Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Global spatial regularity for a regularized elasto–plastic model

Andreas Bumb, Dorothee Knees[1]

submitted: May 28, 2009

[1] Weierstraß–Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: knees@wias-berlin.de

No. 1419
Berlin 2009

2000 *Mathematics Subject Classification.* 35B65, 49N60, 74C10.

Key words and phrases. Global spatial regularity, nonsmooth domain, regularized elasto-viscoplastic model.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

In this note the spatial regularity of weak solutions for a class of elasto-viscoplastic evolution models is studied for nonsmooth domains. The considered class comprises e.g. models which are obtained through a Yosida regularization from classical, rate-independent elasto-plastic models. The corresponding evolution model consists of an elliptic PDE for the (generalized) displacements which is coupled with an ordinary differential equation with a Lipschitz continuous nonlinearity describing the evolution of the internal variable. It is shown that the global spatial regularity of the displacements and the inner variables is exactly determined through the mapping properties of the underlying elliptic operator.

1 Introduction

In this note we study the spatial regularity of weak solutions for a class of elasto-viscoplastic models on nonsmooth domains. The class comprises the Perzyna model and models which arise from a regularization of classical rate-independent elasto-plastic problems.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $S = (0,T)$ a time interval. By $u : S\times\Omega \to \mathbb{R}^d$ we denote the displacement field and by $z : S\times\Omega \to \mathbb{R}^N$ the vector of the internal variables. Assuming small strains, the behavior of the body is described by the quasistatic balance of forces (1.1), Hooke's law (1.2), which relates the stress $\sigma : S\times\Omega \to \mathbb{R}^{d\times d}_{\text{sym}}$ with the elastic part of the strain, and an evolution equation for the internal variable z (1.3):

$$\operatorname{div}_x \sigma + f = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$\sigma = A(\varepsilon(u) - Bz) \qquad \text{in } (0,T)\times\Omega, \tag{1.2}$$

$$\partial_t z = g(\nabla u, z) \qquad \text{in } (0,T)\times\Omega. \tag{1.3}$$

These equations are completed with an initial condition for z and Dirichlet and Neumann boundary conditions for u. The function f is a given volume force density, the tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ denotes the linearized strain tensor, $A \in \operatorname{Lin}(\mathbb{R}^{d\times d}_{\text{sym}}, \mathbb{R}^{d\times d}_{\text{sym}})$ is the fourth order elasticity tensor and the linear mapping $B : \mathbb{R}^N \to \mathbb{R}^{d\times d}_{\text{sym}}$ maps the vector z of internal variables on the plastic strain $\varepsilon_p = Bz$. Throughout the whole paper we assume that the constitutive function $g : \mathbb{R}^{d\times d}\times\mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous.

Equations (1.1)–(1.3) with Lipschitz continuous g typically arise as a regularization of classical elasto-plastic models: Assume that $\tilde g : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ is a multi-valued monotone mapping with $0 \in \tilde g(0)$ and replace (1.3) with the relation $\partial_t z(t) \in \tilde g(B^\top\sigma - Lz)$, where $L \in \operatorname{Lin}(\mathbb{R}^N, \mathbb{R}^N)$ is a symmetric and positive semi definite tensor. Then (1.3) belongs to the class of constitutive relations of monotone type introduced by Alber, [Alb98], where a typical example is the model of elasto-plasticity with linear kinematic or isotropic hardening. Replacing the monotone mapping $\tilde{g}$ with its Yosida approximation leads to the systems which we study here.

Elasto-viscoplastic models with Lipschitz continuous g are extensively studied in the literature, see for example [FHSV01, IS93, DL76] and the references therein, where existence of solutions is established and where numerical schemes for solving (1.1)–(1.3) are discussed. In order to obtain information about convergence rates, global spatial regularity properties of the solutions are needed. In this note we show that regularity results for linear elliptic systems can immediately be carried over to time dependent systems of the type (1.1)–(1.3).

In particular we prove the following global regularity result for weak solutions of (1.1)-(1.3) with Lipschitz continuous g (Theorem 3.2): Assume that the operator of linear elasticity generates an isomorphism between the spaces $H^{1+s}_{\Gamma}(\Omega) \to H^{s-1}_{\Gamma}(\Omega)$ for some $s \in (0, 1]$, where $H^{1+s}_{\Gamma} = \{\, v \in H^{1+s}(\Omega)\, ;\, v\big|_{\Gamma_{\mathrm{Dir}}} = 0 \,\}$. Then, under natural assumptions on the smoothness of the given data, we have

$$u \in W^{1,\infty}(0, T; H^{1+s}(\Omega, \mathbb{R}^d)), \quad z \in W^{1,\infty}(0, T; H^s(\Omega, \mathbb{R}^N)). \tag{1.4}$$

This extends a local regularity result by Miersemann, [Mie80].

The regularity result is obtained by discussing the properties of the fixed point operator, which is used to prove existence of solutions. The smoothness properties in (1.4) fit exactly with the smoothness assumptions in the paper [FHSV01] (if we neglect the contact problem studied there), where convergence rates for numerical schemes are discussed.

It is an open problem whether the regularity result of the present paper can be carried over to solutions of classical elasto-plastic models with a multi-valued monotone constitutive function $\tilde{g}$. The problem is that regularity estimates, which are uniform with respect to the regularization parameter, are not available yet. However, with a different technique (difference quotients in combination with a reflection argument) a global regularity result was recently derived for elasto-plastic models with a multi-valued maximal monotone mapping $\tilde{g}$, [Kne09].

The paper is organized as follows: In Section 2 we study an abstract ordinary differential equation in a Banach space and formulate and prove the regularity result in the abstract setting. In Section 3 we reformulate the abstract result for general elasto-viscoplastic models and give some examples. In the last section, Section 4, we illustrate the influence of the regularity of solutions on the convergence rates of numerical schemes.

The results which we present in this note were derived while A. Bumb was student research assistant at the Weierstrass Institute.

2 An abstract regularity result

In this section we study the "spatial regularity" of solutions of the following ordinary differential equation:

Let Y, Z be Banach spaces. The problem under consideration is: For given $z_0 \in Z$ and $f : S \to Y$ find $z : S \to Z$ with

$$\partial_t z(t) = \mathcal{G}(f(t), z(t)) \qquad \text{for } t \in S, \tag{2.1}$$

$$z(0) = z_0. \tag{2.2}$$

Here, $\mathcal{G} : Y \times Z \to Z$ is a given Lipschitz continuous operator. We denote by $L^p(S; Z)$ and $W^{k,p}(S; Z)$, $k \in \mathbb{N}$ the spaces of functions $z : S \to Z$ which are measurable and p-integrable and which have p-integrable weak derivatives up to order k.

