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# Complete-damage evolution based on energies and stresses


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**Abstract**

The rate-independent damage model recently developed in [BMR09] allows for complete damage, such that the deformation is no longer well-defined. The evolution can be described in terms of energy densities and stresses. Using concepts of parametrized Γ-convergence, we generalize the theory to convex, but non-quadratic elastic energies by providing Γ-convergence of energetic solutions from partial to complete damage under rather general conditions.

# 1 Introduction

There is a rich literature on rate-independent mechanical models for damage in brittle materials, cf. [Ort85, FrM93, DPO94, FrN96, DMT01, MaA01, HaS03], and recently several mathematical approaches [FrM98, FKS99, FrG06] were developed, in particular the abstract theory of rate-independent processes [MiT99, MiT04, Mie05] proved very helpful as it allows one to employ the machinery of incremental minimization.

Here we want to contribute to the models discussed in [MiR06, BMR09, MRZ07]. Let $u : \Omega \to \mathbb{R}^d$ be the displacement and $z : \Omega \to [0,1]$ the damage variable, then the rate-independent system is given by the triple $(\mathcal{F}{\times}\mathcal{Z}, \mathcal{E}, \mathcal{D})$, where $u \in \mathcal{F}$, $z \in \mathcal{Z}$. The energy-storage functional has the form

$$\mathcal{E}_\delta(t,u,z) = \int_\Omega W_\delta(x, \mathbf{e}(u_{\mathrm{D}}(t){+}u)(x), z(x))\,\mathrm{d}x + \mathcal{G}(z), \text{ where } \mathbf{e}(u) = \frac{1}{2}(\nabla u{+}(\nabla u)^{\mathsf{T}}),$$

and the dissipation is $\mathcal{D}(z,\widehat{z}) = \int_\Omega D(x, z(x), \widehat{z}(x))\,\mathrm{d}x$. Here $u_{\mathrm{D}} \in \mathrm{C}^1([0,T]; \mathrm{W}^{1,p}(\Omega))$ prescribes time-dependent boundary displacements on the Dirichlet part $\Gamma_{\mathrm{D}}$ of the total boundary $\partial\Omega$. For $\delta > 0$ the stored-energy density is regularized in the form $W_\delta(e,z) = W(e,z) + \delta|e|^p$, which renders $W_\delta$ coercive while $W$ may be non-coercive for complete damage $z = 0$.

For $\delta > 0$ existence of energetic solutions $(u_\delta, z_\delta)$ is known for general $W$, see [MiR06, ThM09]. The limit passage for $\delta \to 0$ in the sense of Γ-limits was established in [BMR09, MRZ07] under the assumption that $e \mapsto W(x,e,z)$ is quadratic. However, this is not a realistic model, since it implies that damage behaves symmetric under compression and extension. The purpose of this work is to generalize the approach to a much larger class of functionals. For instance, we are able to treat the model

$$W(e,z) = \frac{z}{2}\, e{:}\mathbb{C}{:}e + \frac{c}{2}\big(\min\{0, \operatorname{tr} e\}\big)^{1+\beta}, \quad c > 0 \text{ and } \beta \in \,]0,1]\,.$$

which displays resistance to compression even after complete damage, like powderized concrete.

The difficulty is that $W$ is not coercive, hence in the limit $\delta \to 0$ we are not able to control $u_\delta$, and convergence should only be valid for $z_\delta$. The task is to define a limit equation in terms of $z$. In particular, one needs a replacement of the power of the external forces that provides the limit of

$$\partial_t \mathcal{E}_\delta(t, u_\delta(t), z_\delta(t)) = \int_\Omega \Sigma_\delta{:}\mathbf{e}(\dot u_\mathrm{D})\,\mathrm{d}x \quad \text{with } \Sigma_\delta = \mathrm{D}_e W_\delta(\mathbf{e}(u_\mathrm{D}{+}u_\delta), z_\delta). \tag{1}$$

We will show that it is possible to control the limit of the stresses $\Sigma_\delta$ while in general the strains $e_\delta = \mathbf{e}(u_\mathrm{D}{+}u_\delta)$ will have no limits. Hence, we follow the ideas of [BMR09] to eliminate the elastic variable $u$ completely by defining the reduced functional

$$\mathcal{I}_\delta(t,z) = \min\{\, \mathcal{E}_\delta(t, \widetilde{u}, z) \mid u \in \mathcal{F}\,\} \quad \text{with } \mathcal{F} = \{\, u \in \mathrm{W}^{1,p}(\Omega) \mid u|_{\Gamma_\mathrm{D}} = 0\,\}$$

and to apply the $\Gamma$-convergence theory to the rate-independent systems $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$. Note that a convergence theory for the systems $(\mathcal{F}\times\mathcal{Z}, \mathcal{E}_\delta, \mathcal{D})$ is doomed to fail because of the missing uniform coercivity with respect to $u \in \mathcal{F}$.

However, the total elimination of the displacements, and hence of the strains, leads to missing information on the stresses which is needed to control the limit in (1). Thus, the second important idea in [BMR09] is the introduction of an intermediate functional defined in terms of the boundary displacements $u_\mathrm{D}$. More precisely, we let

$$\mathcal{J}_\delta(e,z) = \min\{ \int_\Omega W_\delta(e{+}\mathbf{e}(u), z)\,\mathrm{d}x + \mathcal{G}(z) \mid u \in \mathcal{F}\,\}.$$

Here $e$ can be taken from all of $\mathbb{E} = \mathrm{L}^p(\Omega, \mathbb{R}^{d\times d}_\mathrm{sym})$, but the minimization with respect to all admissible displacements shows that it depends only on much less information to be extracted from $e$. The point about the definition of $\mathcal{J}_\delta$ is that it provides the formulas

$$\text{(i)}\ \ \mathcal{I}_\delta(t,z) = \mathcal{J}_\delta(\mathbf{e}(u_\mathrm{D}(t)), z) \quad \text{(ii)}\ \ \partial_t \mathcal{I}_\delta(t,z) = \langle \mathrm{D}_e \mathcal{J}_\delta(\mathbf{e}(u_\mathrm{D}(t)), z), \mathbf{e}(\dot u_\mathrm{D}(t))\rangle. \tag{2}$$

In fact, $\mathrm{D}_e\mathcal{J}_\delta(\mathbf{e}(u_\mathrm{D}), z) \in \mathbb{E}^* = \mathrm{L}^{p'}(\Omega, \mathbb{R}^{d\times d}_\mathrm{sym})$ provides the equilibrium stresses associated with the given boundary data $u_\mathrm{D}$ and the damage state $\mathcal{Z}$.

In Section 3 we will discuss the theory of $\Gamma$-convergence for a family of functionals $\mathcal{J}_\delta : \mathbb{E}\times\mathcal{Z} \to \mathbb{R}_\infty$, where the $\Gamma$-convergence is done with $e \in \mathbb{E}$ treated as a parameter, i.e., $\mathcal{J}_\delta(e,\cdot) \xrightarrow{\Gamma} \mathfrak{J}(e,\cdot)$. The main question is how properties of the functions $\mathcal{J}_\delta(\cdot, z) : \mathbb{E} \to \mathbb{R}$ are inherited to the limit $\mathfrak{J}(\cdot, z)$. For this we introduce the notion of *simultaneous* $\Gamma$*-limits* for parametrized families $(\mathcal{J}_\delta(e,\cdot))_{\delta>0}$ by asking that for each two points $e_1$ and $e_2$ and each $z \in \mathcal{Z}$ there exists a recovery sequence $(z_\delta)_{\delta>0}$ such that $\mathcal{J}_\delta(e_j, z_\delta) \to \mathfrak{J}(e_j, z)$ for $j = 1$ and 2. With this condition we are able to coclude that convexity and differentiability with respect to $e$ passes from $\mathcal{J}_\delta(\cdot, z)$ to $\mathfrak{J}(\cdot, z)$. In particular, we provide the following convergence of stresses, which is crucial in the theory of rate-independent systems (cf. [FrM06, Prop. 4.4]):

$$\left.\begin{array}{c} z_\delta \rightharpoonup z_0 \\ \mathcal{J}_\delta(e, z_\delta) \to \mathfrak{J}(e, z_0) \end{array}\right\} \implies \mathrm{D}_e\mathcal{J}_\delta(e, z_\delta) \rightharpoonup \mathrm{D}_e\mathfrak{J}(e, z_0) \text{ in } \mathbb{E}^*.$$

Combining this result with (2ii) we are able to obtain the limit $\partial_t \mathcal{I}_\delta(t, z_\delta) \to \partial_t \mathfrak{I}(t, z_0)$.

