A}$ of A.

(iii) Let $\sigma_0 := \sigma_{ac}(A_0)$. We say that A_0 is strictly ac-minimal if for any extension $\widetilde{A} = \widetilde{A}^*$ of A the parts A_0^{ac} and $\widetilde{A}^{ac} E_{\widetilde{A}}(\sigma_0)$ are unitarily equivalent.

In particular, if A_0 is ac-minimal, then $\sigma_{ac}(\widetilde{A}) \supseteq \sigma_{ac}(A_0)$. Note that an ac-minimal extension of A is not unique. For any two ac-minimal extensions their ac-parts are unitarily equivalent.

We show (cf. Theorem 5.12) that if $n_{\pm}(S_n) < \infty$, then the ac-part A_0^{ac} of any direct sum extension $A_0 = \oplus_{n=1}^{\infty} S_{0n}$ of $A := \oplus_{n=1}^{\infty} S_n$ is ac-minimal. In particular, $\sigma_{ac}(\widetilde{A}) \supseteq \sigma_{ac}(A_0)$ for any $\widetilde{A} = \widetilde{A}^* \in \operatorname{Ext}_A$. This result looks surprising with respect to Theorem 1.1. Indeed, in this case A_0^{ac} is still a part of $\widetilde{A}^{ac}$ for any $\widetilde{A} \in \operatorname{Ext}_A$ though the resolvent difference $K_{\widetilde{A}}$ (see (1.1)) is not even compact. In other words, in this case the ac-spectrum of A_0 (but not its spectral multiplicity) remains stable under (non-additive) compact perturbations $K_{\widetilde{A}}$ though both $\sigma_{ac}(A_0)$ and its multiplicity can only increase, whenever $K_{\widetilde{A}} \notin \mathfrak{S}_{\infty}$.

Moreover, we apply our technique to minimal symmetric non-negative Sturm-Liouville operator A with an unbounded operator potential

$$(Af)(x) = -f''(x) + Tf(x). \tag{1.4}$$

We show that the Friedrichs extension A^F is ac-minimal and under a simple additional assumption is even strictly ac-minimal.

The paper is organized as follows. In Section 2 we give a short introduction into the theory of ordinary and generalized boundary triplets and the corresponding Weyl functions. In Section 3 we express the spectral multiplicity function of the ac-part $\widetilde{A}^{ac}$ of $\widetilde{A} = \widetilde{A}^* (\in \operatorname{Ext}_A)$ by means of the corresponding Weyl function. In Section 4 we apply this technique to prove Theorem 1.2 as well as to give a simple proof of the Kato-Rosenblum theorem.

In Section 5 direct sums of boundary triplets $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for operators S_n^* adjoint to symmetric operators S_n are investigated. We show that though, in general, $\Pi = \oplus_{n=1}^{\infty} \Pi_n$ is not a boundary triplet for the direct sum $A^* := \oplus_{n=1}^{\infty} S_n^*$, it is always possible to modify the triplets Π_n in such a way that a new sequence $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ of boundary triplets for S_n^* satisfies the following properties: $\widetilde{\Pi} = \oplus_{n=1}^{\infty} \widetilde{\Pi}_n$ forms a boundary triplet for A^* such that $S_{0n} := S_n^* \upharpoonright \ker(\Gamma_{0n}) = S_n^* \upharpoonright \ker(\widetilde{\Gamma}_{0n}) =: \widetilde{S}_{0n}$, $n \in \mathbb{N}$. In particular, the corresponding Weyl function $M(\cdot)$ is block-diagonal (see Theorem 5.3). Our spectral applications to direct sums are substantially based on this result. In particular, it is used in proving of Theorem 5.12 mentioned above.

Finally, in Section 6 we apply the technique (and abstract results) to operators (1.4) with bounded and unbounded operator potentials. In particular, we investigate the ac-spectrum of self-adjoint realizations of Schrödinger operator

$$\mathcal{L} = -\left(\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\right) + q(x). \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad q \in L^\infty(\mathbb{R}^n),$$

in $L^2(\mathbb{R}_+ \times \mathbb{R}^n)$, $n \geq 1$. For instance, we show that if $q(\cdot) \geq 0$ and

$$\lim_{|x|\to\infty} \int_{|x-y|\leq 1} |q(y)|dy = 0, \tag{1.5}$$

then the Dirichlet realization L^D is absolutely continuous, strictly ac-minimal and $\sigma(L^D) = \sigma_{ac}(L^D) = [0, \infty)$.

Notations In the following we consider only separable Hilbert spaces which are denoted by $\mathfrak{H}$, $\mathcal{H}$ etc. The symbols $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and $[\mathfrak{H}_1, \mathfrak{H}_2]$ stand for the set of closed densely defined linear operators and the set of bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, respectively. We set $\mathcal{C}(\mathcal{H}) := \mathcal{C}(\mathcal{H}, \mathcal{H})$ and $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. The symbols $\mathrm{dom}\,(\cdot)$, $\mathrm{ran}\,(\cdot)$, $\varrho(T)$ and $\sigma(T)$ stand for the domain, the range, the resolvent set and the spectrum of an operator $T \in \mathcal{C}(\mathcal{H})$, respectively; T^{ac} and $\sigma_{ac}(T)$ stand for the ac-part and the ac spectrum of an operator $T = T^* \in \mathcal{C}(\mathcal{H})$.

$\mathfrak{S}_p(\mathfrak{H})$, $p \in [1, \infty]$, stand for the Schatten-von Neumann ideals in $\mathfrak{H}$. Denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra of the line $\mathbb{R}$ and by $\mathcal{B}_b(\mathbb{R})$ the algebra of bounded subsets in $\mathcal{B}_b(\mathbb{R})$. The Lebesgue measure of a set $\delta \in \mathcal{B}(\mathbb{R})$ is denoted by $|\delta|$.

2 Preliminaries

2.1 Operator measures

Definition 2.1 Let $\mathcal{H}$ be a separable Hilbert space. A mapping $\Sigma(\cdot): \mathcal{B}_b(\mathbb{R}) \to [\mathcal{H}]$ is called an operator (operator-valued) measure if

(i) $\Sigma(\cdot)$ is δ-additive in the strong sense and

(ii) $\Sigma(\delta) = \Sigma(\delta)^* \geq 0$ for $\delta \in \mathcal{B}_b(\mathbb{R})$.

The operator measure $\Sigma(\cdot)$ is called bounded if it extends to the Borel algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, i.e. $\Sigma(\mathbb{R}) \in [\mathcal{H}]$. Otherwise, it is called unbounded. A bounded operator measure $\Sigma(\cdot) = E(\cdot)$ is called orthogonal if, in addition the conditions

(iii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$ for $\delta_1, \delta_2 \in \mathcal{B}(\mathbb{R})$ and $E(\mathbb{R}) = I_{\mathcal{H}}$

are satisfied.

Setting in (iii) $\delta_1 = \delta_2$, one gets that an orthogonal measure $E(\cdot)$ takes its values in the set of orthogonal projections on $\mathcal{H}$. Every orthogonal measure $E(\cdot)$ defines an operator $T = T^* = \int_{\mathbb{R}} \lambda dE(\lambda)$ in $\mathcal{H}$ with $E(\cdot)$ being its spectral measure. Conversely, by the spectral theorem, every operator $T = T^*$ in $\mathcal{H}$ admits the above representation with the orthogonal spectral measure $E =: E_T$.

By Σ^{ac}, Σ^{s}, Σ^{sc} and Σ^{pp} we denote absolutely continuous, singular, singular continuous and pure point parts of the measure Σ, respectively. The Lebesgue decomposition of Σ is given by $\Sigma = \Sigma^{ac} + \Sigma^{s} = \Sigma^{ac} + \Sigma^{sc} + \Sigma^{pp}$.

The operator measure Σ_1 is called subordinated to the operator measure Σ_2, in short $\Sigma_1 \prec \Sigma_2$, if $\Sigma_2(\delta) = 0$ yields $\Sigma_1(\delta) = 0$ for $\delta \in \mathcal{B}_b(\mathbb{R})$. If the measures Σ_1 and Σ_2 are mutually subordinated, then they are called equivalent, in short $\Sigma_1 \sim \Sigma_2$. Note, that there are always exists a scalar measure ρ defined on $\mathcal{B}_b(\mathbb{R})$ such that $\Sigma \sim \rho$, see [27, Remark 2.2]. In particular, there is always a scalar measure such that $\Sigma \prec \rho$.

Usually, with the operator-valued measure $\Sigma(\cdot)$ one associates a distribution operator-valued function $\Sigma(\cdot)$ defined by

$$\Sigma(t) = \begin{cases} \Sigma([0,t)) & t > 0 \\ 0 & t = 0 \\ -\Sigma([t,0)) & t < 0 \end{cases} \tag{2.1}$$

which is called the spectral function of Σ. Clearly, $\Sigma(\cdot)$ is strongly left continuous, $\Sigma(t-0) = \Sigma(t)$, and satisfies $\Sigma(t) = \Sigma(t)^*$, $\quad \Sigma(s) \le \Sigma(t)$, $s \le t$.

Definition 2.2 ([27, Definition 4.5]) Let Σ be an operator measure in $\mathcal{H}$ and let ρ be a scalar measure on $\mathcal{B}(\mathbb{R})$ such that $\Sigma \prec \rho$. Further, let $e = \{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. Let

$$\Sigma_{ij}(t) := \big(\Sigma(t)e_i, e_j\big), \qquad \Psi_{ij}(t) := d\Sigma_{ij}(t)/d\rho,$$
$$\Psi_n^e(t) := \big(\Psi_{ij}(t)\big)_{i,j=1}^{n}, \qquad \Psi^e(t) := \big(\Psi_{ij}(t)\big)_{i,j=1}^{\infty}.$$

We call

$$N_\Sigma^e(t) := \operatorname{rank}(\Psi^e(t)) := \sup_{n\ge 1} \operatorname{rank}(\Psi_n^e(t)) \ (\mathrm{mod}(\rho)) \tag{2.2}$$

and

$$N_\Sigma(t) := \operatorname*{ess\,sup}_{e} N_\Sigma^e(t) \ (\mathrm{mod}(\rho))$$

the multiplicity function and the total multiplicity of Σ, respectively.

By [27, Proposition 4.6] $N_\Sigma^e(\cdot)$ does not depend on the orthogonal basis e. Therefore one always has $N_\Sigma(t) := N_\Sigma^e(t)$ and one can omit the index e in (2.2).

When applying this definition to the absolutely continuous part Σ^{ac} of Σ the scalar measure ρ^{ac} can be chosen to be the Lebesgue measure $|\cdot|$ on $\mathcal{B}(\mathbb{R})$.

The concept of the multiplicity function allows one to introduce the following definitions.

Definition 2.3 Let Σ_1 and Σ_2 be two operator measures.

(i) The operator measure Σ_1 is called spectrally subordinate to the operator measure Σ_2, in short $\Sigma_1 \prec\!\!\prec \Sigma_2$, if $\Sigma_1 \prec \Sigma_2$ and $N_{\Sigma_1}(t) \le N_{\Sigma_2}(t)$ $(\mathrm{mod}(\Sigma_2))$.

(ii) The operator measures Σ_1 and Σ_2 are called spectrally equivalent, in short $\Sigma_1 \approx \Sigma_2$, if $\Sigma_1 \sim \Sigma_2$ and $N_{\Sigma_1}(t) = N_{\Sigma_2}(t)(\mathrm{mod}(\Sigma_2))$.

Crucial for us in the sequel is the following theorem.

Theorem 2.4 *Let T_j be self-adjoint operators acting in $\mathfrak{H}_j$ with corresponding spectral measures $E_{T_j}(\cdot)$, $j = 1,2$. Let $\mathcal{D} \in \mathcal{B}(\mathbb{R})$.*

(i) *$T_1 E_{T_1}(\mathcal{D})$ is a part of $T_2 E_{T_2}(\mathcal{D})$ if and only if $E_{T_1,\mathcal{D}} \prec\!\!\prec E_{T_2,\mathcal{D}}$, where $E_{T_j,\mathcal{D}}(\delta) := E_{T_j}(\delta \cap \mathcal{D})$, $j = 1,2$.*

(ii) *The parts* $T_1E_{T_1}(\mathcal{D})$ *and* $T_2E_{T_2}(\mathcal{D})$ *are unitarily equivalent if and only if* $E_{T_1,\mathcal{D}} \approx E_{T_2,\mathcal{D}}$.

The proof is immediate from [7, Theorem 7.5.1]. For $\mathcal{D} = \mathbb{R}$ Theorem 2.4 gives conditions for T_1 to be unitarily equivalent either to a part of T_2 or to T_2 itself.

2.2 R-Functions

Let $\mathcal{H}$ be a separable Hilbert space. We recall that an operator-valued function $F(\cdot)$ with values in $[\mathcal{H}]$ is called to be a Herglotz, Nevanlinna or R-function [1, 3, 17, 23], if it is holomorphic in $\mathbb{C}_+$ and its imaginary part is non-negative, i.e. $\mathrm{Im}(F(z)) := (2i)^{-1}\big(F(z) - F(z)^*\big) \geq 0,\ z \in \mathbb{C}_+$. In what follows we prefer the notion of R-function. The class of R-functions with values in $[\mathcal{H}]$ will be denoted by $(R_\mathcal{H})$. Any $(R_\mathcal{H})$-function $F(\cdot)$ admits an integral representation

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\Sigma_F, \quad z \in \mathbb{C}_+, \tag{2.3}$$

(see, for instance, [1, 3, 23]), where $C_0 = C_0^*,\ C_1 \geq 0$ and Σ_F is an operator-valued Borel measure on $\mathbb{R}$ satisfying $\int_\mathbb{R}(1+t^2)^{-1}d\Sigma_F \in [\mathcal{H}]$. The integral is understood in the strong sense.

In contrast to spectral measures of self-adjoint operators the measure Σ_F is not necessarily orthogonal. However, the operator-valued measure Σ_F is uniquely determined by the R-function $F(\cdot)$. It is called the spectral measure of $F(\cdot)$. The associated spectral function is denoted by $\Sigma_F(t)$, $t \in \mathbb{R}$, cf. (2.1).

Let us calculate $N_{\Sigma_F^{ac}}(t)$, $t \in \mathbb{R}$. For any Hilbert-Schmidt operator $D \in \mathfrak{S}_2(\mathcal{H})$ satisfying $\ker(D) = \ker(D^*) = \{0\}$ let us consider the modified $R_\mathcal{H}$-function

$$(F^D)(z) := D^*F(z)D, \quad z \in \mathbb{C}_+.$$

For $F^D(\cdot)$ the strong limit $F^D(t) := F^D(t+i0) :=$ s-$\lim_{y\to+\infty} F^D(t+iy)$ exists for a.e. $t \in \mathbb{R}$. We set

$$d_{F^D}(t) := \dim(\mathrm{ran}\,(\mathrm{Im}(F^D)(t))), \quad \text{for a.e.} \quad t \in \mathbb{R}. \tag{2.4}$$

Proposition 2.5 *Let* $F(\cdot) \in (R_\mathcal{H}),\ \ D \in \mathfrak{S}_2(\mathcal{H})$ *and* $\ker(D) = \ker(D^*) = \{0\}$. *Then* $N_{\Sigma_F^{ac}}(t) = d_{F^D}(t)$ *for a.e.* $t \in \mathbb{R}$.

Proof. It follows from (2.3) that

$$\mathrm{Im}(F(\lambda+iy)) = yC_1 + \int_{-\infty}^{\infty} \frac{y}{(t-\lambda)^2+y^2} d\Sigma_F, \qquad \lambda \in \mathbb{R}. \tag{2.5}$$

By Berezanskii-Gel'fand-Kostyuchenko theorem [3, 7] the derivative $\Psi_{D^*\Sigma_F D}(t) := \frac{d}{dt}D^*\Sigma_F(t)D$ exists for a.e. $t \in \mathbb{R}$ and the representation

$$D^*\Sigma_F^{ac}(\delta)D = \int_\delta \Psi_{D^*\Sigma_F D}(t)dt, \qquad \delta \in \mathcal{B}_b(\mathbb{R})$$

holds. Applying the Fatou theorem (see [23]) to (2.5) and using (2.4) we obtain

$$\mathrm{Im}((F^D)(\lambda)) = \pi\Psi_{D^*\Sigma_F D}(\lambda) \quad \text{for a.e.} \quad \lambda \in \mathbb{R}. \tag{2.6}$$

By [27, Corollary 4.7] $N_{\Sigma_F^{ac}}(\lambda) = \mathrm{rank}\,(\Psi_{D^*\Sigma_F D}(\lambda)) = \dim(\overline{\mathrm{ran}\,(\Psi_{D^*\Sigma_F D}(\lambda))})$ for a.e. $\lambda \in \mathbb{R}$. Finally, using (2.6) we get $N_{\Sigma_F^{ac}}(\lambda) = d_{F^D}(\lambda)$ for a.e. $\lambda \in \mathbb{R}$. □

Notice that Proposition 2.5 implies that $D_{F^D}(t)$ does not dependent on D. Assuming the existence of the limit $F(t) :=$ s-$\lim_{y\to+0} F(t+iy)$ for a.e. $t \in \mathbb{R}$, we set

$$d_F(t) := \mathrm{rank}\,(\mathrm{Im}(F(t)) = \dim(\mathrm{ran}\,(\mathrm{Im}(F(t))))$$

for a.e. $t \in \mathbb{R}$. In this case Proposition 2.5 can be modified as follows.

Corollary 2.6 *Let $F(\cdot) \in (R_{\mathcal{H}})$. If the limit $F(t) := \text{s-lim}_{y\to+0} F(t+iy)$ exists for a.e. $t \in \mathbb{R}$, then $N_{\Sigma_F^{ac}}(t) = d_F(t)$ for a.e. $t \in \mathbb{R}$.*

2.3 Boundary triplets and self-adjoint extensions

In this section we briefly recall the basic facts on boundary triplets and the corresponding Weyl functions, cf. [11, 12, 13, 17].

Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim(\ker(A^* \mp i)) \le \infty$.

Definition 2.7 ([17]) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$ are linear mappings, is called an (ordinary) boundary triplet for A^* if the "abstract Green's identity"

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \operatorname{dom}(A^*), \tag{2.7}$$

holds and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Definition 2.8 ([17]) A closed extension A' of A is called a proper extension, in short $A' \in \operatorname{Ext}_A$, if $A \subset A' \subset A^*$;

Two proper extensions A', A'' are called disjoint if $\operatorname{dom}(A') \cap \operatorname{dom}(A'') = \operatorname{dom}(A)$ and transversal if in addition $\operatorname{dom}(A') + \operatorname{dom}(A'') = \operatorname{dom}(A^*)$.

Clearly, any self-adjoint extension $\widetilde{A} = \widetilde{A}^*$ is proper, $\widetilde{A} \in \operatorname{Ext}_A$. A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists whenever $n_+(A) = n_-(A)$. Moreover, the relations $n_\pm(A) = \dim(\mathcal{H})$ and $\ker(\Gamma_0) \cap \ker(\Gamma_1) = \operatorname{dom}(A)$ are valid. Besides, $\Gamma_0, \Gamma_1 \in [\mathfrak{H}_+, \mathcal{H}]$, where $\mathfrak{H}_+$ denotes the Hilbert space obtained by equipping $\operatorname{dom}(A^*)$ with the graph norm of A^*.

With any boundary triplet Π one associates two extensions $A_j := A^* \upharpoonright \ker(\Gamma_j)$, $j \in \{0, 1\}$, which are self-adjoint in view of Proposition 2.9 below. Conversely, for any extension $A_0 = A_0^* \in \operatorname{Ext}_A$ there exists a (non-unique) boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$.

Using the concept of boundary triplets one can parameterize all proper, in particular, self-adjoint extensions of A. For this purpose denote by $\widetilde{\mathcal{C}}(\mathcal{H})$ the set of closed linear relations in $\mathcal{H}$, that is, the set of (closed) linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. The adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of a linear relation Θ in $\mathcal{H}$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is called *symmetric* if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.

The multivalued part $\operatorname{mul}(\Theta)$ of $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ is $\operatorname{mul}(\Theta) = \{h \in \mathcal{H} : \{0, h\} \in \Theta\}$. Setting $\mathcal{H}_\infty := \operatorname{mul}(\Theta)$ and $\mathcal{H}_{\rm op} := \mathcal{H}_\infty^\perp$ we get $\mathcal{H} = \mathcal{H}_{\rm op} \oplus \mathcal{H}_\infty$. This decomposition yields an orthogonal decomposition $\Theta = \Theta_{\rm op} \oplus \Theta_\infty$ where $\Theta_\infty := \{0\} \oplus \operatorname{mul}(\Theta)$ and $\Theta_{\rm op} := \{\{f, g\} \in \Theta : f \in \operatorname{dom}(\Theta), g \perp \operatorname{mul}(\Theta)\}$. For the definition of the inverse and the resolvent set of a linear relation Θ we refer to [14].

Proposition 2.9 *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$(\operatorname{Ext}_A \ni)\ \widetilde{A} \to \Gamma \operatorname{dom}(\widetilde{A}) = \{\{\Gamma_0 f, \Gamma_1 f\} : f \in \operatorname{dom}(\widetilde{A})\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.8}$$

establishes a bijective correspondence between the sets Ext_A and $\widetilde{\mathcal{C}}(\mathcal{H})$. We put $A_\Theta := \widetilde{A}$ where Θ is defined by (2.8). Moreover, the following holds:

(i) $A_\Theta = A_\Theta^*$ *if and only if* $\Theta = \Theta^*$;

(ii) *The extensions* A_Θ *and* A_0 *are disjoint if and only if* $\Theta \in \mathcal{C}(\mathcal{H})$. *In this case* (2.8) *becomes*

$$A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0);$$

(iii) *The extensions* A_Θ *and* A_0 *are transversal if and only if* $\Theta = \Theta^* \in [\mathcal{H}]$.

In particular, $A_j := A^* \upharpoonright \ker(\Gamma_j) = A_{\Theta_j}$, $j \in \{0,1\}$ where $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\}$. Hence $A_j = A_j^*$ since $\Theta_j = \Theta_j^*$. In the sequel the extension A_0 is usually regarded as a reference self-adjoint extension.

2.4 Weyl functions and γ-fields

It is well known that Weyl functions give an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [11, 12, 13] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator A with $n_+(A) = n_-(A)$. Following [11, 12, 13] we recall basic facts on Weyl functions and γ-fields associated with a boundary triplet Π.

Definition 2.10 ([11, 12]) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The functions $\gamma(\cdot) : \varrho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) : \varrho(A_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := \big(\Gamma_0 \upharpoonright \mathfrak{N}_z\big)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1\gamma(z), \qquad z \in \varrho(A_0), \tag{2.9}$$

are called the γ-*field* and the *Weyl function*, respectively, corresponding to Π.

It follows from the identity $\mathrm{dom}(A^*) = \ker(\Gamma_0) \dot{+} \mathfrak{N}_z$, $z \in \varrho(A_0)$, where $A_0 = A^* \upharpoonright \ker(\Gamma_0)$, and $\mathfrak{N}_z := \ker(A^* - z)$, that the γ-field $\gamma(\cdot)$ is well defined and takes values in $[\mathcal{H}, \mathfrak{H}]$. Since $\Gamma_1 \in [\mathfrak{H}_+, \mathcal{H}]$, it follows from (2.9) that $M(\cdot)$ is well defined too and takes values in $[\mathcal{H}]$. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\varrho(A_0)$ and satisfy the following relations (see [12])

$$\gamma(z) = \big(I + (z - \zeta)(A_0 - z)^{-1}\big)\gamma(\zeta), \qquad z, \zeta \in \varrho(A_0), \tag{2.10}$$

and

$$M(z) - M(\zeta)^* = (z - \bar{\zeta})\gamma(\zeta)^*\gamma(z), \qquad z, \zeta \in \varrho(A_0). \tag{2.11}$$

The last identity yields that $M(\cdot)$ is a $R_{\mathcal{H}}$-function, that is, $M(\cdot)$ is a $[\mathcal{H}]$-valued holomorphic function on $\mathbb{C}\backslash\mathbb{R}$ satisfying

$$M(z) = M(\overline{z})^* \qquad \text{and} \qquad \frac{\mathrm{Im}\,(M(z))}{\mathrm{Im}\,(z)} \geq 0, \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

Moreover, it follows from (2.11) that $M(\cdot)$ satisfies $0 \in \varrho(\mathrm{Im}\,(M(z)))$, $z \in \mathbb{C}\backslash\mathbb{R}$.

If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ uniquely up to unitary equivalence (see [13, 22]). Therefore $M(\cdot)$ contains (implicitly) full information on spectral properties of A_0. We recall that a symmetric operator is said to be *simple* if there is no non-trivial subspace which reduces it to a self-adjoint operator.