The following existence theorem is standard for ordinary differential equations in Banach spaces:

Theorem 2.1. *Assume that $\mathcal{G} : Y \times Z \to Z$ is Lipschitz continuous. For every $z_0 \in Z$ and every $f \in W^{k,p}(S; Y)$ with $k \in \{0, 1\}$ and $p \in [1, \infty]$, there exists a unique element $z \in W^{k+1,p}(S, Z)$ solving* (2.1)–(2.2)*.*

The proof of this theorem relies on Banach's fixed point theorem. Since we need the fixed point operator for proving our regularity result, Theorem 2.2 below, we give a short sketch of the proof following the lines in [Sof93].

Proof. Let $p \in [1, \infty]$ and $f \in L^p(S; Y)$. For $\eta \in L^p(S; Z)$ and $t \in S$ let $z_\eta(t) \in W^{1,p}(S; Z)$ be defined through $z_\eta = z_0 + \int_0^t \eta(s)\,\mathrm{d}s$. The fixed point operator is defined in the usual way

$$\mathcal{Q} : L^p(S; Z) \to L^p(S; Z);\ \eta \mapsto \mathcal{G}(f, z_\eta). \tag{2.3}$$

It is shown in [Sof93] that there exists a constant $n_0 \in \mathbb{N}$ such that $(\mathcal{Q} \circ \ldots \circ \mathcal{Q})_{n_0 \text{times}} = \mathcal{Q}^{n_0}$ is a contraction in $L^p(S; Z)$. The generalized Banach fixed point theorem implies that $\mathcal{Q}$ has a unique fixed point $\eta^* \in L^p(S; Z)$ and that for every $\eta \in L^p(S; Z)$ we have $\lim_{n\to\infty} \mathcal{Q}^{n_0 n}(\eta) = \eta^*$. Finally, $z_{\eta^*} \in W^{1,p}(S; Z)$ is the unique solution of (2.1)–(2.2). $\square$

Let now Y_1, Z_1 be further Banach spaces which are continuously embedded in Y and Z, respectively. In addition to the Lipschitz continuity of $\mathcal{G} : Y \times Z \to Z$ we assume that

$$\mathcal{G} : Y_1 \times Z_1 \to Z_1 \tag{2.4}$$

is well defined and bounded, i.e. there is a constant $c_b > 0$ such that for every $y \in Y_1$ and $z \in Z_1$ we have

$$\|\mathcal{G}(y, z)\|_{Z_1} \leq c_b\big(1 + \|y\|_{Y_1} + \|z\|_{Z_1}\big). \tag{2.5}$$

Theorem 2.2. *Let Y, Y_1, Z, Z_1 be as described above and assume in addition that the spaces $L^p(S; Z_1)$, $p \in (1, \infty]$, are sequentially weakly$*$ compact. Let furthermore $\mathcal{G} : Y \times Z \to Z$ be Lipschitz with (2.4)–(2.5). Then for every $z_0 \in Z_1$ the unique solution of (2.1)–(2.2) satisfies*

$$p \in (1, \infty] \text{ and } f \in L^p(S; Y_1) \implies z \in W^{1,p}(S; Z_1).$$

Since $\mathcal{G} : Y_1 \times Z_1 \to Z_1$ is bounded, only, we do not obtain further information on the second time derivative of z, which means that $f \in W^{1,p}(S; Y_1)$ does not imply $z \in W^{2,p}(S; Z_1)$, in general.

Proof. Let $f \in L^p(S; Y_1)$ with $p \in (1, \infty]$. The goal is to show that for every $\eta \in L^p(S; Z_1)$ we have

$$\sup_{n \in \mathbb{N}} \|\mathcal{Q}^n(\eta)\|_{L^p(S; Z_1)} < \infty, \tag{2.6}$$

where $\mathcal{Q}$ is the operator defined in (2.3). Since $L^p(S; Z_1)$ is sequentially weakly$*$ compact, estimate (2.6) implies that the sequence $(\mathcal{Q}^{n_0 n}(\eta))_{n \in \mathbb{N}}$ contains a subsequence which converges weakly$*$ in $L^p(S; Z_1)$ to an element $\widetilde{\eta} \in L^p(S; Z_1)$. Here, n_0 is the number in the proof of Theorem 2.1. From the proof of Theorem 2.1 and the uniqueness of limits it follows that $z_{\widetilde{\eta}}$ is the solution of (2.1)–(2.2). Observe that $z_{\widetilde{\eta}} \in W^{1,p}(S; Z_1)$.

It remains to prove estimate (2.6). Let $w(t) = 1 + \|f(t)\|_{Y_1} + \|z_0\|_{Z_1}$. From (2.5) and the definition of $\mathcal{Q}$ it follows that for almost every $t_0 \in S$ we have

$$\|\mathcal{Q}(\eta(t_0))\|_{Z_1} \le c_b \big(w(t_0) + \int_0^{t_0} \|\eta(t_1)\|_{Z_1} \, \mathrm{d}t_1 \big)$$

and, by induction,

$$\begin{aligned} \|\mathcal{Q}^n(\eta(t_0))\|_{Z_1} &\le c_b^n \int_0^{t_0} \cdots \int_0^{t_{n-1}} \|\eta(t_n)\|_{Z_1} \, \mathrm{d}t_n \ldots \mathrm{d}t_1 \\ &\quad + c_b w(t_0) + c_b^2 \int_0^{t_0} w(t_1) \, \mathrm{d}t_1 + \ldots + c_b^n \int_0^{t_0} \cdots \int_0^{t_{n-2}} w(t_{n-1}) \, \mathrm{d}t_{n-1} \ldots \mathrm{d}t_1 \\ &= S_{1,n}(t_0) + S_{2,n}(t_0). \end{aligned} \tag{2.7}$$

Assume now that $p < \infty$. From (2.7) we obtain with a constant depending on p

$$c_p \|\mathcal{Q}^n(\eta)\|^p_{L^p(S; Z_1)} \le \int_0^T |S_{1,n}(t_0)|^p \, \mathrm{d}t_0 + \int_0^T |S_{2,n}(t_0)|^p \, \mathrm{d}t_0 \tag{2.8}$$

Note that for $\alpha > 0$ we have

$$\int_0^{t_0} \cdots \int_0^{t_{n-2}} t_{n-1}^{\alpha} \, \mathrm{d}t_{n-1} \ldots \mathrm{d}t_1 = t_0^{n-1+\alpha} \prod_{l=1}^{n-1} (l + \alpha)^{-1} \le \frac{t_0^{n-1+\alpha}}{(n-1)!}. \tag{2.9}$$