For the complete-damage problem one easily obtains the simultaneous Γ-convergence by taking Γ-convergence with respect to strong convergence in $\mathrm{W}^{1,p}(\Omega)$, because of the strong continuity of $\mathcal{G}$ and the monotonicity of $z \mapsto W(e,z)$, see Proposition 4.5. The main difficulty is then to establish our main *structural assumption* (see (17)) that weak convergence along so-called *stable sequences* implies strong convergence. In this work we show that this condition holds under the additional assumption $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{C}(\overline{\Omega})$, i.e. $r > d$. However, in Section 6.1 we give arguments in favor of our conjecture, that the strong convergence can also be established for $r \in [1,d]$.

Our main result is formulated in Theorem 2.3 for $r > d$: any family of energetic solutions $z_\delta : [0,T] \to \mathcal{Z}$ for $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$ has a subsequence $(z_{\delta_j})_{j\in\mathbb{N}}$ with $\delta_j \to 0$ and $z_{\delta_j}(t) \to z(t)$ for all $t \in [0,T]$, where $z : [0,T] \to \mathcal{Z}$ is an energetic solution of the complete-damage system given by $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$. The result is based on the abstract theory of Γ-convergence for rate-independent systems developed in [MRS08].

If the uniform differentiability property does not hold, one can still use convexity arguments. If each $\mathcal{J}_\delta(\cdot, z)$ is convex, the parametrized Γ-limit is convex as well. This convexity allows us to characterize the Clarke differential of $\mathfrak{I}(\cdot, z)$ using the left and right partial derivative in $t$:

$$\partial_t^{\mathrm{Cl}} \mathfrak{I}(t,z) = \Big[\partial_t^- \mathfrak{I}(t,z), \partial_t^+ \mathfrak{I}(t,z)\Big], \quad \text{where } \partial_t^\pm \mathfrak{I}(t,z) = \lim_{\varepsilon\to 0^+} \tfrac{\pm 1}{\varepsilon}\big(\mathfrak{I}(t\pm\varepsilon, z) - \mathfrak{I}(t,z)\big).$$

In fact, we have $\partial_t^\pm \mathfrak{I}(t,z) = \pm \sup\{\, \pm\langle \sigma, \mathbf{e}(\dot u_{\mathrm{D}}(t))\rangle \mid \sigma \in \partial_e^{\mathrm{sub}} \mathfrak{I}(\mathbf{e}(u_{\mathrm{D}}(t)), z)\,\}$.

We generalize the notion of energetic solutions [Mie05] to *generalized energetic solutions* by keeping stability (S) and replacing the energy balance by

$$\mathfrak{I}(t, z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0,t]) = \mathfrak{I}(0, z(0)) + \int_0^t p(\tau)\,\mathrm{d}\tau \quad \text{with } p(\tau) \in \partial_\tau^{\mathrm{Cl}} \mathfrak{I}(\tau, z(\tau)),$$

see Definition 6.1. Theorem 6.2 establishes existence of generalized energetic solutions to the rate-independent system $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$.

# 2 Setup of the model

We first discuss the physical setup and provide the existence result for the coercive case $\delta > 0$. Afterwards we discuss the reduction of the problem by eliminating the displacement while keeping the boundary strains $\mathbf{e}_{\mathrm{D}}(t) = \mathbf{e}(u_{\mathrm{D}}(t))$.

## 2.1 Discussion of the coercive model

The body $\Omega \subset \mathbb{R}^d$ is described by a bounded Lipschitz domain. The state of the system is described by the displacement $\widetilde{u} : \Omega \to \mathbb{R}^d$ and the scalar damage variable $z : \Omega \to [0,1]$, where $z = 1$ denotes no damage and $z = 0$ means that the maximal damage has been reached (all microscopic breakable structures are broken). The displacement $\widetilde{u}$ will satisfy time-dependent Dirichlet boundary conditions on $\Gamma_{\mathrm{D}} \subset \partial\Omega$ via $u_{\mathrm{D}} \in \mathrm{C}^1([0,T], \mathrm{W}^{1,p}(\Omega))$ in the form

$$\widetilde{u}(t) = u_{\mathrm{D}}(t) + u(t) \quad \text{with } u(t) \in \mathcal{F} = \{\, v \in \mathrm{W}^{1,p}(\Omega) \mid v|_{\Gamma_{\mathrm{D}}} \equiv 0 \,\}.$$

We also use the infinitesimal strain tensor $\mathbf{e}(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^{\mathsf{T}}\big)$ and set

$$\mathbf{e}_{\mathrm{D}}(t) = \mathbf{e}(u_{\mathrm{D}}(t)) \text{ and } \dot{\mathbf{e}}_{\mathrm{D}}(t) = \mathbf{e}(\dot{u}_{\mathrm{D}}(t)) \text{ where } \dot{} = \partial_t.$$

The stored energy of the system is given via the functional

$$\mathcal{E}(t,u,z) = \int_\Omega W(x, \mathbf{e}_{\mathrm{D}}(t,x){+}\mathbf{e}(u)(x), z(x))\,\mathrm{d}x + \mathcal{G}(z) \tag{3a}$$

$$\text{with } \mathcal{G}(z) = \int_\Omega b(x,z(x)) + G(x,\nabla z(x))^r\,\mathrm{d}x. \tag{3b}$$

Here $b : \Omega \times [0,1] \to \mathbb{R}$ and $G : \Omega \times \mathbb{R}^d \to \mathbb{R}$ are Carathéordory functions satisfying

$$\exists\, C>0\ \forall\,(x,z):\ \ 0 \le b(x,z) \le C, \tag{4a}$$

$$\forall\, x\in\Omega:\ \ z \mapsto b(x,z) \text{ is non-decreasing,} \tag{4b}$$

$$\exists\, C>0,\ r>1\ \forall\,(x,a):\ \ \frac{|a|^r}{C} - C \le G(x,a) \le C|a|^r + C, \tag{4c}$$

$$\forall\, x\in\Omega:\ \ a \mapsto G(x,a) \text{ is strictly convex.} \tag{4d}$$

The function $G$ contains the regularizing term and is typically of the form $\kappa(x)|a|^r$. Thus, the suitable space for the damage states is $\mathcal{Z} = \{\, z \in \mathrm{W}^{1,r}(\Omega) | 0 \le z \le 1 \,\}$. The additional term $b$ is intended to model cohesive effects (or healing), i.e., if the stresses in the material are released then the damage may heal ($\dot{z} > 0$) by using up some energy.