For a fixed $A_0 = A_0^*$ a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ satisfying $\mathrm{dom}(A_0) = \ker(\Gamma_0)$ is not unique. Let $\Pi_j = \{\mathcal{H}_j, \Gamma_0^j, \Gamma_1^j\}$, $j \in \{1,2\}$, be two such triplets. Then the corresponding Weyl functions $M_1(\cdot)$ and $M_2(\cdot)$ are related by

$$M_2(z) = R^* M_1(z) R + R_0, \tag{2.12}$$

where $R_0 = R_0^* \in [\mathcal{H}_2]$ and $R \in [\mathcal{H}_2, \mathcal{H}_1]$ is boundedly invertible.

According to Proposition 2.9 the extensions A_Θ and A_0 are not disjoint whenever $\mathrm{mul}\,(\Theta) \neq \{0\}$. Considering A_Θ and A_0 as extensions of an intermediate extension $S := A_0 \upharpoonright (\mathrm{dom}\,(A_0) \cap \mathrm{dom}\,(A_\Theta))$ we can avoid this inconvenience.

Lemma 2.11 *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*, $M(\cdot)$ *the corresponding Weyl function,* $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ *and* $\Theta = \Theta_{op} \oplus \Theta_\infty$ *its orthogonal decomposition. Further let* $S := A_0 \upharpoonright (\mathrm{dom}\,(A_0) \cap \mathrm{dom}\,(A_\Theta))$. *Then the triplet* $\widehat{\Pi} = \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, *defined by*

$$\widehat{\mathcal{H}} := \mathcal{H}_{\mathrm{op}} = \overline{\mathrm{dom}\,(\Theta)}, \quad \widehat{\Gamma}_0 := \Gamma_0 \upharpoonright \mathrm{dom}\,(S^*), \quad \widehat{\Gamma}_1 := \pi_{\mathrm{op}}\Gamma_1 \upharpoonright \mathrm{dom}\,(S^*),$$

is a boundary triplet for S^*, *where* π_{op} *is the orthogonal projection from* $\mathcal{H}$ *onto* $\mathcal{H}_{\mathrm{op}}$, $A_0 = S^* \upharpoonright \ker(\widehat{\Gamma}_0)$ *and* $A_\Theta = S_{\Theta_{\mathrm{op}}}$. *The corresponding Weyl function is*

$$\widehat{M}(z) := \pi_{\mathrm{op}} M(z) \upharpoonright \mathcal{H}_{\mathrm{op}}, \quad z \in \mathbb{C}_\pm. \tag{2.13}$$

The proof can be found in [10]. Hence without loss of generality we can very often assume that the "coordinate" $\Theta := \Gamma\widetilde{A}$ of an extension $\widetilde{A} = A_\Theta = A_\Theta^* \in \mathrm{Ext}_A$ corresponds to the graph of a self-adjoint operator.

In what follows, without loss of generality, we always assume that the closed symmetric A is simple and, due to Lemma 2.11, the "coordinate" Θ of the extension $A_\Theta = A_\Theta^* \in \mathrm{Ext}_A$ is the graph of a self-adjoint operator.

2.5 Krein type formula for resolvents and comparability

With any boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* and any proper (not necessarily self-adjoint) extension $A_\Theta \in \mathrm{Ext}_A$ it is naturally associated the following (unique) Krein type formula (cf. [11, 12, 13])

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma(\overline{z})^*, \quad z \in \varrho(A_0) \cap \varrho(A_\Theta). \tag{2.14}$$

Formula (2.14) is a generalization of the known Krein formula for resolvents. We note also, that all objects in (2.14) are expressed in terms of the boundary triplet Π (cf. [11, 12, 13]). In other words, (2.14) gives a relation between Krein-type formula for canonical resolvents and the theory of abstract boundary value problems (framework of boundary triplets).

The following result is deduced from formula (2.14) (cf. [12, Theorem 2]).

Proposition 2.12 *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*, $\Theta_i = \Theta_i^* \in \widetilde{\mathcal{C}}(\mathcal{H})$, $i \in \{1, 2\}$. *Then for any Schatten-von Neumann ideal* $\mathfrak{S}_p$, $p \in (0, \infty]$, *and any* $z \in \mathbb{C} \setminus \mathbb{R}$ *the following equivalence holds*

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longleftrightarrow \big(\Theta_1 - z\big)^{-1} - \big(\Theta_2 - z\big)^{-1} \in \mathfrak{S}_p(\mathcal{H})$$

In particular, $(A_{\Theta_1} - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longleftrightarrow \big(\Theta_1 - i\big)^{-1} \in \mathfrak{S}_p(\mathcal{H})$.

If in addition $\Theta_1, \Theta_2 \in [\mathcal{H}]$, *then for any* $p \in (0, \infty]$ *the equivalence holds*

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longleftrightarrow \Theta_1 - \Theta_2 \in \mathfrak{S}_p(\mathcal{H}).$$

2.6 Generalized boundary triplets and proper extensions

In applications the concept of boundary triplets is too restrictive. Here we recall some facts on generalized boundary triplets following [13].

Definition 2.13 ([13, Definition 6.1]) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *generalized boundary triplet for* A^* if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_j : \mathrm{dom}\,(\Gamma_j) \to \mathcal{H}$, $j = 0, 1$ are linear mappings such that $\mathrm{dom}(\Gamma) := \mathrm{dom}(\Gamma_0) \cap \mathrm{dom}(\Gamma_1)$ is a core for A^*, Γ_0 is surjective, $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ is self-adjoint and the following Green's formula holds

$$(A_* f, g) - (f, A_* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \mathrm{dom}(A_*), \tag{2.15}$$

where $A_* := A^* \upharpoonright \mathrm{dom}(\Gamma)$.

By definition, $A_* := A^* \upharpoonright \mathrm{dom}(\Gamma)$ and $A_* \subseteq A^* = \overline{A_*}$ and $(A_*)^* = A$. Clearly, every ordinary boundary triplet is a generalized boundary triplet.

Lemma 2.14 ([13, Proposition 6.1]) *Let A be a densely defined closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*. Then the following assertions are true:*

(i) $\mathfrak{N}_z^* := \mathrm{dom}(A_*) \cap \mathbb{N}_z$ *is dense in* $\mathfrak{N}_z$ *and* $\mathrm{dom}(A_*) = \mathrm{dom}(A_0) + \mathfrak{N}_z^*$*;*

(ii) $\overline{\Gamma_1 \mathrm{dom}(A_0)} = \mathcal{H}$*;*

(iii) $\ker(\Gamma) = \mathrm{dom}(A)$ *and* $\overline{\mathrm{ran}(\Gamma)} = \mathcal{H} \oplus \mathcal{H}$.

Lemma 2.15 *Let A be a densely defined closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*. Then the mapping $\Gamma = \{\Gamma_0, \Gamma_1\}^\top$ is closable and $\overline{\Gamma} \in \mathcal{C}(\mathfrak{H}_+, \mathcal{H})$.*

Proof. The Green's formula can be rewritten as $(A_* f, g) - (f, A_* g) = (J\Gamma f, \Gamma g)$ where $\Gamma := (\Gamma_0, \Gamma_1)^\top$ and $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Let $f_n \in \mathrm{dom}(\Gamma_0) \cap \mathrm{dom}(\Gamma_1) = \mathrm{dom}(A_*)$, $\|f_n\|_{\mathfrak{H}_+} \to 0$ and $\Gamma f_n = \{\Gamma_0 f_n, \Gamma_1 f_n\} \to \{\varphi, \psi\}$ as $n \to \infty$. Hence

$$0 = \lim_{n\to\infty} [(A_* f_n, g) - (f_n, A_* g)] = (J f_\infty, \Gamma g), \quad \text{where} \quad f_\infty := \{\varphi, \psi\}^\top.$$

Since $\mathrm{ran}(\Gamma)$ is dense in $\mathcal{H} \oplus \mathcal{H}$ one has $J f_\infty = 0$. Thus, $\varphi = \psi = 0$ and Γ is closable. □

For any generalized boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ we set $A_j := A^* \upharpoonright \ker(\Gamma_j)$, $j \in \{0, 1\}$. The extensions A_0 and A_1 are disjoint but not necessarily transversal. The latter holds if and only if Π is an ordinary boundary triplet. In general, the extension A_1 is only essentially self-adjoint.

Starting with Definition 2.13, one easily extends the definitions of γ-field and Weyl function to the case of a generalized boundary triplet Π by analogy with Definition 2.10 (cf. [13, Definition 6.2]).

Definition 2.16 *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*. Then the operator valued functions $\gamma(\cdot)$ and $M(\cdot)$ defined by*

$$\gamma(z) := \left(\Gamma_0 \upharpoonright \mathfrak{N}_z^*\right)^{-1} : \ \mathcal{H} \to \mathfrak{N}_z \ \ \textit{and} \ \ M(z) := \Gamma_1 \gamma(z), \ \ z \in \varrho(A_0), \tag{2.16}$$

are called the (generalized) γ-field *and the* Weyl function *associated with the generalized boundary triplet* Π, *respectively.*

It follows from Lemma 2.14(i) that $\gamma(\cdot)$ takes values in $[\mathcal{H}, \mathfrak{H}]$, $\mathrm{ran}(\gamma(z)) = \mathfrak{N}_z^* := \mathrm{dom}(A_*) \cap \mathfrak{N}_z$ and it satisfies the identity similar to that of (2.10) which shows that $\gamma(z)$ is a holomorphic operator valued function on $\varrho(A_0)$.

Further, one has $\mathrm{dom}(M(z)) = \mathcal{H}$ since $\mathrm{ran}\,\gamma(z) \subset \mathrm{dom}(\Gamma_1)$, $z \in \varrho(A_0)$. By (2.16) $M(z)$ is closable since $\gamma(z)$ is bounded and Γ_1 is closable, by Lemma 2.15. Hence, by the closed graph theorem $M(\cdot)$ takes values in $[\mathcal{H}]$. Moreover, it is holomorphic on $\varrho(A_0)$, because so is $\gamma(\cdot)$, and satisfies the relation (2.11). It follows that $\ker(\mathrm{Im}\,M(z)) = \{0\}$, $z \in \mathbb{C}_+$, though the stronger condition $0 \in \varrho(\mathrm{Im}\,M(i))(\Longleftrightarrow \mathrm{ran}(\gamma(i)) = \mathfrak{N}_i)$ is satisfied if and only if Π is an ordinary boundary triplet (in the sense of Definition 2.7).

In the sequel we need the following simple but useful statement.

Proposition 2.17 *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function, $B = B^* \in \mathcal{C}(\mathcal{H})$ and $A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$. Let $\Gamma_1^B := \Gamma_0$ and $\Gamma_0^B := B\Gamma_0 - \Gamma_1$. Then*

(i) *$\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ is a generalized boundary triplet for A^* such that it holds* $\mathrm{dom}\,(A_*) := \mathrm{dom}\,(\Gamma) := \mathrm{dom}\,(A_0) + \mathrm{dom}\,(A_B) \subseteq \mathrm{dom}\,(A^*)$, *$A_*^* = A$;*

(ii) *the corresponding (generalized) Weyl function $M_B(\cdot)$ is*

$$M_B(z) = (B - M(z))^{-1}, \qquad z \in \mathbb{C}_\pm;$$

(iii) *Π_B is an (ordinary) boundary triplet if and only if $B = B^* \in [\mathcal{H}]$. In this case $M_B(\cdot)$ is an ordinary Weyl function in the sense of Definition 2.7.*

Note, an analogon of Proposition 2.9 does not hold for generalized boundary triplets. Nevertheless, since the corresponding Weyl function determines the pair $\{A, A_0\}$ uniquely, up to unitary equivalence, it is possible to describe the spectral properties of A_0 in terms of the (generalized) Weyl function $M(\cdot)$.

3 Weyl function and spectral multiplicity

Throughout of this section A is a densely defined simple closed symmetric operator in $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for A^*, and let $M(\cdot)$ be the corresponding generalized Weyl function. Since $M(\cdot) \in (R_{\mathcal{H}})$ it admits representation (2.3). Since A is densely defined (see [13, 26]), one gets $C_1 = 0$, i.e.

$$M(z) = C_0 + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma_M.$$

Proposition 3.1 *Let A be a densely defined, simple closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for $A_* (\subseteq A^*)$, $A_*^* = A$, and let $M(\cdot)$ be the corresponding Weyl function. If E_{A_0} is the spectral measure of $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, then $\Sigma_M \approx E_{A_0}$ and $\Sigma_M^{ac} \approx E_{A_0}^{ac}$.*

Proof. Alongside $\Sigma_M(\cdot)$ we introduce the bounded operator measure $\Sigma_M^0(\cdot)$,

$$\Sigma_M^0(\delta) = \int_\delta \frac{1}{1 + t^2}\, d\Sigma_M, \qquad \delta \in \mathcal{B}_b(\mathbb{R}).$$

Clearly, $\Sigma_M^0(\cdot) \approx \Sigma_M(\cdot)$. According to [2, formula (2.16)] one has

$$\Sigma_M^0(\delta) = \gamma(i)^* E_{A_0}(\delta) \gamma(i), \qquad \delta \in \mathcal{B}(\mathbb{R}), \tag{3.1}$$

where $\gamma(\cdot)$ is the generalized γ-field of Π. Note, that though formula (3.1) is proved in [2] for ordinary boundary triplets, the proof remains valid for generalized boundary triplets. Due to the simplicity of A one has

$$\mathrm{span}\left\{(A_0 - z)^{-1} \mathrm{ran}\,(\gamma(i)) : \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-\right\} = \mathfrak{H}.$$

Hence the subspace $\mathfrak{N}_i := \overline{\mathfrak{N}_i^*}$, where $\mathfrak{N}_i^* := \mathrm{ran}\,(\gamma(i))$ is cyclic for A_0. Next, let P_i be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{N}_i$. We set $\widetilde{\Sigma}_M^0(\cdot) := P_i E_{A_0}(\cdot) \upharpoonright \mathfrak{N}_i$.

Clearly, $\widetilde{\Sigma}_M^0(\cdot)$ is an operator measure. Since the linear manifold $\mathfrak{N}_i^*$ is cyclic for A_0, one gets from [27, Theorem 4.15] that the measures $\widetilde{\Sigma}_M^0$ and E_{A_0} are spectrally equivalent.

Note that $\Sigma^0_M(\cdot) = \gamma(i)^*\widetilde{\Sigma}^0_M(\cdot)\gamma(i)$. Since $\mathrm{ran}\,(\gamma(i))$ is dense in $\mathfrak{N}_i$, the latter yields $\Sigma^0_M \sim \widetilde{\Sigma}^0_M$. Let $D \in \mathfrak{S}_2(\mathcal{H})$ and $\ker(D) = \ker(D^*) = \{0\}$. We set

$$\Psi_{D^*\Sigma^0_M D}(t) := \frac{dD^*\Sigma^0_M(t)D}{d\rho(t)} \quad \text{and} \quad \Psi_{\widetilde{D}^*\widetilde{\Sigma}^0_M \widetilde{D}}(t) := \frac{d\widetilde{D}^*\widetilde{\Sigma}^0_M(t)\widetilde{D}}{d\rho(t)}$$

where ρ is a scalar measure such that $\widetilde{\Sigma}^0_M \sim \rho$ and $\widetilde{D} := \gamma(i)D : \mathcal{H} \longrightarrow \mathfrak{N}_i$. We note that $\ker(\widetilde{D}) = \ker(\widetilde{D}^*) = \{0\}$. By [27, Corollary 4.7] we have

$$N_{\Sigma^0_M}(t) = \mathrm{rank}\,(\Psi_{D^*\Sigma^0_M D}(t)) \quad \text{and} \quad N_{\widetilde{\Sigma}^0_M}(t) = \mathrm{rank}\,(\Psi_{\widetilde{D}^*\widetilde{\Sigma}^0_M \widetilde{D}}(t))$$

for a.e. $t \in \mathbb{R}$ (mod(ρ)). Since $\Psi_{D^*\Sigma^0_M D}(t) = \Psi_{\widetilde{D}^*\widetilde{\Sigma}^0_M \widetilde{D}}(t)$ for a.e. $t \in \mathbb{R}$ (mod(ρ)) we get $N_{\Sigma^0_M}(t) = N_{\widetilde{\Sigma}^0_M}(t)$ for a.e. $t \in \mathbb{R}$ (mod(ρ)). Hence $\widetilde{\Sigma}^0_M$ and Σ^0_M are spectrally equivalent. Since $\widetilde{\Sigma}^0_M$ and E_{A_0} are spectrally equivalent the measures Σ^0_M and E_{A_0} are spectrally equivalent. This proves the first statement.

The second statement follows from the equality $\Sigma^{0,ac}_M(\delta) = \gamma(i)^* E^{ac}_{A_0}(\delta)\gamma(i)$, $\delta \in \mathcal{B}(\mathbb{R})$ where $\Sigma^{0,ac}_M$ is the absolutely continuous part of Σ^0_M. □

The proof of Proposition 3.1 leads to the following computing procedure for $N_{\Sigma^{ac}_M}(t)$: choosing $D \in \mathfrak{S}_2(\mathcal{H})$ such that $\ker(D) = \ker(D^*) = \{0\}$ we introduce the sandwiched Weyl function $M^D(\cdot)$,

$$(M^D)(z) := D^*M(z)D, \quad z \in \mathbb{C}_+.$$

It turns out that the limit $(M^D)(t) := \text{s-lim}_{y\to+0} M^D(t+iy)$ exists for a.e. $t \in \mathbb{R}$. We define in accordance with (2.13) the function $d_{M^D}(\cdot) : \mathbb{R} \to \mathbb{N} \cup \{\infty\}$,

$$d_{M^D}(t) := \mathrm{rank}\,(\mathrm{Im}(M^D(t))) = \dim(\mathrm{ran}\,(\mathrm{Im}(M^D(t))))$$

which is well-defined for a.e. $t \in \mathbb{R}$.

For a measurable non-negative function $\xi : \mathbb{R} \longrightarrow \mathbb{R}_+$ defined for a.e. $t \in \mathbb{R}$ we introduce its support $\mathrm{supp}\,(\xi) := \{t \in \mathbb{R} : \ \xi(t) > 0\}$. By $\mathrm{cl}_{ac}(\cdot)$ we denote the absolutely continuous closure of a Borel set of $\mathbb{R}$., cf. Appendix.

Proposition 3.2 *Let A be as in Proposition 3.1, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for $A_*(\subseteq A^*)$, $A^*_* = A$, and let $M(\cdot)$ be the corresponding Weyl function. Further, let $E_{A_0}(\cdot)$ be the spectral measure of $A_0 = A_* \upharpoonright \ker(\Gamma_0) = A^*_0$. If $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfies $\ker(D) = \ker(D^*) = \{0\}$, then $N_{E^{ac}_{A_0}}(t) = d_{M^D}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\mathrm{supp}\,(d_{M^D}))$.*

If, in addition, the limit $M(t) := \text{s-lim}_{y\to+0} M(t+iy)$ exists for a.e. $t \in \mathbb{R}$, then $N_{E^{ac}_{A_0}}(t) = d_M(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\mathrm{supp}\,(d_M))$.

Proof. The relation $N_{E^{ac}_{A_0}}(t) = d_{M^D}(t)$ follows from Theorem 2.5 and Theorem 3.1. Further, let $\{g_k\}^N_{k=1}$, $1 \le N \le \infty$, be a total set in $\mathcal{H}$. We set $h_k := Dg_k$. One easily verifies that $\{h_n\}^N_{n=1}$ is a total set. We set $M_{h_n}(z) := (M(z)h_n, h_n)$, $z \in \mathbb{C}_+$. Clearly, $M_{h_n}(z)$ is R-function for every $n \in \{1, 2, \ldots, N\}$ and

$$M_{h_n}(t) := \lim_{y\to+0} M_{h_n}(t+iy) = (M(t)h_n, h_n)$$

exists for a.e. $t \in \mathbb{R}$. Set

$$\Omega_{ac}(M_{h_n}) := \{t \in \mathbb{R} : \ 0 < \mathrm{Im}(M_{h_n}(t)) < \infty\}.$$

Combining [9, Proposition 4.1] with Lemma A.3 we obtain

$$\sigma_{ac}(A_0) = \overline{\bigcup_{k=1}^N \mathrm{cl}_{ac}(\Omega_{ac}(M_{h_n}))} = \mathrm{cl}_{ac}\left(\bigcup_{k=1}^N \Omega_{ac}(M_{h_n})\right). \tag{3.2}$$

If $t \in \operatorname{supp}(d_{M^D})$, then $\operatorname{Im}((M^D)(t)) \neq 0$. Hence $t \in \Omega_{ac}(M_{h_n})$ for some $n \in \{1, 2, \ldots, N\}$. Therefore $\operatorname{supp}(d_{M^D}) \subseteq \bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n})$ which yields

$$\mathrm{cl}_{ac}(\operatorname{supp}(d_{M^D})) \subseteq \mathrm{cl}_{ac}\left(\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n})\right). \tag{3.3}$$

Conversely, if $t \in \Omega_{ac}(M_{h_n}) \cap \mathcal{E}_{M^D}$, where $\mathcal{E}_{M^D} := \{t \in \mathbb{R} : \ \exists\ (M^D)(t)\}$, for some n, then $0 < d_{M^D}(t)$. Hence $\Omega_{ac}(M_{h_n}) \cap \mathcal{E}_{M^D} \subseteq \operatorname{supp}(d_{M^D})$ which yields $\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n}) \cap\ \mathcal{E}_{M^D} \subseteq \operatorname{supp}(d_{M^D})$. Hence

$$\mathrm{cl}_{ac}\left(\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n}) \cap\ \mathcal{E}_M\right) = \mathrm{cl}_{ac}\left(\bigcup_{k=1}^{N} \Omega_{ac}(M_{h_n})\right) \subseteq \mathrm{cl}_{ac}(\operatorname{supp}(d_{M^D}))$$

Combining this equality with (3.2) and (3.3) we obtain $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\operatorname{supp}(d_{M^D}))$. □

Corollary 3.3 *Let A be as in Proposition 3.2, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* and let $M(\cdot)$ be the corresponding Weyl function. Further, let $B = B^* \in \mathcal{C}(\mathcal{H})$, $A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$ and $E_{A_B}(\cdot)$ the spectral measure of A_B. If $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfies $\ker(D) = \ker(D^*) = \{0\}$, then $N_{E^{ac}_{A_B}}(t) = d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\operatorname{supp}(d_{M_B^D}))$.*

If, in addition, the limit $M_B(t) := \text{s-lim}_{y \to +0} M_B(t + iy)$ exists for a.e. $t \in \mathbb{R}$, then $N_{E^{ac}_{A_B}}(t) = d_{M_B}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\operatorname{supp}(d_{M_B}))$.

Proof. By Proposition 2.17 $\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ is a generalized boundary triplet for $A_* := A^* \upharpoonright \operatorname{dom}(A_*)$, $\operatorname{dom}(A_*) = \operatorname{dom}(A_0) + \operatorname{dom}(A_B)$, and $M_B(z) = (B - M(z))^{-1}$, $z \in \mathbb{C}_+$, the corresponding generalized Weyl function. Clearly, $A_B = A_* \upharpoonright \ker(\Gamma_0^B)$. It remains to apply Proposition 3.2. □

This leads to the following theorem.

Theorem 3.4 *Let A be a densely defined closed symmetric operator, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* and let $M(\cdot)$ be the corresponding Weyl function. Further, let $A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$, $B = B^* \in \mathcal{C}(\mathcal{H})$, and $E_{A_B}(\cdot)$ the spectral measure of A_B. Let $D \in \mathfrak{S}_2(\mathcal{H})$ and $\ker(D) = \ker(D^*) = \{0\}$. Then*

(i) *$A_0 E^{ac}_{A_0}(\mathcal{D})$ is a part of $A_B E^{ac}_{A_B}(\mathcal{D})$ if and only if $d_{M^D}(t) \le d_{M_B^D}(t)$ for a.e. $t \in \mathcal{D}$.*

(ii) *$A_0 E^{ac}_{A_0}(\mathcal{D})$ and $A_B E^{ac}_{A_B}(\mathcal{D})$ are unitarily equivalent if and only if $d_{M^D}(t) = d_{M_B^D}(t)$ for a.e. $t \in \mathcal{D}$.*

Proof. Without loss of generality we assume that A is simple since the self-adjoint part of A is contained as a direct summand in any self-adjoint extension of A. We to show that $\Sigma^{ac}_M(\delta) = 0$ for some $\delta \in \mathcal{B}_b(\mathbb{R})$ if and only if $d_{M^D}(t) = 0$ for a.e $t \in \delta$. By the Berezanskii-Gel'fand-Kostyuchenko theorem [3, 7] the derivative $\Psi_{D^*\Sigma_M D}(t) := \frac{d}{dt} D^*\Sigma(t) D$ exists and the relation

$$D^*\Sigma^{ac}_M(\delta \cap \mathcal{D}) D = \int_{\delta \cap \mathcal{D}} \Psi_{D^*\Sigma_M D}(t) dt, \quad \delta \in \mathcal{B}_b,$$

holds. One has $\Sigma^{ac}_M(\delta) = 0$ if and only if $\Psi_{D^*\Sigma_M D}(t) = 0$ for a.e. $t \in \delta$. Since $d_{M^D}(t) = \dim(\operatorname{ran}(\Psi_{D^*\Sigma_M D}(t)))$ for a.e. $t \in \mathbb{R}$ we find that $\Sigma^{ac}_M(\delta \cap \mathcal{D}) = 0$ if and only if $d_{M^D}(t) = 0$ for a.e. $t \in \delta \cap \mathcal{D}$. Similarly we prove that $\Sigma^{ac}_{M_B}(\delta \cap \mathcal{D}) = 0$ if and only if $d_{D^* M_B D}(t) = 0$ for a.e. $t \in \delta \cap \mathcal{D}$.

