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Self-similar rupture of viscous thin films in the strong-slip regime

D. Peschka[1], A. Münch[2], and B. Niethammer[2]

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[1] Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: peschka@wias-berlin.de

[2] Mathematical Institute
University of Oxford
24-29 St Giles'
Oxford, OX1 3LB
UK
E-Mail: muench@maths.ox.ac.uk
niethammer@maths.ox.ac.uk

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider rupture of thin viscous films in the strong-slip regime with small Reynolds numbers. Numerical simulations indicate that near the rupture point viscosity and van-der-Waals forces are dominant and that there are self-similar solutions of the second kind. For a corresponding simplified model we rigorously analyse self-similar behaviour. There exists a one-parameter family of self-similar solutions and we establish necessary and sufficient conditions for convergence to any self-similar solution in a certain parameter regime. We also present a conjecture on the domains of attraction of all self-similar solutions which is supported by numerical simulations.

1 Introduction

Numerical simulations of a variety of models for pinching of jets and thin-film rupture indicate self-similar behaviour near the pinching or rupture point respectively [1–7, and references therein]. While many, formal and rigorous, results on the existence of self-similar solutions and their asymptotic properties are available, there seem to be almost no rigorous results on the convergence of solutions for general data towards these self-similar solutions. In this paper we rigorously study convergence to self-similar rupture profiles in a simple model for viscosity dominated thin films.

The starting point of our analysis is the *strong-slip model* [8], which contains the non-dimensional slip-length b as one parameter. Numerical simulations in the regime of small Reynolds number and large slip length [9] show that before rupture the evolution passes through a self-similar regime, where viscosity and van-der-Waals forces are dominant. The film thickness h and the horizontal velocity u follow the dynamics

$$h(t,y) \sim (t_* - t)^{\alpha} H(\eta) \qquad \text{and} \qquad u(t,y) \sim (t_* - t)^{\beta-1} U(\eta)\,,$$

where $\eta = (y - y_*)/(t_* - t)^{\beta}$ and y^*, t^* denote the point and time of rupture respectively. The scaling exponent α is $1/3$, while β is not determined by the balance of the dominant terms and therefore one speaks of self-similarity of the second kind. In the context of freely suspended sheets this indeterminacy was noted in [10]. Numerical simulations show that inertia cannot be neglected close to the rupture point; for freely suspended sheets this was pointed out in [11]. However, for very viscous liquids and sufficiently large slip this transient regimes persists for a long time

Second-kind similarity solutions also appear in simple models for jet pinch-off which are very similar to the model we study in this paper. Self-similar solutions are studied in [12], while in [13] the existence of countably many self-similar solutions is established numerically. In [14] convergence to these solutions is discussed. Under strong assumptions on the evolution of the jet the authors show that the selection of the self-similar solution in this model is solely determined by the behaviour of the initial data around their minimum. This behaviour is expected, but a rigorous analysis without a priori assumptions on the solutions has still been elusive.

The goal of this paper is to provide such an analysis for viscosity dominated thin film rupture. We note that this analysis is also directly applicable to the model of jet pinch-off considered in [14].

The paper is organized as follows. In Section 2 we introduce the strong slip equation. In the regime where surface tension and inertia can be neglected we can simplify the strong-slip equation to an integro-differential equation following the method in [15]. For this model finite time rupture is established in [16] following ideas of [15, 17] for jet pinch-off. To investigate whether the rupture evolves self-similarly, we introduce self-similar variables and characterize self-similar solutions in Section 3. As described above, we encounter the situation that the scale for the spatial variable is not determined by dimensional analysis. It turns out that for each $\beta > 1/3$ there exists a unique self-similar solution. Equivalently these self-similar solutions can be uniquely characterized their behaviour $H(\eta) = H_0 + H_\rho \eta^\rho$ as $\eta \to 0$

In Section 4 we investigate whether solutions of the time-dependent problem converge to a self-similar shape and, in case they do, which self-similar solution, i.e. which ρ (or β), is selected. As expected, the long-time asymptotics are completely determined by the behaviour of the initial profile $h(0, y)$ at its minimum. We establish a necessary and sufficient condition for convergence to any of the self-similar solutions with $0 < \rho < 3/2$ (for the pinch-off model it would be for $0 < \rho < 2/3$). The precise criterion is that the solution converges if and only if the data are regularly varying at their minimum with index ρ. The corresponding rescaling is the one associated to the self-similar solution up to some slowly varying function given by the initial data. These results are very similar in nature to the dynamics in mean-field models for domain coarsening [18–20] and coagulation [21], where the long-time behaviour depends sensitively on the tail of the initial distribution functions.

We can prove the analogous characterization of domains of attraction for every positive ρ under an additional assumption. Presently we have no proof whether this assumption is satisfied for regularly varying data. Numerical results in Section 5 indicate that it is, but they also show that the situation is much more involved than in the case of $0 < \rho < 3/2$ and the details of the convergence proof must be different.

2 Model and simplification

The starting point for our considerations is the one-dimensional strong-slip model

$$\partial_t h + \partial_y(hu) = 0, \tag{1a}$$

$$Re^*(\partial_t u + u\partial_y u) = \frac{4}{h}\partial_y(h\partial_y u) + \partial_y\big(\partial^2_{yy}h - V(h)\big) - \frac{u}{b\,h}, \tag{1b}$$

which contains the Reynolds number Re^*, the non-dimensional slip-length b, and the van-der-Waals potential $V(h) = A/h^3$. Boundary conditions at $y = 0$ and $y = L$ are $\partial_y h(t,0) = \partial_y h(t,L) = 0$ and $u(t,0) = u(t,L) = 0$. This equation emerges as a limit of the Navier-Stokes equations with a free boundary if the ratio $\varepsilon = H/L$ of typical the height scale $[h] = H$ and the length scale $[y] = L$ is small and if the dimensional slip-length scales as $B = \varepsilon^{-2}b$. In [8] various models with different scalings of B are derived. A model for the dynamics of freely suspended films can be found in [22].

As discussed in [16] large slip-length as well as surface tension do not have a direct influence on the dynamics near the rupture. Hence we neglect the corresponding terms in (1a,1b). Furthermore we consider the case of small Reynolds number and consequently also neglect the effect of inertia. With $L = A = 1$ this leads to the following equations for $h(t,y)$ and $u(t,y)$

$$\partial_t h + \partial_y(hu) = 0, \tag{2a}$$

$$\frac{4}{h}\partial_y(h\partial_y u) - \partial_y V(h) = 0, \tag{2b}$$

for all $y \in (0,1)$, supplemented with boundary conditions $u(t,0) = u(t,1) = 0$ and initial data $h(0,y) = h_0(y)$ for all $y \in [0,1]$. Equation (2a) describes transport of fluid particles, whereas the momentum equation (2b) describes acceleration due to van-der-Waals forces and dissipation due to Trouton viscosity.

For the analysis to come it is convenient to go over to Lagrangian coordinates. We denote the Lagrangian reference coordinate of a fluid particle by x and its trajectory in the Eulerian coordinate system by $y = y(t,x)$. Then the defining relation for $y(t,x)$ is $u(t,y) = \partial_t y(t,x)$ for all $x \in (0,1)$.

Below we rewrite equations (2a,2b) through the so-called stretching variable defined as $s(t,x) = \partial_x y(t,x)$. The transformation $y(t,\cdot)$ preserves the orientation and it maps $(0,1)$ onto itself, which implies $s(t,x) > 0$ and $y(t,1) = \int_0^1 s(t,x)\,dx = 1$. With $\bar{h}(t,x) = h(t,y)$, $\bar{u}(t,x) = u(t,y)$ and (2a) we obtain $\frac{d}{dt}\big(\bar{h}(t,x)\,s(t,x)\big) = 0$. This implies that the product $\bar{h}s$ is constant along characteristic curves, i.e. $c(x) = \bar{h}(t,x)s(t,x)$. Integrating (2b) with respect to y and going over to Lagrangian coordinates yields

$$\partial_t s(t,x) = \frac{3\,c(x)}{8}\left(\frac{1}{\bar{h}^4(t,x)} - \frac{\sigma^2(t)}{\bar{h}^2(t,x)}\right), \tag{3}$$

with a constant of integration $\sigma^2(t)$. We get rid of $c(x)$ by choosing an initial reference frame $y_0(x) = y(0,x)$ where $c(x)$ is a constant function. This is apparently the case if $y_0(x)$ solves the ordinary differential equation $y_0' = c/h_0(y_0)$ with $y_0(0) = 0$. The remaining constant c is determined by the condition $y_0(1) = 1$. Thus, after rescaling time by an appropriate constant, (3) becomes

$$\partial_t s(t,x) = s(t,x)^2\big(s(t,x)^2 - \sigma^2(t)\big), \qquad x \in [0,1]\,, \tag{4a}$$

where finally $\sigma^2(t)$ is determined by the constraint $\int_0^1 s(t,x)\,dx = 1$, which implies

$$\sigma^2(t) = \int_0^1 s(t,x)^4\,dx \quad \Big/ \quad \int_0^1 s(t,x)^2\,dx. \tag{4b}$$

Equations (4a) and (4b) have to be supplemented with initial conditions

$$s(0,x) = s_0(x) \geq 0 \qquad \text{for all } x \in [0,1]\,, \tag{4c}$$

where $\int_0^1 s_0(x)\,dx = 1$.