The stored energy density $W : \Omega{\times}\mathbf{E}_d{\times}[0,1] \to \mathbb{R}$, where $\mathbf{E}_d = \mathbb{R}^{d\times d}_{\mathrm{sym}}$, is a Carathéordory function satisfying

$$\forall\,(x,z)\in\Omega:\quad W(x,\cdot,z) \in \mathrm{C}^1(\mathbf{E}_d), \tag{5a}$$

$$\exists\, C>0\ \forall\,(x,e,z):\quad 0 \le W(x,e,z) \le C|e|^p + C, \tag{5b}$$

$$\forall\,(x,z):\quad e \mapsto W(x,e,z) \text{ is convex,} \tag{5c}$$

$$\forall\,(x,e):\quad z \mapsto W(x,e,z) \text{ is nondecreasing,} \tag{5d}$$

$$\exists\, c_1,c_2\ \forall\,(x,e,z):\ |\partial_e W(x,e,z)| \le c_1(W(x,e,z){+}c_2)^{1-1/p}. \tag{5e}$$

Condition (5d) means that the material becomes weaker if damage increases, and (5e) is called "stress control", since it allows us to control the size of the stresses in terms of the energy alone, uniformly in $(x,z)$. A typical function $W$ has the form

$$W(x,e,z) = W^0(x,e) + a(z)W^1(x,e),$$

where $W_0$ and $W_1$ are smooth and convex, $W^0$ may be non-coercive while $W^1$ is coercive, $a(z) \ge cz^\alpha$ and $a'(z) \ge 0$. As above we set $W_\delta(e,z) = W(e,z) + \delta|e|^p$ and define $\mathcal{E}_\delta$ as in (3a) with $W$ replaced by $W_\delta$.

For the time-dependent Dirichlet boundary data we impose

$$u_{\mathrm{D}} \in \mathrm{C}^1([0,T], \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \text{ with } p \text{ from (5)}. \tag{6}$$

Finally we describe the dissipation functional $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$ via

$$\mathcal{D}(z_0, z_1) = \int_\Omega D(x, z_0(x), z_1(x))\,\mathrm{d}x,$$

where $D : \Omega \times [0,1]^2 \to [0,\infty]$ is a normal integrand. For each $x$, $D$ satisfies the triangle inequality and the coercivity $D(x,z,\widetilde{z}) \geq c|z-\widetilde{z}|$. The typical choice is $D(x,z,\widetilde{z}) = \delta_+(z-\widetilde{z})$ for $\widetilde{z} \leq z$ and $\delta_-(\widetilde{z}-z)$ for $z \leq \widetilde{z}$, where $\delta_+ \in (0,\infty)$ and $\delta_- \in (0,\infty]$. Here $\delta_- = \infty$ is the *unidirectional case* that enforces that damage can only increase, thus healing is forbidden. The latter can only take place if $\delta_- + b'(z) < 0$ for some $z \in [0,1]$. We refer to [SHS06], where healing is modeled under the name cohesion.

With these functionals we define notion of energetic solution for the rate-independent system $(\mathcal{Q}, \mathcal{E}_\delta, \mathcal{D})$, where $\mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ (see [MiT99, MiT04] and the surveys [Mie05, MiR08]). A mapping $q = (u,z) : [0,T] \to \mathcal{Q}$ is called **energetic solution** if $\tau \mapsto \partial_\tau \mathcal{E}_\delta(\tau, q(\tau))$ lies in $\mathrm{L}^1((0,T))$ and if for all $t \in [0,T]$ we have **stability (S)** and **energy balance (E)**:

$$\begin{aligned}
&\textbf{(S)} \quad \forall\, \widetilde{q} = (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}: \quad \mathcal{E}_\delta(t, q(t)) \leq \mathcal{E}_\delta(t, \widetilde{q}) + \mathcal{D}(z(t), \widetilde{z});\\
&\textbf{(E)} \quad \mathcal{E}_\delta(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0,t]) = \mathcal{E}_\delta(0, q(0)) + \int_0^t \partial_\tau \mathcal{E}_\delta(\tau, q(\tau))\,\mathrm{d}\tau.
\end{aligned} \tag{7}$$

Here $\mathrm{Diss}_{\mathcal{D}}(z, [r,s])$ is defined to be the supremum of $\sum_1^N \mathcal{D}(z(t_{j-1}, z(t_j))$ over all finite partitions $r \leq t_0 < t_1 \cdots t_N \leq s$. For each $q \in \mathcal{Q}$ the power of the external forces $\partial_t \mathcal{E}_\delta(t,q)$ is well defined by using (5e).

For non-coercive problems (i.e. $\delta = 0$), where $u$ is no longer well-defined and we cannot guarantee $q \in \mathcal{Q}$. It is the main problem how to define this partial derivative $\partial_t \mathcal{E}(t,q)$. Thus, it is an open problem whether under the above assumption a general existence result holds. However, the coercive case $\delta > 0$ was solved under more general assumptions including unilateral constraints and volume forces, cf. [MiR06, ThM09]. The following result provides existence in the case where the growth rate $r$ for the regularizing term just needs to satisfy $r > 1$. Originally [MiR06] used the embedding $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{C}(\overline{\Omega})$, which leads to the assumption $r > d$. In [ThM09] a new construction of the joint recovery sequence allowed for the generalization to all $r > 1$.

**Theorem 2.1** *If the above assumption hold with $p$, $r > 1$ and if $\delta > 0$, then for all stable initial states $q^0 \in \mathcal{Q}$ (i.e., (S) holds at $t = 0$ with $q(0)$ replaced by $q^0$) there exists an energetic solution $q_\delta : [0,T] \to \mathcal{Q}$ of the rate-independent system $(\mathcal{Q}, \mathcal{E}_\delta, \mathcal{D})$ with $q(0) = q_0$, $q \in \mathrm{L}^\infty([0,T], \mathrm{W}^{1,p}(\Omega) \times \mathrm{W}^{1,r}(\Omega))$, and $z \in \mathrm{BV}([0,T], \mathrm{L}^1(\Omega))$.*

In general one cannot expect more regularity of the solutions with respect to time. In particular, the solution may have jumps. In [ThM09] convexity conditions on $(e,z) \mapsto W(e,z)$ are discussed which imply simple continuity, Hölder or Lipschitz continuity.

## 2.2 Reduction by eliminating the displacements

The approach for solving non-coercive problems was indicated already in [MiR06] and finally solved in [BMR09] under the additional assumption that $W$ is quadratic: $W(x,e,z) = \frac{z}{2}e{:}\mathbb{C}{:}e$; however more general quadratic forms $\frac{1}{2}e{:}\mathbb{C}(z){:}e+g(z){:}e+\gamma(z)$ would work equally well. The main idea is to approximate the non-coercive case with a coercive one by setting

$$W_\delta(x,e,z) = W(x,e,z) + \delta(1+|e|^2)^{p/2}. \tag{8}$$

Then for each $\delta > 0$ there is a solution $q_\delta = (u_\delta, z_\delta)$ of the rate-independent energetic system $(\mathcal{Q}, \mathcal{E}_\delta, \mathcal{D})$. Moreover, using the stress control (5e) it is not difficult to show that there exists $C > 0$ such that for all $\delta \in (0,1)$ and all $t \in [0,T]$ we have $\mathcal{E}_\delta(t, q_\delta(t)) + \mathrm{Diss}_{\mathcal{D}}(z_\delta, [0,t]) \leq C$.

Now, using the theory of $\Gamma$-convergence of rate-independent energetic systems [MRS08] it is then possible to pass to the limit in the reduced system, where the displacement $u$ is minimized out. The latter step is essential, since it is not to be expected that $u_\delta$ or $\mathbf{e}(u_\delta)$ converges in any reasonably sense. In regions where $z = 0$ holds we may have $W(x,e,0) = 0$ for a large and possibly unbounded set of strains $e \in \mathbf{E}_d$ due to the missing coercivity.