Therefore, the first term in (2.8) can be estimated as follows using Hölder's inequality and $p^{-1} + (p')^{-1} = 1$

$$\int_0^T |S_{1,n}(t_0)|^p \, \mathrm{d}t_0 \le c_b^{np} \|\eta\|_{L^p(S;Z_1)}^p \int_0^T \left(\int_0^{t_0} \dots \int_0^{t_{n-2}} t_{n-1}^{\frac{1}{p'}} \, \mathrm{d}t_{n-1} \dots \mathrm{d}t_1 \right)^p \mathrm{d}t_0$$
$$\le \frac{1}{pn} \|\eta\|_{L^p(S;Z_1)}^p \left(\frac{(c_b T)^n}{(n-1)!} \right)^p. \tag{2.10}$$

The right hand side in (2.10) tends to zero for $n \to \infty$. Furthermore, again with Hölder's inequality and estimate (2.9), we have

$$\begin{aligned} S_{2,n}(t_0) &= c_b w(t_0) + \sum_{l=1}^{n-1} c_b^{l+1} \int_0^{t_0} \dots \int_0^{t_{l-1}} w(t_l) \, \mathrm{d}t_l \dots \mathrm{d}t_1 \\ &\le c_b w(t_0) + \|w\|_{L^p(S)} c_b^2 T^{\frac{1}{p'}} \sum_{l=1}^{n-1} \frac{(c_b T)^{l-1}}{(l-1)!} \\ &\le c_b w(t_0) + c_b^2 T^{\frac{1}{p'}} \|w\|_{L^p(S)} \exp(c_b T). \end{aligned}$$

This implies

$$\|S_{2,n}\|_{L^p(S)} \le c \|w\|_{L^p(S)} (1 + \exp(c_b T)) \tag{2.11}$$

with a constant c which is independent of n. Putting together estimates (2.10) and (2.11) proves (2.6) for $p < \infty$. The case $p = \infty$ can be treated similarly with obvious modifications. □

A special case of the previous theorem is the following:

Let H, V, Z, Y be Banach spaces and assume that $V \subset H$ is a closed subspace. Let furthermore $\mathcal{A} : V \to Y$ be a linear and continuous isomorphism and let $\mathcal{B} : Z \to Y$ and $\mathcal{G} : H \times Z \to Z$ be Lipschitz continuous operators. We consider the following problem: Find $u : S \to V$ and $z : S \to Z$ such that

$$\mathcal{A}u(t) + \mathcal{B}(z(t)) = f(t), \tag{2.12}$$
$$\partial_t z(t) = \mathcal{G}(u(t) + h(t), z(t)), \tag{2.13}$$
$$z(0) = z_0 \tag{2.14}$$

for some given $f \in W^{k,p}(S;Y)$, $h \in W^{k,p}(S;H)$ and $z_0 \in Z$. Let $V_1 \subset H_1, Z_1, Y_1$ be Banach spaces, which are continuously embedded in V, H, Z and Y, and assume that $L^p(S;Z_1)$ is sequentially weakly$*$ compact for every $p \in (1, \infty]$.

Corollary 2.3. *Assume in addition that $\mathcal{A} : V_1 \to Y_1$ is an isomorphism. Moreover, suppose that $\mathcal{B} : Z_1 \to Y_1$ and $\mathcal{G} : H_1 \times Z_1 \to Z_1$ are bounded operators satisfying $\|\mathcal{B}(z)\|_{Y_1} \le c_1(1 + \|z\|_{Z_1})$ and $\|\mathcal{G}(u,z)\|_{Z_1} \le c_2(1 + \|u\|_{H_1} + \|z\|_{Z_1})$ for all $u \in H_1$ and $z \in Z_1$.*

Then, for every $f \in L^p(S;Y_1)$, $h \in L^p(S;H_1)$ with $p \in (1,\infty]$ and for every $z_0 \in Z_1$, there exist unique elements $u \in L^p(S;V_1)$ and $z \in W^{1,p}(S;Z_1)$, which solve (2.12)–(2.14)*. If $f \in W^{1,p}(S;Y_1)$, then $u \in W^{1,p}(S;V_1)$.*

Proof. We set $\widetilde{Y} = Y \times H$ and define $\widetilde{\mathcal{G}} : \widetilde{Y} \times Z \to Z$ by

$$\widetilde{\mathcal{G}}((f,h),z) = \mathcal{G}(\mathcal{A}^{-1}(f - \mathcal{B}(z)) + h(t), z).$$

Then the pair $(u,z) : S \to (V,Z)$ is a solution to problem (2.12)–(2.14) if and only if z solves $\partial_t z(t) = \widetilde{\mathcal{G}}((f(t),h(t)),z(t))$ and $u(t) = \mathcal{A}^{-1}(f(t) - B(z(t)))$. Corollary 2.3 is now a consequence of Theorem 2.2 since the operator $\widetilde{\mathcal{G}}$ satisfies the assumptions of Theorem 2.2 with respect to the space $\widetilde{Y}_1 \times Z_1$, where $\widetilde{Y}_1 = Y_1 \times H_1$. □

3 Application to elasto-viscoplasticity

3.1 Notation and basic assumptions

As an application of Corollary 2.3 we discuss the case, where the operator $\mathcal{A}$ in (2.12) represents a linear, elliptic differential operator of second order. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N denote the Dirichlet and Neumann boundary, respectively. It is assumed that Γ_D is not empty. The spaces H, V, Z and Y are chosen as

$$\begin{gathered} H = H^1(\Omega,\mathbb{R}^m), \quad V = \{\, u \in H^1(\Omega,\mathbb{R}^m)\,;\, u\big|_{\Gamma_D} = 0 \,\}, \\ Z = L^2(\Omega,\mathbb{R}^N), \quad \widetilde{Z} = L^2(\Omega,\mathbb{R}^{m\times d}), \quad Y = V' \text{ (the dual of } V). \end{gathered} \tag{3.1}$$

Let the bilinear form $a : H \times H \to \mathbb{R}$ be defined by $a(u,v) = \int_\Omega A\nabla u : \nabla v \,\mathrm{d}x$, where the following assumptions on the coefficient matrix A shall be satisfied:

A1 $A \in L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^{m\times d}, \mathbb{R}^{m\times d}))$ and satisfies $\sum_{i,j=1}^d \sum_{\alpha,\beta=1}^m A_{ij}^{\alpha\beta}(x)\xi_i\xi_j\eta_\alpha\eta_\beta \geq c_a\,|\xi|^2\,|\eta|^2$ for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^d$, $\eta \in \mathbb{R}^m$. Moreover, the induced bilinear form $a : V \times V \to \mathbb{R}$ is V-elliptic, i.e. there is a constant $c_A > 0$ such that for every $v \in V$ we have $a(v,v) \geq c_A \|v\|^2_{H^1(\Omega)}$.