We make now our assumptions on the initial data s_0 precise. For large times the behaviour of the solution $s(t,x)$ of (4a-4c) is determined by its behaviour around the maximum $s_{\max}(t) = \text{esssup}_{x\in(0,1)} s(t,x)$. The main technical assumption in this paper is that we only consider decreasing initial data s_0, which is certainly justified if h_0 is symmetric around the rupture point and can be justified even for a wider range of initial data. Otherwise we only require s_0 to be right-continuous such that $s_{\max}(t) = s(t,0)$ for all $t \geq 0$ and for convenience we assume that $s_0(x) > 0$ for all $x \in [0,1)$. Since the right-hand side of (4a) is locally Lipschitz and contains x only as a parameter we obtain existence and uniqueness of solutions locally in time. It is easily seen that $s_{\max}(t)$ is strictly increasing in time if $s_0(x)$ is not almost everywhere constant. In [16] we showed that if $s_0(x) < s_{\max}(0)$ for all $x > 0$, then $s_{\max}(t) \to \infty$ as $t \to t_*$ where t_* can be finite or infinite.

From now on we also assume that $s_0(x) \leq s_{\max}(0) - cx^\rho$ for some $c, \rho > 0$. As established in [16] this implies that there is blow-up of $\max s$ in finite time t_*. Our goal in this paper is to study whether this blow-up occurs in a self-similar fashion.

3 Scalings and self-similar solutions

3.1 Scalings of the solution

Roughly speaking, by scaling we denote the rate by which one zooms into the graph of the function $s(t,x)$ at $x = 0$. If that graph converges to something nontrivial under that zooming process one speaks of self-similarity. Commonly one would seek self-similar solutions of (4a-4c) with powerlaw-type scalings

$$s(t,x) = \big(t_* - t\big)^{-\alpha}\varphi\big(\eta\big) \tag{5}$$

with $\eta = x\big(t_* - t\big)^{-\beta}$ and $\theta(t) = \sigma^2(t)(t_* - t)^{2/3}$. By t_* we denote the blow-up time and $\alpha, \beta > 0$ and the profile φ have to be determined. By plugging (5) into (4a) we see that $\alpha = 1/3$ and φ must satisfy

$$\beta\varphi'(\eta) = \varphi^4 - \theta\varphi^2 - \tfrac{1}{3}\varphi\,. \tag{6}$$

Next, we try to determine β using the constraint, which should be valid for a self-similar solution as $t \to t_*$. This means

$$1 = \lim_{t\to t_*} \big(t_* - t\big)^{\beta - 1/3} \int_0^{(t_*-t)^{-\beta}} \varphi(\eta)\, d\eta. \tag{7}$$

This suggests to choose $\beta = 1/3$, but then equation (6) implies that $\varphi(\eta) \sim \eta^{-1}$ as $\eta \to \infty$, which is inconsistent with (7). Hence, instead we need to look for second-kind similarity solutions, and in fact we will see that for any $\beta > 1/3$ there exists one.

To proceed with our analysis it will be more convenient to rescale $s(t,x)$ with $s_{\max}(t)$ instead of $(t^* - t)^{-\alpha}$. More precisely, we introduce new functions ψ and φ as follows.

Definition 3.1. (Height scaling) For any solution $s(t,x)$ of (4a-4c) define the normalized solution $\psi(\tau,x)$ and the new time-scale τ via

$$\psi(\tau,x) = \frac{s(t,x)}{s_{\max}(t)} \qquad \text{and} \qquad \tau = \log\left(\frac{s_{\max}(t)}{s_{\max}(0)}\right). \tag{8}$$

The corresponding initial data are $\psi_0(x) = \frac{s_0(x)}{s_{\max}(0)}$.

This definition of τ is meaningful since $s_{\max}(t)$ is strictly increasing and unbounded. To study the structure near the blow-up we also need to rescale the spatial variable. This scale is not determined a priori but has to be found as part of the solution.

Definition 3.2. (Similarity scaling) Let $\lambda : [0,\infty) \to [1,\infty)$ be a measurable function with $\lambda(0) = 1$ and $\lambda(\tau) \to \infty$ as $\tau \to \infty$. We call any such $\lambda(\tau)$ a rescaling.

For any solution $s(t,x)$ of (4a-4c) define the normalized and rescaled solution $\varphi(\tau,\eta)$ via

$$\varphi(\tau,\eta) = \frac{s(t,x)}{s_{\max}(t)} = \psi(\tau,x), \qquad \text{where} \qquad \eta = x\,\lambda(\tau)\,, \tag{9}$$

and the initial data are $\varphi_0(\eta) = \frac{s_0(\eta)}{s_{\max}(0)}$.

Two such rescalings $\lambda(\tau)$ and $\bar\lambda(\tau)$ are said to be *equivalent*, if there exists a positive number C such that

$$\lim_{\tau\to\infty}\left(\frac{\lambda(\tau)}{\bar\lambda(\tau)}\right) = C.$$

3.2 Solution formulas

The solution of the integro-differential equation (4a-4c) in the height scaling (8) solves

$$\partial_\tau \psi = K(\tau)\big(\psi^4 - \psi^2\big) + \psi^4 - \psi = f\big(K(\tau),\psi\big)\,, \tag{10}$$

where $K(\tau) = \theta(\tau)\big(1-\theta(\tau)\big)^{-1}$ and

$$\theta(\tau) = \frac{\int_0^1 \psi(\tau,x)^4\,dx}{\int_0^1 \psi(\tau,x)^2\,dx}. \tag{11}$$

This form of $K(\tau)$ is equivalent to the constraint (4b) and ensures that

$$1 = \int_0^1 s(t,x)\,dx = s_{\max}(0)\,e^{\tau}\int_0^1 \psi(\tau,x)\,dx \tag{12}$$

is still fulfilled. To integrate (10) we introduce for any given $K_* \geq 0$ the function

$$H(\xi) = \int_{1/2}^{\xi} \frac{dr}{f(K_*,r)}.$$

where f is as in (10). This function is strictly decreasing, $H(1/2) = 0$ and

$$H(\xi) \to \begin{cases} -\infty & as \xi \to 1 \\ \infty & as \xi \to 0 \end{cases}.$$

Next we insert the solution in the height scaling (8) into H_{K_*} and compute the derivative with respect to τ

$$\frac{d}{d\tau}H(\psi(\tau,x)) = \frac{\partial_\tau \psi}{f(K_*,\psi)} = \frac{f\big(K(\tau),\psi(\tau,x)\big)}{f\big(K_*,\psi(\tau,x)\big)}.$$

Integrating in time this gives

$$\begin{aligned} H\big(\psi(\tau,x)\big) - H\big(\psi(\tau_0,x)\big) &= \tau - \tau_0 + \int_{\tau_0}^{\tau}\left(\frac{f\big(K(t),\psi(t,x)\big)}{f\big(K_*,\psi(t,x)\big)} - 1\right)dt \\ &= \tau - \tau_0 + \int_{\tau_0}^{\tau}\big(K(t)-K_*\big)g\big(\psi(t,x)\big)\,dt \end{aligned} \tag{13}$$

for all $x \in (0,1)$ and $0 \leq \tau_0 \leq \tau$ with g defined as

$$g(\xi) = \frac{\xi(\xi+1)}{1+(K_*+1)\xi(\xi+1)} \in \Big[0, \frac{2}{2K_*+3}\Big]. \tag{14}$$

Note that g is bounded and does not depend on $K(\tau)$. It is convenient to study rescaled solutions in this formulation since convergence of $K(\tau) \to K_*$ and the behaviour of solutions near $x = 0$ are encoded separately in the integral over $(K - K_*)g$ and in H respectively.