(i) Since by assumption $d_{M^D}(t) \le d_{M_B^D}(t)$ for a.e. $t \in \mathcal{D}$, one gets by the considerations above that $\Sigma^{ac}_M(\delta \cap \mathcal{D}) \prec \Sigma^{ac}_{M_B}(\delta \cap \mathcal{D})$. By Theorem 2.5 we have $N_{\Sigma^{ac}_M}(t) = d_{M^D}(t)$ and $N_{\Sigma^{ac}_{M_B}}(t) = d_{M_B^D}(t)$ for a.e $t \in \mathbb{R}$. Hence $N_{\Sigma^{ac}_M}(t) \le N_{\Sigma^{ac}_{M_B}}(t)$ for a.e. $t \in \mathcal{D}$ which proves that the restricted measures $\Sigma^{ac}_M(\cdot \cap \mathcal{D})$ is spectrally subordinated to $\Sigma^{ac}_{M_B}(\cdot \cap \mathcal{D})$, cf. Definition 2.3(i). Since $\Sigma^{ac}_M \approx E^{ac}_{A_0}$ and

$\Sigma^{ac}_{M_B} \approx E^{ac}_{A_B}$, by Theorem 3.1, we get that $E^{ac}_{A_0}(\cdot \cap \mathcal{D})$ is spectrally subordinated to $E^{ac}_{A_B}(\cdot \cap \mathcal{D})$. Applying Theorem 2.4(i) we complete the proof.

(ii) If $d_{M^D}(t) = d_{D^* M_B D}(t)$ for a.e. $t \in \mathcal{D}$, then $\Sigma^{ac}_M(\cdot \cap \mathcal{D}) \sim \Sigma^{ac}_{M_B}(\cdot \cap \mathcal{D})$. By Theorem 2.5, $N_{\Sigma^{ac}_M}(t) = d_{M^D}(t)$ and $N_{\Sigma^{ac}_{M_B}}(t) = d_{M^D_B}(t)$ for a.e $t \in \mathbb{R}$ which implies that the operator measures $\Sigma^{ac}_M(\cdot \cap \mathcal{D})$ and $\Sigma^{ac}_{M_B}(\cdot \cap \mathcal{D})$ are spectrally equivalent, cf. Definition 2.3(ii). By Theorem 3.1, $E^{ac}_{A_0}(\cdot \cap \mathcal{D})$ and $E^{ac}_{A_B}(\cdot \cap \mathcal{D})$ are spectrally equivalent. Applying Theorem 2.4(ii) we prove that the absolutely continuous parts $A_0 E^{ac}_{A_0}(\mathcal{D})$ and $A_B E^{ac}_{A_B}(\mathcal{D})$ are unitarily equivalent. □

Theorem 3.4 reduces the problem of unitary equivalence of ac-parts of certain self-adjoint extensions of A to investigation of the functions $d_{M^D}(\cdot)$ and $d_{M^D_B}(\cdot)$.

Corollary 3.5 *Let A be as in Theorem 3.4. If the self-adjoint extensions $\widetilde{A}$ and $\widetilde{A}'$ of A are ac-minimal, then their ac-parts are unitarily equivalent.*

4 Unitary equivalence

4.1 Preliminaries

In what follows we assume that A is a densely defined simple closed symmetric operator in $\mathfrak{H}$. By A_0 we denote a self-adjoint extension of A which is fixed. Alongside A_0 we consider $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$. Usually we assume that

$$(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}). \tag{4.1}$$

It is known (see [12] that there exists a boundary triplet $\Pi := \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. Of course, the boundary triplet Π is not uniquely determined by the assumption $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. If Π_1 and Π_2 are two such boundary triplets of A^*, then their Weyl functions $M_1(\cdot)$ and $M_2(\cdot)$ are related by (2.12) (cf. [12]).

Fix a boundary triplet $\Pi := \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 = A^* \ker(\Gamma_0)$. By Proposition 2.9 there is a linear relation $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ such that $\widetilde{A} = A_\Theta$. In general, Θ is not the graph of an operator, $\Theta \notin \mathcal{C}(\mathcal{H})$. However, let us assume that Θ is the graph an operator B. By Proposition 2.12 we get that $(B - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$, that means, that B is a self-adjoint operator with discrete spectrum. Hence, $\varrho(B) \cap \mathbb{R} \neq \emptyset$. In what follows we assume without loss of generality that $0 \in \varrho(B)$. According to the polar decomposition we have $B^{-1} = DJD$ where

$$D := |B|^{-1/2} = D^* \in \mathfrak{S}_\infty(\mathfrak{H}) \quad \text{and} \quad J := \mathrm{sign}(B) = J^* = J^{-1}. \tag{4.2}$$

Clearly, $D \in \mathfrak{S}_\infty(\mathcal{H})$, $\ker(D) = \{0\}$, and D commutes with J. We set

$$G(z) := J - M^D(z), \quad z \in \mathbb{C}_+, \tag{4.3}$$

$M^D(z) := DM(z)D$, $z \in \mathbb{C}_+$, as usually. Obviously, $M^D(z)$ and $-G(z)$ are R-functions. We have $\ker(G(z)) = \{0\}$ for every $z \in \mathbb{C}_+$. Indeed, if $G(z)f = 0$, then $Jf = DM(z)Df$. Hence, $\mathrm{Im}(M(z)Df, Df) = \mathrm{Im}(Jf, f) = 0$ which yields $Df = 0$ or $f = 0$. Since J is a Fredholm operator satisfying $\ker(J) = \ker(J^*) = \{0\}$ we find by [20, Theorem 5.26] that $G(z)$ is boundedly invertible for $z \in \mathbb{C}_+$. We set $T(z) := G(z)^{-1}$, $z \in \mathcal{C}_+$ and note that $T(\cdot)$ is a Nevanlinna function because so is $M^D(\cdot)$. Moreover, $T(z) - J = T(z)M^D(z)J \in \mathfrak{S}_\infty(\mathfrak{H})$ for $z \in \mathbb{C}_+$.

4.2 Trace class perturbations: Rosenblum-Kato theorem

Here we apply the Weyl function technique in order to obtain a simple and quite different proof of the classical Rosenblum-Kato theorem. In fact, we prove a generalization of the Rosenblum-Kato theorem due to Birman and Krein [6] which includes non-additive (trace class) perturbations.

Our proof demonstrates the main idea of the proof of more general results contained in the next subsection.

Theorem 4.1 *Let A_0 and $\widetilde{A}$ be self-adjoint operators in $\mathfrak{H}$ satisfying*

$$(\widetilde{A}-i)^{-1}-(A_0-i)^{-1}\in\mathfrak{S}_1(\mathfrak{H}). \tag{4.4}$$

Then the absolutely continuous parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A}$ and A_0, respectively, are unitarily equivalent.

Proof. To include the operators $\widetilde{A}^{ac}$ and A_0^{ac} in the framework of extension theory we set

$$A:=A_0\upharpoonright \operatorname{dom}(A),\quad \operatorname{dom}(A)=\{f\in\operatorname{dom}(\widetilde{A})\cap\operatorname{dom}(A_0):\ A_0f=\widetilde{A}f\}.$$

Obviously, we have $A:=\widetilde{A}\upharpoonright\operatorname{dom}(A)$. Clearly, A is a closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices and $A_0,\widetilde{A}\in\operatorname{Ext}_A$.

First we assume that A is densely defined. Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a (ordinary) boundary triplet for A^*, such that $A_0:=A^*\upharpoonright\ker(\Gamma_0)$, and $M(\cdot)$ the corresponding Weyl function. By definition $\widetilde{A}=\widetilde{A}^*\in\operatorname{Ext}_A$ and $\widetilde{A}$ and A_0 are disjoint, that is, $\operatorname{dom}(A)=\operatorname{dom}(A_0)\cap\operatorname{dom}(\widetilde{A})$. Hence, by Proposition 2.9(ii), there exists an operator $B=B^*\in\mathcal{C}(\mathcal{H})$ such that $\widetilde{A}=A_B$.

It follows from (2.14) and (4.4) that $M_B(z):=(B-M(z))^{-1}\in\mathfrak{S}_1(\mathcal{H})$ for $z\in\mathbb{C}_+$. In accordance with [5, Lemma 2.4], see also [31], the limits $M_B(t):=\lim_{y\to+0}M_B(t+iy)$ exist in $\mathfrak{S}_2(\mathcal{H})$, for a.e $t\in\mathbb{R}$. By Theorem 3.4 it is it suffices to calculate the multiplicity function $d_{M_B}(t):=\operatorname{rank}(M_B(t))=\dim(\operatorname{ran}(\operatorname{Im}(M_B(t))))$.

It follows from (4.2) and (4.3) that

$$\begin{aligned}T(z)=G(z)^{-1}&=\big(J-M^D(z)\big)^{-1}=\big(J-DM(z)D\big)^{-1}\\&=D^{-1}\big(D^{-1}JD^{-1}-M(z)\big)^{-1}D^{-1}=|B|^{1/2}\big(B-M(z)\big)^{-1}|B|^{1/2},\ \ z\in\mathbb{C}_+.\end{aligned} \tag{4.5}$$

Combining this relation with (4.2) yields

$$M_B(z):=(B-M(z))^{-1}=DT(z)D,\qquad z\in\mathbb{C}_+.$$

In turn, this equality implies

$$\operatorname{Im}(M_B(z))=DT(z)^*\operatorname{Im}(M^D(z))T(z)D,\qquad z\in\mathbb{C}_+. \tag{4.6}$$

Moreover, since $M^D(z)\in\mathfrak{S}_1(\mathcal{H})$ and $T(z)-J\in\mathfrak{S}_1$ for $z\in\mathbb{C}_+$, by [5, Lemma 2.4] (see also [31]), for a.e $t\in\mathbb{R}$ and $y\to0$ there exist the limits $M^D(t)$ and $T(t)$ in $\mathfrak{S}_2(\mathcal{H})$-norm of the Nevanlinna operator functions $M^D((t+iy))$ and $T(t+iy)$, respectively. Therefore passing to the limit in (4.6) as $y\to0$ we get

$$\operatorname{Im}(M_B(t))=DT(t)^*\operatorname{Im}(M^D(t))T(t)D\quad\text{for a.e.}\quad t\in\mathbb{R}. \tag{4.7}$$

Therefore we find

$$\begin{aligned}d_{M_B}(t)&=\dim(\operatorname{ran}(\operatorname{Im}(M_B(t))))\\&=\dim(\operatorname{ran}(\sqrt{\operatorname{Im}(M_B(t))}))=\dim(\operatorname{ran}(\sqrt{\operatorname{Im}(M^D(t))}T(t)D)).\end{aligned} \tag{4.8}$$

Since $(J-M^D(t))T(t)=T(t)(J-M^D(t))=I$ for a.e. $t\in\mathbb{R}$, we find $\operatorname{ran}(T(t))=\mathcal{H}$ for a.e. $t\in\mathbb{R}$. Combining this relation with $\overline{\operatorname{ran}(D)}=\mathcal{H}$ and (4.8) we obtain

$$d_{M_B}(t)=\dim(\operatorname{ran}(\sqrt{\operatorname{Im}(M^D(t))}))=\dim(\operatorname{ran}(\operatorname{Im}(M^D(t))))=d_{M^D}(t) \tag{4.9}$$

for a.e. $t\in\mathbb{R}$. Applying Theorem 3.4(ii) we complete this part of the proof.

If A is not densely defined one can repeat the above reasonings applying only the boundary triplet technique for non-densely defined symmetric operators developed in [13, 26]. It turns out that the proof above can easily be carried over to this case. $\square$

In the following corollary we show that in proving of unitary equivalence of A_0 and $\widetilde{A} \in \mathrm{Ext}_A$ it suffices to restrict the consideration to disjoint extensions.

Corollary 4.2 *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*, and let $M(\cdot)$ be the corresponding Weyl function. Let also $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $\mathcal{D} \in \mathcal{B}(\mathbb{R})$.*

(i) If $A_0^{ac}E_{A_0}(\mathcal{D})$ is a part of $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ for any extension $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ disjoint with A_0, then $A_0^{ac}E_{A_0}(\mathcal{D})$ is a part of $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ for any extension $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$.

(ii) If $A_0^{ac}E_{A_0}(\mathcal{D})$ is unitarily equivalent to $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ for any extension $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ disjoint with A_0, then $A_0^{ac}E_{A_0}(\mathcal{D})$ is unitarily equivalent to the absolutely continuous part $\widetilde{A}^{ac}E_{\widetilde{A}}(\mathcal{D})$ of any extension $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$.

Proof. By Proposition 2.9 an extension $\widetilde{A} \in \mathrm{Ext}_A$ which is not disjoint with A_0 admits a representation $\widetilde{A}_\Theta$ with $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H}) \setminus \mathcal{C}(\mathcal{H})$. However, Θ admits a decomposition $\mathcal{H} = \mathcal{H}_{\rm op} \oplus \mathcal{H}_\infty$, $\Theta = \Theta_{\rm op} \oplus \Theta_\infty$ where $\Theta_{\rm op}$ is the graph of the operator $B_{\rm op} = B_{\rm op}^* \in \mathcal{C}(\mathcal{H}_{\rm op})$ (cf. Section 2). Denoting by $\pi_{\rm op}$ the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\rm op}$ and $M_{\rm op}(z) := \pi_{\rm op} M(z) \upharpoonright \mathcal{H}_{\rm op}$, we get $(\Theta - M(z))^{-1} = (B_{\rm op} - M_{\rm op}(z))^{-1}\pi_{\rm op}$. Therefore formula (2.14) takes the form

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(B_{\rm op} - M_{\rm op}(z))^{-1}\pi_{\rm op}\gamma(\overline{z})^*, \quad z \in \mathbb{C}_\pm.$$

Choose an operator $B_\infty = B_\infty^* \in \mathcal{C}(\mathcal{H}_\infty)$ such that $(B_\infty - i)^{-1} \in \mathfrak{S}_1(\mathcal{H}_\infty)$ and put $B = B_{\rm op} \oplus B_\infty$. It follows from Proposition 2.12 that

$$(A_\Theta - z)^{-1} - (A_B - z)^{-1} \in \mathfrak{S}_1(\mathfrak{H}),$$

since $(B_\infty - i)^{-1} \in \mathfrak{S}_1(\mathcal{H}_\infty)$. By Theorem 4.1 the absolutely continuous parts A_Θ^{ac} and A_B^{ac} of A_Θ and A_B, respectively, are unitarily equivalent.

(i) Since by assumption $A_0^{ac}E_{A_0}(\mathcal{D})$ is a part of $A_B^{ac}E_{A_B}(\mathcal{D})$ and A_B^{ac} is unitarily equivalent to A_Θ^{ac} we get that $A_0^{ac}E_{A_0}(\mathcal{D})$ is a part of $A_\Theta^{ac}E_{A_\Theta}(\mathcal{D})$.

(ii) Since, by assumption, $A_0^{ac}E_{A_0}(\mathcal{D})$ is unitarily equivalent to $A_B^{ac}E_{A_B}(\mathcal{D})$ and A_B^{ac} is unitarily equivalent to A_Θ, we get that $A_0^{ac}E_{A_0}(\mathcal{D})$ is unitarily equivalent to $A_\Theta^{ac}E_{A_\Theta}(\mathcal{D})$. $\square$

4.3 Compact non-additive perturbations

Here we generalize the Rosenblum-Kato theorem for the case of compact perturbations. To this end we assume that the maximal normal function

$$m^+(t) := \sup_{0<y\leq 1} \|M(t+iy)\|$$

is finite for a.e. $t \in \mathbb{R}$. This is the case if and only if the normal limits $M(t) := \text{w-lim}_{y\to+0} M(t+iy)$ exist and are bounded operators for a.e. $t \in \mathbb{R}$. Indeed, let $D = D^*$ be a Hilbert-Schmidt operator such that $\ker(D) = \{0\}$ and let $M^D(z) := DM(z)D$, $z \in \mathbb{C}_+$. Since the limit $M^D(t) := \text{o-lim}_{y\to+0} M^D(t+iy)$ exists and is a bounded operator for a.e. $t \in \mathbb{R}$, see [5, 31], we find that

$$\lim_{y\to+0}(M(t+iy)Df, Dg) = (M^D(t)f, g), \quad f, g \in \mathcal{H}, \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Hence the limit $\lim_{y\to+0}(M(t+iy)h, k)$ exists for a.e. $t \in \mathbb{R}$ and $h, k \in \mathrm{ran}(D)$ which yields the existence of $M(t) := \text{w-lim}_{y\to+0} M(t+iy)$ for a.e. $t \in \mathbb{R}$. The converse statement is obvious.

Now we are ready to prove the main result of this section.

Theorem 4.3 *Let A be a densely defined, closed symmetric operator in $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*, and let $M(\cdot)$ be the corresponding Weyl function. Let $\widetilde{A}$ be a self-adjoint extension of A and $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. If the maximal normal function $m^+(t)$ is finite for a.e. $t \in \mathbb{R}$ and condition (4.1) is satisfied, then the absolutely continuous parts $\widetilde{A}^{ac}$ and A_0^{ac} of $\widetilde{A}$ and A_0, respectively, are unitarily equivalent.*

Proof. We divide the proof into several steps.

(i) First we assume that the extensions $\widetilde{A}$ and A_0 are disjoint, that is $\widetilde{A} = A_B$ where $B = B^* \in \mathcal{C}(\mathcal{H})$. We define the operator $D \in \mathfrak{S}_\infty(\mathcal{H})$ in accordance with (4.2), $D := |B|^{-1/2}$, and investigate the function $M^D(z) := M^D(z) := DM(z)D$, $z \in \mathbb{C}_+$. Let $M^D(t) := DM(t)D$. Since the (weak) limit $M(t) :=$ w-$\lim_{y\to+0} M(t+iy)$ exists for a.e. $t \in \mathbb{R}$, by [31, Lemma 6.1.4], the following limit exists

$$\text{o-}\lim_{y\to+0} \|M^D(t+iy) - M^D(t)\| = 0 \qquad \text{for a.e.} \quad t \in \mathbb{R}. \tag{4.10}$$

Let $\delta_a := \{t \in \mathbb{R} : \|M(t)\| \le a\}$. Since $D = D^*$ is a a non-negative compact operator, it admits the spectral decomposition

$$D = \sum_{l\in\mathbb{N}} \mu_l Q_l$$

where $\{\mu_l\}_{l=1}^\infty$, is the decreasing sequence of eigenvalues of D, $\{Q_l\}_{l\in\mathbb{N}}$ the corresponding sequence of eigenprojections, $\dim\{Q_l\} < \infty$.

Since $\mu_l \to 0$ as $l \to \infty$, there exists a number $l_a \in \mathbb{N}$ such that $\mu_{l_a} < 1/\sqrt{2a}$. We put $\mathcal{H}_1 := \bigoplus_{l=l_a+1}^\infty Q_l\mathcal{H}$ and $\mathcal{H}_2 := \bigoplus_{l=1}^{l_a} Q_l\mathcal{H}$. Clearly, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\dim(\mathcal{H}_2) < \infty$. Moreover, the operator D admits the following decomposition $D = D_1 \oplus D_2$ where

$$D_1 := \sum_{l=l_a+1}^\infty \mu_l Q_l \quad \text{and} \quad D_2 := \sum_{l=1}^{l_a} \mu_l Q_l.$$

Since $\mu_{l_a} < 1/\sqrt{2a}$, we have $\|D_1\| < 1/\sqrt{2a}$. Hence

$$\|D_1 M(t) D_1\| < 1/2, \qquad t \in \delta_a. \tag{4.11}$$

Denote by P_1 and P_2 the orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Note that $P_1 J = JP_1$ and $P_2 J = JP_2$.

(ii) Our next aim is to show that the operator function $G(z) := J - M^D(z)$ is invertible in $\mathbb{C}_+$ and that $T(z) := G(z)^{-1}$ has the limits $T(t) :=$ s-$\lim_{y\to+0} T(t+iy)$ for a.e. $t \in \delta_a$. For this purpose we consider the decompositions

$$M^D(z) := \Bigl(D_i M(z) D_j\Bigr)_{i,j=1}^2 := \begin{pmatrix} M_{11}^D(z) & M_{12}^D(z) \\ M_{21}^D(z) & M_{22}^D(z) \end{pmatrix} : \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{H}_2 \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{H}_2 \end{matrix},$$

$z \in \mathbb{C}_+$, and

$$G(z) = J - M^D(z) = \begin{pmatrix} J_1 - M_{11}^D(z) & -M_{12}^D(z) \\ -M_{21}^D(z) & J_2 - M_{22}^D(z) \end{pmatrix}, \qquad z \in \mathbb{C}_+,$$

where $J_1 := JP_1$ and $J_2 := JP_2$.

(ii)$_1$ Let us prove that $\ker(J_1 - M_{11}^D(z)) = \{0\}$ for $z \in \mathbb{C}_+$. Indeed, from $0 = J_1 g - M_{11}^D(z) g = J_1 g - D_1 M(z) D_1 g$ one gets that $0 = \mathrm{Im}(M_{11}^D(z)g, g) = (\mathrm{Im}(M(z)D_1 g, D_1 g)$. Hence $0 = D_1 g = Dg$ which yields $g = 0$. Since $0 \in \varrho(J_1)$ and $M_{11}^D(\cdot) \in \mathfrak{S}_\infty$, we obtain that the operator $J_1 - M_{11}^D(z) = J_1(I_1 - J_1 M_{11}^D(z))$ is boundedly invertible for every $z \in \mathbb{C}_+$. Since $M_{11}^D(z)$ is a $R_{\mathcal{H}_1}$-function, we get that $\Xi(z) := (J_1 - M_{11}^D(z))^{-1}$, $z \in \mathbb{C}_+$, is a $R_{\mathcal{H}_1}$-function too.

(ii)$_2$ We show that for a.e. $t \in \delta_a$, $a > 0$, the limit $\Xi(t) := \text{o-lim}_{y\to+0} \Xi(t+iy)$ exists in the operator norm and the following representation holds

$$\Xi(t) = (J_1 - M^D_{11}(t))^{-1}. \tag{4.12}$$

First we note that $J_1 - M^D_{11}(z) = J_1(I_1 - J_1 M^D_{11}(z)$. Using (4.11) we get $\|J_1 M^D_{11}(t)\| < 1$ for $t \in \delta_a$. Hence the inverse operator $(I_1 - J_1M^D_{11}(t))^{-1}$ exists for $t \in \delta_a$. Using $(J_1 - M^D_{11}(t))^{-1} = (I_1 - J_1M^D_{11}(t))^{-1}J_1$ we find that the inverse operator $(J_1 - M^D_{11}(t))^{-1}$ exist for $t \in \delta_a$. Since $M^D_{11}(z)$ has limits $M^D_{11}(t)$ for a.e. $t \in \mathbb{R}$ one gets that $J_1M^D_{11}(t) = \text{o-lim}_{y\to+0} J_1M^D_{11}(t+iy)$ for a.e. $t \in \mathbb{R}$. Fix any such $t_0 \in \delta_a$. Then due to estimate (4.11) there exists $\eta = \eta(t_0)$ such that $\sup_{y\in(0,\eta)} \|J_1M^D_{11}(t_0+iy)\| \le 1/2$. Therefore, the family $\{\|(I_1 - J_1M^D_{11}(t_0+iy))^{-1}\|\}_{y\in(0,\eta)}$ is uniformly bounded for any fixed $t_0 \in \delta_a$. Using this fact and (4.10) we can pass to the limit as $y \to 0$ in the identity

$$\begin{aligned}&(I_1 - J_1M^D_{11}(t_0+iy))^{-1} - (I_1 - J_1M^D_{11}(t_0))^{-1}\\ &\quad = (I_1 - J_1M^D_{11}(t_0+iy))^{-1}(J_1M^D_{11}(t_0+iy)) - J_1M^D_{11}(t_0))(I_1 - J_1M^D_{11}(t_0))^{-1}.\end{aligned}$$

We obtain $\text{o-lim}_{y\to+0}((I_1 - J_1M^D_{11}(t+iy))^{-1} = (I_1 - J_1M^D_{11}(t))^{-1}$ for a.e. $t \in \delta_a$ which yields the existence of $\Xi(t) := \text{o-lim}_{y\to+0} \Xi(t+iy)$ and proves representation (4.12).