The Lax-Milgram Lemma guarantees that the operator $\mathcal{A} : V \to V'$, which is defined by

$$\langle \mathcal{A}u, v\rangle = a(u,v) \quad \text{for every } u,v \in V, \tag{3.2}$$

is an isomorphism. Concerning the constitutive functions B and g, we assume

A2 $B : \Omega \times \mathbb{R}^N \to \mathbb{R}^{m\times d}$ is a Carathéodory function for which there exists a constant $L_B > 0$ such that for every $x, y \in \Omega$ and every $z_1, z_2 \in \mathbb{R}^N$ we have

$$\begin{aligned} |B(x,z_1) - B(x,z_2)| &\leq L_B\,|z_1 - z_2|\,, \\ |B(x,z_1) - B(y,z_1)| &\leq L_B(1 + |z_1|)\,|x - y|\,. \end{aligned}$$

A3 $g : \Omega \times \mathbb{R}^{m\times d} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function for which there exists a constant $L_g > 0$ such that for every $x, y \in \Omega$, $z_1, z_2 \in \mathbb{R}^N$ and $a_1, a_2 \in \mathbb{R}^{m\times d}$ we have

$$|g(x, a_1, z_1) - g(y, a_1, z_1)| \le L_g(1 + |a_1| + |z_1|)\,|x - y|\,,$$
$$|g(x, a_1, z_1) - g(x, a_2, z_2)| \le L_g\big(\,|a_1 - a_2| + |z_1 - z_2|\,\big).$$

The operators $\operatorname{Div} : \widetilde{Z} \to Y$, $\tilde{\mathcal{B}} : \widetilde{Z} \to Z$, $\mathcal{B} : Z \to Y$ and $\mathcal{G} : H \times Z \to Z$ are defined via

$$\langle \operatorname{Div} \eta, u\rangle = -\int_\Omega \eta(x) : \nabla u(x)\,\mathrm{d}x \quad \text{for } u \in V, \eta \in \widetilde{Z}, \tag{3.3}$$
$$\tilde{\mathcal{B}}(z)(x) = B(x, z(x)) \quad \text{for } z \in Z, \tag{3.4}$$
$$\mathcal{B}(z) = \operatorname{Div} \tilde{\mathcal{B}}(z) \quad \text{for } z \in Z, \tag{3.5}$$
$$\mathcal{G}(u, z)(x) = g(x, \nabla u(x), z(x)) \quad \text{for every } u \in H, z \in Z. \tag{3.6}$$

It is easily checked that $\mathcal{B}$ and $\mathcal{G}$ are well defined and Lipschitz continuous provided that A2 and A3 are satisfied. For the data we assume

A4 $f \in L^p(S; Y)$, $H_D \in L^p(S; H)$ with $p \in (1, \infty]$, $z_0 \in Z$.

The function H_D can be interpreted as an extension of the Dirichlet datum to the entire domain Ω. The Neumann datum is included in f. With f and H_D we associate the function $F \in L^p(S; Y)$ via $\langle F(t), v\rangle = \langle f(t), v\rangle - a(H_D(t), v)$ for every $v \in V$.

The problem under consideration is: Find $u \in L^p(S; V)$ and $z \in W^{1,p}(S; Z)$ such that

$$\mathcal{A}u(t) - \mathcal{B}(z(t)) = F(t), \tag{3.7}$$
$$\partial_t z(t) = \mathcal{G}(u(t) + H_D(t), z(t)), \tag{3.8}$$
$$z(0) = z_0. \tag{3.9}$$

Problem (3.7)–(3.9) contains the model (1.1)–(1.3) as well as the models in [FHSV01, IS93] as special cases. If conditions A1–A4 hold, then Theorem 2.1 guarantees the existence of unique elements $u \in L^p(S; V)$ and $z \in W^{1,p}(S; Z)$ which solve (3.7)–(3.9). Under suitable regularity assumptions on the elliptic operator $\mathcal{A}$, higher regularity properties can be derived on the basis of Corollary 2.3. This will be explained in the next section.

3.2 Regularity in Sobolev–Slobodeckij spaces

We will now investigate the higher spatial regularity of u and z in Sobolev–Slobodeckij spaces. For $s > 0$ we denote by $H^s(\Omega)$ the usual Sobolev–Slobodeckij spaces and refer to [Gri85] for a definition.

Proposition 3.1. *Assume A2 and A3. For every $s \in [0, 1]$ the operators*

$$\tilde{\mathcal{B}} : H^s(\Omega) \to H^s(\Omega), \qquad \mathcal{G} : H^{1+s}(\Omega) \times H^s(\Omega) \to H^s(\Omega)$$

are well defined and there exist constants $c_B, c_g > 0$ such that for every $z \in H^s(\Omega)$ and $u \in H^{1+s}(\Omega)$ we have

$$\big\|\tilde{\mathcal{B}}(z)\big\|_{H^s(\Omega)} \leq c_B(1 + \|z\|_{H^s(\Omega)}), \qquad \|\mathcal{G}(u,z)\|_{H^s(\Omega)} \leq c_g(1 + \|u\|_{H^{1+s}(\Omega)} + \|z\|_{H^s(\Omega)}). \tag{3.10}$$

Proof. For $s = 1$ the assertion follows from the Lipschitz continuity of B and g. The case $s \in (0,1)$ is then a consequence of Tartar's interpolation theorem for nonlinear operators, [Tar72]. □

Our final assumption concerns the regularity property of $\mathcal{A}$:

A5 There exists $s \in (0,1]$ and a Hilbert space $Y_s \subset Y$ (continuous embedding) such that $\mathcal{A} : V \cap H^{1+s}(\Omega) \to Y_s$ is an isomorphism and such that the restriction of Div to $H^s(\Omega)$ is well defined and continuous as an operator $\mathrm{Div} : H^s(\Omega) \to Y_s$.

The space Y_s depends strongly on the smoothness of the coefficient matrix A, the smoothness of $\partial\Omega$ and the type of the boundary conditions. If $\partial\Omega$ is $C^{1,1}$–smooth, if $A \in C^{0,1}(\overline{\Omega}, \mathrm{Lin}(\mathbb{R}^{m\times d}, \mathbb{R}^{m\times d})$ satisfies A1 and if $\partial\Omega = \Gamma_D$, then classical regularity and interpolation results, [Neč67, Tri78], guarantee that A5 holds for every $s \in (0,1]$ with $Y_s = (H^{1-s}_{\partial\Omega}(\Omega))'$, where for $\delta \geq 0$

$$H^\delta_{\partial\Omega}(\Omega) := \begin{cases} H^\delta(\Omega) & \text{if } \delta - \frac{1}{2} < 0, \\ \{\, u \in H^{\frac{1}{2}}(\mathbb{R}^d)\,;\, \mathrm{supp}\, u \subset \overline{\Omega}\,\} & \text{if } \delta = \frac{1}{2}, \\ \{\, u \in H^\delta(\Omega)\,;\, u\big|_\Gamma = 0\,\} & \text{if } \delta - \frac{1}{2} > 0. \end{cases}$$

Subsequent to the next theorem we give further examples, where A5 is valid.