We will repeatedly use that the leading order singular behaviour of H as $\psi \to 1$ is

$$H(\psi) = \tfrac{1}{2K_*+3}\log(1-\psi) + \mathcal{O}(1)\,, \tag{15}$$

which follows from the fact that

$$\frac{1}{2K_*+3}\log\left(\frac{1-\psi}{1/2}\right) - H(\psi) = \int_{1/2}^{\psi}\left(\frac{1}{(2K_*+3)(\xi-1)} - \frac{1}{f(K_*,\xi)}\right)d\xi \tag{16}$$

and that the integrand on the right-hand side is bounded as $s \to 1$.

Now we consider solutions $\varphi(\tau,\eta)$ under the similarity scaling. For further reference notice that the functions $\theta(\tau)$ and $K(\tau)$ defined in (11) are independent of the rescaling $\lambda = \lambda(\tau)$ and can also be written as $K = \theta(1-\theta)^{-1}$ where

$$\theta(\tau) = \frac{\int_0^\lambda \varphi(\tau,\eta)^4 \, d\eta}{\int_0^\lambda \varphi(\tau,\eta)^2 \, d\eta}. \tag{17}$$

Notice that $K(\tau)$ is independent from any particular choice of $a(\tau)$. Recalling (9), equation (13) implies that

$$\begin{aligned} H\big(\varphi(\tau,\eta)\big) = \quad & \tau - \tau_0 + H\left(\varphi\left(\tau_0, \eta\tfrac{\lambda(\tau_0)}{\lambda(\tau)}\right)\right) \\ & + \textstyle\int_{\tau_0}^{\tau} \big(K(t) - K_*\big)\; g\Big(\varphi\left(t, \eta\tfrac{\lambda(t)}{\lambda(\tau)}\right)\Big)\, dt. \end{aligned} \tag{18}$$

for all $\eta \in \big(0, \lambda(\tau)\big)$ and $0 \le \tau_0 \le \tau$. Notice that this formulation is weaker than

$$\partial_\tau \varphi + (\partial_\tau \log \lambda)\, \eta \partial_\eta \varphi = f\big(K(\tau), \varphi\big)\,, \tag{19}$$

which holds for sufficiently smooth φ and λ.

3.3 Self-similar solutions

Self-similar solutions are time-independent rescaled solutions $\varphi_*(\eta)$ of (4a-4c) with constant $K(\tau) = K_*$. That is, in view of (18) they satisfy

$$H\big(\varphi_*(\eta)\big) = \tau + H\Big(\varphi_*\Big(\frac{\eta}{\lambda(\tau)}\Big)\Big) \tag{20}$$

for all $\eta > 0$ and for some rescaling $\lambda(\tau)$.

If we assume that φ_* and λ are differentiable, which is justified a-priori, then we find from (19) that $\partial_\tau \log \lambda(\tau)$ is constant. We denote this constant by γ and see $\lambda(\tau) = e^{\gamma\tau}$. Since φ_* is decreasing and $f < 0$ we deduce that $\gamma > 0$ and that φ_* solves

$$\gamma\eta \frac{d\varphi_*}{d\eta} = K_*\big(\varphi_*^2 - \varphi_*^2\big) + \varphi_*^4 - \varphi_* = f(K_*, \varphi_*)\,. \tag{21}$$

Notice that for any $K_* \in \mathbb{R}$ equation (21) also has the two homogeneous solutions $\varphi_*(\eta) = 1$ and $\varphi_*(\eta) = 0$ for all $\eta > 0$. The latter is discontinuous for $\eta \to 0$. These are obviously not relevant solutions and correspond to rescalings which either grow to fast, such that one sees only $\varphi_* = 1$ in the limit, or too slowly in the other case.

Equation (21) is equivalent to

$$H(\varphi_*(\eta)) = \frac{1}{\gamma} \log \eta + C \tag{22}$$

for some constant $C \in \mathbb{R}$. The undetermined constant C in (22) is due to the invariance of equation (21) under a constant rescaling of η which is also valid for equation (19) and is also related to the equivalence of rescalings $\lambda(\tau)$.

In the following we will call two stationary solutions φ_*^1 and φ_*^2 equivalent if there exists $c > 0$ such that $\varphi_*^1(c\eta) = \varphi_*^2(\eta)$ for all $\eta > 0$. Sometimes it is convenient to fix one member of each equivalence class, for example by requiring that

$$\varphi_*(1/2) = 1/2. \tag{23}$$

Then equation (22) becomes

$$H(\varphi_*(\eta)) = \frac{1}{\gamma} \log\left(\frac{\eta}{1/2}\right).$$

Before we proceed we collect some properties of a stationary solution $\varphi_*(\eta)$. First, (22) implies that φ_* is decreasing and satisfies $\lim_{\eta\to 0} \varphi_*(\eta) = 1$ and $\lim_{\eta\to\infty} \varphi_*(\eta) = 0$. Furthermore, as $\eta \to \infty$ equation (21) implies that

$$\varphi_*(\eta) \sim \frac{1}{\eta^{1/\gamma}} \qquad \text{as } \eta \to \infty\,. \tag{24}$$

Using Taylor's expansion $f(K,\xi) = -(2K+3)(1-\xi) + O\big((1-\xi)^2\big)$, we also find that every solution of (21) satisfies

$$\varphi_*(\eta) \sim 1 - C\eta^n \quad \text{for } \eta \to 0 \qquad \text{where} \quad \gamma n = 2K_* + 3\,.$$

Next, we have to ask which combinations of γ and K_* are meaningful. We first notice, that the constraint (12) poses further restrictions on γ. As discussed before, we request that (12) is valid in the limit as $\tau \to \infty$, that is

$$\frac{1}{s_{\max}(0)} = \lim_{\tau\to\infty} e^{-(\gamma-1)\tau} \int_0^{e^{\gamma\tau}} \varphi_*(\eta)\, d\eta\,. \tag{25}$$

Thus a nontrivial self-similar solution which satisfies (12) can only exist if $\gamma \geq 1$. Furthermore, due to (24), the choice $\gamma = 1$ leads to a contradiction with (25). Hence we need $\gamma > 1$ for a self-similar solution. Instead of working with (25), which is difficult to deal with due to the divergences involved, we will in the following request that a self-similar solution satisfies the appropriate analogue of (17).

Definition 3.3. (Exact similarity solutions) A function φ_*, which for some $\gamma > 1$ satisfies (22) for all $\eta > 0$ and for which

$$\overline{K}(K_*, \gamma) = K_*\,, \tag{26a}$$

where

$$\overline{K}(K_*, \gamma) = \frac{\int_0^\infty \varphi_*^4(\eta)\, d\eta}{\int_0^\infty \big(\varphi_*^2(\eta) - \varphi_*^4(\eta)\big)\, d\eta}, \tag{26b}$$

is called an *exact self-similar solution* of (4a-4c). In (26b) we use the convention that $\overline{K} = 0$ if the second moment of φ_* is infinite.

In Definition 3.3 we set $\overline{K} = 0$ if $\int_0^\infty \varphi_*^2 \, d\eta = \infty$. This is is motivated by the following Lemma.

Lemma 3.1. *Assume that $\varphi(\tau, \eta)$ converges pointwise to a self-similar solution φ_* and that $\int_0^{\lambda(\tau)} \varphi^2 \, d\eta \to \infty$ as $\tau \to \infty$. Then $K(\tau) \to 0$ as $\tau \to \infty$.*

Proof. We know that a self-similar solution in the sense of Definition 3.3 is decreasing and satisfies $\varphi_*(\eta) \in (0, 1)$ for all $\eta > 0$. Thus we can choose for an arbitrary $\varepsilon > 0$ numbers η_0 and τ_0 such that $\varepsilon/2 < \varphi(\tau, \eta_0) < \varepsilon$ for all $\tau > \tau_0$. Then we have

$$\theta(\tau) = \frac{\int_0^{\lambda(\tau)} \varphi^4 \, d\eta}{\int_0^{\lambda(\tau)} \varphi^2 \, d\eta} \leq \frac{\int_0^{\eta_0} d\eta + \int_{\eta_0}^{\lambda(\tau)} \varphi^4 \, d\eta}{\int_0^{\eta_0} (\varepsilon/2)^2 \, d\eta + \int_{\eta_0}^{\lambda(\tau)} \varphi^2 \, d\eta} \leq \frac{\eta_0 + \varepsilon^2 \int_{\eta_0}^{\lambda(\tau)} \varphi^2 \, d\eta}{\eta_0 \varepsilon^2/4 + \int_{\eta_0}^{a(\tau)} \varphi^2 \, d\eta}$$
$$\to \varepsilon^2 \qquad \text{as } \tau \to \infty.$$

Since ε was arbitrary, this implies in particular $K(\tau) = \theta(\tau)(1 - \theta(\tau))^{-1} \to 0$ as $\tau \to \infty$. ☐

Our main result en existence and uniqueness of self-similar solutions is the following.