(ii)$_3$ Next we set

$$\Delta(z) := M^D_{22}(z) + M^D_{21}(z)(J_1 - M^D_{11}(z))^{-1}M^D_{12}(z), \quad z \in \mathbb{C}_+.$$

and show that the function $T_2(\cdot) := (J_2 - \Delta(\cdot))^{-1}$ is $R_{\mathcal{H}_2}$-function.

Clearly, $\Delta(\cdot)$ is holomorphic in $\mathbb{C}_+$ and it acts in a finite dimensional Hilbert space $\mathcal{H}_2$. Since $\det(J_2 - \Delta(\cdot))$ is also holomorphic in $\mathbb{C}_+$, the determinant $\det(J_2 - \Delta(\cdot))$ has only a discrete set of zeros in $\mathbb{C}_+$. Hence the inverse operator $T_2(\cdot) := (J_2 - \Delta(\cdot))^{-1}$ exists for $z \in \Omega \subset \mathbb{C}_+$ where $\mathbb{C}_+ \setminus \Omega$ is at most countable discrete set, that is, $T_2(\cdot)$ is meromorphic in $\mathbb{C}_+$.

As we just mentioned the inverse operator $(J_2 - \Delta(z))^{-1}$ exists for $z \in \Omega \subset \mathbb{C}_+$. Choose any $z \in \Omega$. Then, by the Frobenius formula,

$$T(z) := (J - M^D(z))^{-1} = \begin{pmatrix} T_1(z) & \Xi(z)M^D_{12}(z)T_2(z)\\ T_2(z)M^D_{21}(z)\Xi(z) & T_2(z)\end{pmatrix} \tag{4.13}$$

where

$$T_1(z) := \Xi(z) + \Xi(z)M^D_{12}(z)T_2(z)M^D_{21}(z)\Xi(z). \tag{4.14}$$

Hence

$$T_2(z) = P_2T(z) \upharpoonright \mathcal{H}_2, \qquad z \in \Omega.$$

Since $T(\cdot)$ is a $R_{\mathcal{H}}$-function, we get that $\operatorname{Im}(T_2(z)) > 0$ for $z \in \Omega$. Since in addition $T_2(\cdot)$ is meromorphic in $\mathbb{C}_+$, we conclude that it is holomorphic. Thus, $T_2(\cdot) = (J_2 - \Delta(\cdot))^{-1}$ is $R_{\mathcal{H}_2}$-function, too.

(ii)$_4$ In this step we show that for any $a > 0$ the limit $T(t) := \text{o-lim}_{y\to+0} T(t+iy)$ exists in the operator norm for a.e. $t \in \delta_a$. Since $T_2(\cdot)$ is the matrix $R_{\mathcal{H}_2}$-function, the limit $T_2(t) = \text{o-lim}_{y\to+0} T_2(t+iy)$ exists for a.e. $t \in \mathbb{R}$. Besides, (4.10) yields

$$\lim_{y\to+0} \|M^D_{12}(t+iy) - M^D_{12}(t)\| = 0 \quad \text{and} \quad \lim_{y\to+0} \|M^D_{21}(t+iy) - M^D_{21}(t)\| = 0$$

for a.e. $t \in \mathbb{R}$. Combining these relations with (4.12) and (4.14) yields the existence of the limit $T_1(t) := \text{o-lim}_{y\to+0} T_1(t+iy)$ for a.e $t \in \delta_a$. Finally, combining all these relations with the block-matrix representation (4.13) we complete the proof of (ii).

(iii) Using the results of (ii) we are now going to complete the proof of the theorem. We set $\delta_n := \{t \in \mathbb{R} : \ m^+(t) \le n\}$ and note that $\bigcup_{n\in\mathbb{N}} \delta_n$ differs from $\mathbb{R}$ by a set of Lebesgue measure zero. By step (ii) the limit $T(t) := \text{o-}\lim_{y\to+0} T(t+iy)$ exists for a.e. $t \in \bigcup_{n\in\mathbb{N}} \delta_n$ in the operator norm. Hence the limit $T(t) := \text{o-}\lim_{y\to+0} T(t+iy)$ exist for a.e. $t \in \mathbb{R}$. Combining this fact with (4.10) we can pass to the limit in the identity $(J - M^D(t+iy))T(t+iy) = I$ as $y \to 0$. We get

$$(J - M^D(t))T(t) = T(t)(J - M^D(t)) = I \qquad \text{for a.e.} \quad t \in \mathbb{R} \tag{4.15}$$

The rest of the proof is similar to that of Theorem 4.1. First we assume that $\widetilde{A}$ is disjoint with A_0, hence, it admits a representation $\widetilde{A} = A_B$ with $B \in \mathcal{C}(\mathcal{H})$. Therefore, setting $M_B(\cdot) := (B - M(\cdot))^{-1}$ and assuming without loss of generality that $0 \in \varrho(B)$ we arrive at the representation (4.7) with $D = |B|^{-1/2}$ for a.e. $t \in \mathbb{R}$. Moreover, (4.15) yields $\operatorname{ran}(T(t)) = \mathcal{H}$ for a.e. $t \in \mathbb{R}$. Therefore arguing as in (4.8) and (4.9) we obtain

$$\begin{aligned} d_{M_B}(t) &= \dim(\operatorname{ran}(\sqrt{\operatorname{Im}(M^D(t))}\,)) = \dim(\operatorname{ran}(\sqrt{\operatorname{Im}(M(t))}D\,)) \\ &= \dim(\operatorname{ran}(\sqrt{\operatorname{Im}(M(t))}\,)) = \dim(\operatorname{ran}(\operatorname{Im}(M(t)))) = d_M(t) \end{aligned}$$

for a.e. $t \in \mathbb{R}$. Applying Theorem 3.4(ii) we complete the proof.

Finally, we apply Corollary 4.2 to extend the proof for extensions $\widetilde{A}$ not disjoint with A_0. □

Remark 4.4 The result as well as the proof remains valid if A is non-densely defined. In this case it suffices to use the boundary triplet technique for non-densely defined operators developed in [13, 26], cf. proof of Theorem 4.1. However, the assumptions on the Weyl function are indispensable.

The following result is immediate from Theorem 3.4(ii) and Theorem 4.3.

Corollary 4.5 *Let the assumptions of Theorem 4.3 be satisfied and let*

$$\mathcal{F} := \{t \in \mathbb{R} : m^+(t) < \infty\}. \tag{4.16}$$

If condition (4.1) *holds, then the parts* $\widetilde{A}^{ac}E_{\widetilde{A}^{ac}}(\mathcal{F})$ *and* $A_0^{ac}E_{A_0^{ac}}(\mathcal{F})$ *of* $\widetilde{A}$ *and* A_0*, respectively, are unitarily equivalent.*

Remark 4.6 Let us define the invariant maximal normal function

$$\mathfrak{m}^+(t) := \sup_{y\in(0,1]} \left\| \operatorname{Im}(M(i))^{-1/2} \left(M(t+iy) - \operatorname{Re}(M(i))\right) \operatorname{Im}(M(i))^{-1/2} \right\|, \tag{4.17}$$

for $t \in \mathbb{R}$. For Weyl functions one easily proves that $m^+(t)$ is finite if and only if $\mathfrak{m}^+(t)$ is finite.

(i) The quantity $\mathfrak{m}^+(t)$ has the advantage that it is invariant: Let A be a densely defined closed symmetric operator, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ a boundary triplet for A^*, and $M(\cdot)$ the corresponding Weyl function. Further, let $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be another boundary triplet for A^* with the Weyl function $\widetilde{M}(\cdot)$ and let $A_0 := A^* \upharpoonright \ker(\Gamma_0) = A^* \upharpoonright \ker(\widetilde{\Gamma}_0)$. In this case $M(\cdot)$ and $\widetilde{M}(\cdot)$ are related by (2.12) However, $\widetilde{\mathfrak{m}}^+(t) = \mathfrak{m}^+(t)$ for $t \in \mathbb{R}$, where $\mathfrak{m}^+(t)$ is obtained by replacing in (4.17) $M(\cdot)$ by $\widetilde{M}(\cdot)$.

(ii) Further, if the Weyl function $M(\cdot)$ satisfies $M(i) = i$, then $m^+(t) = \mathfrak{m}^+(t)$ for $t \in \mathbb{R}$.

(iii) Let π be an orthogonal projection onto a subspace $\widehat{\mathcal{H}}$ of $\mathcal{H}$. If $\mathfrak{m}^+(t)$ is finite, then the invariant maximal normal function $\widehat{\mathfrak{m}}^+(t)$, obtained from (4.17) replacing $M(\cdot)$ by $\widehat{M}(\cdot) := \pi M(\cdot) \upharpoonright \widehat{\mathcal{H}}$, is also finite and satisfies $\widehat{\mathfrak{m}}^+(t) \le \mathfrak{m}^+(t)$ for $t \in \mathbb{R}$.

5 Direct sums of symmetric operators

5.1 Boundary triplets for direct sums

Let S_n be a closed densely defined symmetric operators in $\mathfrak{H}_n$, $n_+(S_n) = n_-(S_n)$, and let $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. Let

$$A := \bigoplus_{n=1}^{\infty} S_n, \quad \mathrm{dom}\,(A) := \bigoplus_{n=1}^{\infty} \mathrm{dom}\,(S_n). \tag{5.1}$$

Clearly, A is a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$ with $n_\pm(A) = \infty$. Consider the direct sum $\Pi := \oplus_{n=1}^{\infty} \Pi_n =: \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of (ordinary) boundary triplets defined by

$$\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n, \quad \Gamma_0 := \bigoplus_{n=1}^{\infty} \Gamma_{0n} \quad \text{and} \quad \Gamma_1 := \bigoplus_{n=1}^{\infty} \Gamma_{1n}. \tag{5.2}$$

Clearly,

$$A^* = \bigoplus_{n=1}^{\infty} S_n^*, \quad \mathrm{dom}\,(A^*) = \bigoplus_{n=1}^{\infty} \mathrm{dom}\,(S_n^*). \tag{5.3}$$

We note that the Green's identity

$$(S_n^* f_n, g_n) - (f_n, S_n^* g_n) = (\Gamma_{1n} f_n, \Gamma_{0n} g_n)_{\mathcal{H}_n} - (\Gamma_{0n} f_n, \Gamma_{1n} g_n)_{\mathcal{H}_n},$$

$f_n, g_n \in \mathrm{dom}\,(S_n^*)$, holds for every S_n^*, $n \in \mathbb{N}$. This yields the Green's identity (2.15) for $A_* := A^* \upharpoonright \mathrm{dom}\,(\Gamma)$, $\mathrm{dom}\,(\Gamma) := \mathrm{dom}\,(\Gamma_0) \cap \mathrm{dom}\,(\Gamma_1) \subseteq \mathrm{dom}\,(A^*)$, that is, for $f = \oplus_{n=1}^{\infty} f_n$, $g = \oplus_{n=1}^{\infty} g_n \in \mathrm{dom}\,(\Gamma)$ we have

$$(A_* f, g) - (f, A_* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \mathrm{dom}\,(\Gamma), \tag{5.4}$$

where A^* and Γ_j are defined by (5.3) and (5.2), respectively. However, the Green's identity (5.4) cannot be extended to $\mathrm{dom}\,(A^*)$ in general, since $\mathrm{dom}\,(\Gamma)$ is smaller than $\mathrm{dom}\,(A^*)$ generically. It might even happen that Γ_j are not bounded as mappings from $\mathrm{dom}\,(A^*)$ equipped with the graph norm into $\mathcal{H}$. Counterexamples for the direct sum $\Pi = \oplus_{n=1}^{\infty} \Pi_n$, which does not form a boundary triplet, firstly appeared in [21]).

In this section we show that it is always possible to modify the boundary triplets Π_n in such a way that a new sequence $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ of boundary triplets for S_n^* satisfies the following properties: $\widetilde{\Pi} = \oplus_{n=1}^{\infty} \widetilde{\Pi}_n$ forms a boundary triplet for A^* and the following relations hold

$$\widetilde{S}_{0n} := S_n^* \upharpoonright \ker\,(\widetilde{\Gamma}_{0n}) = S_n^* \upharpoonright \ker\,(\Gamma_{0n}) =: S_{0n}, \quad n \in \mathbb{N}. \tag{5.5}$$

Hence $\widetilde{A}_0 := \oplus_{n=1}^{\infty} \widetilde{S}_{0n} = \oplus_{n=1}^{\infty} S_{0n} =: A_0$. We note that the existence of a boundary triplet $\Pi' = \{\mathcal{H}, \Gamma_0', \Gamma_1'\}$ for A^* satisfying $\ker\,(\Gamma_0') = \mathrm{dom}\,(A_0)$ is known (see [17, 12]). However, we emphasize that in applications we need a special form (5.2) of a boundary triplet for A^* because it leads to the block-diagonal form of the corresponding Weyl function (cf. Sections 5.2, 5.3 below).

We start with a simple technical lemma.

Lemma 5.1 *Let S be a densely defined closed symmetric operator with equal deficiency indices, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ a boundary triplet for S^*, and $M(\cdot)$ the corresponding Weyl function. Then there exists a boundary triplet $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for S^* such that $\ker\,(\widetilde{\Gamma}_0) = \ker\,(\Gamma_0)$ and the corresponding Weyl function $\widetilde{M}(\cdot)$ satisfies $\widetilde{M}(i) = i$.*

Proof. Let $M(i) = Q + iR^2$ where $Q := \mathrm{Re}(M(i))$, $R := \sqrt{\mathrm{Im}(M(i))}$. We set

$$\widetilde{\Gamma}_0 := R\Gamma_0 \quad \text{and} \quad \widetilde{\Gamma}_1 := R^{-1}(\Gamma_1 - Q\Gamma_0). \tag{5.6}$$

A straightforward computation shows that $\widetilde{\Pi} := \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary triplet for A^*. Clearly, $\ker(\widetilde{\Gamma}_0) = \ker(\Gamma_0)$. The Weyl function $\widetilde{M}(\cdot)$ of $\widetilde{\Pi}$ is given by $\widetilde{M}(\cdot) = R^{-1}(M(\cdot) - Q)R^{-1}$ which yields $\widetilde{M}(i) = i$. $\square$

If S is a densely defined closed symmetric operator in $\mathfrak{H}$, then by the first v. Neumann formula the direct decomposition $\mathrm{dom}\,(S^*) = \mathrm{dom}\,(S) \dotplus \mathfrak{N}_i \dotplus \mathfrak{N}_{-i}$ holds where $\mathfrak{N}_{\pm i} := \ker(S^* \mp i)$. Equipping $\mathrm{dom}\,(S^*)$ with the inner product

$$(f,g)_+ := (S^*f, S^*g) + (f,g), \quad f,g \in \mathrm{dom}\,(S^*), \tag{5.7}$$

one obtains a Hilbert space denoted by $\mathfrak{H}_+$. The first v. Neumann formula leads to the following orthogonal decomposition

$$\mathfrak{H}_+ = \mathrm{dom}\,(S) \oplus \mathfrak{N}_i \oplus \mathfrak{N}_{-i}.$$

Lemma 5.2 *Let S be as in Lemma 5.1, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a (ordinary) boundary triplet for S^*, and $M(\cdot)$ the corresponding Weyl function. If $M(i) = i$, then the operator $\Gamma : \mathfrak{H}_+ \longrightarrow \mathcal{H} \oplus \mathcal{H}$, $\Gamma := (\Gamma_0, \Gamma_1)^\top$ is a contraction. Moreover, Γ isometrically maps $\mathfrak{N} := \mathfrak{N}_i \oplus \mathfrak{N}_{-i}$ onto $\mathcal{H}$.*

Proof. We show that

$$\|\Gamma(f + f_i + f_{-i})\|^2_{\mathcal{H}\oplus\mathcal{H}} = \|f_i + f_{-i}\|^2_+ \tag{5.8}$$

where $f \dotplus f_i \dotplus f_{-i} \in \mathrm{dom}\,(S) \dotplus \mathfrak{N}_i \dotplus \mathfrak{N}_{-i} = \mathrm{dom}\,(S^*)$. Indeed, since $\mathrm{dom}\,(S) = \ker(\Gamma_0) \cap \ker(\Gamma_1)$, we find

$$\|\Gamma(f + f_i + f_{-i})\|^2_{\mathcal{H}\oplus\mathcal{H}} = \|\Gamma_0(f_i + f_{-i})\|^2_{\mathcal{H}} + \|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}}.$$

Clearly,

$$\|\Gamma_j(f_i + f_{-i})\|^2_{\mathcal{H}} = \|\Gamma_j f_i\|^2 + 2\,\mathrm{Re}((\Gamma_j f_i, \Gamma_j f_{-i})) + \ \|\Gamma_j f_{-i}\|^2, \quad j \in \{0,1\}. \tag{5.9}$$

Using $\Gamma_1 f_i = M(i)\Gamma_0 f_i = i\Gamma_0 f_i$ and $\Gamma_1 f_{-i} = M(-i)\Gamma_0 f_{-i} = -i\Gamma_0 f_{-i}$ we obtain

$$\|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}} = (\Gamma_0 f_i, \Gamma_0 f_i) - 2\,\mathrm{Re}((\Gamma_0 f_i, \Gamma_0 f_{-i})) + (\Gamma_0 f_{-i}, \Gamma_0 f_{-i}) \tag{5.10}$$

Taking a sum of (5.9) and (5.10) we get

$$\|\Gamma_0(f_i + f_{-i})\|^2_{\mathcal{H}} + \|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}} = 2\|\Gamma_0 f_i\|^2_{\mathcal{H}} + 2\|\Gamma_0 f_{-i}\|^2_{\mathcal{H}}. \tag{5.11}$$

Combining equalities $\Gamma_1 f_{\pm i} = \pm i\Gamma_0 f_{\pm i}$ with Green's identity (2.7) we obtain $\|\Gamma_0 f_i\|_{\mathcal{H}} = \|f_i\|$ and $\|\Gamma_0 f_{-i}\|_{\mathcal{H}} = \|f_{-i}\|$. Therefore (5.11) takes the form

$$\|\Gamma_0(f_i + f_{-i})\|^2_{\mathcal{H}} + \|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}} = 2\|f_i\|^2 + 2\|f_{-i}\|^2. \tag{5.12}$$

A straightforward computation shows $\|f_i + f_{-i}\|^2_+ = 2\|f_i\|^2 + 2\|f_{-i}\|^2$ which together with (5.12) proves (5.8). Since $\|f_i + f_{-i}\|^2_+ \le \|f\|^2_+ + \|f_i + f_{-i}\|^2_+ = \|f + f_i + f_{-i}\|^2_+$, we get from (5.8) that Γ is a contraction.

Obviously, Γ is an isometry from $\mathfrak{N}$ into $\mathcal{H} \oplus \mathcal{H}$. Since Π is a boundary triplet for S^*, $\mathrm{ran}\,(\Gamma) = \mathcal{H} \oplus \mathcal{H}$. Hence Γ is an isometry from $\mathfrak{N}$ onto $\mathcal{H} \oplus \mathcal{H}$. $\square$

Passing to direct sum (5.1), we equip $\mathrm{dom}\,(A_n^*)$ and $\mathrm{dom}\,(A^*)$ with the graph's norms and obtain the Hilbert spaces $\mathfrak{H}_{+n}$ and $\mathfrak{H}_+$, respectively. Clearly, the corresponding inner products $(f,g)_{+n}$ and $(f,g)_+$ are defined by (5.7) with S replaced by S_n and A, respectively. Obviously, $\mathfrak{H}_+ = \bigoplus_{n=1}^\infty \mathfrak{H}_{+n}$.

Theorem 5.3 *Let $\{S_n\}_{n=1}^\infty$ be a sequence of densely defined closed symmetric operators, $\mathrm{dom}\,(S_n) \subset \mathfrak{H}_n$, $n_+(S_n) = n_-(S_n)$, and let $S_{0n} = S_{0n}^* \in \mathrm{Ext}_{S_n}$. Further, let A and A_0 be given by* (5.1) *and*

$$A_0 := \bigoplus_{n=1}^\infty S_{0n}, \tag{5.13}$$

respectively. Then there exist boundary triplets $\Pi_n := \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$, $n \in \mathbb{N}$, and the direct sum $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ defined by (5.2) forms an ordinary boundary triplet for A^* satisfying $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. Moreover, the corresponding Weyl function $M(\cdot)$ and the γ-field $\gamma(\cdot)$ are given by

$$M(z) = \bigoplus_{n=1}^{\infty} M_n(z) \quad \text{and} \quad \gamma(z) = \bigoplus_{n=1}^{\infty} \gamma_n(z) \tag{5.14}$$

where $M_n(\cdot)$ and $\gamma_n(\cdot)$ are the Weyl functions and the γ-field corresponding to Π_n, $n \in \mathbb{N}$. In addition, the condition $M(i) = iI$ holds.

Proof. For every $S_{0n} = S_{0n}^* \in \operatorname{Ext} S_n$ there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* such that $S_{0n} := A_n^* \upharpoonright \ker(\Gamma_{0n})$ (see [12]). By Lemma 5.1 we can assume without loss of generality that the corresponding Weyl function $M_n(\cdot)$ satisfies $M_n(i) = i$. By Lemma 5.2 the mapping $\Gamma^n := (\Gamma_{0n}, \Gamma_{1n})^\top : \mathfrak{H}_{+n} \longrightarrow \mathcal{H}_n \oplus \mathcal{H}_n$, is contractive for each $n \in \mathbb{N}$. Hence $\|\Gamma_j\| = \sup_n \|\Gamma_{jn}\| \le 1, j \in \{0,1\}$, where Γ_0 and Γ_1 are defined by (5.2). It follows that the mappings Γ_0 and Γ_1 are well-defined on $\operatorname{dom}(\Gamma) = \operatorname{dom}(A^*) = \bigoplus_{n=1}^{\infty} \operatorname{dom}(S_n^*)$. Thus, the Green's identity (5.4) holds for all $f, g \in \operatorname{dom}(A^*)$.

Further, we set $\mathfrak{N}_{\pm in} := \ker(S_n^* \mp i)$, $\mathfrak{N}_n := \mathfrak{N}_{in} \dot{+} \mathfrak{N}_{-in}$, $\mathfrak{N}_{\pm i} := \ker(A^* \mp i)$ and $\mathfrak{N} := \mathfrak{N}_i \dot{+} \mathfrak{N}_{-i}$. By Lemma 5.2 the restriction $\Gamma^n \upharpoonright \mathfrak{N}_n$ is an isometry from $\mathfrak{N}_n$, regarded as a subspace of $\mathfrak{H}_{+n}$, onto $\mathcal{H}_n \oplus \mathcal{H}_n$. Since $\mathfrak{N}$ regarded as a subspace of $\mathfrak{H}_+$ admits the representation $\mathfrak{N} = \bigoplus_{n=1}^{\infty} \mathfrak{N}_n$, the restriction $\Gamma \upharpoonright \mathfrak{N}$, $\Gamma := \bigoplus_{n=1}^{\infty} \Gamma^n$, isometrically maps $\mathfrak{N}$ onto $\mathcal{H} \oplus \mathcal{H}$. Hence $\operatorname{ran}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$. Equalities (5.14) are immediate from Definition 2.10. □

Remark 5.4 Kochubei [21] proved that $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ forms a boundary triplet whenever any pair $\{S_n, S_{0n}\}$, $S_{0n} := S_n^* \upharpoonright \ker(\Gamma_{0n})$, $n \in \mathbb{N}$, is unitarily equivalent to $\{S_1, S_{01}\}$.

Recall, that for any non-negative symmetric operator A the set of its non-negative self-adjoint extensions $\operatorname{Ext}_A(0,\infty)$ is non-empty (see [1, 20]). The set $\operatorname{Ext}_A(0,\infty)$ contains the Friedrichs (the biggest) extension A^F and the Krein (the smallest) extension A^K. These extensions are uniquely determined by the following extremal property in the class $\operatorname{Ext}_A(0,\infty)$:

$$(A^F + x)^{-1} \le (\widetilde{A} + x)^{-1} \le (A^K + x)^{-1}, \quad x > 0, \quad \widetilde{A} \in \operatorname{Ext}_A(0,\infty).$$

Corollary 5.5 *Assume conditions of Theorem 5.3. Let $S_n \ge 0$, $n \in \mathbb{N}$, and let S_n^F and S_n^K be the Friedrichs and the Krein extensions of S_n, respectively. Then*

$$A^F = \oplus_{n=1}^{\infty} S_n^F \qquad \text{and} \qquad A^K = \oplus_{n=1}^{\infty} S_n^K. \tag{5.15}$$

Proof. Let us prove the second of relations (5.15). The first one is proved similarly. By Theorem 5.3 there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* such that $S_n^K = S_{0n}$ and $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ is a boundary triplet for A^*.