The following regularity theorem is a direct consequence of Corollary 2.3 and Proposition 3.1.

Theorem 3.2. *Assume A1–A5 for some $p \in (1,\infty]$ and $s \in (0,1]$. Let furthermore $f \in L^p(S; Y_s)$, $H_D \in L^p(S; H^{1+s}(\Omega))$ and $z_0 \in H^s(\Omega)$. Then the unique solution (u,z) of (3.7)–(3.9) satisfies*

$$u \in L^p(S; H^{1+s}(\Omega)), \qquad z \in W^{1,p}(S; H^s(\Omega)).$$

If in addition $f \in W^{1,p}(S; Y_s)$ and $H_D \in W^{1,p}(S; H^{1+s}(\Omega))$, then $u \in W^{1,p}(S; H^{1+s}(\Omega))$.

This theorem shows that regularity results for linear elliptic operators can immediately be carried over to the viscous models. Note that in the scale of Sobolev–Slobodeckij spaces we may at best expect $u(t) \in H^2(\Omega)$ and $z(t) \in H^1(\Omega)$, since for $s > 1$ and an arbitrary Lipschitz continuous function g one cannot guarantee in general that $\mathcal{G}(H^{1+s}(\Omega) \times H^s(\Omega)) \subset H^s(\Omega)$.

Example 3.3 (Scalar case)**.** This example relies on regularity results by Dauge for scalar elliptic equations on polyhedral domains [Dau88]. Assume that $m = 1$ and that the coefficient matrix A is constant and satisfies A1. For simplicity we restrict ourself to two and three space dimensions. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded, polyhedral domain with Lipschitz boundary (i.e. $\partial\Omega$ coincides locally with the graph of a Lipschitz continuous function). We denote the faces of Ω with $\Gamma_i, 1 \leq i \leq L$, and assume that $\partial\Omega = \cup_{1\leq i\leq L}\overline{\Gamma_i}$ and that every Γ_i is an open subset of a $d-1$ dimensional hyperplane. Moreover, we assume that the opening angle $\measuredangle(\Gamma_i,\Gamma_j) \neq \pi$ for $i \neq j$ if $\overline{\Gamma_i}\cap\overline{\Gamma_j} \neq \emptyset$. Finally, we assume that for every i we have $\Gamma_i \subset \Gamma_D$ or $\Gamma_i \subset \Gamma_N$. Let $I_D = \{\, i\,;\, \Gamma_i \subset \Gamma_D \,\}$ and $I_N = \{\, i\,;\, \Gamma_i \subset \Gamma_N \,\}$. From these assumptions it follows that the type of the boundary conditions does not change within a face of the polyhedron Ω.

For $s \in (0,1)\backslash\{\frac{1}{2}\}$ we define analogously to [Dau88, p. 194]

$$V^{1-s} = \{\, v \in H^{1-s}(\Omega)\,;\, v\big|_{\Gamma_D} = 0 \,\}, \quad Y_s = (V^{1-s})' \quad \text{if } s < \tfrac{1}{2}, \tag{3.11}$$

$$V^{1-s} = H^{1-s}(\Omega), \quad Y_s = (V^{1-s})' \times \prod_{i\in I_N} H^{s-\frac{1}{2}}(\Gamma_i) \quad \text{if } s > \tfrac{1}{2}. \tag{3.12}$$

From Theorem 23.3 in [Dau88] it follows that there exists $s \in (0,1)\backslash\{\frac{1}{2}\}$ such that $\mathcal{A} : V \cap H^{1+s}(\Omega) \to Y_s$ is an isomorphism, and thus condition A5 holds for this particular s. The optimal s depends on the opening angles of Ω, the boundary conditions and the coefficient matrix A and can be calculated from a nonlinear eigenvalue problem, see e.g. [MNP91, Dau88]. For example, if Ω is a two dimensional polygon and the interior opening angle between Dirichlet and Neumann boundary satisfies $\measuredangle(\Gamma_D,\Gamma_N) < \pi$ and if A is symmetric, then A5 holds for every $s \in (0,\frac{1}{2})$, see the estimates of eigenvalues in [Kne04]. If again A is symmetric and if $\Omega \subset \mathbb{R}^d$ is a polyhedral domain with Lipschitz boundary and with $\partial\Omega = \Gamma_D$, then there exists $\delta \in (0,\frac{1}{2})$ such that A5 is satisfied for $s = \frac{1}{2} + \delta$, [KM88].

Corollary 3.4. *Under the above assumptions on Ω, Γ_D, Γ_N and $s \in (0,1)\backslash\{\frac{1}{2}\}$ it follows that for every $f \in L^p(S;Y_s)$, $H_D \in L^p(S;H^{1+s}(\Omega))$ and $z_0 \in H^s(\Omega)$ we have $u \in L^p(S;H^{1+s}(\Omega))$ and $z \in W^{1,p}(S;H^s(\Omega))$.*

Finally, if $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded convex domain with $\partial\Omega = \Gamma_D$ and if u is scalar, then $\mathcal{A} : H^1_0(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$ is an isomorphism [Gri85] and we have Corollary 3.4 with $s = 1$ and $Y_s = L^2(\Omega)$.

Example 3.5 (Elasto-viscoplasticity)**.** Here, we consider system (1.1)–(1.3) with a Lipschitz continuous function $g : \mathbb{R}^N \to \mathbb{R}^N$ and with $\partial\Omega = \Gamma_D$. Assume that the material tensors A and B from (1.1)–(1.3) satisfy: $A \in \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})$, A symmetric and positive definite and $B \in \mathrm{Lin}(\mathbb{R}^N;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. We define $\mathcal{A} : H^1_0(\Omega) \to H^{-1}(\Omega) = (H^1_0(\Omega))'$ via $\langle \mathcal{A}u, v\rangle = -\int_\Omega A\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x$ for $u, v \in H^1_0(\Omega)$. Condition A1 is satisfied due to Korn's inequality.

Assume that $\partial\Omega$ is smooth enough such that for some $s \in (0,1]$ the mapping $\mathcal{A} : V \cap H^{1+s}(\Omega) \to H^{s-1}(\Omega)$ is an isomorphism. For example, if $\partial\Omega$ is $C^{1,1}$–smooth, then one may choose $s = 1$. If Ω is a two or three dimensional polyhedral domain with Lipschitz boundary and if A is the coefficient matrix for isotropic elasticity, then again from the work by Dauge in combination with [KM88] it follows that there is a $\delta \in (0, \frac{1}{2}]$ such that we may choose $s = \frac{1}{2} + \delta$. Like in Example 3.3, the optimal s depends on the opening angles near the edges and vertices.