Theorem 3.2. *For any $\gamma > 1$, or equivalently for any $\rho > 0$, there exists a self-similar solution φ_* which is unique up to equivalence.*

For given $\gamma > 1$ and $K_* \geq 0$ we find a φ_* via (22) and impose (23). For such a solution consider $\overline{K}(K_*, \gamma)$ as in (26b) and find a unique fixed point K_* of $\overline{K}(K_*, \gamma)$.

Proof.

1. Proof for $K_* = 0$ ($\gamma \geq 2$ or $0 < \rho \leq 3/2$)

   We can easily integrate (21) explicitly using (22). For $K_* = 0$ we find that

   $$H\big(\varphi_*(\eta)\big) = \frac{1}{3} \log \left(\frac{s^3 - 1}{s^3} \right) \Bigg|_{1/2}^{\varphi_*(\eta)}$$

   such that $\varphi_*(\eta) = \big(1 + c_0 \eta^{3/\gamma}\big)^{-1/3}$, where the choice of c_0 ensures $\varphi_*(1/2) = 1/2$.

   For φ_* to be a self-similar solution we need $\overline{K}(0, \gamma) = 0$, more precisely $\int_0^\infty \varphi_*^2(\eta) \, d\eta = \infty$. This is the case for $\gamma \geq 2$. Solutions of (21) with same γ are equivalent and the relation between γ and ρ is $\gamma\rho = 3$. This proves Theorem 3.2 for $\gamma \geq 2$ or $\rho \in (0, 3/2]$ respectively.

2. Proof for $K_* > 0$ ($\gamma \in (1, 2)$ or $\rho > 3/2$):

   A rigorous proof can be found in [9]. The main idea is to show $\overline{K} > K$ for sufficiently large K and fixed γ. As this proof is somewhat lengthy we only present a numerical solution to the problem. Figure 1 shows that for any

$\gamma \in (1,2)$ the function $\overline{K}(K_*,\gamma)$ is positive at $K_* = 0$ and grows slowly with increasing K_*. Since $\overline{K}$ depends continuously on $\bar{K}$ there exists a (unique) fixed point $K_* > 0$. The relation between ρ and γ is one-two-one because of $\gamma\rho = 2K_* + 3$ and since $\overline{K}$ in Figure 1 is decreasing with γ. $\square$

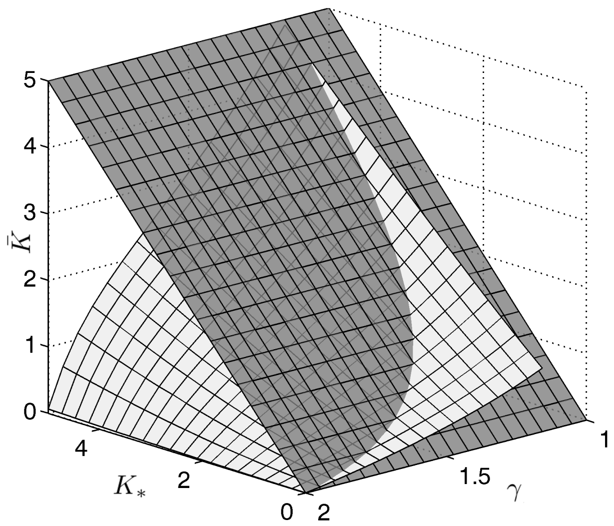

Figure 1: $\bar{K}(K_*,\gamma)$ surface by numerical quadrature (bright) and K plane (dark); Self-similar solutions with $\gamma \in (1,2)$ lie on the intersection curve of both surfaces.

4 Convergence to self-similar solutions

Now we study the dynamics of solutions $s(t,x)$ of (4a-4c). What we prove is that, for a large class of initial data, solutions evolve approximately self-similar. That is to say that the corresponding $\varphi(\tau,\eta)$ of (9) converges to some $\varphi_*(\eta)$ as $\tau \to \infty$ for some yet unknown scaling $\lambda(\tau)$. As mentioned earlier, there are the two trivial $\varphi_* = 1$ and $\varphi_* = 0$ a.e., which are meaningless if one wants to consider convergence to self-similar solutions.

In the following lemma we show that convergence of φ to a self-similar solution implies uniqueness of the corresponding scalings up to the equivalence.

Lemma 4.1. *Let $s(t,x)$ be a solution of (4a-4c) with decreasing initial data and for some $\lambda(\tau)$ let $\varphi(\tau,\eta)$ be the associated similarity scaling. Assume $\varphi(\tau,\eta)$ converges (pointwise) to a nonconstant continuous function $\varphi_*(\eta)$ as $\tau \to \infty$. Then this scale transformation λ is unique up to equivalence.*

Furthermore, one can always select a rescaling $\bar{\lambda}(\tau)$ such that the corresponding rescaled solution solution rescaled with $\bar{\lambda}(\tau)$ fulfils $\varphi(\tau,1/2) = 1/2$ for all $\tau \geq 0$.

Proof. First note that φ converges uniformly. Assume that there exist two rescalings λ_i, such that the corresponding φ_i converge ($i = 1,2$). Both solutions are related via $\varphi_2(\tau,\eta) = \varphi_1(\tau,\eta\lambda_1(\tau)/\lambda_2(\tau))$. Then if $(\lambda_1/\lambda_2)(\tau_n) \to 0$ for a subsequence $\tau_n \to \infty$, we would have $\varphi_2(\tau_n,\eta) \to 1$ which gives a contradiction since the solution is constant. Similarly we can exclude that $(\lambda_1/\lambda_2)(\tau_n) \to \infty$ for a subsequence $\tau_n \to \infty$, since this would give the second trivial solution $\varphi_*(\eta) = 0$ for $\eta > 0$.

Now assume that $(\lambda_1/\lambda_2)(\tau)$ remains bounded, but does not converge. Then there are at least two different accumulation points α_1 and α_2. If φ_1^* denotes the limit of φ_1 and φ_2^* the limit of φ_2, we find $\varphi_2^*(\eta) = \varphi_1^*(\alpha_1\eta) = \varphi_1^*(\alpha_2\eta)$ for all $\eta > 0$. Due to the properties of the limit functions this is only possible if $\alpha_1 = \alpha_2$ which proves uniqueness up to equivalence of the rescaling.

The second part of the Lemma follows by choosing $\bar{\lambda}(\tau) = \frac{\lambda(\tau)}{2\chi(\tau,1/2)}$ where $\chi(\tau,\cdot)$ denotes the (generalized) inverse of $\psi(\tau,\cdot)$ for any $\tau \geq 0$. □

Our main result, Theorem 4.3, gives a necessary and sufficient criterion for convergence to any of the self-similar solution under the assumption that $K(\tau)$ converges sufficiently fast. If the data satisfy $\varphi_0(\eta) \leq 1 - c\eta^\rho$ for $0 < \rho < 3/2$ we can also prove fast convergence of $K(\tau) \to 0$ (cf. Proposition 4.4), such that we are able to completely characterize the domain of attraction of the self-similar solutions for $\rho \in (0,3/2)$ (cf. Corollary 4.5).

4.1 Weak characterization of scalings

Before we prove our main result, we first give a weak characterization of scale transformations λ, i.e., for a large class of initial data we characterize for which λ there can never be convergence. Assuming that $K(\tau)$ converges and that the initial data are bounded from above or from below by some power law behaviour, i.e., $\varphi_0(\eta) \leq 1 - c\eta^\rho$ or $\varphi_0(\eta) \geq 1 - c\eta^\rho$, then a solution does not converge to a self-similar solution, but rather to one of the two trivial homogeneous states, if the rescaling is not suitable.

Theorem 4.2. *Consider a solution $s(t,x)$ of (4a-4c) and assume $K(\tau) \to K_*$. Furthermore let $\varphi(\tau,\eta)$ be defined as in* (9) *and assume for some $\gamma > 1$ that $\gamma\tau - \log\lambda(\tau) = o(\tau)$ as $\tau \to \infty$.*

1. *If the initial data satisfy*

$$\varphi_0(\eta) \geq 1 - \alpha\eta^\rho \qquad \text{for some } \alpha, \rho > 0$$

and $\rho > \frac{2K_*+3}{\gamma}$, then $\varphi(\tau,\eta) \to 1$ for all $\eta \geq 0$.