To define the reduced problem we use the strict convexity (5c) to find that $\mathcal{E}_\delta(t,\cdot,z)$ has a unique minimizer $u = U_\delta(t,z) \in \mathcal{F}$. With this we have

$$\mathcal{I}_\delta(t,z) = \int_\Omega W_\delta(x, \mathbf{e}_{\mathrm{D}}(t){+}\mathbf{e}(U_\delta(t,z)), z)\,\mathrm{d}x + \mathcal{G}(z).$$

A classical argument [KnM08, KMZ08] shows that $\partial_t \mathcal{I}_\delta(t,z) = \partial_t \mathcal{E}_\delta(t, U_\delta(t,z), z)$.

While the limit of the energy $\mathcal{I}_\delta(t,z_\delta)$ along energetic solutions $q_\delta$ can be understood in the sense of $\Gamma$-limits, it is nontrivial to control the power

$$\begin{aligned}&\partial_t \mathcal{I}_\delta(t,z_\delta) = \textstyle\int_\Omega \sigma_\delta(t){:}\dot{\mathbf{e}}_{\mathrm{D}}(t)\,\mathrm{d}x \text{ with}\\ &\sigma_\delta(t,x) = \partial_e W(x, \mathbf{e}_{\mathrm{D}}(t,x){+}\mathbf{e}(u_\delta(t))(x), z_\delta(t,x)).\end{aligned}$$

The main observation is that the stress-control assumption (5e) and the usual energy a priori estimates provide bounds for $\sigma_\delta$ in $\mathrm{L}^{p/(p-1)}(\Omega, \mathbf{E}_d)$ that are independent of $\delta > 0$.

The essential idea to make the limit tractable is to introduce an auxiliary functional in which it is possible to keep control over the $\Gamma$-limit. Denote by $\mathbb{E} = \mathrm{L}^p(\Omega; \mathbf{E}_d)$ the strain space, and for $(e,z) \in \mathbb{E} \times \mathcal{Z}$ let

$$\begin{aligned}&\mathcal{J}_\delta(e,z) = \mathcal{V}_\delta(e,z) + \mathcal{G}(z) \text{ with}\\ &\mathcal{V}_\delta(e,z) = \min\{\, \textstyle\int_\Omega W_\delta(x, e{+}\mathbf{e}(u), z)\,\mathrm{d}x \mid u \in \mathcal{F}\,\}.\end{aligned} \tag{9}$$

In fact, the functional $\mathcal{V}_\delta$ should not be considered as a functional on $\mathbb{E}$ but rather on $\mathbb{B} = \{\, u|_{\partial\Omega} \mid u \in \mathcal{F}\,\}$, since all the other information is minimized out. Moreover, for fixed $z \in \mathcal{Z}$, the mapping $e \mapsto \mathcal{V}_\delta(e,z)$ is convex and differentiable with

$$\mathrm{D}_e \mathcal{V}_\delta(e,z) = \partial_e W(x, e{+}\mathbf{e}(V(e,z)), z) \in \mathbb{E}^* = \mathrm{L}^{p/(p-1)}(\Omega; \mathbf{E}_d),$$

where $V(e,z) \in \mathcal{F}$ is the unique minimizer in (9). This shows that $\sigma = \mathrm{D}_e\mathcal{V}_\delta(e,z)$ is in fact an equilibrium stress, and thus satisfies $\operatorname{div}\sigma = 0$ in $\Omega$ and $\sigma\,\nu = 0$ on $\partial\Omega\backslash\Gamma_{\mathrm{D}}$.

The importance of the functional $\mathcal{V}_\delta$ is that on the one hand it is possible to do the $\Gamma$-limit for $\delta \to 0$ and keep some of the main features and that on the other hand, by construction the reduced functional $\mathcal{I}_\delta$ and its partial derivative with respect to $t$ can be easily expressed:

$$\mathcal{I}_\delta(t,z) = \mathcal{V}_\delta(\mathbf{e}_\mathrm{D}(t),z)+\mathcal{G}(z) \text{ and } \partial_t\mathcal{I}_\delta(t,z) = \langle \mathrm{D}_e\mathcal{V}_\delta(\mathbf{e}_\mathrm{D}(t),z), \dot{\mathbf{e}}_\mathrm{D}(t)\rangle.$$

Thus, we have found a way to express the energies in terms of the damage alone and we still have control over the equilibrium stresses $\mathrm{D}_e\mathcal{V}_\delta(\mathbf{e}_\mathrm{D}(t),z)$ that are needed to control the power generated by the boundary data $u_\mathrm{D}(t)$.

## 2.3 The main convergence and existence result

In this subsection we provide convergence results of (subsequences of) energetic solutions for $(\mathcal{Z},\mathcal{I}_\delta,\mathcal{D})$ to solutions of the complete damage problem $(\mathcal{Z},\mathfrak{I},\mathcal{D})$. Here $\mathfrak{I}$ is the parametrized $\Gamma$-limit $\mathfrak{I}(t,\cdot) = \Gamma\text{-}\lim_{\delta\to0^+}\mathcal{I}_\delta(t,\cdot)$. The main difficulty in the limit procedure is to show the convergence of the power

$$\partial_t\mathcal{I}_\delta(t,z_\delta(t)) \to \partial_t\mathfrak{I}(t,z(t)),$$

for which it is necessary to know that $\mathfrak{I}(\cdot,z) \in \mathrm{C}^1([0,T])$. For this we will show that $\mathfrak{V}(e,\cdot) = \Gamma\text{-}\lim_{\delta\to0^+}\mathcal{V}_\delta(e,\cdot)$ exists and is differentiable with respect to $e \in \mathbb{E}$.

For this, we need an additional uniform differentiability assumption on the the stored energy density $W$, which reads as follows:

$$\begin{aligned} &\exists\, C>0\ \exists\, \beta \in\, ]0,\min\{1,p{-}1\}]\ \forall\, e_0,e_1 \in \mathbf{E}_d\ \forall\, z\in[0,1]: \\ &W(x,e_0,z)+W(x,e_1,z)-2W(x,\tfrac12(e_0{+}e_1),z) \\ &\qquad \le C\big(1+W(x,\tfrac12(e_0{+}e_1),z)+|e_1{-}e_0|^p\big)^{1-(1+\beta)/p}\,|e_1{-}e_0|^{1+\beta}, \end{aligned} \tag{10}$$

where $p$ is as in (5). It is easy to construct nontrivial examples fulfilling this condition, because it is additive in the following sense: If the nonnegative densities $W_1,\dots,W_k$ satisfy (10) with the same $p$, $\beta$, and $C_1,\dots,C_k$, respectively, then the sum $W=\sum_1^k W_j$ satisfies the condition as well with $C=\sum_1^k C_j$.

**Example 2.2** *We list a few examples of uniformly differentiable functions:*

$$\textit{(i)}\ \ \tfrac12 e{:}\mathbb{C}{:}e, \quad \textit{(ii)}\ \ \min\{0,\operatorname{tr}e\}^q, \quad \textit{(iii)}\ \ |e|^q.$$

*For (i) we can take any $\beta \in\, ]0,1]$ and $p \ge 1{+}\beta$. For (ii) and (iii) the condition* (10) *is satisfied if and only if $1 \le 1{+}\beta \le q \le p$.*

The main result is restricted to the case $r>d$, which provides the helpful embedding $\mathrm{W}^{1,r}(\Omega) \subset \mathrm{C}(\overline{\Omega})$. However, in Section 6 we discuss possibilities of generalizations.