Fix any $x_2 \in \mathbb{R}_+$ and put $C_2 := \|M(-x_2)\|$. Then any $h = \oplus_{n=1}^{\infty} h_n \in \mathcal{H}$ can be decomposed by $h = h^{(1)} \oplus h^{(2)}$ with $h^{(1)} \in \oplus_{n=1}^{p}\mathcal{H}_n$ and $h^{(2)} \in \oplus_{n=p+1}^{\infty}\mathcal{H}_n$ such that $\|h^{(2)}\| < C_2^{-1/2}$. Hence $|(M(-x_2)h^{(2)}, h^{(2)})| < 1$. Due to the monotonicity of $M(\cdot)$ we get

$$\Big(M(-x)h^{(2)}, h^{(2)}\Big) > \Big(M(-x_2)h^{(2)}, h^{(2)}\Big) > -1, \qquad x \in (0, x_2).$$

Since $S_{0n} = S_n^K$, the Weyl function $M_n(\cdot)$ satisfies

$$\lim_{x \downarrow 0} \Big(M_n(-x)g_n, g_n\Big) = +\infty, \qquad g_n \in \mathcal{H}_n \setminus \{0\}, \tag{5.16}$$

cf. [12, Proposition 4]. Since $M(\cdot) = \oplus_{n=1}^{\infty} M_n(\cdot)$ is block-diagonal, cf. (5.14), we get from (5.16) that for any $N > 0$ there exists $x_1 > 0$ such that

$$\left(M(-x)h^{(1)}, h^{(1)}\right) = \sum_{n=1}^{p} \left(M_n(-x)h_n, h_n\right) > N \quad \text{for} \quad x \in (0, x_1). \tag{5.17}$$

Combining (5.16) with (5.17) and using the diagonal form of $M(\cdot)$, we get

$$(M(-x)h, h) = (M(-x)h^{(1)}, h^{(1)}) + (M(-x)h^{(2)}, h^{(2)}) > N - 1$$

for $0 < x \leq \min(x_1, x_2)$. Thus, $\lim_{x \downarrow 0}(M(-x)h, h) = +\infty$ for $h \in \mathcal{H} \setminus \{0\}$. Applying [12, Proposition 4] we prove the second relation of (5.15). □

Remark 5.6 Another proof can be obtained by using characterization of A^F and A^K by means of the respective quadratic forms.

5.2 Summands with arbitrary equal deficiency indices

Here we apply Theorem 4.3 to direct sums of symmetric operators (5.1), allowing summands S_n to have arbitrary (finite or infinite) equal deficiency indices. We start with a simple general proposition.

Proposition 5.7 *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of densely defined closed symmetric operators, $\operatorname{dom}(S_n) \subset \mathfrak{H}_n$, $n_+(S_n) = n_-(S_n)$, and let $S_{0n} = S_{0n}^* \in \operatorname{Ext}_{S_n}$. Further, let A and A_0 be given by* (5.1) *and* (5.13)*, respectively. If $\widetilde{A}$ is a self-adjoint extension of A such that condition* (4.1) *is satisfied, then*

$$\sigma_{ac}(A_0) = \overline{\bigcup \sigma_{ac}(S_{0n})} \subseteq \sigma(\widetilde{A}) \quad \text{and} \quad \sigma_{ac}(\widetilde{A}) \subseteq \overline{\bigcup \sigma(S_{0n})} = \sigma(A_0). \tag{5.18}$$

Proof. By the Weyl theorem, condition (4.1) yields $\sigma_{\rm ess}(\widetilde{A}) = \sigma_{\rm ess}(A_0)$. Hence

$$\overline{\bigcup \sigma_{ac}(S_{0n})} = \sigma_{ac}(A_0) \subseteq \sigma_{\rm ess}(A_0) = \sigma_{\rm ess}(\widetilde{A}) \subseteq \sigma(\widetilde{A})$$

and

$$\sigma_{ac}(\widetilde{A}) \subseteq \sigma_{\rm ess}(\widetilde{A}) = \sigma_{\rm ess}(A_0) \subseteq \sigma(A_0) = \overline{\bigcup \sigma(S_{0n})}.$$

□

Our further considerations are substantially based on Theorem 5.3.

Theorem 5.8 *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of densely defined closed symmetric operators, $\operatorname{dom}(S_n) \subset \mathfrak{H}_n$, $n_+(S_n) = n_-(S_n)$, and let $S_{0n} = S_{0n}^* \in \operatorname{Ext}_{S_n}$. Further, let $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be an ordinary boundary triplet for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$, $n \in \mathbb{N}$, and let $M_n(\cdot)$ be the corresponding Weyl function. Moreover, let $\mathfrak{m}_n^+(t)$, $n \in \mathbb{N}$, be the invariant maximal normal function obtained from* (4.17) *by replacing $M(\cdot)$ by $M_n(\cdot)$. If $\sup_{n \in \mathbb{N}} \mathfrak{m}_n^+(t) < +\infty$ for a.e. $t \in \mathbb{R}$, then for any self-adjoint extension $\widetilde{A}$ of A defined by* (5.1)*, which satisfies the condition* (4.1)*, the absolutely continuous parts $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent. In particular, instead of* (5.18) *we have $\sigma_{ac}(A_0) = \sigma_{ac}(\widetilde{A})$.*

Proof. Let $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ be a boundary triplet for $S^*{}_n$, $n \in \mathbb{N}$, defined according to (5.6), that is $\widetilde{\Gamma}_{0n} := R_n \Gamma_{0n}$ and $\widetilde{\Gamma}_{1n} := R_n^{-1}\big(\Gamma_{1n} - \operatorname{Re}(M_n(i))\Gamma_{0n}\big)$, where $R_n := \sqrt{\operatorname{Im} M_n(i))}$. The corresponding Weyl function $\widetilde{M}_n(\cdot)$ is

$$\widetilde{M}_n(z) = R_n^{-1}\big(M_n(z) - \operatorname{Re} M_n(i)\big)R_n^{-1}, \quad n \in \mathbb{N}.$$

Since $\widetilde{M}_n(i) = i,\ n \in \mathbb{N}$, by Theorem 5.3, $\widetilde{\Pi} = \oplus_{n=1}^{\infty}\widetilde{\Pi}_n =: \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary triplet for $A^* = \oplus_{n=1}^{\infty} S_n{}^*$ satisfying $A^* \upharpoonright \ker \widetilde{\Gamma}_0 = A_0 := \oplus_{n=1}^{\infty} S_{0n}$. By definition of $\mathfrak{m}_n^+(\cdot)$ and due to Remark 4.6 one has $\mathfrak{m}_n^+(t) = \widetilde{m}_n^+(t) := \sup_{y\in(0,1]} \|\widetilde{M}_n(t+iy)\|$ for $t \in \mathbb{R}$, $n \in \mathbb{N}$. Since $A_0 = \oplus_{n=1}^{\infty} S_{0n}$ we get that $\widetilde{m}^+(t) = \sup_n m_n^+(t)$, where $\widetilde{m}^+(t) := \sup_{y\in(0,1]} \|\widetilde{M}(t+iy)\|$, $t \in \mathbb{R}$. Since, by assumption, the maximal normal function $\widetilde{m}^+(t)$ is finite, we obtain from Theorem 4.3 that $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent. $\square$

Corollary 5.9 *Let the assumptions of Theorem 5.8 be satisfied and let*

$$\mathcal{N} := \{t \in \mathbb{R} : \sup_{n\in\mathbb{N}} \mathfrak{m}_n^+(t) < \infty\}. \tag{5.19}$$

If condition (4.1) *holds, then the parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\mathcal{N})$ and $A_0^{ac} E_{A_0}(\mathcal{N})$ of the operators $\widetilde{A}$ and A_0, respectively, are unitarily equivalent.*

Let T and T' be densely defined closed symmetric operators in $\mathfrak{H}$ and let T_0 and T_0' be self-adjoint extensions of T and T', respectively. It is said that the pairs $\{T, T_0\}$ and $\{T', T_0'\}$ are unitarily equivalent if there exists a unitary operator U in $\mathfrak{H}$ such that $T' = UTU^{-1}$ and $T_0' = UT_0U^{-1}$.

Corollary 5.10 *Assume the conditions of Theorem 5.8. Let also the pairs $\{S_n, S_{0n}\}$, $n \in \mathbb{N}$, be unitarily equivalent to the pair $\{S_1, S_{01}\}$. If the maximal normal function $m_1^+(t)$ is finite for a.e. $t \in \mathbb{R}$ and condition* (4.1) *is satisfied, then the absolutely continuous parts $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent.*

Proof. Since the symmetric operators S_n are unitarily equivalent, we assume without loss of generality that $\mathcal{H}_n = \mathcal{H}$ for each $n \in \mathbb{N}$. Let U_n be a unitary operator such that $A_1 = U_n S_n U_n^{-1}$ and $A_{01} = U_n S_{0n} U_n^{-1}$. A straightforward computation shows that $\Pi_n' := \{\mathcal{H}, \Gamma_{0n}', \Gamma_{1n}'\}$, $\Gamma_{0n}' := \Gamma_{01} U_n$ and $\Gamma_{1n}' := \Gamma_{1n} U_n$, defines a boundary triplet for S_n^*. The Weyl function $M_n'(\cdot)$ corresponding to Π_n' is $M_n'(z) = M_1(z)$. Hence $\mathfrak{m}_n^+(\cdot) = \mathfrak{m}_n'^+(\cdot)$ and $\mathfrak{m}_1^+(t) = \mathfrak{m}_n'^+(t)$ for $t \in \mathbb{R}$, where $\mathfrak{m}_n^+(t)$ and $\mathfrak{m}_n'^+(t)$ are the invariant maximal normal functions corresponding to the triplets Π_n and Π_n', respectively. By Remark 4.6(i), $\mathfrak{m}_1^+(t) = \mathfrak{m}_n^+(t)$ for $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Applying Theorem 5.8 we complete the proof. $\square$

5.3 Finite deficiency summands: ac-minimal extensions

Here we improve the previous results assuming that $n_\pm(S_n) < \infty$. First, we show that extensions $A_0 = \oplus_{n=1}^{\infty} S_{0n} (\in \operatorname{Ext}_A)$ of the form (5.13) posses a certain spectral minimality property. To this end we start with the following lemma.

Lemma 5.11 *Let H be a bounded non-negative self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let L be a bounded operator in $\mathfrak{H}$. Then*

(i) $\dim(\overline{\operatorname{ran}(H)}) = \dim(\overline{\operatorname{ran}(\sqrt{H})})$.

(ii) *If $L^*L \le H$, then* $\dim(\overline{\operatorname{ran}(L)}) \le \dim(\overline{\operatorname{ran}(H)})$.

(iii) *If P is an orthogonal projection, then* $\dim(\overline{\operatorname{ran}(PHP)}) \le \dim(\overline{\operatorname{ran}(H)})$.

Proof. The assertion (i) is obvious.

(ii) If $L^*L \le H$, then there is a contraction C such that $L = C\sqrt{H}$. Hence $\dim(\overline{\operatorname{ran}(L)}) = \dim(\overline{\operatorname{ran}(C\sqrt{H})}) \le \dim(\overline{\operatorname{ran}(\sqrt{H})}) = \dim(\overline{\operatorname{ran}(H)})$.

(iii) Clearly, $\dim(\overline{\operatorname{ran}(PHP)}) \le \dim(\overline{\operatorname{ran}(HP)}) \le \dim(\overline{\operatorname{ran}(H)})$. $\square$

Theorem 5.12 *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of densely defined closed symmetric operators, $\mathrm{dom}\,(S_n) \subset \mathfrak{H}_n$, with $n_+(S_n) = n_-(S_n) < \infty$, $n \in \mathbb{N}$ and let $S_{0n} = S_{0n}^* \in \mathrm{Ext}_{S_n}$. Let also A and A_0 be given by* (5.1) *and* (5.13)*, respectively. Then A_0 is ac-minimal, in particular, $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$.*

Proof. By Theorem 5.3 there is a sequence of boundary triplets $\Pi_n := \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$, $n \in \mathbb{N}$, for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$, $n \in \mathbb{N}$, and the direct sum $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} = \bigoplus_{n=1}^{\infty} \Pi_n$ of the form (5.1) is a boundary triplet for A^* satisfying $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. By Proposition 2.9, any $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ admits a representation $\widetilde{A} = A_\Theta$ with $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$. By Corollary 4.2(i), we can assume that $\widetilde{A}$ and A_0, are disjoint, that is $\Theta = B = B^* \in \mathcal{C}(\mathcal{H})$. Consider the generalized Weyl function $M_B(\cdot) := (B - M(\cdot))^{-1}$, where $M(\cdot) = \bigoplus_{n=1}^{\infty} M_n(\cdot)$, cf. (5.14). Clearly,

$$\mathrm{Im}\,(M_B(z)) = M_B(z)^* \mathrm{Im}\,(M(z)) M_B(z), \quad z \in \mathbb{C}_+.$$

Denote by P_N, $N \in \mathbb{N}$, the orthogonal projection from $\mathcal{H}$ onto the subspace $\mathcal{H}_N := \bigoplus_{n=1}^{N} \mathcal{H}_n$. Setting $M_B^{P_N}(z) := P_N M_B(z) \upharpoonright \mathcal{H}_N$, and taking into account the block-diagonal form of $M(\cdot)$ and the inequality $\mathrm{Im}\,(M(z)) > 0$ we obtain

$$\begin{aligned} \mathrm{Im}\,(M_B^{P_N}(z)) &= \mathrm{Im}\,(P_N M_B(z) P_N) \\ &= P_N M_B(z)^* \mathrm{Im}\,(M(z)) M_B(z) P_N \geq M_B^{P_N}(z)^* \mathrm{Im}\,(M^{P_N}(z)) M_B^{P_N}(z), \end{aligned} \tag{5.20}$$

where $M^{P_N}(z) := P_N M(z) \upharpoonright \mathcal{H}_N = \bigoplus_{n=1}^{N} M_n(z)$. Since P^N is a finite dimensional projection the limits $M_B^{P_N}(t) := \text{s-lim}_{y \to +0} M_B^{P_N}(t + iy)$ and $M^{P_N}(t) := \text{s-lim}_{y \to +0} M^{P_N}(t + iy)$ exist for a.e. $t \in \mathbb{R}$. From (5.20) we get

$$\mathrm{Im}\,(M_B^{P_N}(t)) \geq M_B^{P_N}(t)^* \mathrm{Im}\,(M^{P_N}(t)) M_B^{P_N}(t) \quad \text{for a.e.} \quad t \in \mathbb{R}. \tag{5.21}$$

Since $M_B(\cdot)$ is a generalized Weyl function, it is a strict $R_{\mathcal{H}}$-function, that is, $\ker(\mathrm{Im}\,(M_B(z))) = \{0\}$, $z \in \mathbb{C}_+$. Therefore, $M_B^{P_N}(\cdot)$ is also strict. Hence $0 \in \varrho(M_B^{P_N}(z))$, $z \in \mathbb{C}_+$, and $G_N(\cdot) := -(M_B^{P_N}(\cdot))^{-1}$ is strict. Since both $G_N(\cdot)$ and $M_B^{P_N}(\cdot)$ are matrix-valued R-functions, the limits $M_B^{P_N}(t + i0) := \lim_{y \to +0} M_B^{P_N}(t + iy)$ and $G_N(t + i0) := \lim_{y \to +0} G_N(t + iy)$ exist for a.e. $t \in \mathbb{R}$. Therefore, passing to the limit in the identity $M_B^{P_N}(t + iy) G_N(t + iy) = -I$ as $y \to 0$, we get $M_B^{P_N}(t + i0) G_N(t + i0) = -I$ for a.e. $t \in \mathbb{R}$. Hence $M_B^{P_N}(t) := M_B^{P_N}(t + i0)$ is invertible for a.e. $t \in \mathbb{R}$.

Further, combining (5.21) with Lemma 5.11(ii) we get

$$\dim\left(\overline{\mathrm{ran}\,\left(\sqrt{\mathrm{Im}\,M^{P_N}(t)} M_B^{P_N}(t)\right)}\right) \leq d_{M_B^{P_N}}(t) \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Since $M_B^{P_N}(t)$ is invertible for a.e. $t \in \mathbb{R}$, we find

$$d_{M^{P_N}}(t) := \dim\left(\overline{\mathrm{ran}\,\left(\sqrt{\mathrm{Im}\,M^{P_N}(t)}\right)}\right) \leq d_{M_B^{P_N}}(t) \quad \text{for a.e.} \quad t \in \mathbb{R}. \tag{5.22}$$

Let $D_N = P_N \oplus D_0$ where $D_0 \in \mathfrak{S}_2(\mathcal{H}_N^{\perp})$ and satisfy $\ker(D_0) = \ker(D_0^*) = \{0\}$. Then $\ker(D_N) = \ker(D_N^*) = \{0\}$ and $P_N = P_N D_N = D_N P_N$. By Lemma 5.11(iii), $d_{M^{P_N}}(t) \leq d_{M_B^{D_N}}(t)$ for a.e. $t \in \mathbb{R}$. Further, for any $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfying $\ker(D) = \ker(D^*) = \{0\}$, $d_{M_B^D}(t) = d_{M_B^{D_N}}(t)$ for a.e. $t \in \mathbb{R}$. Combining this equality with (5.22) we get $d_{M^{P_N}}(t) \leq d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$ and $N \in \mathbb{N}$. Since

$$d_{M^{P_N}}(t) = \sum_{n=1}^{N} d_{M_n(t)} \quad \text{and} \quad d_{M^D}(t) = \sum_{n=1}^{\infty} d_{M_n}(t) \tag{5.23}$$

for a.e. $t \in \mathbb{R}$, we finally prove that $d_{M^D}(t) \leq d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$. One completes the proof by applying Theorem 3.4(i). $\square$

Corollary 5.13 *Let the assumptions of Theorem 5.12 be satisfied and let $S_n \geq 0$, $n \in \mathbb{N}$. Further, let A and A be given by (5.1) and (5.13), respectively. Then the Friedrichs and the Krein extensions A^F and A^K of A are ac-minimal. In particular, $(A^F)^{ac}$ and $(A^K)^{ac}$ are unitarily equivalent.*

Proof. Combining Theorem 5.12 and Corollary 5.5 yields the assertion. $\square$

Corollary 5.14 *Let the assumptions of Theorem 5.12 be satisfied and let*

$$\mathcal{D} := \{t \in \mathbb{R} : \sum_{n\in\mathbb{N}} d_{M_n}(t) = \infty\}. \tag{5.24}$$

If, in addition, condition (4.1) holds, then the parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\mathcal{D})$ and $A_0^{ac} E_{A_0}(\mathcal{D})$ of the operators $\widetilde{A}$ and A_0, respectively, are unitarily equivalent.

Proof. By (5.23) and (5.24), $d_{M^{\mathcal{D}}}(t) = +\infty$ for a.e. $t \in \mathcal{D}$. Applying Theorem 5.12 and Theorem 2.4(ii) we complete the proof. $\square$

Corollary 5.15 *Let the assumptions of Theorem 5.12 be satisfied and let $\mathcal{N}$ and $\mathcal{D}$ be given by (5.19) and (5.24), respectively. If condition (4.1) is valid, then the parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\mathcal{D} \cup \mathcal{N})$ and $A_0^{ac} E_{A_0}(\mathcal{D} \cup \mathcal{N})$ are unitarily equivalent.*

Proof. By Corollary 5.9, the parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\mathcal{N})$ and $A_0^{ac} E_{A_0}(\mathcal{N})$ are unitarily equivalent. Corollary 5.14 yields the unitary equivalence of the parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\mathcal{D})$ and $A_0^{ac} E_{A_0}(\mathcal{D})$. Hence the parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\mathcal{D} \cup \mathcal{N})$ and $A_0^{ac} E_{A_0}(\mathcal{D} \cup \mathcal{N})$ are unitarily equivalent too. $\square$

Corollary 5.16 *Assume conditions of Theorem 5.12. Then $\overline{\bigcup_{n\in\mathbb{N}} \sigma_{ac}(S_{0n})} \subseteq \sigma_{ac}(\widetilde{A})$. If, in addition, condition (4.1) is valid and the extensions S_{0n}, $n \in \mathbb{N}$, are purely absolutely continuous, then*

$$\sigma_{ac}(\widetilde{A}) = \overline{\bigcup_{n\in\mathbb{N}} \sigma_{ac}(S_{0n})}. \tag{5.25}$$

Proof. The first statement immediately follows from Theorem 5.12. Relation (5.25) is implied by Proposition 5.7. $\square$

Corollary 5.17 *Assume the conditions of Theorem 5.12. Let also the pairs $\{S_n, S_{0n}\}$, $n \in \mathbb{N}$, be pairwise unitarily equivalent. If condition (4.1) holds , then for any $\widetilde{A} \in \mathrm{Ext}_A$ the ac-parts $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent.*

Remark 5.18 (i) For the special case $n_\pm(S_n) = 1$, $n \in \mathbb{N}$, Theorem 5.12 complements [2, Corollary 5.4] where the inclusion $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ was proved. Moreover, Corollary 5.17 might be regarded as a substantial generalization of [2, Theorem 5.6(i)] to the case $n_\pm(S_n) > 1$. However, in the case $n_\pm(S_n) = 1$, Corollary 5.17 is implied by [2, Theorem 5.6(i)] where the unitary equivalence of $\widetilde{A}^{ac} = \widetilde{A}_B^{ac}$ and A_0^{ac} was proved under the weaker assumption that B is purely singular. Indeed, by Proposition 2.12 condition (4.1) with $\widetilde{A} = A_B$ is equivalent to discreteness of B.

(ii) The inequality $N_{E^{ac}_{A_0}}(t) \leq N_{E^{ac}_{\widetilde{A}}}(t)$ in Theorem 5.12 might be strict even for $t \in \sigma_{ac}(A_0)$. Indeed, assume that (α, β) is a gap for all except for the operators $S_1, \ldots, S_N$. Set $S_1 := \oplus_{n=1}^N S_n$ and $S_2 := \oplus_{n=N+1}^\infty S_n$. Then $n_\pm(S_2) = \infty$ and (α, β) is a gap for S_2. By [8] there exists $\widetilde{S}_2 = \widetilde{S}_2^* \in \mathrm{Ext}_{S_2}$ having ac-spectrum within (α, β) of arbitrary multiplicity. Moreover, even for operators $A = \oplus_{n=1}^\infty S_n$ satisfying assumptions of Corollary 5.17 with $n_\pm(S_n) = 1$ the inclusion $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ might be strict whenever condition (4.1) is violated, cf. [8] or [2, Theorem 4.4] which guarantees the appearance of prescribed spectrum either within one gap or within several gaps of A_0.

6 Sturm-Liouville operators with operator potentials

6.1 Bounded operator potentials

Let $\mathcal{H}$ be a separable Hilbert space. As usual, $L^2(\mathbb{R}_+, \mathcal{H}) := L^2(\mathbb{R}_+) \otimes \mathcal{H}$ stands for the Hilbert space of (weakly) measurable vector-functions $f(\cdot) : \mathbb{R}_+ \to \mathcal{H}$ satisfying $\int_{\mathbb{R}_+} \|f(t)\|^2_{\mathcal{H}} dt < \infty$. Denote also by $W^{2,2}(\mathbb{R}_+, \mathcal{H}) := W^{2,2}(\mathbb{R}_+) \otimes \mathcal{H}$ the Sobolev space of vector-functions taking values in $\mathcal{H}$.

Let $T = T^* \geq 0$ be a bounded operator in $\mathcal{H}$. Denote by $A := A_{\min}$ the minimal operator generated on $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$ by a differential expression $\mathcal{A} = -\frac{d^2}{dx^2} \otimes I_{\mathcal{H}} + I_{L^2(\mathbb{R}_+)} \otimes T$. It is known (see [17, 28]) that A is given by

$$(Af)(x) = -f''(x) + Tf(x), \quad f \in \operatorname{dom}(A) = W_0^{2,2}(\mathbb{R}_+, \mathcal{H}), \tag{6.1}$$

where $W_0^{2,2}(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) : \ f(0) = f'(0) = 0\}$.