2. *Conversely, if the initial data satisfy*

$$\varphi_0(\eta) \leq 1 - \alpha\eta^\rho \qquad \text{for some } \alpha, \rho > 0$$

and $\rho < \frac{2K_*+3}{\gamma}$, then $\varphi(\tau,\eta) \to 0$ for all $\eta > 0$.

Remark. Rescalings which satisfy the assumption in the previous Theorem are for example $\lambda(\tau) = e^{\gamma\tau}$, but also $\lambda(\tau) = e^{\gamma\tau}(1+\tau)^k$ with $k \in \mathbb{R}$.

It will be immediately apparent from the proof, that the statements can be slightly generalized. The first statement remains true if the positive part fulfils $\big(\gamma\tau - \log\lambda(\tau)\big)_+ = o(\tau)$, while the second remains valid if $\big(\gamma\tau - \log\lambda(\tau)\big)_- = o(\tau)$ as $\tau \to \infty$.

Proof. (Part 1.) For given positive $\varepsilon < \frac{\rho\gamma}{2K_*+3} - 1$ choose τ_1 such that $\sup_{\tau\geq\tau_1} |K(\tau) - K_*| \frac{2}{2K_*+3} < \varepsilon$. Then $\big|\int_0^{\tau_1}(K(t)-K_*)g\,dt\big| \leq C(\tau_1)$ due to (14), where the value of the expression $C(\tau_1)$ does not depend on τ. Now we use (18) for $\tau_0 = 0$ and (15) to find

$$
\begin{aligned}
H(\varphi(\tau,\eta)) \qquad & \leq \tau + H\Big(\varphi_0\Big(\eta\tfrac{1}{\lambda(\tau)}\Big)\Big) + \varepsilon\tau + C(\tau_1) \\
&= \tau(1+\varepsilon) + \tfrac{1}{2K_*+3}\log\Big(\eta^\rho\Big(\tfrac{1}{\lambda(\tau)}\Big)^\rho\Big) + C(\tau_1) \\
&= \tau(1+\varepsilon) - \tfrac{\rho}{2K_*+3}\log\big(\lambda(\tau)\big) + C(\tau_1) \\
&= \tau\Big(1 - \tfrac{\rho\gamma}{2K_*+3} + \varepsilon\Big) + \tfrac{\rho}{2K_*+3}\left(\gamma\tau - \log\lambda\right) + C(\tau_1)\,.
\end{aligned}
$$

By assumption the second term is of order $o(\tau)$ as $\tau \to \infty$, whereas the first one converges to $-\infty$ linearly in τ. Hence $\varphi(\tau,\eta) \to 1$ as $\tau \to \infty$ for all $\eta > 0$.

The proof of Part 2. is entirely similar. □

4.2 Criterion for convergence

In the remainder of this paper we will denote for any $\rho \in \mathbb{R}$ the corresponding self-similar solution by φ_*^ρ and the corresponding value of K_* by K_*^ρ.

Our main result is the following. If for given ρ we have $K(\tau) \to K_*^\rho$ sufficiently fast, then $\varphi(\tau,\eta)$ converges to $\varphi_*^\rho(\eta)$ if and only if the data are regularly varying at zero with power ρ, that is, more precisely if $1 - \varphi_0(\eta) \sim \big(\eta L(\eta)\big)^\rho$ for a slowly varying function L.

We say that function L is slowly varying at zero if

$$
\lim_{x\to 0}\frac{L(tx)}{L(x)} = 1 \qquad \text{for all } t > 0\,.
$$

Typical examples of slowly varying functions are $\log x$, $\log(\log x)$ etc. and all powers of logarithms. We refer to the book [23] for a full characterization of slowly varying functions, as well as further generalizations and examples.

Theorem 4.3. *Let $s(t,x)$ be a solution of* (4a)*, let $\rho > 0$ and assume $K(\tau) \to K_*^\rho$ as $\tau \to \infty$ such that $\int_0^\infty |K(\tau) - K_*^\rho|\,d\tau < \infty$.*

Then there exists a rescaling $\lambda(\tau)$ (9) *such that with $\varphi(\tau,\eta)$ as in* (9) *we have that $\varphi(\tau,\eta) \to \varphi_*^\rho(\eta)$ for all $\eta > 0$ as $\tau \to \infty$ if and only if the data satisfy*

$$
1 - \varphi_0(\eta) \sim \big(\eta L(\eta)\big)^\rho \qquad \text{as } \eta \to 0 \tag{27a}
$$

with slowly varying function L. The rescaling is equivalent to $\lambda(\tau)$ implicitly defined as

$$\varphi_0\left(\frac{1}{\lambda(\tau)}\right) = 1 - e^{-(2K_*^\rho+3)\tau}\,. \tag{27b}$$

Remark. Notice first that (27b) defines a proper rescaling since φ_0 is decreasing and $\varphi_0(0) = 1$. Equation (27b) implies that with $\gamma = \frac{2K_*^\rho+3}{\rho}$ we have

$$\frac{1}{\lambda(\tau)}L\Big(\frac{1}{\lambda(\tau)}\Big) \sim e^{-\gamma\tau} \qquad \text{as } \tau \to \infty$$

which can be rewritten, using the de Bruijn conjugate $L^\#$ of L. This is a slowly varying function which satisfies $L(x)L^\#\big(xL(x)\big) \sim 1$ as $x \to 0$ and comes into play for example if one wants to invert slowly varying functions. With this definition and the inversion formula (see Chapter 1.5.7 of [23]), we can write λ as

$$\frac{1}{\lambda(\tau)} \sim e^{-\gamma\tau}L^\#\big(e^{-\gamma\tau}\big) \qquad \text{as } \tau \to \infty\,.$$

Thus we see that $\lambda(\tau)$ is essentially the time scale for the self-similar solution up to a slowly varying correction which is given by the data.

Convergence for rescalings equivalent to $e^{-\gamma\tau}$ only occurs if $1-\varphi_0(\eta) \sim c\eta^\rho$ as $\eta \to 0$, that is if the data behave exactly as a power law.

Proof. Let $\lambda(\tau)$ be a rescaling and $\varphi(\tau, \eta)$ as in (9). Within this proof we denote for convenience K_*^ρ by K_*.

1. The solution formula (18) for $\tau_0 = 0$ implies

$$H\big(\varphi(\tau,\eta)\big) = \tau + H\Big(\varphi_0\Big(\tfrac{\eta}{\lambda(\tau)}\Big)\Big) + \int_0^\tau \big(K(t) - K_*\big)\, g\Big(\varphi\Big(t, \eta\tfrac{\lambda(t)}{\lambda(\tau)}\Big)\Big)\, dt\,.$$

   We first argue that

$$I(\tau) = \int_0^\tau \big(K(t) - K_*\big)\, g\Big(\varphi\Big(t, \eta\tfrac{\lambda(t)}{\lambda(\tau)}\Big)\Big)\, dt \to g(1)\int_0^\infty \big(K(t) - K_*\big)\, dt \tag{28}$$

   as $\tau \to \infty$. Indeed, for any $\tau_0 > 0$ we write $I(\tau) = \int_0^{\tau_0} \cdots + \int_{\tau_0}^{\tau} \cdots = I_1(\tau) + I_2(\tau)$. Due to the assumptions on $K(\tau)$ and the boundedness of g (cf. (14)), we can for any given $\varepsilon > 0$ choose τ_0 so large such that $|I_2(\tau)| < \varepsilon$ for all $\tau > \tau_0$. Furthermore, since $\lambda(\tau) \to \infty$ as $\tau \to \infty$ we find

$$g(\varphi(t, \eta\lambda(t)/\lambda(\tau))) \overset{\tau\to\infty}{\to} g(\varphi(1,0)) = g(1)$$

   for all $0 \le t < \tau$. Since $|(K(t) - K_*)g(\varphi(t, \eta\lambda(t)/\lambda(\tau)))| \le C|K(\tau) - K_*|$ and since $|K(t) - K_*|$ is integrable by assumption, Lebesgue's Dominated Convergence Theorem implies that $I_1(\tau) \to g(1)\int_0^{\tau_0}(K(t) - K_*)\, dt$ as $\tau \to \infty$. This proves (28).