Corollary 3.6. *Under the above assumptions and with $z_0 \in H^s(\Omega)$, $f \in L^p(S; H^{s-1}(\Omega))$ and $H_D \in L^p(S; H^{1+s}(\Omega))$ we have $u \in L^p(S; H^{1+s}(\Omega))$ and $z \in W^{1,p}(S; H^s(\Omega))$.*

Let us note that the viscous models studied by Sofonea et al., see for example [FHSV01, IS93], can be reformulated in the form of (1.1)–(1.2) with an evolution law of the type $\partial_t z = g(\nabla u, z)$ with a Lipschitz continuous function g. Therefore, Corollary 3.6 is valid for these models.

Example 3.7 (Smooth inclusions)**.** Let $\Omega_1, \Omega \subset \mathbb{R}^d$ be bounded domains with $C^{1,1}$-smooth boundaries and $\Omega_1 \Subset \Omega$. Let $\Omega_2 = \Omega \backslash \overline{\Omega}_1$ with $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$. Assume that the coefficients A and B satisfy A1 and A2 and that their restrictions to the subdomains Ω_i are constant. Consider the spaces $H = H^1(\Omega)$, $V = H^1_0(\Omega)$, $Z = L^2(\Omega)$ and define $H_1 = \{ u \in H^1(\Omega) ; u\big|_{\Omega_i} \in H^2(\Omega_i) \}$, $V_1 = V \cap H_1$, $Z_1 = \{ z \in Z ; z\big|_{\Omega_i} \in H^1(\Omega_i) \}$ and $Y_1 = L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)$. Let furthermore $\mathcal{A} : V \to V'$ and $\mathcal{B} : Z \to V'$ be defined as in (3.2) and (3.5). Observe that $\mathcal{A} : V_1 \to Y_1$ and $\mathcal{B} : Z_1 \to Y_1$ are well defined and bounded. From regularity theory for elliptic problems with smooth inclusions it follows that $\mathcal{A} : V_1 \to Y_1$ is an isomorphism. Hence, Corollary 2.3 is applicable to the time dependent problem (3.7)–(3.9). We refer to [NS94, NS99, CDN99, Kne04, Nic93] for more information about the regularity theory of elliptic problems with nonsmooth coefficients.

4 Example: Convergence rates based on regularity results

In this section we illustrate how the regularity of solutions affects the convergence rates of numerical schemes. First, we provide an estimate concerning the convergence rate when discretizing the problem with a standard FE-method in space and an implicit Euler scheme in time. The predicted convergence rate is then verified for an explicit example.

4.1 An error Estimate

At first we derive an estimate for a semi-discrete version of (3.7)–(3.9). Let $H_h \subset H$, $V_h = V \cap H_h$ and $Z_h \subset Z$ be closed subspaces of H, V and Z, where the spaces H, V and Z are chosen as in (3.1). In the notation of Section 3.1 the discrete model reads: Find

$u_h \in L^p(S;V_h)$, $z_h \in W^{1,p}(S;Z_h)$ such that for $t \in S$

$$\mathcal{A}_h u_h(t) - \mathcal{B}_h(z_h(t)) = F_h(t) \quad \text{in } V_h', \tag{4.1}$$

$$\partial_t z_h(t) = \mathcal{G}(u_h(t) + H_{D,h}(t), z_h(t)) \quad \text{in } Z_h, \tag{4.2}$$

$$z_h(0) = z_{0,h}, \tag{4.3}$$

where the operators $\mathcal{A}_h$ and $\mathcal{B}_h$ are defined as

$$\langle \mathcal{A}_h u_h, v_h \rangle_{V_h', V_h} := \langle \mathcal{A} u_h, v_h \rangle_{V',V} \quad \text{for } u_h,\ v_h \in V_h,$$

$$\langle \mathcal{B}_h(z_h), v_h \rangle_{V_h', V_h} := -\int_\Omega \tilde{\mathcal{B}}(z_h) : \nabla v_h \, \mathrm{d}x \quad \text{for } z_h \in Z_h,\ v_h \in V_h.$$

In addition to A1–A4 we assume that the nonlinear operator $\mathcal{G}$ has the mapping property $\mathcal{G}(V_h, Z_h) \subset Z_h$. For example this is guaranteed if V_h consists of continuous and piecewise affine functions and if the elements of Z_h are piecewise constant. The discretized data shall satisfy

A6 $f_h \in W^{1,p}(S;V_h')$, $H_{D,h} \in W^{1,p}(S;H_h)$, $\langle F_h(t), v\rangle := \langle f_h(t), v\rangle - a(H_{D,h}, v)$ for $v \in V_h$, and $z_{0,h} \in Z_h$.

Theorem 2.1 implies the existence and uniqueness of solutions $u_h \in W^{1,p}(S;V_h)$ and $z_h \in W^{2,p}(S;Z_h)$ of the discretized problem (4.1)–(4.3). Let

$$r(t,h) := \|z_0 - z_{0,h}\|_Z + \inf_{v_h \in V_h} \|u(t) - v_h\|_V + \|F(t) - F_h(t)\|_{V_h'} + \int_0^t \inf_{v_h \in V_h} \|u(\tau) - v_h\|_V + \|H_D(\tau) - H_{D,h}(\tau)\|_H \, \mathrm{d}\tau. \tag{4.4}$$

The next proposition is an application of Cea's Lemma:

Proposition 4.1. *There exists a constant $\kappa > 0$ such that for a.e. $t \in S$ and all V_h and Z_h the solutions (u,z) of (3.7)–(3.9) and (u_h, z_h) of (4.1)–(4.3) satisfy:*

$$\|(u - u_h)(t)\|_V + \|(z - z_h)(t)\|_Z \le \kappa\, r(t,h).$$

Proof. By Cea's Lemma it follows from relations (3.7) and (4.1) that for a.e. $t \in S$ it holds

$$\|(u - u_h)(t)\|_V \le c_1 \big(\|(z - z_h)(t)\|_Z + \|F(t) - F_h(t)\|_{V_h'} + \inf_{v \in V_h} \|u(t) - v\|_V \big), \tag{4.5}$$

and the constant c_1 is independent of t and the subspaces V_h and Z_h.