The operator A is closed, symmetric and non-negative. It can be proved similarly to [9, Example 5.3] that A is simple. The adjoint operator A^* is given by [17, Theorem 3.4.1]

$$(A^*f)(x) = -f''(x) + Tf(x), \quad f \in \operatorname{dom}(A^*) = W^{2,2}(\mathbb{R}_+, \mathcal{H}). \tag{6.2}$$

By [25, Theorem 1.3.1] the trace operators $\Gamma_0, \ \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$,

$$\Gamma_0 f = f(0) \quad \text{and} \quad \Gamma_1 f = f'(0), \quad f \in \operatorname{dom}(A^*), \tag{6.3}$$

are well defined Moreover, the deficiency subspace $\mathfrak{N}_z(A)$ is

$$\mathfrak{N}_z(A) = \{e^{ix\sqrt{z-T}}h : \ h \in \mathcal{H}\}, \qquad z \in \mathbb{C}_\pm. \tag{6.4}$$

Lemma 6.1 *A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, with Γ_0 and Γ_1 defined by* (6.3)*, forms a boundary triplet for A^*. The corresponding Weyl function $M(\cdot)$ is*

$$M(z) = i\sqrt{z-T} = i\int \sqrt{t+iy-\lambda}\, dE_T(\lambda), \quad z = t+iy \in \mathbb{C}_+. \tag{6.5}$$

Proof. One obtains the Green formula integrating by parts. The surjectivity of the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is immediate from (6.3) and [25, Theorem 1.3.2]. Formula (6.5) is implied by (6.4). $\square$

Lemma 6.2 *Let T be a bounded non-negative self-adjoint operator in $\mathcal{H}$ and let A and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be defined by* (6.1) *and* (6.3)*, respectively. Then*

(i) *the invariant maximal normal function $\mathfrak{m}^+(t)$ of the Weyl function $M(\cdot)$ is finite for all $t \in \mathbb{R}$ and satisfies*

$$\mathfrak{m}^+(t) \leq 2(1+t^2)^{1/4}, \quad t \in \mathbb{R}. \tag{6.6}$$

(ii) *The limit $M(t+i0) :=$ s-$\lim_{y\to+0} M(t+iy)$ exists, is bounded and equals*

$$M(t+i0) = i\int_{\mathbb{R}} \sqrt{t-\lambda}\, dE_T(\lambda) \quad \textit{for any} \ \ t \in \mathbb{R}. \tag{6.7}$$

(iii) $d_M(t) = \dim(\operatorname{ran}(E_T([0,t))))$ *for any $t \in \mathbb{R}$.*

Proof. (i) It is immediate from (6.5) and definition (4.17) of $\mathfrak{m}^+(\cdot)$ that

$$\mathfrak{m}^+(t) \le \sup_{y\in(0,1]} \sup_{\lambda\ge 0} \left| \frac{\sqrt{t+iy-\lambda} - \mathrm{Re}\,(\sqrt{i-\lambda})}{\mathrm{Im}\,(\sqrt{i-\lambda})} \right|.$$

Clearly, $\sqrt{i-\lambda} = (1+\lambda^2)^{1/4} e^{i(\pi-\varphi)/2}$ where $\varphi := \arccos\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$. Hence

$$\left| \frac{\mathrm{Re}(\sqrt{i-\lambda})}{\mathrm{Im}(\sqrt{i-\lambda})} \right| = \tan(\frac{\varphi}{2}) = \frac{1}{\lambda + \sqrt{1+\lambda^2}} \le 1, \quad \lambda \ge 0.$$

Furthermore, we have

$$\left| \frac{\sqrt{t+iy-\lambda}}{\mathrm{Im}(\sqrt{i-\lambda})} \right| \le \sqrt{2} \sqrt{\frac{\sqrt{(\lambda-t)^2+y^2}}{\lambda+\sqrt{1+\lambda^2}}} \le 2^{3/4}(1+t^2)^{1/4}$$

for $\lambda \ge 0$, $t \in \mathbb{R}$ and $y \in (0,1]$ which yields (6.6).

(ii) From (6.5) we find $M(t) := M(t+i0) := \text{s-lim}_{y\to+0}\, i\sqrt{t+iy-T} = i\sqrt{t-T}$, for any $t \in \mathbb{R}$, which proves (6.7). Clearly, $M(t) \in [\mathcal{H}]$ since $T \in [\mathcal{H}]$.

(iii) It follows that $\mathrm{Im}(M(t)) = \sqrt{t-T}E_T([0,t))$, which yields $d_M(t) = \dim(\mathrm{ran}\,(\mathrm{Im}(M(t)))) = \dim(\mathrm{ran}\,(E_T([0,t))))$. $\square$

With the operator $A = A_{\min}$ it is naturally associated a (closable) quadratic form $\mathfrak{t}'_F[f] := (Af, f)$, $\mathrm{dom}\,(\mathfrak{t}') = \mathrm{dom}\,(A)$. Its closure $\mathfrak{t}_F$ is given by

$$\mathfrak{t}_F[f] := \int_{\mathbb{R}_+} \left\{ \|f'(x)\|^2_{\mathcal{H}} + \|\sqrt{T} f(x)\|^2_{\mathcal{H}} \right\} dx, \tag{6.8}$$

$f \in \mathrm{dom}\,(\mathfrak{t}_F) = W^{1,2}_0(\mathbb{R}_+, \mathcal{H})$, where $W^{1,2}_0(\mathbb{R}_+, \mathcal{H}) := \{W^{1,2}(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}$. By definition, the Friedrichs extension A^F of A is a self-adjoint operator associated with $\mathfrak{t}_F$. Clearly, $A^F = A^* \upharpoonright (\mathrm{dom}\,(A^*) \cap \mathrm{dom}\,(\mathfrak{t}_F))$.

Theorem 6.3 *Let $T \ge 0$, $T = T^* \in [\mathcal{H}]$, and $t_0 := \inf \sigma(T)$. Let A be defined by* (6.1) *and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ the boundary triplet for A^* defined by* (6.3)*. Then*

(i) *the Friedrichs extension A^F coincides with A_0 that is*

$$\mathrm{dom}\,(A^F) = \mathrm{dom}\,(A^*) \cap \mathrm{dom}\,(\mathfrak{t}_F) = \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) :\ f(0) = 0\} = \mathrm{dom}\,(A_0),$$

and A^F corresponds to the Dirichlet problem. Moreover, A^F is absolutely continuous, $A^F = (A^F)^{ac}$, and $\sigma(A^F) = \sigma_{ac}(A^F) = [t_0, \infty)$.

(ii) *the Krein extension A^K is given by*

$$\mathrm{dom}\,(A^K) = \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) :\ f'(0) + \sqrt{T} f(0) = 0\}. \tag{6.9}$$

Moreover, $\ker\,(A^K) = \mathfrak{H}_0 := \overline{\mathfrak{H}'_0}$, $\mathfrak{H}'_0 := \{e^{-x\sqrt{T}}h :\ h \in \mathrm{ran}\,(T^{1/4})\}$ *and the restriction $A^K \upharpoonright \mathrm{dom}\,(A^K) \cap \mathfrak{H}_0^\perp$ is absolutely continuous, that is $\mathfrak{H}_0^\perp = \mathfrak{H}^{ac}(A^K)$ and $A^K = 0_{\mathfrak{H}_0} \oplus (A^K)^{ac}$.*

(iii) *The extension $A_1 := A^* \upharpoonright \ker\,(\Gamma_1)$, coincides with A^N,* $\mathrm{dom}\,(A^N) := \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) : f'(0) = 0\}$, *i.e. A_1 corresponds to the Neumann boundary condition. Moreover, A^N is absolutely continuous $(A^N)^{ac} = A^N$ and $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.*

(iv) *The operators A^F, $(A^K)^{ac}$ and A^N are unitarily equivalent.*

Proof. (i) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet defined in Lemma 6.1. We show that $A^F = A_0 := A^* \upharpoonright \ker(\Gamma_0)$. It follows from (6.2) and (6.3) that $\mathrm{dom}(A_0) = \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}$. Since $\mathrm{dom}(A_0) \subset W_0^{1,2}(\mathbb{R}_+, \mathcal{H}) = \mathrm{dom}(\mathfrak{t}_F)$, we have $A_0 = A^F$ (see [1, Section 8] and [20, Theorem 6.2.11]).

It follows from (6.7) and [9, Theorem 4.3] that $\sigma_p(A_0) = \sigma_{sc}(A_0) = \emptyset$. Hence A_0 is absolutely continuous. Taking into account Lemma 6.2(iii) and Proposition 3.2 we get $\sigma(A_0) = \sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\mathrm{supp}\,(d_M)) = [t_0, \infty)$ which proves (i).

(ii) By [12, Proposition 5] A^K is defined by $A^K = A^* \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0)$. It follows from (6.5) that $M(0) = -\sqrt{T}$. Therefore, A^K is defined by (6.9).

It follows from the extremal property of the Krein extension that $\ker(A^K) = \ker(A^*)$. Clearly, $f_h(x) := \exp(-x\sqrt{T})h \in L^2(\mathbb{R}_+, \mathcal{H})$, $h \in \mathrm{ran}\,(T^{1/4})$, since

$$\int_0^\infty \|\exp(-x\sqrt{T})h\|^2_{\mathcal{H}} dx = \int_0^{\|T\|} d\rho_h(t) \int_0^\infty e^{-2x\sqrt{t}} dx = \int_0^{\|T\|} \frac{1}{2\sqrt{t}} d\rho_h(t) < \infty,$$

where $\rho_h(t) := \big(E_T(t)h, h\big)$. Thus, $\mathfrak{H}_0' \subset \ker(A^*)$. It is easily seen that $\mathfrak{H}_0'$ is dense in $\mathfrak{H}_0$. To investigate the rest of the spectrum of A^K consider the Weyl function $M_K(\cdot)$ corresponding to A^K. It follows from (6.5) and Proposition 2.17 that

$$\begin{aligned} M_K(z) &= M_{-\sqrt{T}}(z) = -\big(\sqrt{T} + M(z)\big)^{-1} \\ &= -(\sqrt{T} + i\sqrt{z-T})^{-1} = \frac{1}{z}(i\sqrt{z-T} - \sqrt{T}) = -\frac{2\sqrt{T}}{z} + \Phi(z). \end{aligned}$$

where $\Phi(z) := \frac{1}{z}[i\sqrt{z-T} + \sqrt{T}]$. It follows that for $t > 0$

$$\mathrm{Im}\, M_K(t + i0) = \mathrm{Im}\, \Phi(t + i0) = t^{-1}\sqrt{t - T} E_T([0, t)). \tag{6.10}$$

Hence, by [9, Theorem4.3], $\sigma_p(A^K) \cap (0, \infty) = \sigma_{sc}(A^K) \cap (0, \infty) = \emptyset$. It follows from (6.10) that $\mathrm{Im}\,(M_K(t + i0)) > 0$ for $t > t_0$. By Proposition 3.2 $\sigma_{ac}(A^K) = [t_0, \infty)$. Further, it follows from (6.7) and (6.10) that for any $t > t_0$

$$\begin{aligned} d_M(t) &= \mathrm{rank}\,(\mathrm{Im}(M(t))) \\ &= \mathrm{rank}\,(E_T([0, t))) = \mathrm{rank}\,(\mathrm{Im}(M_K(t))) = d_{M_K}(t) \end{aligned}$$

Combining this equality with $\sigma_{ac}(A^K) = \sigma_{ac}(A^F) = [t_0, \infty)$, we conclude that A^F and $(A^K)^{ac}$ are unitarily equivalent.

(iii) By Proposition 2.17 the Weyl function corresponding to $A_1 = A^* \upharpoonright \ker(\Gamma_1 - 0\Gamma_0)$ is

$$M_0(z) := (0 - M(z))^{-1} = i(z - T)^{-1/2} = i \int \frac{1}{\sqrt{z - \lambda}} dE_T(\lambda), \quad z \in \mathbb{C}_+.$$

Since $M_0(\cdot)$ is regular within $(-\infty, t_0)$, we have $(-\infty, t_0) \subset \varrho(A_1)$. Further, let $\tau > t_0$. We set $\mathcal{H}_\tau := E_T([t_0, \tau))\mathcal{H}$ and note that for any $h \in \mathcal{H}_\tau$ and $t > \tau$

$$\big(M_0(t + i0)h, h\big) = i\big((t - T)^{-1/2}h, h\big) = i \int_{t_0}^\tau \frac{1}{\sqrt{t - \lambda}} d(E_T(\lambda)h, h). \tag{6.11}$$

Hence for any $h \in \mathcal{H}_\tau \setminus \{0\}$ and $t > \tau$

$$0 < (t - t_0)^{-1/2}\|h\|^2 \le \mathrm{Im}\,(M_0(t + i0)h, h) = \int_{t_0}^\tau (t - \lambda)^{-1/2} d(E_T(\lambda)h, h) < \infty.$$

By [9, Proposition 4.2], $\sigma_{ac}(A_1) \supseteq [\tau, \infty)$ for any $\tau > t_0$, which yields $\sigma_{ac}(A_1) = [t_0, \infty)$. It remains to show that A_1 is purely absolutely continuous. Since $M_0(t + i0) \notin [\mathcal{H}]$ we cannot apply

[9, Theorem 4.3] directly. Fortunately, to investigate the ac-spectrum of A_1 we can use [9, Corollary 4.7]. For any $t \in \mathbb{R}$, $y > 0$, and $h \in \mathcal{H}$ we set

$$V_h(t+iy) := \operatorname{Im}(M_0(t+iy)h, h) = \int \operatorname{Im}\left(\frac{1}{\sqrt{\lambda - t - iy}}\right) d(E_T(\lambda)h, h).$$

Obviously, one has

$$V_h(t+iy) \le \int \frac{1}{((\lambda-t)^2 + y^2)^{1/4}} d(E_T(\lambda)h, h), \quad t \in \mathbb{R}, \quad y > 0, \quad h \in \mathcal{H}.$$

Hence

$$V_h(t+iy)^p \le \|h\|^{2(p-1)} \int \frac{1}{((\lambda-t)^2 + y^2)^{p/4}} d(E_T(\lambda)h, h), \quad p \in (1, \infty).$$

We show that for $p \in (1,2)$ and $-\infty < a < b < \infty$

$$C_p(h; a, b) := \sup_{y \in (0,1]} \int_a^b V_h(t+iy)^p \, dt < \infty.$$

Clearly,

$$\begin{aligned}\int_a^b V_h(t+iy)^p dt &\le \|h\|^{2(p-1)} \int_0^{\|T\|} d(E(\lambda)h, h) \int_a^b \frac{1}{((\lambda-t)^2 + y^2)^{p/4}} dt \\ &= \|h\|^{2(p-1)} \int_0^{\|T\|} d(E(\lambda)h, h) \int_{a-\lambda}^{b-\lambda} \frac{1}{(t^2 + y^2)^{p/4}} dt.\end{aligned}$$

Note, that for $p \in (1,2)$ and $-\infty < a < b < \infty$

$$\int_{a-\lambda}^{b-\lambda} \frac{1}{(t^2+y^2)^{p/4}} dt \le \int_{a-\|T\|}^{b} \frac{1}{t^{p/2}} dt =: \varkappa_p(b, a - \|T\|) < \infty,$$

Hence $C_p(h; a, b) \le \varkappa_p(b, a - \|T\|)\|h\|^{2p} < \infty$ for $p \in (1,2)$, $-\infty < a < b < \infty$ and $h \in \mathcal{H}$. By [9, Corollary 4.7], A_1 is purely absolutely continuous on any bounded interval (a, b). Hence A_1 is purely absolutely continuous.

(iv) It follows from (6.7) and (6.10) that $d_M(t) = d_{M_K}(t) = \operatorname{rank}(\sqrt{t-T})$ for $t > t_0$. Combining this equality with $\sigma_{ac}(A^K) = \sigma_{ac}(A^F) = [t_0, \infty)$, we conclude that A^F and $(A^K)^{ac}$ are unitarily equivalent.

Passing to A_1, we assume that $1 \le \dim(\operatorname{ran}(E_T([0,s)))) = p_1 < \infty$ for some $s > 0$. Let λ_k, $k \in \{1, \ldots, p\}$, $p \le p_1$, be the set of distinct eigenvalues within $[0, s)$. Since $M_0(t+iy)E_T([0,t))$ is the $p \times p$ matrix-function, the limit $M_0(t+i0)E_T([0,t))$ exists for $t \in [0,s) \setminus \bigcup_{k=1}^p \{\lambda_k\}$. It follows from (6.11) that

$$\operatorname{Im}(M_0(t)) = |T - t|^{-1/2} E_T([0,t)), \qquad t \in [0,s) \setminus \bigcup_{k=1}^p \{\lambda_k\}.$$

This yields

$$d_{M_0(t))} := \dim(\operatorname{ran}(\operatorname{Im}(M_0(t)))) = \dim(\operatorname{ran}(E_T([0,t)))) = d_M(t)$$

for a.e $t \in [0,s) \setminus \bigcup_{k=1}^p \{\lambda_k\}$, that is, for a.e. $t \in [0,s)$.

If $\dim(E_T([t_0,s))) = \infty$, then there exists a point $s_0 \in (0,s)$, such that $\dim(E_T([0,s_0])) = \infty$ and $\dim(E_T([0,s))) < \infty$ for $s \in [0, s_0)$. For any $t \in (s_0, s)$ choose $\tau \in (s_0, t)$ and note that $\dim(\operatorname{ran}(E_T([0,\tau)))) = \infty$. We set $\mathcal{H}_\tau := E_T([0,\tau))\mathcal{H}$ and $\mathcal{H}_\infty := E_T([\tau,\infty))\mathcal{H}$ as well as $T_\tau := TE_T([0,\tau))$ and $T_\infty := TE_T([\tau,\infty))$. Further, we choose Hilbert-Schmidt operators D_τ and D_∞ in $\mathcal{H}_\tau$ and $\mathcal{H}_\infty$, respectively, such that $\ker(D_\tau) = \ker(D_\tau^*) = \ker(D_\infty) = \ker(D_\infty^*) = \{0\}$.

According to the decomposition $\mathcal{H} = \mathcal{H}_\tau \oplus \mathcal{H}_\infty$ we have $M_0 = M_\tau \oplus M_\infty$, $D = D_\tau \oplus D_\infty$ and $d_{M_0^D}(t) = d_{M_\tau^{D_\tau}}(t) + d_{M_\infty^{D_\infty}}(t)$ for a.e. $t \in [0,\infty)$. Hence $d_{M_0^D}(t) \geq d_{M_\tau^{D_\tau}}(t)$ for a.e. $t \in [0,\infty)$. Clearly, $M_\tau(t+iy) = i(t+iy-T_\tau)^{-1/2}$. If $t > \tau$, then $t \in \varrho(T_\tau)$ and $M(t) := \text{s-lim}_{y\to+0} M(t+i0)$ exists and

$$M_\tau(t) := \operatorname*{s-lim}_{y\to 0} M_\tau(t+iy) = i(t-T_\tau)^{-1/2}E_T([0,\tau)).$$

Hence $d_{M_\tau^{D_\tau}}(t) = \dim(\operatorname{ran}(E_T([0,\tau)))) = \infty$ for $t > s_0$. Hence $d_{M_0^D}(t) = d_M(t) = \infty$ for a.e. $t > s_0$ which yields $d_{M_0^D}(t) = d_M(t)$ for a.e. $t \in [0,\infty)$. Using Theorem 3.4(ii) we obtain that A_0^{ac} and A_1^{ac} are unitarily equivalent which shows A_0 and A_1 are unitarily equivalent. $\square$

Next we describe the spectral properties of any self-adjoint extension of A. In particular, we show that the Friedrichs extension A^F of A is ac-minimal, though A does not satisfy conditions of Theorem 5.12.

Theorem 6.4 *Let $T \geq 0$, $T = T^* \in [\mathcal{H}]$, and $t_1 := \inf \sigma_{\rm ess}(T)$. Let also A be the symmetric operator defined by* (6.1) *and $\widetilde{A} = \widetilde{A}^* \in \operatorname{Ext}_A$. Then*

(i) *the absolutely continuous part $\widetilde{A}^{ac}E_{\widetilde{A}}([t_1,\infty))$ of $\widetilde{A}E_{\widetilde{A}}([t_1,\infty))$ is unitarily equivalent to $A^F E_{A^F}([t_1,\infty)) = (A^F)^{ac}E_{A^F}([t_1,\infty))$;*

(ii) *the Friedrichs extension A^F is ac-minimal and $\sigma_{ac}(A^F) \subseteq \sigma_{ac}(\widetilde{A})$;*

(iii) *the absolutely continuous part $\widetilde{A}^{ac}$ of $\widetilde{A}$ is unitarily equivalent to A^F whenever either $(\widetilde{A} - i)^{-1} - (A^F - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ or $(\widetilde{A} - i)^{-1} - (A^K - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$.*

Proof. By Corollary 4.2 it suffices to assume that the extension $\widetilde{A} = \widetilde{A}^*$ is disjoint with A_0, that is, by Proposition 2.9(ii) it admits a representation $\widetilde{A} = A_B$ with $B \in \mathcal{C}(\mathcal{H})$.

(i) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* defined by (6.3). In accordance with Theorem 3.4 we calculate $d_{M_B^K}(t)$ where $M_B(\cdot) := (B - M(\cdot))^{-1}$ is the generalized Weyl function of the extension A_B. Clearly,

$$\operatorname{Im}(M_B(z)) = M_B(z)^* \operatorname{Im}(M(z))M_B(z), \quad z \in \mathbb{C}_+. \tag{6.12}$$

Since $\operatorname{Re}(\sqrt{z-\lambda}) > 0$ for $z = t+iy,\ y > 0$, it follows from (6.5) that

$$\operatorname{Im}(M(z)) = \int_{[0,\infty)} \operatorname{Re}(\sqrt{z-\lambda})\, dE_T(\lambda) \geq \int_{[0,\tau)} \operatorname{Re}(\sqrt{z-\lambda})\, dE_T(\lambda), \tag{6.13}$$

where $z = t+iy$. It is easily seen that

$$\operatorname{Re}(\sqrt{z-\lambda}) \geq \sqrt{t-\lambda} \geq \sqrt{t-\tau}, \quad \lambda \in [0,\tau), \quad t > \tau. \tag{6.14}$$

Combining (6.12) with (6.13) and (6.14) we get

$$\operatorname{Im}(M_B(t+iy)) \geq \sqrt{t-\tau} M_B(t+iy)^* E_T([0,\tau))M_B(t+iy), \quad t > \tau > 0.$$

Let Q be a finite-dimensional orthogonal projection, $Q \leq E_T([0,\tau))$. Hence

$$\operatorname{Im}(M_B(t+iy)) \geq \sqrt{t-\tau} M_B(t+iy)^* Q M_B(t+iy), \quad t > \tau > 0, \quad y > 0.$$

Setting $\mathcal{H}_1 = \operatorname{ran}(Q)$, $\mathcal{H}_2 := \operatorname{ran}(Q^\perp)$, and choosing $K_2 \in \mathfrak{S}_2(\mathcal{H}_2)$ and satisfying $\ker(K_2) = \ker(K_2^*) = \{0\}$, we define a Hilbert-Schmidt operator $K := Q \oplus K_2 \in \mathfrak{S}_2(\mathcal{H})$. Clearly, $\ker(K) = \ker(K^*) = \{0\}$ and,

$$\begin{aligned}\operatorname{Im}(K^* M_B(t+iy)K) \geq& \\ &\sqrt{t-\tau} K^* M_B(t+iy)^* Q M_B(t+iy)K, \quad t > \tau > 0.\end{aligned} \tag{6.15}$$

Since $M_B(\cdot) \in (R_{\mathcal{H}})$ and $Q,\ K \in \mathfrak{S}_2(\mathcal{H})$, the limits

$$\begin{aligned} K^*M_B(t)^*Q &:= \text{s-}\lim_{y\to+0} K^*M_B(t+iy)^*Q \quad \text{and} \\ (QM_BK)(t) &:= \text{s-}\lim_{y\to+0} QM_B(t+iy)K \end{aligned}$$

exist for a.e. $t \in \mathbb{R}$ (see [5]). Therefore passing to the limit as $y \to 0$ in (6.15), we arrive at the inequality

$$\mathrm{Im}(M_B^K(t)) \geq \sqrt{t-\tau}(K^*M_B(t)^*Q)(QM_BK(t)), \quad t > \tau > 0, \quad y > 0.$$

It follows that

$$\dim(\mathrm{ran}\ ((QM_BK)(t))) \leq \dim(\mathrm{ran}\ \big(\mathrm{Im}\, M_B^K(t)\big)) = d_{M_B^K}(t), \quad t > \tau. \tag{6.16}$$

We set $\widetilde{M}_B^Q(z) := QM_B(z)Q \upharpoonright \mathcal{H}_1$. Since $\dim(\mathcal{H}_1) < \infty$ the limit $\widetilde{M}_B^Q(t) := \text{s-lim}_{y\to+0} \widetilde{M}_B^Q(t+iy)$ exists for a.e. $t \in \mathbb{R}$. Since $(QM_BK)(t) \upharpoonright \mathcal{H}_1 = \mathrm{ran}\ \Big((\widetilde{M}_B^Q)(t)\Big)$, (6.16) yields the inequality

$$\dim(\mathrm{ran}\ \Big(\widetilde{M}_B^Q(t)\Big)) \leq \dim(\mathrm{ran}\ ((QM_BK)(t))) \leq d_{M_B^K}(t) \tag{6.17}$$

for a.e. $t \in [\tau, \infty)$.

Since $\dim(\mathcal{H}_1) < \infty$ and $\ker(\widetilde{M}_B^Q(z)) = \{0\}, z \in \mathbb{C}$, we easily get by repeating the corresponding reasonings of the proof of Theorem 5.12 that $\mathrm{ran}\ \Big(\widetilde{M}_B^Q(t)\Big) = \mathcal{H}_1$ for a.e. $t \in \mathbb{R}$. Therefore (6.17) yields $\dim(\mathcal{H}_1) \leq d_{M_B^K}(t)$ for a.e. $t \in [\tau, \infty)$.