2. Next we recall (15) and (16) to write

$$H\Big(\varphi_0\Big(\tfrac{\eta}{\lambda(\tau)}\Big)\Big) = \tfrac{1}{2K_*+3}\log\Big(1-\varphi_0\Big(\tfrac{\eta}{\lambda(\tau)}\Big)\Big) + \tfrac{\log 2}{2K_*+3} + \int_{1/2}^{\varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)} \Big\{\tfrac{1}{(2K_*+3)(\xi-1)} - \tfrac{1}{f(K_*,\xi)}\Big\}\,d\xi\,. \tag{29}$$

Since the integrand in the last term is bounded as $s \to 1$ we find that

$$\int_{1/2}^{\varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)} \Big\{\frac{1}{(2K_*+3)(\xi-1)} - \frac{1}{f(K_*,\xi)}\Big\}\,d\xi \to \int_{1/2}^{1} \dots\, d\xi = \text{const.} \tag{30}$$

as $\tau \to \infty$. Thus, (28), (29) and (30) imply

$$H\big(\varphi(\tau,\eta)\big) = \tau + \frac{1}{2K_*+3}\log\Big(1-\varphi_0\Big(\frac{\eta}{\lambda(\tau)}\Big)\Big) + \omega(\tau)\,, \tag{31}$$

with some function $\omega(\tau)$ that satisfies $\omega(\tau) \to \mathit{const.}$ as $\tau \to \infty$ with a constant independent of η.

3. Now assume that $\varphi_0(\cdot)$ satisfies (27a) and choose $\lambda(\tau)$ as in (27b). Then (31) gives

$$\begin{aligned} H\big(\varphi(\tau,\eta)\big) &= \tfrac{1}{2K_*+3}\log\left(\frac{1-\varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)}{1-\varphi_0\left(\frac{1}{\lambda(\tau)}\right)}\right) + \omega(\tau) \\ &= \tfrac{\rho}{2K_*+3}\left(\log\eta + \log\left(\frac{L\left(\frac{\eta}{\lambda(\tau)}\right)}{L\left(\frac{1}{\lambda(\tau)}\right)}\right)\right) + \omega(\tau)\,. \end{aligned} \tag{32}$$

Since L is slowly varying at zero we find

$$\lim_{\tau\to\infty} \log\left(\frac{L\left(\frac{\eta}{\lambda(\tau)}\right)}{L\left(\frac{1}{\lambda(\tau)}\right)}\right) \to 0$$

and thus

$$H\big(\varphi(\tau,\eta)\big) \to \frac{\rho}{2K_*+3}\log\eta + C \qquad \text{as } \tau \to \infty,$$

which indeed implies that $\varphi(\tau,\eta) \to \varphi_*^\rho(\eta)$ for all $\eta > 0$ as $\tau \to \infty$ (recall (22)).

4. Conversely, assume that $\varphi(\tau,\eta) \to \varphi_*^\rho(\eta)$ as $\tau \to \infty$ for all $\eta > 0$. This implies, using (22) and (31), that

$$\tau + \frac{1}{2K_*+3}\log\Big(1-\varphi_0\Big(\frac{\eta}{\lambda(\tau)}\Big)\Big) \to \frac{\rho}{2K_*+3}\log\eta + C \tag{33}$$

as $\tau \to \infty$ for all $\eta > 0$ and some constant $C \in \mathbb{R}$. In particular, for $\eta = 1$ we obtain

$$\tau + \frac{1}{2K_*+3}\log\Big(1-\varphi_0\Big(\frac{1}{\lambda(\tau)}\Big)\Big) \to C \tag{34}$$

as $\tau \to \infty$ for all $\eta > 0$. This implies also that $\lambda(\tau)$ is indeed equivalent to the choice in (27b). Subtracting (34) from (33) we find

$$\frac{1}{2K_* + 3} \log \left(\frac{1 - \varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)}{1 - \varphi_0\left(\frac{1}{\lambda(\tau)}\right)} \right) \to \frac{\rho}{2K_* + 3} \log \eta + C \tag{35}$$

as $\tau \to \infty$ for all $\eta > 0$.

Define $L(\eta) = \frac{(1-\varphi_0(\eta))^{1/\rho}}{\eta}$. Then (35) is equivalent to

$$\log \left(\frac{L\left(\frac{\eta}{\lambda(\tau)}\right)}{L\left(\frac{1}{\lambda(\tau)}\right)} \right) \to C$$

as $\tau \to \infty$ for all $\eta > 0$. Choosing $\eta = 1$ we find that this constant must be zero and hence L is slowly varying.

□

4.3 Fast convergence of $K(\tau)$ for non-flat data

An important issue in the proof of convergence to a self-similar solution is establishing convergence of $K(\tau)$. In general, we have yet very little control over $K(\tau)$. In the following proposition, however, which relies on a simple comparison argument, we show that $K(\tau) \to 0$ exponentially fast if the data are bounded above by $1 - cx^\rho$ for some $\rho \in (0, 3/2)$.

Proposition 4.4. *Assume that $\psi_0(x) \le 1 - cx^\rho$ for all $x \in (0,1)$ for some $c > 0$ and $\rho \in (0, 3/2)$. Then there exists a $C > 0$ such*

$$0 < K(\tau) \le C \begin{cases} \exp\left(\tau\left(2 - \frac{3}{\rho}\right)\right) & \frac{3}{4} < \rho < \frac{3}{2} \\ \exp(-2\tau) & 0 < \rho \le \frac{3}{4} \end{cases}.$$

Proof. Because of $\psi \in [0,1]$ the derivative fulfils $\partial_\tau \psi \le \psi^4 - \psi^2$ and hence

$$\psi(\tau, x) \le \bar{\psi}(\tau, x) \equiv \left(1 + \frac{1 - \psi_0(x)^3}{\psi_0(x)^3} \exp(3\tau)\right)^{-1/3},$$

where $\bar{\psi}$ is the solution of $\partial_\tau \bar{\psi} = \bar{\psi}^4 - \bar{\psi}$ with initial data $\bar{\psi}_0(x) = \psi_0(x)$. Furthermore one can easily check that

$$\begin{aligned} \psi(\tau, x)^4 \le \bar{\psi}(\tau, x)^4 \quad &\le \left(1 + (1 - \psi_0(x)^3)\, e^{3\tau}\right)^{-4/3} \\ &\le \left(1 + \left(1 - (1 - cx^\rho)^3\right) e^{3\tau}\right)^{-4/3} \\ &\le \left(1 + cx^\rho e^{3\tau}\right)^{-4/3} \end{aligned}$$

By integrating Equation (10) one finds $\|\psi(\tau,\cdot)\|_1 = \|\psi_0\|_1\, e^{-\tau}$ and by applying the Cauchy-Schwarz inequality $\|\psi^2(\tau,\cdot)\|_1 \geq \|\psi_0\|_1^2\, e^{-2\tau}$. Using the abbreviation $a = e^{-3\tau/\rho}$ the function $\theta(\tau)$ can be estimated from above as follows

$$\begin{aligned}\theta(\tau) &\leq \frac{\int_0^1 \bar{\psi}(\tau,x)^4\,dx}{\|\psi_0\|_1^2\, e^{-2\tau}} \leq \frac{e^{2\tau}}{\|\psi_0\|_1^2}\left(\int_0^a dx + \int_a^1 (1+cx^\rho e^{3\tau})^{-4/3} dx\right)\\ &= \frac{\exp\left(\tau\left(2-\frac{3}{\rho}\right)\right)}{\|\psi_0\|_1^2}\left(1+\int_1^{1/a}(1+cy^\rho)^{-4/3}\,dy\right)\end{aligned}$$

For $\rho > 3/4$ the integral converges, whereas for $\rho \leq 3/4$ the integrand can be bounded by a multiple of $y^{4\rho/3}$ (depending on c) and hence it grows proportional to $\exp(\tau(3/\rho-4))$. Thus $\theta(\tau) \leq C\exp\left(\tau\max(2-\frac{3}{\rho},-2)\right)$ and in the same manner for $K(\tau) = \theta(1-\theta)^{-1}$. □

Corollary 4.5. *Assume that $\varphi_0(\eta) \leq 1 - cx^\mu$ for some $0 < \mu < 3/2$. Then there exists a rescaling $\lambda(\tau)$ such that $\varphi(\tau,\eta) \to \varphi_*^\rho(\eta)$ for all $\eta > 0$ as $\tau \to \infty$ if and only if the data satisfy* (27a) *for some $0 < \rho \leq \mu$.*

Proof. This is a consequence of Theorem 4.3 and Proposition 4.4. □

5 Numerical examples

Whether a solution $\varphi(\tau,\eta)$ of the time-dependent problem converges to a self-similar solution is fully understood if $\varphi_0(\eta) \leq 1 - c\eta^\rho$ for some $c > 0$ and $0 < \rho < 3/2$. In the general case, however, it is not clear under which conditions the associated $K(\tau)$ converges, and if, whether it does so sufficiently fast. In order to shed some light on this issue we present the results of numerical simulations in this section. They confirm the conjecture that convergence to the self-similar solution φ_*^ρ occurs if and only if the data satisfy that $1 - \varphi_0$ is regularly varying with power ρ.