**Theorem 2.3 ($\Gamma$-convergence)** *Let the assumptions of Section 2.1 and* (10) *hold with $r>d$. For $\delta>0$ consider energetic solutions $z_\delta : [0,T] \to \mathcal{Z}$ of $(\mathcal{Z},\mathcal{I}_\delta,\mathcal{D})$, then there* *exists a subsequence* $(z_{\delta_j})_{j\in\mathbb{N}}$ *with* $\delta_j \to 0^+$ *and an energetic solution* $z : [0,T] \to \mathcal{Z}$ *of* $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$ *such that the following holds for all* $t \in [0,T]$*:*

$$\begin{aligned}
&\text{(i)} && z_{\delta_j}(t) \to z(t) \ \textit{in } \mathrm{W}^{1,r}(\Omega),\\
&\text{(ii)} && \mathrm{Diss}_{\mathcal{D}}(z_{\delta_j}, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(z, [0,t]),\\
&\text{(iii)} && \mathcal{I}_{\delta_j}(t, z_{\delta_j}(t)) \to \mathfrak{I}(t, z(t)),\\
&\text{(iv)} && \mathrm{D}_e\mathcal{V}_{\delta_j}(\mathbf{e}_{\mathrm{D}}(t), z_{\delta_j}(t)) \rightharpoonup \mathrm{D}_e\mathfrak{V}(\mathbf{e}_{\mathrm{D}}(t), z(t)).
\end{aligned}$$

*Moreover, for each stable* $z_0 \in \mathcal{Z}$*, i.e.* $\mathfrak{I}(0, z_0) \le \mathfrak{I}(0, \widetilde{z}) + \mathcal{D}(z_0, \widetilde{z})$ *for all* $\widetilde{z} \in \mathcal{Z}$*, there exists at least one energetic solution* $z : [0,T] \to \mathcal{Z}$ *for the complete damage problem* $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$*.*

The proof of this result, which is given in Section 5, follows closely the theory developed in [MiR06, BMR09], and thus relies on the abstract theory of Γ-convergence for rate-independent systems developed in [MRS08].

# 3 Parametrized Γ-convergence

In this section we consider general reflexive Banach spaces $\mathbb{E}$ and $Z$ and assume that $\mathcal{Z}$ is a weakly closed subset of $Z$. We now discuss sequences of functionals $\mathcal{J}_\delta : \mathbb{E} \times \mathcal{Z} \to \mathbb{R}$ and their parametrized Γ-limits $\mathfrak{J}(e,\cdot) = \Gamma\text{-}\lim_{\delta\to 0^+} \mathcal{J}_\delta(e,\cdot)$. Here $e \in \mathbb{E}$ is treated as a fixed parameter, and Γ-convergence in $\mathcal{Z}$ is meant with respect to the strong convergence, viz.

$$\textit{liminf estimate:} \qquad z_\delta \to z \implies \mathfrak{J}(e,z) \le \liminf_{\delta\to 0^+} \mathcal{J}_\delta(e, z_\delta), \tag{11a}$$

$$\textit{recovery sequence:} \qquad \forall\, z \in \mathcal{Z}\ \exists\, (z_\delta)_{\delta>0} : \quad z_\delta \to z \text{ and } \mathcal{J}_\delta(e, z_\delta) \to \mathfrak{J}(e,z). \tag{11b}$$

The following example shows that natural properties of the functionals $\mathcal{J}_\delta(\cdot, z)$ may be lost for parametrized Γ-limits.

**Example 3.1 (Convexity)** *We consider* $\mathbb{E} = \mathbb{R}$, $\mathcal{Z} = \mathbb{R}$ *and the functionals*

$$\mathcal{J}_\delta(e,z) = |e - g(z/\delta)| + 1 - g(z/\delta)^2 \quad \textit{with } g(t) = \max\{-1, \min\{t, 1\}\}.$$

*Clearly, each* $\mathcal{J}_\delta(\cdot, z)$ *is convex. The parametrized* Γ*-limit exists and reads*

$$\mathfrak{J}(e,z) = \begin{cases} |e - \mathrm{sign}(z)| & \textit{for } z \neq 0, \\ |1 - |e|| & \textit{for } z = 0. \end{cases}$$

*For* $z \neq 0$ *we can take constant recovery sequences* $z_\delta = z$*. For* $z = 0$*, the recovery sequences will depend on* $e$*: for* $e > 0$ *we choose* $z_\delta = \delta$ *and find* $\mathcal{J}_\delta(e, z_\delta) = |e-1|$*, while for* $e < 0$ *let* $z_\delta = -\delta$ *obtaining* $\mathcal{J}_\delta(e, z_\delta) = |e+1|$*.*

The following definition is made to avoid the problem of different recovery sequences at different points.

**Definition 3.2** *The family* $(\mathcal{J}_\delta)_{\delta>0}$ *has the* simultaneous $\Gamma$-limit $\mathfrak{J}:\mathbb{E}\times\mathcal{Z}\to\mathbb{R}$, *if* (11a) *holds and for each* $R>0$ *there exists* $\widehat{R}>0$ *such that*

$$\begin{aligned}&\forall\, z\in\mathcal{Z} \text{ with } \|z\|\leq R\ \forall\, e_1,e_2\in\mathbb{E}\ \exists\,(z_\delta)_{\delta>0} \text{ with } \sup_{\delta>0}\|z_\delta\|\leq\widehat{R}:\\ &z_\delta\to z \text{ and } \mathcal{J}_\delta(e_j,z_\delta)\to\mathfrak{J}(e_j,z) \text{ for } j=1,2.\end{aligned}\tag{12}$$

The point of simultaneous $\Gamma$-convergence is that there must exist recovery sequences that work at each pair of two points $e_1$ and $e_2$ simultaneously. This condition will allow us to inherit, from the family $\mathcal{J}_\delta$ to the parametrized $\Gamma$-limit, all properties that can be formulated in terms of finitely many function evaluations.

**Proposition 3.3 (Convexity)** *If all* $\mathcal{J}_\delta(\cdot,z)$ *are convex and* $\mathfrak{J}$ *is the simultaneous* $\Gamma$-*limit of* $(\mathcal{J}_\delta)_{\delta>0}$ *for* $\delta\to0$, *then* $\mathfrak{J}(\cdot,z):\mathbb{E}\to\mathbb{R}$ *is convex for each* $z\in\mathcal{Z}$.

**Proof:** For arbitrary $e_0,e_1$ and $\theta\in\,]0,1[$ we define $e_\theta=(1{-}\theta)e_0+\theta e_1$. Then, convexity of $\mathcal{J}_\delta(\cdot,z_\delta)$ gives

$$\mathcal{J}_\delta(e_\theta,z_\delta)\leq(1{-}\theta)\mathcal{J}_\delta(e_0,z_\delta)+\theta\mathcal{J}_\theta(e_1,z_\delta).$$

By the assumption of 2-simultaneous $\Gamma$-convergence, we may assume that $z_\delta\to z$ recovers the $\Gamma$-limit at $e_0$ and $e_1$. Thus, we conclude

$$\begin{aligned}\mathfrak{J}(e_\theta,z)&\leq\liminf_{\delta\to0^+}\mathcal{J}_\delta(e_\theta,z_\delta)\ \leq\ \liminf_{\delta\to0^+}\Big((1{-}\theta)\mathcal{J}_\delta(e_0,z_\delta)+\theta\mathcal{J}_\theta(e_1,z_\delta)\Big)\\&=(1{-}\theta)\mathfrak{J}(e_0,z)+\theta\mathfrak{J}(e_1,z),\end{aligned}$$

which is the desired convexity. ∎

We formulate a quantitative notation of continuous differentiability. We say that $\mathcal{J}:\mathbb{E}\times\mathcal{Z}\to R$ is *$\beta$-differentiable*, if all $\mathcal{J}(\cdot,z)$ lie in $\mathrm{C}^1(\mathbb{E})$ and for all $R>0$ there exists a constant $C_R>0$ such that for all $e_0,e_1\in\mathbb{E}$, $z\in\mathcal{Z}$ with $\|e_0\|+\|e_1\|+\|z\|\leq R$ we have

$$\|\mathrm{D}_e\mathcal{J}(e_1,z)-\mathrm{D}_e\mathcal{J}(e_0,z)\|_{\mathbb{E}^*}\leq C_R\|e_1{-}e_0\|^\beta.\tag{13}$$

We say that the family $(\mathcal{J}_\delta)_{\delta>0}$ is *uniformly $\beta$-differentiable* if the constant $C_R$ can be chosen independently of $\delta>0$.