If $\tau > t_1$, then $\dim(E_T([0,\tau))\mathcal{H}) = \infty$ and the dimension of a projection $Q \leq E_T([0,\tau))$ can be arbitrary. Thus, $d_{M_B^K}(t) = \infty$ for a.e. $t > \tau$. Since $\tau > t_1$ is arbitrary we get $d_{M_B^K}(t) = \infty$ for a.e. $t > t_1$. By Theorem 3.4(ii) the operator $\widetilde{A}^{ac}E_{\widetilde{A}}([t_1,\infty))$ is unitarily equivalent to $A_0E_{A_0}([t_1,\infty))$.

(ii) If $\tau \in (t_0, t_1)$, then $\dim(E_T([0,\tau))\mathcal{H}) =: p(\tau) < \infty$. Hence, $\dim(Q\mathcal{H}) \leq p(\tau)$ which shows that $d_{M_B^K}(t) \geq p(\tau)$ for a.e. $t \in (\tau, t_1)$. Since τ is arbitrary, we obtain $d_{M_B^K}(t) \geq p(\tau)$ for a.e. $t \in [0, t_1)$. Using Theorem 3.4(i) we prove (ii).

(iii) By Lemma 6.2 the invariant maximal normal function $\mathfrak{m}^+(t)$ is finite for $t \in \mathbb{R}$. By Theorem 4.3 $\widetilde{A}^{ac}$ and $(A^F)^{ac}$ are unitarily equivalent. Similarly we prove that $\widetilde{A}^{ac}$ and $(A^K)^{ac}$ are unitarily equivalent. To complete the proof it remains to apply Theorem 6.3(iv). □

Corollary 6.5 *Let the assumptions of Theorem 6.4 be satisfied. If* $\dim(\mathcal{H}) = \infty$ *and* $t_0 := \inf\sigma(T) = \inf\sigma_{\rm ess}(T) =: t_1$, *then the Friedrichs extension* A^F *is strictly ac-minimal.*

Remark 6.6 Let $\dim(E_T([t_0,t_1))\mathcal{H}) = \infty$. Then there are self-adjoint extensions $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ of A such that $\sigma_{ac}(\widetilde{A}) = \sigma(A^F) = \sigma_{ac}(A^F)$ but $\widetilde{A}$ is not unitarily equivalent to A^F.

6.2 Unbounded operator potentials

In this subsection we consider the differential expression (6.1) with unbounded $T = T^* \geq 0$, $T \in \mathcal{C}(\mathcal{H})$,

$$(\mathcal{A}_T f)(x) = -\frac{d^2}{dx^2}f(x) + Tf(x). \tag{6.18}$$

The minimal operator $A := A_{T,\min}$ is defined as the closure of the operator A'_T generated on $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$ by expression (6.18) on the domain

$$\mathcal{D}'_0 := \left\{ \sum_{1\leq j\leq k} \phi_j(x)h_j :\ \phi_j \in W_0^{2,2}(\mathbb{R}_+),\ h_j \in \mathrm{dom}\,(T), \ \ k \in \mathbb{N} \right\}, \tag{6.19}$$

that is $A'_T f = \mathcal{A}_T f,\ \mathrm{dom}\,(A'_T) = \mathcal{D}'_0$. Clearly, A is non-negative, since $T \geq 0$ and $A_{T,min} := \overline{A'_T}$ coincides with A defined by (6.1) provided that T is bounded.

Let $\mathcal{H}_T$ be the Hilbert space which is obtained equipping the set $\mathrm{dom}\,(T)$ with the graph norm of T. Following [25] we introduce the Hilbert spaces $W^{k,2}_T(\mathbb{R}_+;\mathcal{H}) := W^{k,2}(\mathbb{R}_+;\mathcal{H}) \cap L^2(\mathbb{R}_+,\mathcal{H}_T)$, $k \in \mathbb{N}$, equipped with the Hilbert norms

$$\|f\|^2_{W^{k,2}_T} = \int_{\mathbb{R}_+} \left(\|f^{(k)}(t)\|^2_{\mathcal{H}} + \|f(t)\|^2_{\mathcal{H}} + \|Tf(t)\|^2_{\mathcal{H}}\right) dt.$$

Obviously, we have $W^{2,2}_{0,T}(\mathbb{R}_+,\mathcal{H}) := \{f \in W^{2,2}_T(\mathbb{R}_+;\mathcal{H}) :\ f(0) = f'(0) = 0\} \subseteq \mathrm{dom}\,(A_{T,\min})$.

Lemma 6.7 *Let $T = T^*$ be a non-negative operator in $\mathcal{H}$. Then* $\mathrm{dom}\,(A_{T,\min})$ *and* $W^{2,2}_{0,T}(\mathbb{R}_+,\mathcal{H})$ *coincide algebraically and topologically.*

Proof. Obviously, for any $f \in \mathcal{D}'_0$ we have

$$\begin{aligned}\|\mathcal{A}_T f\|^2_{\mathfrak{H}} &= \int_{\mathbb{R}_+} \|f''(x)\|^2_{\mathcal{H}}\, dx \\ &+ \int_{\mathbb{R}_+} \|Tf(x)\|^2_{\mathcal{H}} dx - 2\mathrm{Re}\left\{\int_{\mathbb{R}_+} (f''(x), Tf(x))_{\mathcal{H}}\, dx\right\}.\end{aligned}$$

Integrating by parts we find

$$\int_{\mathbb{R}_+} (f''(x), Tf(x))\, dx = -\int_{\mathbb{R}_+} \left\|\sqrt{T} f'(x)\right\|^2_{\mathcal{H}} dx.$$

Combining these equalities yields

$$\|\mathcal{A}_T f\|^2_{\mathfrak{H}} = \int_{\mathbb{R}_+} \|f''(x)\|\, dx + \int_{\mathbb{R}_+} \|Tf(x)\|^2 dx + 2\int_{\mathbb{R}_+} \left\|\sqrt{T} f'(x)\right\|^2_{\mathcal{H}} dx$$

for any $f \in \mathcal{D}'_0$. Hence

$$\|f\|^2_{W^{2,2}_T} \leq \|\mathcal{A}_T f\|^2_{\mathfrak{H}} + \|f\|^2, \quad f \in \mathcal{D}'_0.$$

Furthermore, by the Schwartz inequality,

$$2\left|\mathrm{Re}\left\{\int_{\mathbb{R}_+} (f'(x), Tf(x))_{\mathcal{H}}\, dx\right\}\right| \leq \|f\|^2_{W^{2,2}_T}, \quad f \in \mathcal{D}'_0.$$

which gives

$$\|\mathcal{A}_T f\|^2_{\mathfrak{H}} + \|f\|^2 \leq 2\|f\|^2_{W^{2,2}_T}, \quad f \in \mathcal{D}'_0.$$

Thus, we arrive at the two-sided estimate

$$\|f\|^2_{W^{2,2}_T} \leq \|\mathcal{A}_T f\|^2_{\mathfrak{H}} + \|f\|^2_{\mathfrak{H}} \leq 2\|f\|^2_{W^{2,2}_T} \qquad f \in \mathcal{D}'_0.$$

Since $\mathcal{D}'_0$ is dense in $W^{2,2}_{0,T}$ too, we obtain that $\mathrm{dom}\,(A_{T,\min})$ coincides with $W^{2,2}_{0,T}$ algebraically and topologically. □

Since A is non-negative it admits the Friedrichs extension A^F and the Krein extension A^K. We define the extension A^N as the self-adjoint operator associated with the closed quadratic form $\mathfrak{t}_N$,

$$\mathfrak{t}_N[f] := \int_0^\infty \left\{\|f'(x)\|^2_{\mathcal{H}} + \|\sqrt{T} f(x)\|^2_{\mathcal{H}}\right\} dx = \|u\|^2_{W^{1,2}_{\sqrt{T}}} - \|u\|^2_{L^2(\mathbb{R}_+,\mathcal{H})}, \tag{6.20}$$

$\mathrm{dom}\,(\mathfrak{t}_N) := W^{1,2}_{\sqrt{T}}(\mathbb{R}_+,\mathcal{H})$. The definition makes sense for $T \in [\mathcal{H}]$. In this case $A^N = A_1$ with A_1 defined in Theorem 6.3(iii).

We also put $\mathfrak{t}_F := \mathfrak{t}_N \upharpoonright \mathrm{dom}\,(\mathfrak{t}_F)$, $\mathrm{dom}\,(\mathfrak{t}_F) := \{f \in W^{1,2}_{\sqrt{T}}(\mathbb{R}_+;\mathcal{H}) :\ f(0) = 0\}$.

Proposition 6.8 *Let $T = T^* \in \mathcal{C}(\mathcal{H})$, $T \geq 0$, and let $A := A_{T,\min}$ be defined by (6.18)-(6.19). Let also $\mathcal{H}_n := \operatorname{ran}\big(E_T([n-1,n))\big)$, $T_n := TE_T([n-1,n))$ and let S_n be the closed minimal symmetric operator defined by (6.1) in $\mathfrak{H}_n := L^2(\mathbb{R}_+, \mathcal{H}_n)$ with T replaced by T_n. Then*

(i) *the following decompositions hold*

$$A = \bigoplus_{n=1}^{\infty} S_n, \quad A^F = \bigoplus_{n=1}^{\infty} S_n^F, \quad A^K = \bigoplus_{n=1}^{\infty} S_n^K, \quad A^N = \bigoplus_{n=1}^{\infty} S_n^N. \tag{6.21}$$

(ii) *The domain $\operatorname{dom}(A^F)$ equipped with the graph norm is a closed subspace of $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$,*

$$\operatorname{dom}(A^F) = \{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}.$$

(iii) *The domain $\operatorname{dom}(A^N)$ equipped with the graph norm is a closed subspace of $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$, $\operatorname{dom}(A^N) = \{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) : f'(0) = 0\}$.*

Proof. (i) We introduce the sets

$$\mathcal{D}_0'' := \left\{ \sum_{1 \leq j \leq k} \phi_j(x) h_j : \ \phi_j \in W_0^{2,2}(\mathbb{R}_+), \ h_j \in \mathcal{H}_n, \ k, n \in \mathbb{N} \right\}$$

and $\mathcal{D}_{0n}'' := \{f \in \mathcal{D}_0'' : f(x) \in \mathcal{H}_n\}$, $n \in \mathbb{N}$. Obviously, we have $\mathcal{D}_0'' = \bigoplus_{n=1}^{\infty} \mathcal{D}_{0n}'' \subseteq \mathcal{D}_0'$. Setting $A_T'' := A_T' \upharpoonright \mathcal{D}_0''$ we find $\overline{A_T''} = \overline{A_T'} = A_{T,\min}$. Moreover, setting $A_n'' := A_n \upharpoonright \mathcal{D}_{0n}''$, $n \in \mathbb{N}$, we have $\overline{A_n''} = A_n$, $n \in \mathbb{N}$. Since $A_T'' = \bigoplus_{n=1}^{\infty} A_n'' \subseteq A_T'$, we obtain

$$A_{T,min} = \overline{A_T''} = \bigoplus_{n=1}^{\infty} \overline{A_n''} = \bigoplus_{n=1}^{\infty} A_n \subseteq A_{T,min}$$

which proves the first relation of (6.21). The second and the third relations are implied by Corollary 5.5.

To prove the last relation of (6.21) we set $S^N := \bigoplus_{n=1}^{\infty} S_n^N$. Since $S_n^N = (S_n^N)^* \in \operatorname{Ext}_{S_n}$ and $A = \oplus_{n=1}^{\infty} S_n$, S^N is a self-adjoint extension of A, $S^N \in \operatorname{Ext}_A$. Let $f = \oplus_{n=1}^{\infty} f_n \in \operatorname{dom}(S^N)$. Then integrating by parts we obtain

$$\begin{aligned}(S^N f, f) = \sum_1^{\infty} (S_n^N f_n, f_n) &= \sum_{n=1}^{\infty} \int_0^{\infty} \left\{ \|f_n'(x)\|_{\mathcal{H}_n}^2 + \|\sqrt{T}_n f_n(x)\|_{\mathcal{H}_n}^2 \right\} dx \\ &= \int_0^{\infty} \left\{ \|f'(x)\|_{\mathcal{H}}^2 + \|\sqrt{T} f(x)\|_{\mathcal{H}}^2 \right\} dx = \mathfrak{t}_N[f].\end{aligned}$$

Since, by definition, A^N is associated with the quadratic form $\mathfrak{t}_N$, the last equality yields $S^N \subset A^N$. Hence $S^N = A^N$, since both S^N and A^N are self-adjoint extensions of A.

(ii) Following the reasoning of Lemma 6.7 we find

$$\|f_n\|_{W_{T_n}^{2,2}}^2 \leq \|S_n^F f_n\|_{\mathfrak{H}_n}^2 + \|f_n\|_{\mathfrak{H}_n}^2 \leq 2\|f_n\|_{W_{T_n}^{2,2}}^2, \qquad n \in \mathbb{N}, \tag{6.22}$$

where $f_n \in \operatorname{dom}(S_n^F) = \{g_n \in W^{2,2}(\mathbb{R}_+, \mathcal{H}_n) : g_n(0) = 0\}$. Using representation (6.21) for A^F and setting $f^m := \oplus_{n=1}^{m} f_n$, $f_n \in \operatorname{dom}(F_n)$, we obtain from (6.22)

$$\|f^m\|_{W_T^{2,2}}^2 \leq \|A^F f^m\|_{\mathfrak{H}}^2 + \|f^m\|_{\mathfrak{H}}^2 \leq 2\|f^m\|_{W_T^{2,2}}^2, \quad m \in \mathbb{N}. \tag{6.23}$$

Since the set $\{f^m = \oplus_{n=1}^{m} f_n : \ f_n \in \operatorname{dom}(S_n^F), \ m \in \mathbb{N}\}$, is a core for A^F, inequality (6.23) remains valid for $f \in \operatorname{dom}(A^F)$. This shows that $\operatorname{dom}(A^F) = \{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}$. Moreover, due to (6.23) the graph norm of A^F and the norm $\|\cdot\|_{W_T^{2,2}}$ restricted to $\operatorname{dom}(A^F)$ are equivalent.

(iii) Similarly to (6.22) one gets

$$\|f_n\|^2_{W^{2,2}_{T_n}} \le \|S_n^N f_n\|^2_{\mathfrak{H}_n} + \|f_n\|^2 \le 2\|f_n\|^2_{W^{2,2}_{T_n}}$$

for $f_n \in \operatorname{dom}(S_n^N) = \{g_n \in W^{2,2}(\mathbb{R}_+, \mathcal{H}_n) : g_n'(0) = 0\}$, $n \in \mathbb{N}$. It remains to repeat the reasonings of (ii) $\square$

To extend Theorem 6.3 to the case of unbounded operators $T = T^* \ge 0$ we first construct a boundary triplet for A^*, using Theorem 5.3 and representation (6.21) for A.

Lemma 6.9 *Assume conditions of Proposition 6.8. Then there is a sequence of boundary triplets* $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ *for* S_n^* *such that* $\Pi := \oplus_{n=1}^\infty \widehat{\Pi}_n =: \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ *forms an ordinary boundary triplet for* A^*. *Moreover,* $A^F = A^* \upharpoonright \ker(\widehat{\Gamma}_0)$ *and the corresponding Weyl function is*

$$\widehat{M}(z) = \frac{i\sqrt{z-T} + \operatorname{Im}(\sqrt{i-T})}{\operatorname{Re}(\sqrt{i-T})}. \qquad z \in \mathbb{C}_+, \tag{6.24}$$

Proof. For any $n \in \mathbb{N}$ we define a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* with Γ_{0n}, Γ_{1n} defined by (6.3). By Theorem 6.3(i) $S_n^F = S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$ and by Lemma 6.1 the corresponding Weyl function is $M_n(z) = i\sqrt{z - T_n}$.

Following Lemma 5.1, cf. (5.6), we define a sequence of regularized boundary triplets $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ for S_n^* by setting $R_n := (\operatorname{Re}(\sqrt{i-T_n}))^{1/2}$, $Q_n := -\operatorname{Im}(\sqrt{i-T_n})$ and

$$\widehat{\Gamma}_{0n} := R_n\Gamma_{0n}, \quad \widehat{\Gamma}_{1n} := R_n^{-1}(\Gamma_{1n} - Q_n\Gamma_{0n}), \qquad n \in \mathbb{N}. \tag{6.25}$$

Hence $S_n^F = S_{0n}$ and the corresponding Weyl function $\widehat{M}_n(\cdot)$ is given by

$$\widehat{M}_n(z) = \frac{i\sqrt{z-T_n} + \operatorname{Im}(\sqrt{i-T_n})}{\operatorname{Re}(\sqrt{i-T_n})}, \qquad z \in \mathbb{C}_+, \qquad n \in \mathbb{N}. \tag{6.26}$$

By Theorem 5.3 the direct sum $\widehat{\Pi} := \bigoplus_{n=1}^\infty \widehat{\Pi}_n = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ forms a boundary triplet for A^* and the corresponding Weyl function is

$$\widehat{M}(z) = \bigoplus_{n\in\mathbb{N}} \widehat{M}_n(z) \qquad z \in \mathbb{C}_+. \tag{6.27}$$

Combining (6.27) with (6.26) we arrive at (6.24).

Combining Theorem 5.3 (cf. (5.13)) with Corollary 5.5 we get

$$A_0 = A^* \upharpoonright \ker(\widehat{\Gamma}_0) = \bigoplus_{n=1}^\infty S_n^* \upharpoonright \ker(\widehat{\Gamma}_{0n}) = \bigoplus_{n=1}^\infty S_{0n} = \bigoplus_{n=1}^\infty S_n^F = A^F \tag{6.28}$$

which proves the second assertion. $\square$

Next we generalize Theorem 6.3 to the case of unbounded operator potentials.

Theorem 6.10 *Let* $T = T^* \ge 0$, $t_0 := \inf\sigma(T)$, *and* $A := A_{T,min}$, *cf.* (6.18)-(6.19). *Let also* $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ *be the boundary triplet for* A^* *defined by Lemma 6.9 and* $\widehat{M}(\cdot)$ *the corresponding Weyl function (cf.* (6.24)*). Then*

(i) *The Friedrichs extension* A^F *coincides with* $A_0 := A^* \upharpoonright \ker(\widehat{\Gamma}_0)$. *Moreover,* A^F *is absolutely continuous,* $A^F = (A^F)^{ac}$, *and* $\sigma(A^F) = \sigma_{ac}(A^F) = [t_0, \infty)$.

(ii) *The Krein extension* A^K *is given by* $A_{B^K} := A^* \upharpoonright \ker(\Gamma_1 - B^K\Gamma_0)$, *where*

$$B^K = \frac{1}{\sqrt{2}\sqrt{T} + \sqrt{T + \sqrt{1+T^2}}} \frac{1}{\sqrt{T + \sqrt{1+T^2}}}. \tag{6.29}$$

Moreover, $\ker(A^K) = \mathfrak{H}_0 := \overline{\mathfrak{H}_0'}$, $\mathfrak{H}_0' := \{e^{-x\sqrt{T}}h : h \in \mathrm{ran}(T^{1/4})\}$, *the restriction* $A^K \upharpoonright \mathrm{dom}(A^K) \cap \mathfrak{H}_0^\perp$ *is absolutely continuous, and* $A^K = 0_{\mathfrak{H}_0} \oplus (A^K)^{ac}$.

(iii) *The extension* A^N *is given by* $A^N = A^* \upharpoonright \ker(\widehat{\Gamma}_1 - B^N\widehat{\Gamma}_0)$ *where* $B^N := \sqrt{T + \sqrt{1+T^2}}$. *Moreover,* A^N *is absolutely continuous,* $A^N = (A^N)^{ac}$ *and* $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.

(iv) *The operators* A^F, $(A^K)^{ac}$ *and and* A^N *are unitarily equivalent.*

Proof. (i) This statement is implied by combining Theorem 6.3 with (6.28).

(ii) Using the polar decomposition $i - \lambda = \sqrt{1+\lambda^2}e^{i\theta(\lambda)}$ with $\theta(\lambda) = \pi - \arctan(1/\lambda)$, $\lambda \geq 0$ we get

$$\mathrm{Re}(\sqrt{i-T}) = \int_0^\infty \sqrt[4]{1+\lambda^2}\cos(\theta(\lambda)/2)dE_T(\lambda). \tag{6.30}$$

Setting $\varphi(\lambda) = \arctan(1/\lambda)$, $\lambda \geq 0$ and noting that $\cos(\varphi(\lambda)) = \lambda(1+\lambda^2)^{-1/2}$, we find $\cos(\theta(\lambda)/2) = 2^{-1/2}(1+\lambda^2)^{-1/4}(\lambda + \sqrt{1+\lambda^2})^{-1/2}$. Substituting this expression in (6.30) yields

$$\mathrm{Re}(\sqrt{i-T}) = 2^{-1/2}(T + \sqrt{1+T^2}\,)^{-1/2}. \tag{6.31}$$

Similarly, taking into account $\sin(\theta(\lambda)/2) = \cos(\varphi(\lambda)/2)$ and $\cos(\varphi(\lambda)/2) = 2^{-1/2}(1+\lambda^2)^{-1/4}(\lambda + \sqrt{1+\lambda^2})^{1/2}$, we get

$$\mathrm{Im}(\sqrt{i-T}) = \int_0^\infty \sqrt[4]{1+\lambda^2}\cos(\varphi(\lambda)/2)dE_T(\lambda) = \frac{1}{\sqrt{2}}\sqrt{T + \sqrt{1+T^2}}. \tag{6.32}$$

It follows from (6.24) with account of (6.31) and (6.32) that $M(0) := \text{s-lim}_{x\to+0} M(-x) =: B^K$ where B^K is defined by (6.29). Therefore, by [12, Proposition 5(iv)] the Krein extension A^K is given by $A_{B^K} := A^* \upharpoonright \ker(\Gamma_1 - B^K\Gamma_0)$. The second statement follows from Proposition 6.8 and Theorem 6.3(ii).

(iii) It is easily seen that in the boundary triplet $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ defined by (6.25) the extension A_n^N admits a representation $A_n^N = A_{B_n}$ where $B_n := \sqrt{T_n + \sqrt{1+T_n^2}}$, $n \in \mathbb{N}$. By Proposition 6.8, $A^N = \oplus_{n=1}^\infty A_n^N = A_{B^N}$ where $B^N = \oplus_{n=1}^\infty B_n$. The remaining part of (iii) follows from the representation $A^N = \bigoplus_{n=1}^\infty A_n^N$ and Theorem 6.3(iii).

(iv) The assertion follows from Theorem 6.3(iv) and (6.21). □

Next we generalize Theorem 6.4 to the case of unbounded $T \geq 0$.

Theorem 6.11 *Let* $T = T^* \geq 0$ *and* $t_1 := \inf\sigma_{\mathrm{ess}}(T)$. *Further, let* $A := A_{T,\min}$, *cf.* (6.18)-(6.19), *and* $\widetilde{A} = \widetilde{A}^* \in Ext_A$. *Then*

(i) *the absolutely continuous part* $\widetilde{A}^{ac}E_{\widetilde{A}}([t_1,\infty))$ *is unitarily equivalent to the part* $A^F E_{A^F}([t_1,\infty)) = (A^F)^{ac}E_{A^F}([t_1,\infty))$;

(ii) *the Friedrichs extension* A^F *is ac-minimal and* $\sigma_{ac}(A^F) \subseteq \sigma_{ac}(\widetilde{A})$;

(iii) *the ac-part* $\widetilde{A}^{ac}$ *of* $\widetilde{A}$ *is unitarily equivalent to* A^F *if either* $(\widetilde{A} - i)^{-1} - (A^F - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ *or* $(\widetilde{A} - i)^{-1} - (A^K - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$.

Proof. By Corollary 4.2 it suffices to assume that the extension $\widetilde{A} = \widetilde{A}^*$ is disjoint with A_0, that is, it admits a representation $\widetilde{A} = A_B$ with $B \in \mathcal{C}(\mathcal{H})$.

(i) We consider the boundary triplet $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ defined in Lemma 6.9. By Proposition 2.17 the generalized Weyl function corresponding to the generalized boundary triplet $\widehat{\Pi}_B$ is defined by $\widehat{M}_B(z) = (B - \widehat{M}(z))^{-1}$, $z \in \mathbb{C}_+$, where $\widehat{M}(z)$ is given by (6.24). Clearly,

$$\mathrm{Im}(\widehat{M}_B(z)) = \widehat{M}_B(z)^* \,\mathrm{Im}(\widehat{M}(z))\,\widehat{M}_B(z), \qquad z \in \mathbb{C}_+. \tag{6.33}$$

It follows from (6.24) that $\left(\mathrm{Re}(\sqrt{i-T})\right)^{-1} \geq \sqrt{2}$. Therefore (6.31) yields

$$\mathrm{Im}(\widehat{M}(z)) \geq \sqrt{2}\,\mathrm{Im}(M(z)), \quad z \in \mathbb{C}_+, \quad \text{where} \quad M(z) = i\sqrt{z-T}, \tag{6.34}$$

cf. (6.5). Following the line of reasoning of the proof of Theorem 6.4(i) we obtain from (6.34) that $d_{\widehat{M}^D}(t) = \infty$ for a.e. $t \in [t_1, \infty)$, where $D = D^* \in \mathfrak{S}_2(\mathcal{H})$ and $\ker D = \{0\}$. Moreover, it follows from (6.33) that $d_{\widehat{M}^D_B}(t) = d_{\widehat{M}^D}(t) = \infty$ for a.e. $t \in [t_1, \infty)$. One completes the proof by applying Theorem 3.4.