5.1 Initial data with $\rho < 3/2$

In order to test our numerical scheme we consider first the regime which is covered by our results in Section 4. First we investigate the convergence rate of $K(\tau)$ with initial data $\varphi_0(\eta) = 1 - \eta^\rho$ with $1/2 \leq \rho < 3/2$.

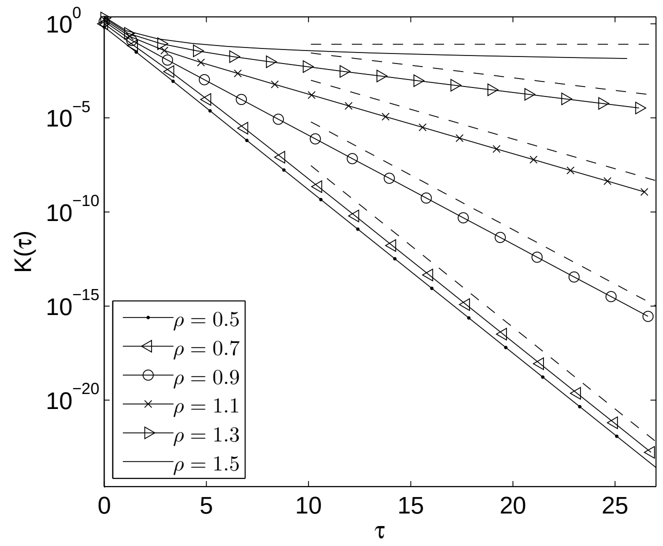

Figure 2: $K(\tau)$ for solutions of $\varphi(\tau,\eta)$ with initial data $\varphi_0(\eta) = 1 - \eta^\rho$

For $\rho < 3/2$ we know that $K(\tau)$ converges to zero exponentially fast, where K decays at least as $K(\tau) \le C\exp\big(\max(2 - 3/\rho, -2)\big)$. In Figure 2 the actual convergence rates (solid lines) are compared with this bound (dashed lines) and we found that $K(\tau)$ actually converges with the predicted rate.

Since $K(\tau)$ converges exponentially fast, φ converges to a self-similar solution if and only if $1 - \varphi_0$ is regularly varying at zero. We present three examples for such behaviour, namely for initial data

$$\varphi_0(\eta) = 1 + \eta(\log(\eta) - 1), \tag{36a}$$
$$\varphi_0(\eta) = 1 - \eta\big(\sin(\log\eta) + 2\big), \tag{36b}$$
$$\varphi_0(\eta) = 1 - \eta. \tag{36c}$$

While $1 - \varphi_0$ in (36a) and (36c) is regularly varying, this is not the case for (36b). Corresponding to our theoretical results Figure 3 shows convergence for (36a) and non-convergence for (36b). In addition, Figure 4 indicates that the rate of convergence $|\varphi - \varphi_*|$ as $\tau \to \infty$ is faster for the example (36c), which is supported by the fact that the rescaled solution (dotted curves) tend to be closer to the exact solution (full line) for comparable times τ in Figure 3 (top) and Figure 4.

5.2 Initial data $\rho > 3/2$

For $\rho < 3/2$ convergence $K(\tau) \to 0$ is known due to Proposition 4.4. Now we present three more examples where $\rho = 2$. Qualitatively similar results are obtained for other $\rho > 3/2$. This part indicates in which cases one can expect convergence of $K(\tau)$ and what the limit and the rate of convergence will be.

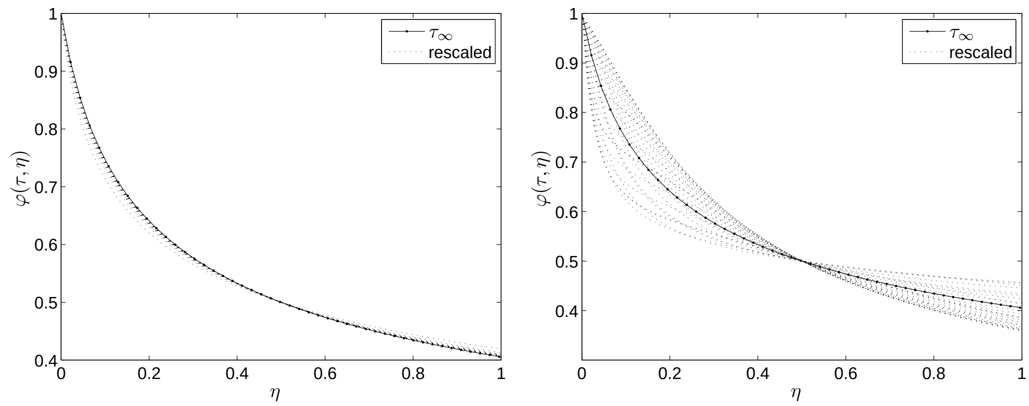

Figure 3: $\varphi(\tau,\eta)$ at various times showing convergence for (36a) (left), but no convergence due to oscillations in (36b) (right); The full line shows the self-similar solution with $n = 1$ and $K_* = 0$ and the scaling $a(\tau)$ is such that $\varphi(\tau, 1/2) = 1/2$ at all times.

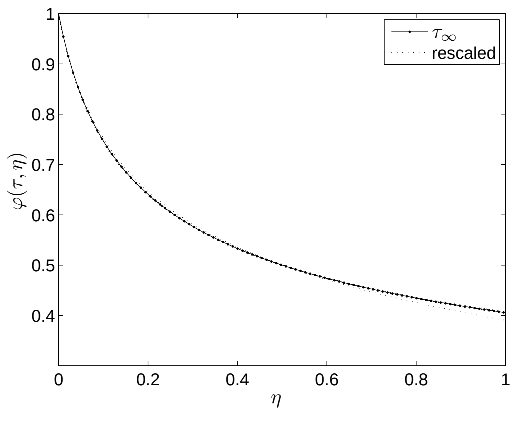

Figure 4: $\varphi(\tau,\eta)$ at various times showing convergence for initial data (36c); compared to (36a) the convergence-rate $|\varphi - \varphi_*|_\infty$ as $\tau \to \infty$ seems faster

We choose

$$\varphi_0(\eta) \quad = 1 + \eta^2(\log(\eta) - 1) \tag{37a}$$

$$\varphi_0(\eta) \quad = 1 - \eta^2\big(\sin(\log\eta) + 2\big) \tag{37b}$$

$$\varphi_0(\eta) \quad = 1 - \eta^2 \tag{37c}$$

We expect convergence for (37a) and (37c) and no convergence for (37b). In addition we show the convergence-rates for $K(\tau)$ in these three cases.

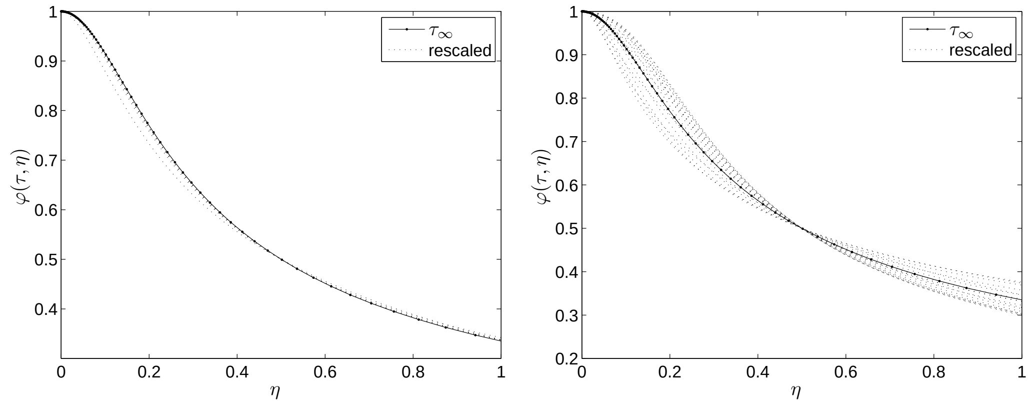

Figure 5: $\varphi(\tau, \eta)$ at various times showing convergence for (37a) (left), but no convergence due to oscillations in (37b) (right); The full line shows the self-similar solution with $\rho = 2$, the scaling $a(\tau)$ is such that $\varphi(\tau, 1/2) = 1/2$ at all times.