Multiplying (3.8) and (4.2) with $(z - z_h)(t)$ (here we use the assumption that $\mathcal{G}(V_h, Z_h) \subset Z_h$) and using the Lipschitz continuity of $\mathcal{G}$, estimate (4.5) and Young's inequality, we arrive at

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|(z - z_h)(t)\|_Z^2 \le c_2 \big(\inf_{v \in V_h} \|u(t) - v\|_V^2 + \|H_D(t) - H_{D,h}(t)\|_H^2 + \|(z - z_h)(t)\|_Z^2 \big).$$

The Gronwall Lemma now leads to the desired result. □

For the time discretization we use an implicit Euler method. Let $N \in \mathbb{N}$ be the number of time steps, $\triangle t_N = T/N$ the time step size, and $t_k^N = k\triangle t_N$. For given $z_{0,h} \in Z_h$ let $u_h^{0,N} := \mathcal{A}_h^{-1}(F(0) + \mathcal{B}_h(z_{0,h}))$. For $1 \le k \le N$ the pair $(u_h^{k,N}, z_h^{k,N}) \in V_h \times Z_h$ is defined as the solution of

$$\mathcal{A}_h u_h^{k,N} - \mathcal{B}_h(z_h^{k,N}) = F_h(t_k^N) \quad \text{in } V_h', \tag{4.6}$$

$$\tfrac{1}{\triangle t_N}(z_h^{k,N} - z_h^{k-1,N}) = \mathcal{G}(u_h^{k,N} + H_{D,h}(t_k^N), z_h^{k,N}) \quad \text{in } Z_h. \tag{4.7}$$

The next proposition gives the full error estimate:

Proposition 4.2. *Let $f \in W^{1,p}(S;V')$, $H_D \in W^{1,p}(S;H)$ for some $p \in (1,\infty]$ and assume that A6 is satisfied. There exist constants $\kappa > 0$ and $N_0 \in \mathbb{N}$ such that for all $N > N_0$, $1 \le k \le N$ and all solutions $(u_h^{k,N}, z_h^{k,N})_{1\le k\le N} \subset V_h \times Z_h$ it holds with $r(t,h)$ from (4.4):*

$$\left\|u(t_k^N) - u_h^{k,N}\right\|_V + \left\|z(t_k^N) - z_h^{k,N}\right\|_Z \le \kappa\big(\triangle t_N \left\|z_h\right\|_{W^{2,\infty}(S;Z)} + r(t_k^N, h)\big), \tag{4.8}$$

and there exists a constant $c > 0$ such that for all V_h and Z_h it holds $\|z_h\|_{W^{2,\infty}(S;Z)} \le c$.

Proof. The last statement of the proposition follows from Proposition 4.1 and the Lipschitz continuity of $\mathcal{G}$. In order to prove (4.8) observe that it holds

$$\begin{aligned} &\left\|u(t_k) - u_h^k\right\|_V + \left\|z(t_k) - z_h^k\right\|_Z \\ &\le \left\|u(t_k) - u_h(t_k)\right\|_V + \left\|z(t_k) - z_h(t_k)\right\|_Z + \left\|u_h(t_k) - u_h^k\right\|_V + \left\|z_h(t_k) - z_h^k\right\|_Z. \end{aligned} \tag{4.9}$$

The first two terms are estimated in Proposition 4.1. From (4.1) and (4.6) it follows that

$$\left\|u_h(t_k^N) - u_h^{k,N}\right\|_V \le c_1 \left\|z_h(t_k^N) - z_h^{k,N}\right\|_Z, \tag{4.10}$$

and the constant is independent of h and N. From the Lipschitz continuity of $\mathcal{G}$ and estimate (4.10) it follows by standard arguments that the discretization error

$$R\left(t_k^N, \triangle t_N\right) := (\triangle t_N)^{-1}\left(z_h(t_k^N) - z_h(t_{k-1}^N)\right) - \mathcal{G}\big(u_h^{k,N} + H_{D,h}(t_k^N), z_h(t_k^N)\big)$$

satisfies

$$\left\|R(t_k^N, \triangle t_N)\right\|_Z \le c_2\big(\triangle t_N \left\|z_h\right\|_{W^{2,\infty}(S;Z)} + \left\|z_h(t_k^N) - z_h^{k,N}\right\|_Z\big). \tag{4.11}$$

Hence, taking into account relation (4.7), we obtain with (4.11) and for $\triangle t_N < C^{-1}$, where $C = C_{\mathrm{Lip}(\mathcal{G})} + c_2$, that

$$\left\|z_h(t_k^N) - z_h^{k,N}\right\|_Z \le (1 - \triangle t_N C)^{-1}\Big(\left\|z_h(t_{k-1}^N) - z_h^{k-1,N}\right\|_Z + c_2(\triangle t_N)^2 \left\|z_h\right\|_{W^{2,\infty}(S;Z)}\Big).$$

After recursion this yields

$$\left\|z_h(t_k^N) - z_h^{k,N}\right\|_Z \le \exp(CT)\Big(\left\|z_h(0) - z_h^0\right\|_Z + \triangle t_N\, c_2 \left\|z_h\right\|_{W^{2,\infty}(S;Z)}\Big).$$

Combining the last estimate with (4.9) and (4.10) finishes the proof. □

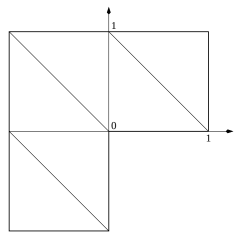

Figure 1: Domain Ω

4.2 The numerical example

Let $\Omega \subset [-1,1]^2$ be an L-shaped domain, see Fig. 1. The problem under consideration is to find $u : S \times \Omega \to \mathbb{R}$, $z : S \times \Omega \to \mathbb{R}$ such that:

$$\operatorname{div}\left(\nabla u(t,x) - z(t,x)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) = 0 \qquad \text{in } S \times \Omega, \tag{4.12}$$

$$\partial_t z(t,x) = g(\partial_1 u - 2z), \quad \text{in } S \times \Omega, \tag{4.13}$$

$$u(t,x) = h_D(t,x) \qquad \text{on } \partial\Omega, \tag{4.14}$$

$$z(0,x) = z_0(x) \qquad \text{in } \Omega, \tag{4.15}$$

where $h_D(t,x) = \tilde{h}(t)\, r^{\frac{2}{3}} \sin\left(\frac{2}{3}\phi\right)$, $\tilde{h}(t) = \max\{0, (t-2)^3\} \sin(\frac{\pi t}{2})$ and (r,ϕ) denote polar coordinates. Observe that $h_D \in W^{1,\infty}(S; H^{\frac{3}{2}}(\partial\Omega))$. The function $g : \mathbb{R} \to \mathbb{R}$ is chosen as $g(s) = \max\{0; s-1\} + \min\{0; s+1\}$. For this setting obviously the assumptions A1 to A4 are fulfilled and Proposition 4.1 and 4.2 can be applied.

As already discussed in Example 3.3 there exists $s \in [0,1]$ such that the Laplace operator with Dirichlet boundary conditions is an isomorphism between the spaces $H^{1+s}(\Omega) \cap H^1_0(\Omega) \to Y_s$ with Y_s as in (3.11). For the considered L-shaped domain, we may choose $s = \frac{2}{3} - \delta$ for arbitrary $\delta > 0$, [Gri85]. Hence the solution of (4.12)–(4.15) has the regularity $u \in W^{1,\infty}(S; H^{1+\frac{2}{3}-\delta}(\Omega))$ and $z \in W^{1,\infty}(S; H^{\frac{2}{3}-\delta}(\Omega))$ for every $\delta > 0$.