The importance of this notion is that it can be equivalently formulated by using function values only and avoiding the derivative. This equivalence is a standard exercise in Banach-space analysis.

**Lemma 3.4** *A function* $\mathcal{J}:\mathbb{E}\times\mathcal{Z}\to R$ *is* $\beta$-differentiable *if and only if for all* $R>0$ *there exists a constant* $\widehat{C}_R>0$ *such that for all* $\theta\in\,]0,1[$, $e_0,e_1\in\mathbb{E}$, $z\in\mathcal{Z}$ *with* $\|e_0\|,\|e_1\|,\|z\|\leq R$ *we have*

$$|\mathcal{J}(e_\theta,z)-(1{-}\theta)\mathcal{J}(e_0,z)-\theta\mathcal{J}(e_1)|\leq\widehat{C}_R\theta(1{-}\theta)\|e_1{-}e_0\|^{1+\beta}.\tag{14}$$

We note that going from (13) to (14) one can estimate $\widehat{C}_R\leq C_*C_{2R}$, where $C_*$ is a universal constant. Similarly, one can estimate $C_R\leq C_*\widehat{C}_{2R}$ for the opposite implication.

**Proposition 3.5** *If the family $(\mathcal{J}_\delta)_{\delta>0}$ is uniformly $\beta$-differentiable and if $\mathfrak{J}$ is the simultaneous $\Gamma$-limit of this family, then $\mathfrak{J}$ is also $\beta$-differentiable.*

**Proof:** It suffices to show that $\mathfrak{J}$ satisfies (14). We first note that this estimate holds uniformly in $\delta$ for all $\mathcal{J}_\delta$. For a given $R>0$ we choose $\widehat{R}$ according to Definition 3.2. First choose a simultaneous recovery sequence $z_\delta \to z$ for the points $e_0$ and $e_1$. Then,

$$\begin{aligned}&\mathfrak{J}(e_\theta,z)-(1-\theta)\mathfrak{J}(e_0,z)-\theta\mathfrak{J}(e_1,z)\\&\leq \liminf_{\delta\to0^+}\Big(\mathcal{J}_\delta(e_\theta,z_\delta)-(1-\theta)\mathcal{J}_\delta(e_0,z_\delta)-\theta\mathcal{J}_\theta(e_1,z_\delta)\Big)\ \leq\ \widehat{C}_{\widehat{R}}\theta(1-\theta)\|e_1-e_0\|^{1+\beta}.\end{aligned}$$

The opposite estimate is obtained by multiplying with $-1$ and choosing a recovery sequence for the point $e_\theta$:

$$\begin{aligned}&(1-\theta)\mathfrak{J}(e_0,z)+\theta\mathfrak{J}(e_1,z)-\mathfrak{J}(e_\theta,z)\\&\leq \liminf_{\delta\to0^+}\Big((1-\theta)\mathcal{J}_\delta(e_0,z_\delta)+\theta\mathcal{J}_\theta(e_1,z_\delta)-\mathcal{J}_\delta(e_\theta,z_\delta)\Big)\ \leq\ \widehat{C}_{\widehat{R}}\theta(1-\theta)\|e_1-e_0\|^{1+\beta}.\end{aligned}$$

This proves (14) with $C_R=\widehat{C}_{\widehat{R}}$. ∎

For convex functions the notion of uniform differentiability can be simplified as one estimate in (14) holds automatically. Moreover, it suffices to reduce to the case $\theta=1/2$ (cf. [Zăl02]), i.e. one can replace (14) by

$$0\leq\mathcal{J}(e_0,z)+\mathcal{J}(e_1,z)-2\mathcal{J}(\tfrac12(e_0+e_1),z)\leq C_R\|e_0-e_1\|^{1+\beta}. \tag{15}$$

**Proposition 3.6** *Assume that the family $(\mathcal{J}_\delta)_{\delta>0}$ is uniformly $\beta$-differentiable and that all $\mathcal{J}_\delta(\cdot,z)$ are convex. Moreover, assume that $\mathfrak{J}$ is the simultaneous $\Gamma$-limit of this family, then $\mathfrak{J}$ is $\beta$-differentiable and each $\mathfrak{J}(\cdot,z)$ is convex. Moreover, we have the following* convergence of stresses*:*

$$\left.\begin{array}{r}z_\delta\to z \text{ in } \mathbb{E}\\ \mathcal{J}_\delta(e,z_\delta)\to\mathfrak{J}(e,z)\end{array}\right\}\implies \mathrm{D}_e\mathcal{J}_\delta(e,z_\delta)\to\mathrm{D}_e\mathfrak{J}(e,z) \text{ in } \mathbb{E}^*. \tag{16}$$

**Proof:** The results on $\beta$-differentiability and convexity for $\mathfrak{J}$ are already established above. The convergence of stresses follows from the differentiability, which means that the subdifferential $\partial_e\mathfrak{J}$ is a singleton containing $\mathrm{D}_e\mathfrak{J}$. In fact, $\Sigma_\delta=\mathrm{D}_e\mathcal{J}_\delta(e,z_\delta)$ is bounded in $\mathbb{E}^*$, and we may choose a subsequence $\delta_j\to0^+$ such that $\Sigma_{\delta_j}\rightharpoonup\Sigma_0$ in $\mathbb{E}^*$ and $\mathcal{J}_{\delta_j}(\widetilde{e},z_{\delta_j})\to J(\widetilde{e})$ for all $\widetilde{e}\in\mathbb{E}$. The latter pointwise convergence follows from Arzela-Ascoli's theorem because of the uniform Lipschitz continuity of the $\mathcal{J}_\delta(\cdot,z_\delta)$ on all balls $B_R(e)$, $R\in N$.

As $J:\mathbb{E}\to R$ is the pointwise limit of a the family $(\mathcal{J}_{\delta_j}(\cdot,z_{\delta_j}))_j\in N$, which is convex and uniformly $\beta$-differentiable, $J$ has these properties as well. By construction we also have $J(e)=\mathfrak{J}(e,z)$ and $J(\widetilde{e})\geq\mathfrak{J}(\widetilde{e},z)$. This implies $\Sigma_*=\mathrm{D}J(e)=\mathrm{D}_e\mathfrak{J}(e,z)$.