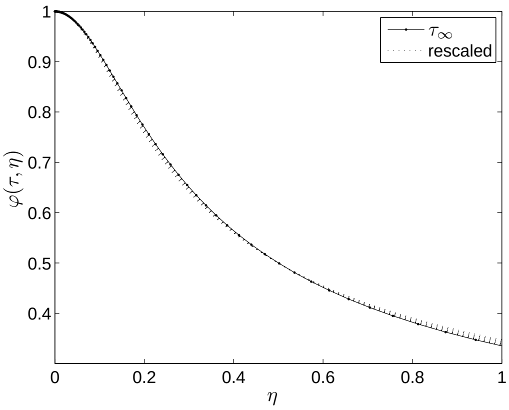

Figure 6: $\varphi(\tau, \eta)$ for initial data (37c); compared to (37a) the convergence-rate seems faster

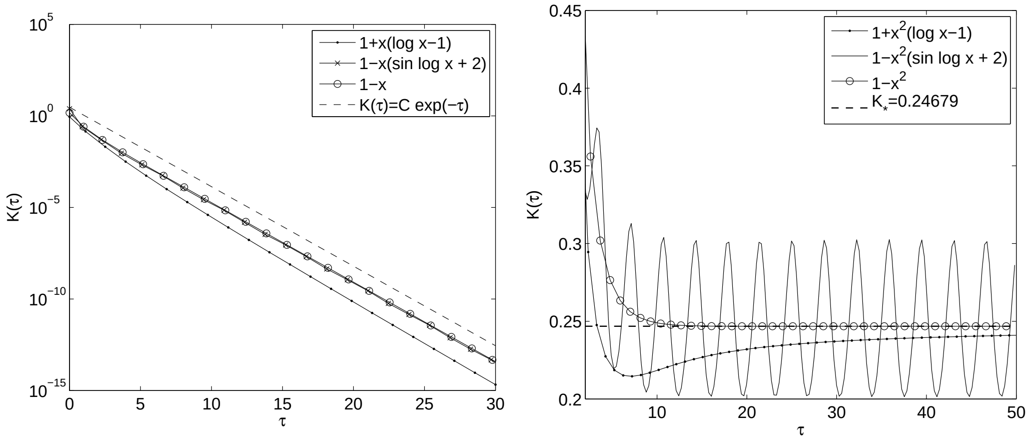

Figure 7: While there is convergence for all initial data (even not regularly varying ones) in case (36a-36c) (left), there is fast convergence for (37c), convergence for (37a) and no convergence for (37b) (right)

A closer look reveals that convergence for (37c) is exponentially fast, while for (37a) it is much slower.

6 Conclusions

We studied the structure of inertialess thin-film rupture, where van-der-Waals forces and viscosity are the dominant driving forces. For a simplified model we established for each $\rho > 0$ the existence of a self-similar solution. In terms of the stretching variable φ each of them is characterized by its behaviour near the singularity, i.e. by a number ρ such that

$$\varphi_*^\rho(\eta) = 1 - c\eta^\rho + o(\eta^\rho)$$

and by a corresponding value of K_*^ρ. The solution is unique up to a rescaling of the spatial scale by a constant.

The main purpose of this paper was to rigorously study convergence to these self-similar solutions. For $\rho \in (0, 3/2)$ where $K_*^\rho = 0$, we can completely characterize their domains of attraction. It turns out that for given data φ_0 there exists a spatial rescaling λ such that the solution converges to φ_*^ρ if and only $1 - \varphi_0$ is regularly varying at zero with power ρ. In original variables the spatial rescaling is asymptotically equivalent to $(t_* - t)^{-\beta} L((t_* - t)^{-\beta})$ with $\beta = \beta(\rho) \in (1/3, \infty)$ and for a slowly varying function L given by the data. It is worth pointing out that this corresponds to a pure power law only if $L(s) \sim const$ as $s \to \infty$ which is the case if and only if the data behave like an exact power law at the origin.

We can prove the result described above for all φ_*^ρ under the assumption that K converges to K_*^ρ sufficiently fast. It is easy to prove by a comparison principle that this is true for $\rho < 3/2$, for $\rho > 3/2$ a proof is still lacking. Numerical simulations indicate that this convergence is satisfied if and only if the data are regularly varying. This is strikingly different to the case $\rho < 3/2$ where K converges fast to $K_*^\rho = 0$ whenever the data are bounded above by $1 - c\eta^\rho$ for some $\rho < 3/2$ independent of whether they are regularly varying or not.

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References

[1] P. Constantin, T.F. Dupont, R.E. Goldstein, L.P. Kadanoff, M.J. Shelley, and S.-M. Zhou. Droplet breakup in a model of the Hele-Shaw cell. *Phys. Rev. E*, 47(6):4169–4181, Jun 1993.

[2] J. Eggers. Universal pinching of 3d axisymmetric free-surface flow. *Phys. Rev. Lett.*, 71(21):3458–3460, Nov 1993.

[3] J. Eggers. Nonlinear dynamics and breakup of free-surface flows. *Rev. Mod. Phys.*, 69(3):865–930, Jul 1997.

[4] T.P. Witelski and A.J. Bernoff. Stability of self-similar solutions for van der Waals driven thin film rupture. *Physics of Fluids*, 11(9):2443–2445, 1999.

[5] W.W. Zhang and J.R. Lister. Similarity solutions for van der Waals rupture of a thin film on a solid substrate. *Phys. Fluids*, 11(9):2454–2462, 1999.

[6] J. Eggers and E. Villermaux. Physics of liquid jets. *Reports on progress in physics*, 73(3), 2008.

[7] J. Eggers and M.A. Fontelos. The role of self-similarity in singularities of partial differential equations. *Nonlinearity*, 22(1):R1–R44, 2009.

[8] A. Münch, B. Wagner, and T.P. Witelski. Lubrication models with small to large slip lengths. *Journal of Engineering Mathematics*, 53:359–383, Dec 2005.

[9] D. Peschka. Self-similar rupture of thin liquid films with slippage. *PhD thesis, Humboldt Universität zu Berlin*, 2008.

[10] M.P. Ida and M.J. Miksis. Thin film rupture. *Appl. Math. Lett.*, 9(3):35–40, 1996.

[11] D. Vaynblat, J.R. Lister, and T.P. Witelski. Rupture of thin viscous films by van der Waals forces: Evolution and self-similarity. *Physics of Fluids*, 13(5): 1130–1140, 2001.

[12] D.T. Papageorgiou. On the breakup of viscous liquid threads. *Phys. Fluids*, 7 (7):1529–1544, 1995.

[13] M.P. Brenner, J.R. Lister, and H.A. Stone. Pinching threads, singularities and the number $0.0304\cdots$. *Phys. Fluids*, 8(11):2827–2836, 1996.

[14] M. Renardy. A comment on self-similar breakup for inertialess Newtonian liquid jets. *IMA J. Appl. Math.*, 70(3):353–358, 2005.

[15] M. Renardy. Finite time breakup of viscous filaments. *Z. Angew. Math. Phys.*, 52(5):881–887, 2001.

[16] D. Peschka, A. Münch, and B. Niethammer. Thin film rupture for large slip. *to appear.*

[17] M. Renardy. Corrigendum for: "Finite time breakup of viscous filaments" [Z. Angew. Math. Phys. **52** (2001), no. 5, 881–887. *Z. Angew. Math. Phys.*, 58(5): 904–905, 2007.

[18] B. Niethammer and R.L. Pego. Non-self-similar behavior in the LSW theory of Ostwald ripening. *J. Statist. Phys.*, 95(5-6):867–902, 1999.

[19] J. Carr. Stability of self-similar solutions in a simplified LSW model. *Phys. D*, 222(1-2):73–79, 2006.

[20] G. Menon, B. Niethammer, and R.L. Pego. Dynamics and self-similarity in min-driven clustering. 2008.

[21] G. Menon and R.L. Pego. Approach to self-similarity in Smoluchowski's coagulation equations. *Comm. Pure Appl. Math.*, 57(9):1197–1232, 2004.

[22] T. Erneux and S.H. Davis. Nonlinear rupture of free films. *Physics of Fluids A: Fluid Dynamics*, 5(5):1117–1122, 1993.

[23] N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular Variation*, volume 27. Encyclopedia of Mathematics and its Applications, 1989.