In order to compute the solution of (4.12)–(4.15) numerically the domain Ω is discretized with a sequence of regular triangulations $\mathcal{T}_h$ in triangles (see Fig. 1 for the initial mesh). To reveal the influence of the regularity of the solution on the convergence rate, the meshes are not refined towards the origin, where the solution develops a singularity.

The spaces H_h and Z_h are chosen as $H_h = \{\, v \in H^1(\Omega)\,;\, \forall \tau \in \mathcal{T}_h \ v\big|_\tau \in \mathcal{P}_1(\tau)\,\}$, where $\mathcal{P}_1(\tau)$ consists of the affine functions on τ, and $Z_h = \{\, z \in L^2(\Omega)\,;\, \forall \tau \in \mathcal{T}_h \ z\big|_\tau = \text{const}\,\}$. With this choice, the mapping property $\mathcal{G}(V_h, Z_h) \subset Z_h$ of Section 4.1 is valid.

Combining the error estimate (4.8) with estimates for the interpolation error of H^{1+s}-functions (see e.g. [BS94]) and assuming that $z(0) = z_h(0) = z_h^0$, we obtain the following estimate for the convergence rate

$$\max_{1 \le k \le N} \left(\left\| u(t_k^N) - u_h^{k,N} \right\|_V + \left\| z(t_k^N) - z_h^{k,N} \right\|_Z \right) \le \kappa(\triangle t_N + h^s). \tag{4.16}$$

Here $h = \max_{\tau \in \mathcal{T}_h} \operatorname{diam}(\tau)$ is the mesh size, $\triangle t_N$ the time step size and, in the above example, $s = \frac{2}{3} - \delta$ with $\delta > 0$ arbitrary.

Elements of h_i	$\left\| u_{h_i}^{N_i} - u_{h_{i+1}}^{N_{i+1}} \right\|_{H^1(\Omega)}$	$\left\| z_{h_i}^{N_i} - z_{h_{i+1}}^{N_{i+1}} \right\|_{L^2(\Omega)}$	s_i
12	3.0620	0.8459	0.5861
48	2.0336	0.5697	0.7382
192	1.2028	0.3578	0.6919
768	0.7539	0.2122	0.6838
3072	0.4694	0.132	0.6286
12288	0.2958	0.0932	0.6901
49152	0.1858	0.0553	

Table 1: Computed convergence rate s_i

In the experiment we consider a sequence of step-sizes $h_i = 2^{-(i+1)}h_1$ and choose $\Delta t_i = h_i^s$ with $s = \frac{2}{3}$ in order to obtain similar convergence rates in space and time. Since an explicit solution of the problem is not known we approximate the convergence rate s for $k = N$ through the expression

$$s_i \ln 2 = \ln \left(\frac{\left\| u_{h_i}^{N_i} - u_{h_{i+1}}^{N_{i+1}} \right\|_{H^1(\Omega)} + \left\| z_{h_i}^{N_i} - z_{h_{i+1}}^{N_{i+1}} \right\|_{L^2(\Omega)}}{\left\| u_{h_{i+1}}^{N_{i+1}} - u_{h_{i+2}}^{N_{i+2}} \right\|_{H^1(\Omega)} + \left\| z_{h_{i+1}}^{N_{i+1}} - z_{h_{i+2}}^{N_{i+2}} \right\|_{L^2(\Omega)}} \right). \tag{4.17}$$

Tabular 1 shows the values of s_i for our example calculated with Comsol Script. The resulting convergence rate s_i is $\approx \frac{2}{3}$, which coincides with the predicted rate.

References

[Alb98] H.-D. Alber. *Materials with memory. Initial-boundary value problems for constitutive equations with internal variables*, volume 1682 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1998.

[BS94] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer Verlag Inc., New York, 1994.

[CDN99] M. Costabel, M. Dauge, and S. Nicaise. Singularities of Maxwell interface problems. *Math. Mod. Numer. Anal.*, 33:627–649, 1999.

[Dau88] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains. Smoothness and Asymptotic Expansion*, volume 1341 of *Lecture Notes in Mathematics.* Springer-Verlag, 1988.

[DL76] G. Duvaut and J. L. Lions. *Inequalities in mechanics and physics*, volume 219 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, 1976.

[FHSV01] J.R. Férnandez, W. Han, M. Sofonea, and J.M. Viaño. Variational and numerical analysis of a frictionless contact problem for elastic-viscoplastic materials with internal state variables. *Q. J. Mech. Appl. Math.*, 54(4):501–522, 2001.

[Gri85] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman Publishing Inc, Boston, 1985.

[IS93] I.R. Ionescu and M. Sofonea. *Functional and numerical methods in viscoplasticity.* Oxford University Press, 1993.

[KM88] V. A. Kozlov and V. G. Maz'ya. Spectral properties of the operator bundles generated by elliptic boundary-value problems in a cone. *Func. Anal. Appl.*, 22:114–121, 1988.

[Kne04] D. Knees. On the regularity of weak solutions of quasi-linear elliptic transmission problems on polyhedral domains. *Z. Anal. Anwend.*, 23(3):509–546, 2004.

[Kne09] D. Knees. Global spatial regularity for time dependent elasto-plasticity and related problems. WIAS-Preprint No. 1395, Weierstrass Institute for Applied Analysis and Stochastics, Berlin, 2009.

[Mie80] E. Miersemann. Zur Regularität der quasistatischen elasto-viskoplastischen Verschiebungen und Spannungen. *Math. Nachrichten*, 96:293–299, 1980.

[MNP91] V.G. Maz'ya, S. A. Nazarov, and B. A. Plamenevsky. *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten I,II.* Akademie-Verlag, Berlin, 1991.

[Neč67] J. Nečas. *Les Méthodes Directes en Théorie des Équations Elliptiques.* Masson et Cie. and Éditeurs, Paris, 1967.

[Nic93] S. Nicaise. *Polygonal Interface Problems*, volume 39 of *Methoden und Verfahren der mathematischen Physik.* Peter Lang Verlag, 1993.

[NS94] S. Nicaise and A.-M. Sändig. General interface problems I. *Math. Methods Appl. Sci.*, 17:327–361, 1994.

[NS99] S. Nicaise and A.-M. Sändig. Transmission problems for the Laplace and elasticity operators: Regularity and boundary integral formulation. *Math. Methods Appl. Sci.*, 9:855–898, 1999.

[Sof93] M. Sofonea. Error estimates of a numerical method for a class of nonlinear evolution equations. *Rev. Colomb. Mat.*, 27(3-4):253–265, 1993.

[Tar72] L. Tartar. Interpolation non linéaire et régularité. *J. Funct. Anal.*, 9:469–489, 1972.

[Tri78] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* North Holland Publishing Company, Amsterdam, New York, Oxford, 1978.