Moreover, convexity implies $\mathcal{J}_\delta(\widetilde{e},z_\delta)\geq\mathcal{J}_\delta(e,z_\delta)+\langle\Sigma_\delta,\widetilde{e}-e\rangle$, and passing to the limit $\delta_j\to0$ gives $J(\widetilde{e})\geq J(e)+\langle\Sigma_0,\widetilde{e}-e\rangle$. Thus, we conclude $\Sigma_0=\mathrm{D}J(e)$. In turn, this implies $\Sigma_\delta\rightharpoonup\mathrm{D}J(e)=\mathrm{D}_e\mathfrak{J}(e,z)$ (no subsequence), which is the desired result. ∎

# 4 The complete-damage problem via Γ-convergence

Before we can apply the abstract theory of the previous section, we have to deal with the fact that $\mathcal{J}_\delta : (e,z) \mapsto \mathcal{V}_\delta(e,z) + \mathcal{G}(z)$ is defined by minimizing $\mathcal{E}_\delta$ with respect to $u \in \mathcal{F}$. Hence, $\mathcal{V}_\delta$ is only defined implicitly, which makes is more difficult to check convexity and $\beta$-differentiability.

## 4.1 Convexity and differentiability for the reduced damage functionals

We recall the definition of $\mathcal{J}_\delta(e,z) = \mathcal{V}_\delta(e,z) + \mathcal{G}(z)$, where

$$\mathcal{V}_\delta(e,z) = \min\{\, \mathcal{W}_\delta(e{+}\mathbf{e}(u),z) \mid u \in \mathcal{F}\,\} \ \text{ with } \mathcal{W}_\delta(e,z) = \int_\Omega W_\delta(x,e(x),z(x))\,\mathrm{d}x$$

with $W_\delta(x,e,z) = W(x,e,z) + \delta|e|^p$, where $W$ satisfies (5), which includes the convexity condition (5c). Since in this section we treat the dependence on $e$ only, we omit the constant term $\mathcal{G}(z)$ that always cancels in convexity and differentiability conditions.

For $\delta > 0$ the stored-energy density $W_\delta$ is strictly convex with respect to $e \in \mathbf{E}_d$. Moreover, for $\delta > 0$ we have the coercivity $W_\delta(x,e,z) \geq \delta|e|^p$ which implies that there exists for each $z \in \mathcal{Z}$ and each $e \in \mathbb{E}$ a unique $u = U_\delta(e,z)$ such that

$$\mathcal{V}_\delta(e,z) = \mathcal{W}_\delta(e{+}\mathbf{e}(U_\delta(e,z)),z), \quad U_\delta(e,z) \in \mathcal{F}.$$

In particular, we have $\mathcal{V}_\delta(e{+}\mathbf{e}(\widehat{u}),z) = \mathcal{V}_\delta(e,z)$ for all $\widehat{u} \in \mathcal{F}$, because of $U_\delta(e{+}\mathbf{e}(\widehat{u}),z) = U_\delta(e,z) - \widehat{u}$. This shows that $\mathcal{V}_\delta(\cdot,z) : \mathbb{E} \to \mathbb{R}$ is highly degenerate and should be considered as a functional on $\mathbb{E}/_{\mathbf{e}(\mathcal{F})}$.

**Lemma 4.1 (Convexity of $\mathcal{V}_\delta$)** *Let $W$ satisfy* (5)*. Then, the functionals $\mathcal{V}_\delta(\cdot,z) : \mathbb{E} \to \mathbb{R}$ are convex and satisfy the estimates $0 \leq \mathcal{V}_\delta(e,z) \leq C(1{+}\|e\|^p) + \delta\|e\|^p$.*

**Proof:** For arbitrary $\theta \in \,]0,1[$, $e_0$, $e_1 \in \mathbb{E}$ and $z \in \mathcal{Z}$ we have

$$\begin{aligned}\mathcal{V}_\delta(e_\theta,z) &= \mathcal{W}_\delta(e_\theta{+}U_\delta(e_\theta,z),z) \leq \mathcal{W}_\delta(e_\theta{+}(1{-}\theta)U_\delta(e_0,z) + \theta U_\delta(e_1,z),z)\\ &= \mathcal{W}_\delta((1{-}\theta)[e_0{+}U_\delta(e_0,z)] + \theta[e_1{+}U_\delta(e_1,z)],z)\\ &\overset{\text{convex}}{\leq} (1{-}\theta)\mathcal{W}_\delta(e_0{+}U_\delta(e_0,z),z) + \theta\mathcal{W}_\delta(e_1{+}U_\delta(e_1,z),z)\\ &= (1{-}\theta)\mathcal{V}_\delta(e_0,z) + \theta\mathcal{V}_\delta(e_1,z).\end{aligned}$$

This is the desired convexity.

For the estimates we first derive $0 \leq \mathcal{W}_0(e,z) \leq C(1{+}\|e\|^p)$, which follows easily by integration. We then use $0 \leq \mathcal{W}_\delta(e{+}U_\delta(e,z),z) = \mathcal{V}_\delta(e,z) \leq \mathcal{W}_0(e,z) + \delta\|e\|^p$. ∎

To obtain uniform $\beta$-differentiability of $\mathcal{V}_\delta$ in the form (15), we use the additional uniform differentiability condition (10) on the energy density $W$. It is easy to derive the corresponding condition for the functional $\mathcal{W}_\delta$, but is is essential that the condition is also stable under the reduction from $\mathcal{W}_\delta$ to $\mathcal{V}_\delta$.

**Proposition 4.2** *Let $W$ satisfy (5) and (10). Then, for each $R > 0$ there exists a constant $C_R > 0$ such that for all $\delta \in \,]0,1]$, $e_0$, $e_1 \in \mathbb{E}$, and $z \in \mathcal{Z}$, we have*

$$\mathcal{V}_\delta(e_0, z) + \mathcal{V}_\delta(e_1, z) - 2\mathcal{V}_\delta(e_{1/2}, z) \leq C_R \|e_1 - e_0\|^{1+\beta}.$$

**Proof:** We note that $W_\delta$ satisfies all the assumptions uniformly for $\delta \in [0,1]$. Integration of (10) for $W_\delta$ and using Hölder's inequality gives, for $e_0$, $e_1 \in \mathbb{E}$ and $z \in \mathcal{Z}$,

$$\mathcal{W}_\delta(e_0, z) + \mathcal{W}_\delta(e_1, z) - 2\mathcal{W}_\delta(e_{1/2}, z) \leq C\big(|\Omega| + \mathcal{W}_\delta(e_{1/2}, z) + \|e_1 - e_0\|^p\big)^{1-(1+\beta)/p} \|e_1 - e_0\|^{1+\beta}.$$

The corresponding inequality for $\mathcal{V}_\delta$ follows by using the minimization properties. With $E_\delta = \mathbf{e}(U_\delta(e_{1/2}, z))$ we have $\mathcal{V}_\delta(e_{1/2}, z) = \mathcal{W}_\delta(e_{1/2} + E_\delta, z)$ and find

$$\begin{aligned}
&\mathcal{V}_\delta(e_0, z) + \mathcal{V}_\delta(e_1, z) - 2\mathcal{V}_\delta(e_{1/2}, z) \\
&\leq \mathcal{W}_\delta(e_0 + E_\delta, z) + \mathcal{W}_\delta(e_1 + E_\delta, z) - 2\mathcal{W}_\delta(e_{1/2} + E_\delta, z) \\
&\leq C\big(|\Omega| + \mathcal{V}_\delta(e_{1/2}, z) + \|e_1 - e_0\|^p\big)^{1-(1+\beta)/p} \|e_1 - e_0\|^{1+\beta},
\end{aligned}$$

which provides the desired estimate after exploiting Lemma 4.1. ∎

## 4.2 Parametrized Γ-convergence for the damage functional

We now consider the Γ-limit for $\delta \to 0$ and work with the functional $\mathcal{J}_\delta : (e, z) \mapsto \mathcal{V}_\delta(e, z) + \mathcal{G}(z)$ again. For applying the abstract theory it is necessary to derive simultaneous Γ-limits. The main positive result was obtained in [BMR09] for the case that the $\mathcal{G}$ dominates the $\mathrm{L}^r$ norm of $\nabla z$ with $r > d$, where $d$ is the space dimension.