(ii) To prove (ii) we use again estimates (6.34) and follow the proof of Theorem 6.4(ii). We complete the proof by applying Theorem 3.4.

(iii) The Weyl function $\widehat{M}(\cdot)$ is given by (6.24). Taking into account (6.27) one obtains $\sup_{n\in\mathbb{N}} \mathfrak{m}_n^+ < \infty$, where $\mathfrak{m}_n^+$ is the maximal normal invariant function defined by (4.17). Indeed, this follows from (6.6) because this estimate shows that $\mathfrak{m}_n^+$ does not depend on $n \in \mathbb{N}$. Applying Theorem 4.3 and Remark 4.6 we complete the proof.

To prove the second statement we note that the operator B^K defined by (6.29) is bounded. Therefore, by Proposition 2.17 a triplet $\widehat{\Pi}_{B^K} := \{\mathcal{H}, \widehat{\Gamma}_0^{B^K}, \widehat{\Gamma}_1^{B^K}\}$ with $\widehat{\Gamma}_1^{B^K} := \widehat{\Pi}_0$ and $\widehat{\Gamma}_0^{B^K} := B^K\widehat{\Gamma}_0 - \widehat{\Gamma}_1$, is a boundary triplet for A^* such that $A^K := A^* \upharpoonright \ker(\widehat{\Gamma}_0^{B^K})$. The corresponding Weyl function is

$$\widehat{M}_{B^K}(z) = (B^K - \widehat{M}(z))^{-1}, \quad z \in \mathbb{C}_+.$$

Inserting expression (6.29) into this formula we get

$$\widehat{M}_{B^K}(z) = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{T} + i\sqrt{z-T}}\frac{1}{\sqrt{T+\sqrt{1+T^2}}} = \frac{1}{z\sqrt{2}}\frac{\sqrt{T} - i\sqrt{z-T}}{\sqrt{T+\sqrt{1+T^2}}}.$$

It follows that the limit $\widehat{M}_{B^K}(t + i0)$ exists for any $t \in \mathbb{R} \setminus \{0\}$ and

$$\widehat{M}_{B^K}(t) := \text{s-}\lim_{y\to *0} M_{B^K}(t+iy) = -\frac{1}{t\sqrt{2}}\frac{\sqrt{T} - i\sqrt{t-T}}{\sqrt{T+\sqrt{1+T^2}}}.$$

Clearly, $\widehat{M}_{B^K}(t) \in [\mathcal{H}]$ for any $t \in \mathbb{R}\setminus\{0\}$. By Theorem 4.3 the ac-parts of $\widetilde{A}$ and A^K are unitarily equivalent whenever $(\widetilde{A} - i)^{-1} - (A^K - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$. This completes the proof. □

Finally, we generalize Corollary 6.5 to unbounded operator potentials.

Corollary 6.12 *Assume conditions of Theorem 6.11. If* $\dim(\mathcal{H}) = \infty$ *and* $t_0 := \inf\sigma(T) = \inf\sigma_{\mathrm{ess}}(T) =: t_1$*, then the Friedrichs extension* A^F *and the Krein extension* A^K *are strictly ac-minimal.*

6.3 Application

In this subsection we apply previous results of this section to Schroedinger operator in the half-plane. To this end we denote by $L = L_{\min}$ the minimal elliptic operator associated in $L^2(\Omega)$, $\Omega :=$

$\mathbb{R}_+ \times \mathbb{R}^n$, with the differential expression

$$\mathcal{L} := -\Delta + q(x) := -\Big(\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\Big) + q(x), \quad (t,x) \in \Omega, \tag{6.35}$$

where $q = \bar{q} \in L^\infty(\mathbb{R})$, $x := (x_1, \ldots, x_n)$ and $n \geq 1$.

Recall that $L_{\min}$ is the closure of $\mathcal{L}$, defined on $C_0^\infty(\Omega)$. It is known that $\operatorname{dom}(L_{\min}) = H_0^2(\Omega)$. Clearly, L is symmetric. The maximal operator $L_{\max}$ is then defined by $L_{\max} = (L_{\min})^*$. We emphasize that $H^2(\Omega) \subset \operatorname{dom}(A_{\max}) \subset H^2_{loc}(\Omega)$, while $\operatorname{dom}(L_{\max}) \neq H^2(\Omega)$.

Next we define the trace mappings $\gamma_j \colon C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$, $j \in \{0,1\}$, by setting $\gamma_0 u := u \restriction \partial\Omega$ and $\gamma_1 u := \gamma_0(\partial u/\partial n)$ where n stands for the interior normal to $\partial\Omega$. Denote by $D_{\mathcal{L}}(\Omega)$ the domain $\operatorname{dom}(L_{\max})$ equipped with the graph norm. It is known (see [25, 18]) that γ_j can be extended by continuity to the operators mapping $D_{\mathcal{L}}(\Omega)$ continuously onto $H^{-j-1/2}(\partial\Omega)$, $j \in \{0,1\}$.

Let us define the following extensions of $L_{\min}$ (realizations of $\mathcal{L}$):

(i) $L^D f := \mathcal{L}[f]$, $f \in \operatorname{dom}(L^D) := \{\varphi \in H^2(\mathbb{R}_+ \times \mathbb{R}) : \ \gamma_0\varphi = 0\}$;

(ii) $L^N f := \mathcal{L}[f]$, $f \in \operatorname{dom}(L^N) := \{\varphi \in H^2(\mathbb{R}_+ \times \mathbb{R}) : \ \gamma_1\varphi = 0\}$;

(iii) $L^K f := \mathcal{L}[f]$, $f \in \operatorname{dom}(L^K) := \{\varphi \in \operatorname{dom}(A_{\max}) : \gamma_1\varphi + \Lambda\gamma_0\varphi = 0\}$, where $\Lambda := \sqrt{-\Delta_x + q(\cdot)} : \ H^{-1/2}(\partial\Omega) \to H^{-3/2}(\partial\Omega)$.

To treat the operator $L_{\min}$ as the Sturm-Liouville operator with (unbounded) operator potential we denote by T the (closed) minimal operator associated in $\mathcal{H} := L^2(\mathbb{R}^n)$ with the Schrödinger expression

$$-\Delta_x + q(x) := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + q(x). \tag{6.36}$$

Since $q = \bar{q} \in L^\infty(\mathbb{R})$, the operator T is self-adjoint and semibounded. Moreover, $T \geq 0$ if $q(\cdot) \geq 0$. Let $A := A_{T,\min}$ be the minimal operator associated with (6.18) where T is defined by (6.36).

Proposition 6.13 *Let $q(\cdot) \in L^\infty(\mathbb{R})$, $q(\cdot) \geq 0$, and let T be the minimal (self-adjoint) operator associated in $L^2(\mathbb{R})$ with the Schrödinger expression* (6.36)*. Let also $t_0 := \inf\sigma(T)$ and $t_1 := \inf\sigma_{\text{ess}}(T)$. Then:*

(i) *the operator $A_{T,\min}$ coincides with $L = L_{\min}$ and $\operatorname{dom}(A_{T,\min}) = H_0^2(\Omega)$;*

(ii) *the Friedrichs extension A^F coincides with L^D, hence L^D is absolutely continuous, $\sigma(L^D) = \sigma_{ac}(L^D) = [t_0, \infty)$ and $N_{L^D}(t) = \infty$ for a.e. $t \in [t_0, \infty)$;*

(iii) *the Krein extension A^K coincides with L^K, in particular, L^K admits the decomposition $L^K = 0_{\mathcal{H}_0} \oplus (L^K)^{ac}$, $\mathcal{H}_0 := \ker(L^K)$, and $\sigma_{ac}(L^K) = [t_0, \infty)$;*

(iv) *the extension A^N defined by* (6.21)*, coincides with L^N, in particular, L^N is absolutely continuous and $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$;*

(v) *the operators L^D, L^N, and (L^K) are ac-minimal, in particular, L^D, L^N, and $(L^K)^{ac}$ are pairwise unitarily equivalent. If, in addition, $t_0 = t_1$, then the operators L^D, L^N, and (L^K) are strictly ac-minimal;*

(vi) *if $\widetilde{L}$ is a self-adjoint extension of L and $(\widetilde{L} - i)^{-1} - (L^D - i)^{-1} \in \mathfrak{S}_\infty$, then $\widetilde{L}^{ac}$ and L^D are unitarily equivalent. If $\widetilde{L}$ satisfies $(\widetilde{L} - i)^{-1} - (L^K - i)^{-1} \in \mathfrak{S}_\infty$, then $\widetilde{L}^{ac}$ and L^D are unitarily equivalent.*

Proof. (i) We introduce the set

$$\mathcal{D}'_\infty := \left\{ \sum_{1\le j\le k} \phi_j(x)h_j(\xi): \ \phi_j \in C_0^\infty(\mathbb{R}_+), \ h_j \in C_0^\infty(\mathbb{R}), \ k \in \mathbb{N} \right\}$$

We note that $\mathcal{D}'_\infty \subseteq \mathcal{D}'_0$ and $\mathcal{D}'_\infty \subseteq C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$. Moreover, $A_{T,\min} \upharpoonright \mathcal{D}'_\infty = L \upharpoonright \mathcal{D}'_\infty$. Since $\mathcal{D}'_\infty$ is a core for both $A_{T,\min}$ and $L_{\min}$, we have $A_{T,\min} = L_{\min}$.

It is clear (after applying the Fourier transform) that $\operatorname{dom}(T) = \operatorname{dom}(\Delta_x) = H^2(\mathbb{R}^n)$. Therefore, by Lemma 6.7, $\operatorname{dom}(A_{T,\min}) = W^{2,2}_{0,T} = H^2_0(\Omega)$.

(ii) Since $\operatorname{dom}(T) = H^2(\mathbb{R}^n)$, we have $W^{2,2}_T(\mathbb{R}_+;\mathcal{H}) = H^2(\Omega)$. Therefore by Proposition 6.8 $A^F = L^D$. The second assertion follows from Theorem 6.10(i).

(iii) It is proved in [12, Section 9.7] that L^K is the Krein extension of $L_{\min}$. The rest of the statements is implied by Theorem 6.10(ii).

(iv) The equality $A^N = L^N$ is immediate from Proposition 6.8(iii). The second statement follows from Theorem 6.10(iii).

(v) By Theorem 6.11(ii) the extension $L^D(= A^F)$ is ac-minimal. By Theorem 6.10(iv) L^D, $L^N(= A^N)$, and $(L^K)^{ac}(= (A^K)^{ac})$ are pairwise unitarily equivalent, hence L^N, and L^K are ac-minimal too. The last statement is immediate from Corollary 6.12.

(vi) The statement is immediate from (ii), (iii) and Theorem 6.11(iii). □

Remark 6.14 Let T be the (closed) minimal non-negative operator associated in $\mathcal{H} := L^2(\mathbb{R}^n)$ with general uniformly elliptic operator

$$-\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_j} + q(x), \quad a_{jk} \in C^1(\overline{\Omega}), \quad q \in C(\overline{\Omega}) \cap L^\infty(\Omega), \tag{6.37}$$

where the coefficients $a_{jk}(\cdot)$ are bounded with their C^1-derivatives, $q \ge 0$. If the coefficients have some additional "good" properties, then $\operatorname{dom}(T) = H^2(\mathbb{R}^n)$ algebraically and topologically. By Lemma 6.7, $\operatorname{dom}(A_{T,\min}) = W^{2,2}_{0,T}(\mathbb{R}_+,\mathcal{H}) = H^{2,2}_0(\Omega)$ and Proposition 6.13 remains valid with T in place of the Schrödinger operator (6.36).

Note also that the Dirichlet and the Neumann realizations L^D and L^N are always self-adjoint ((cf. [25, Theorem 2.8.1], [18])).

Corollary 6.15 *Assume the conditions of Proposition 6.13. If, in addition,*

$$\lim_{|x|\to\infty} \int_{|x-y|\le 1} |q(y)|dy = 0, \tag{6.38}$$

then the operators L^D, L^N, and (L^K) are strictly ac-minimal,

$$\sigma(L^D) = \sigma_{ac}(L^D) = \sigma_{ac}(L^K) = \sigma(L^N) = \sigma_{ac}(L^N) = [0,\infty),$$

and $N_{E_{L^D}}(t) = N_{E_{L^N}}(t) = N_{E^{ac}_{L^K}}(t) = \infty$ for a.e. $t \in [0,\infty)$.

Proof. By [16, Section 60] condition (6.38) yields the equality $\sigma_c(T) = \mathbb{R}_+$, in particular $0 \in \sigma_c(T)$ and $t_1 = 0$. Since $q \ge 0$, we have $0 \le t_0 \le t_1 = 0$, that is $t_0 = t_1 = 0$. It remains to apply Proposition 6.13 (ii)-(v). □

Remark 6.16 Condition (6.38) is satisfied whenever $\lim_{|x|\to\infty} q(x) = 0$. Thus, in this case the conclusions of Corollary 6.15 are valid. However, it might happen that $\sigma(L^F) = \sigma_{ac}(L^K) = \sigma(L^N) = [t_0,\infty)$, $t_0 > 0$ though $\inf q(x) = 0$.

A Appendix: Absolutely continuous closure

Let us recall some basic facts of the ac-closure of a Borel set of $\mathbb{R}$ introduced in[9], see also [15].

Definition A.1 ([9]) *Let $\delta \in \mathcal{B}(\mathbb{R})$. The set $\mathrm{cl}_{ac}(\delta)$ defined by*

$$\mathrm{cl}_{ac}(\delta) := \{x \in \mathbb{R} : \ |(x-\varepsilon, x+\varepsilon) \cap \delta| > 0 \ \ \forall\, \varepsilon > 0\}.$$

is called the absolutely continuous closure of the Borel set $\delta \in \mathcal{B}(\mathbb{R})$.

Obviously, two Borel sets $\delta_1, \delta_2 \in \mathcal{B}(\mathbb{R})$ have the same ac-closure if their symmetric difference $\delta_1 \triangle \delta_2$ has Lebesgue measure zero. Moreover, the set $\mathrm{cl}_{ac}(\delta)$ is always closed and $\mathrm{cl}_{ac}(\delta) \subseteq \overline{\delta}$. In particular, if we have two measurable non-negative functions ξ_1 and ξ_2 which differ only on a set of Lebesgue measure zero, then $\mathrm{cl}_{ac}(\mathrm{supp}\,(\xi_1)) = \mathrm{cl}_{ac}(\mathrm{supp}\,(\xi_2))$.

Lemma A.2 *If $\delta \in \mathcal{B}(\mathbb{R})$, then $|\delta \setminus \mathrm{cl}_{ac}(\delta)| = 0$.*

Proof. Since $\mathrm{cl}_{ac}(\delta)$ is closed the set $\Delta := \mathbb{R} \setminus \mathrm{cl}_{ac}(\delta)$ is open. The open set Δ is decomposed as $\Delta = \bigcup_{l=1}^{L} \Delta_l$, $1 \le L \le \infty$, where $\Delta_l = (a_l, b_l)$ are disjoint open intervals. We set $\Delta_l = \delta \cap \Delta_l$, $l = 1, 2, \ldots, L$. Obviously,

$$\delta \setminus \mathrm{cl}_{ac}(\delta) = \delta \cap \Delta = \bigcup_{l=1}^{L} \Delta_l.$$

We note that $\Delta_l \cap \mathrm{cl}_{ac}(\delta) = \emptyset$, $l = 1, 2, \ldots, L$. Hence for each $t \in \Delta_l$ there is a sufficiently small neighborhood $\mathcal{O}_t$ such that $|\mathcal{O}_t \cap \delta| = 0$. If η is sufficiently small, then $[a_l + \eta, a_l - \eta] \subseteq (a_l, b_l)$ and $\{\mathcal{O}_t\}_{t \in \Delta_l}$ forms a covering of $[a_l + \eta, a_l - \eta]$. Since $[a_l + \eta, a_l - \eta]$ is compact we can chosen a finite covering $\{\mathcal{O}_{t_m}\}_{m=1}^{M}$ of $[a_l + \eta, a_l - \eta]$. By $[a_l + \eta, a_l - \eta] \subseteq \bigcup_{m=1}^{M} \mathcal{O}_{t_m}$ we find $|[a_l + \eta, a_l - \eta] \cap \delta| = 0$ for each sufficiently small $\eta > 0$. Using that we get

$$\begin{gathered} |(a_l, b_l) \cap \delta| = |(a_l, a_l + \eta) \cap \delta| + |(b_l - \eta, b_l) \cap \delta| = \\ |(a_l, a_l + \eta) \cap \delta| + |(b_l - \eta, b_l) \cap \delta| \le 2\eta \end{gathered}$$

for sufficiently small $\eta > 0$. Hence $|\Delta_l| = |(a_l, b_l) \cap \delta| = 0$ which yields that $|\delta \setminus \mathrm{cl}_{ac}(\delta)| = 0$. □

Lemma A.3 *If $\{\delta_k\}_{k \in \mathbb{N}}$, $\delta_k \subseteq \mathbb{R}$, is a sequence of Borel subsets, then*

$$\mathrm{cl}_{ac}(\delta) = \overline{\bigcup_{k \in \mathbb{N}} \mathrm{cl}_{ac}(\delta_k)}, \quad \delta = \bigcup_{k \in \mathbb{N}} \delta_k. \tag{A.1}$$

Proof. We set $\widehat{\delta}_k = \delta_k \cap \mathrm{cl}_{ac}(\delta_k)$ and $\Delta_k := \delta_k \setminus \mathrm{cl}_{ac}(\delta_k)$. We have $\delta = \widehat{\delta} \cup \Delta$, where $\widehat{\delta} := \bigcup_{k \in \mathbb{N}} \widehat{\delta}_k$ and $\Delta := \bigcup_{k \in \mathbb{N}} \Delta_k$. By Lemma A.2, $|\Delta_k| = 0$, $k \in \mathbb{N}$, which yields $|\Delta| = 0$. Hence $\mathrm{cl}_{ac}(\delta) = \mathrm{cl}_{ac}(\widehat{\delta})$. Similarly one gets $\mathrm{cl}_{ac}(\delta_k) = \mathrm{cl}_{ac}(\widehat{\delta}_k)$, $k \in \mathbb{N}$. Notice that $\widehat{\delta}_k \subseteq \mathrm{cl}_{ac}(\widehat{\delta}_k)$, $k \in \mathbb{N}$. We have

$$\mathrm{cl}_{ac}(\widehat{\delta}) \supseteq \bigcup_{k \in \mathbb{N}} \mathrm{cl}_{ac}(\widehat{\delta}_k) \supseteq \bigcup_{k \in \mathbb{N}} \widehat{\delta}_k = \widehat{\delta}.$$

Hence

$$\mathrm{cl}_{ac}(\widehat{\delta}) = \overline{\mathrm{cl}_{ac}(\widehat{\delta})} \supseteq \overline{\bigcup_{k \in \mathbb{N}} \mathrm{cl}_{ac}(\widehat{\delta}_k)} \supseteq \overline{\widehat{\delta}} \supseteq \mathrm{cl}_{ac}(\widehat{\delta})$$

which yields $\mathrm{cl}_{ac}(\widehat{\delta}) = \overline{\bigcup_{k \in \mathbb{N}} \mathrm{cl}_{ac}(\widehat{\delta}_k)}$. Since $\mathrm{cl}_{ac}(\widehat{\delta}) = \mathrm{cl}_{ac}(\delta)$ and $\mathrm{cl}_{ac}(\widehat{\delta}_k) = \mathrm{cl}_{ac}(\delta_k)$, $k \in \mathbb{N}$, we prove (A.1). □

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References

[1] N. I. Achieser and I. M. Glasmann. *Theorie der linearen Operatoren im Hilbert-Raum.* Verlag Harri Deutsch, Thun, eighth edition, 1981.

[2] S. Albeverio, J. F. Brasche, M. M. Malamud, and H. Neidhardt. Inverse spectral theory for symmetric operators with several gaps: scalar-type Weyl functions. *J. Funct. Anal.*, 228(1):144–188, 2005.

[3] Yu. M. Berezanskiĭ. *Expansions in eigenfunctions of selfadjoint operators.* Translated from the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman. Translations of Mathematical Monographs, Vol. 17. American Mathematical Society, Providence, R.I., 1968.

[4] M. Š. Birman. Existence conditions for wave operators. *Izv. Akad. Nauk SSSR Ser. Mat.*, 27:883–906, 1963.

[5] M. Š. Birman and S. B. Èntina. Stationary approach in abstract scattering theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:401–430, 1967.

[6] M. Š. Birman and M. G. Kreĭn. On the theory of wave operators and scattering operators. *Dokl. Akad. Nauk SSSR*, 144:475–478, 1962.

[7] M. Š. Birman and M. Z. Solomjak. *Spectral theory of selfadjoint operators in Hilbert space.* Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.

[8] J. F. Brasche. Spectral theory for self-adjoint extensions. In *Spectral theory of Schrödinger operators*, volume 340 of *Contemp. Math.*, pages 51–96. Amer. Math. Soc., Providence, RI, 2004.

[9] J. F. Brasche, M. M. Malamud, and H. Neidhardt. Weyl function and spectral properties of self-adjoint extensions. *Integral Equations Operator Theory*, 43(3):264–289, 2002.

[10] V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo. Generalized resolvents of symmetric operators and admissibility. *Methods Funct. Anal. Topology*, 6(3):24–55, 2000.

[11] V. A. Derkach and M. M. Malamud. On the Weyl function and Hermite operators with lacunae. *Dokl. Akad. Nauk SSSR*, 293(5):1041–1046, 1987.

[12] V. A. Derkach and M. M. Malamud. Generalized resolvents and the boundary value problems for Hermitian operators with gaps. *J. Funct. Anal.*, 95(1):1–95, 1991.

[13] V. A. Derkach and M. M. Malamud. The extension theory of Hermitian operators and the moment problem. *J. Math. Sci.*, 73(2):141–242, 1995. Analysis. 3.

[14] A. Dijksma and H. S. V. de Snoo. Symmetric and selfadjoint relations in Kreĭn spaces. I. volume 24 of *Oper. Theory Adv. Appl.*, pages 145–166. Birkhäuser, Basel, 1987.

[15] F. Gesztesy, K. A. Makarov, and M. Zinchenko. Essential closures and AC spectra for reflectionless CMV, Jacobi, and Schrödinger operators revisited. *Acta Appl. Math.*, 103(3):315–339, 2008.

[16] I. M. Glazman. *Direct methods of qualitative spectral analysis of singular differential operators.* Israel Program for Scientific Translations, Jerusalem, 1966.

[17] V. I. Gorbachuk and M. L. Gorbachuk. *Boundary value problems for operator differential equations*, volume 48 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991.

[18] G. Grubb. *Distributions and operators*, volume 252 of *Graduate Texts in Mathematics*. Springer, New York, 2009.

[19] T. Kato. Perturbation of continuous spectra by trace class operators. *Proc. Japan Acad.*, 33:260–264, 1957.

[20] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.

[21] A. N. Kochubei. Symmetric operators and nonclassical spectral problems. *Mat. Zametki*, 25(3):425–434, 477, 1979.

[22] M. G. Kreĭn and G. K. Langer. The defect subspaces and generalized resolvents of a Hermitian operator in the space Π_κ. *Funkcional. Anal. i Priložen*, 5(3):54–69, 1971.

[23] M. G. Kreĭn and A. A. Nudel′man. *The Markov moment problem and extremal problems*. American Mathematical Society, Providence, R.I., 1977.

[24] S. T. Kuroda. On a theorem of Weyl-von Neumann. *Proc. Japan Acad.*, 34:11–15, 1958.

[25] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*. Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 181.

[26] M. M. Malamud. On a formula for the generalized resolvents of a non-densely defined Hermitian operator. *Ukraïn. Mat. Zh.*, 44(12):1658–1688, 1992.

[27] M. M. Malamud and S. M. Malamud. Spectral theory of operator measures in a Hilbert space. *Algebra i Analiz*, 15(3):1–77, 2003.

[28] F. S. Rofe-Beketov. Selfadjoint extensions of differential operators in a space of vector-valued functions. *Teor. Funkciĭ Funkcional. Anal. i Priložen. Vyp.*, 8:3–24, 1969.

[29] M. Rosenblum. Perturbation of the continuous spectrum and unitary equivalence. *Pacific J. Math.*, 7:997–1010, 1957.

[30] J. v. Neumann. *Charakterisierung des Spektrums eines Integraloperators*. Paris: Hermann & Cie., 1935.

[31] D. R. Yafaev. *Mathematical scattering theory*, volume 105 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992.