We generalize this result in several aspects by reducing it to the minimal structural assumption. For this we introduce the stable sets

$$\mathcal{S}_\delta(t) = \{\, z \in \mathcal{Z} \mid \infty > \mathcal{I}_\delta(t, z) \leq \mathcal{I}_\delta(t, \widetilde{z}) + \mathcal{D}(z, \widetilde{z}) \text{ for all } \widetilde{z} \in \mathcal{Z} \,\}.$$

We define the parametrized Γ-limit $\mathfrak{V}(e, \cdot) = \Gamma\text{-}\lim_{\delta \to 0^+} \mathcal{V}_\delta(e, \cdot)$ with respect to the strong topology of $\mathcal{Z}$, which exists by the monotonicity, see [Bra02]. The following example, which is inspired by [BoV88, Ex. 3] and further discussed in [BMR09], shows that in general $\mathfrak{V}$ is strictly smaller than $\mathcal{V}_0(e, z) = \lim_{\delta \to 0^+} \mathcal{V}_\delta(e, z)$.

**Example 4.3** *Consider $\Omega = \,]-1, 1[$ and the energy $\mathcal{J}_\delta(e, z) = \int_\Omega \frac{\delta + z}{2}(e + u')^2 \,\mathrm{d}x + \mathcal{G}(z)$. Then, $\mathcal{V}_\delta(e, z) = \big(\int_\Omega e \,\mathrm{d}x\big)^2 / \int_\Omega \frac{2}{\delta + z} \,\mathrm{d}x$. Clearly, the pointwise limit $\mathcal{V}_0$ is obtained by letting $\delta = 0$. However, the Γ-limit $\mathfrak{V}(e, \cdot)$ in $\mathrm{W}^{1,r}(\Omega)$ satisfies*

$$\mathfrak{V}(e, z) = \mathcal{V}_0(e, z) \text{ for } \min z > 0 \quad \text{and } \mathfrak{V}(e, z) = 0 \text{ for } \min z = 0.$$

*For $\alpha \in \big]1 - \frac{1}{r}, 1\big[$ let $z_\alpha(x) = |x|^\alpha$, then $z_\alpha \in \mathcal{Z}$ and $0 = \mathfrak{V}(e, z) < \mathcal{V}_0(e, z) = \frac{1-\alpha}{4}\big(\int_\Omega e \,\mathrm{d}x\big)^2$.*

Since $\mathcal{G} : \mathcal{Z} \to \mathbb{R}$ is continuous, we also have the following parametrized Γ-limits:

$$\Gamma\text{-}\lim_{\delta \to 0^+} \mathcal{J}_\delta(e, \cdot) = \mathfrak{J}(e, \cdot) = \mathfrak{V}(e, \cdot) + \mathcal{G}(\cdot), \quad \Gamma\text{-}\lim_{\delta \to 0^+} \mathcal{I}_\delta(t, \cdot) = \mathfrak{I}(t, \cdot) = \mathfrak{V}(e_\mathrm{D}(t), \cdot) + \mathcal{G}(\cdot).$$

We also set $\mathfrak{S}(t) = \{\, z \in \mathcal{Z} \mid \infty > \mathfrak{I}(t,z) \leq \mathfrak{I}(t,\widetilde{z}) + \mathcal{D}(z,\widetilde{z}) \text{ for all } \widetilde{z} \in \mathcal{Z} \,\}$.

In generalizing the approach in [BMR09] for the $\Gamma$-convergence we replace the condition "$r > d$" there by the following

$$\begin{array}{l} \textit{structural assumption:} \\ \big(z_j \in \mathcal{S}_{\delta_j}(t), \delta_j \to 0, \text{ and } z_j \rightharpoonup z\big) \quad \Longrightarrow \quad z_j \to z. \end{array} \tag{17}$$

The following result shows that the structural assumption holds for $r > d$, where we use the monotonicity of $W(e,\cdot)$ and the embedding $\mathrm{W}^{1,r}(\Omega) \subset \mathrm{C}(\overline{\Omega})$. Other sufficient conditions will be discussed in Section 6.1.

**Proposition 4.4 (Structural assumption)** *Let the assumptions of Section 2.1 hold.*
*(A) If $(z_j)_{j\in\mathbb{N}}$ is as in* (17)*, then we have $\mathcal{V}_{\delta_j}(e_{\mathrm{D}}(t),z_j) \to \mathfrak{V}(e_{\mathrm{D}}(t),z)$ and $z \in \mathfrak{S}(t)$.*
*(B) If $r > d$, then the structural assumption* (17) *holds and we have*

$$\mathfrak{V}(e,z) = \lim_{\rho\to 0^+}\big(\lim_{\delta\to 0^+} \mathcal{V}_\delta\big(e,\max\{0,z-\rho\}\big)\big) \quad \textit{and} \quad \big(\widetilde{z}_\delta \rightharpoonup \widetilde{z} \;\Rightarrow\; \mathfrak{V}(e,\widetilde{z}) \leq \liminf_{\delta\to 0^+} \mathcal{V}_\delta(e,\widetilde{z})\big),$$

*i.e. the $\Gamma$-convergence is even a Mosco convergence, cf. [Mos67].*

**Proof:** Ad (A). We abbreviate $e = e_{\mathrm{D}}(t)$, let $v = \limsup_{j\to\infty} \mathcal{V}_{\delta_j}(e,z_j)$, and conclude $\limsup_{j\to\infty} \mathcal{I}_{\delta_j}(t,z_j) = v + \mathcal{G}(z)$. Using the stability of $z_j$ we obtain

$$\mathcal{I}_{\delta_j}(t,z_j) \leq \mathcal{I}_{\delta_j}(t,\widehat{z}_j) + \mathcal{D}(z_j,\widehat{z}_j),$$

where we choose $\widehat{z}_j$ as a recovery sequence for $\widehat{z}$, i.e. $\widehat{z}_j \to \widehat{z}$ and $\mathcal{I}_{\delta_j}(t,\widehat{z}_j) \to \mathfrak{I}(t,\widehat{z})$. In the unidirectional case we may restrict to the case $\widehat{z} \leq z$ and assume $\widehat{z}_j \leq z_j$ (by taking the recovery sequence $\widetilde{z}_j = \min\{z_j,\widehat{z}_j\}$ if necessary). Thus we may pass to the limit $j \to \infty$ and obtain

$$\mathfrak{I}(t,z) \leq \limsup_{j\to\infty} \mathcal{I}_{\delta_j}(t,z_j) = v + \mathcal{G}(z) \leq \mathfrak{I}(t,\widehat{z}) + \mathcal{D}(z,\widehat{z}).$$

This proves the stability $z \in \mathfrak{S}(t)$.

Moreover, we may take $\widehat{z} = z$ and conclude $v \leq \mathfrak{I}(t,z) - \mathcal{G}(z) = \mathfrak{V}(e,z)$. Since $\mathfrak{V}(e,z) \leq v$ by the definition of the $\Gamma$-limit we are done.

Ad (B). We first show that the double limit in the formula for $\mathfrak{V}$ exists. For this, we define the function $V(\rho,\delta,e,z) = \mathcal{V}_\delta\big(e,\max\{0,z-\rho\}\big)$. Since $W_\delta(e,z)$ is nondecreasing in $\delta$ and in $z$, $V(\rho,\delta,e,z)$ is nondecreasing in $\delta$ and nonincreasing in $\rho$. For fixed $z$ and $\rho >$ the limit $V^0(\rho,e,z) = \lim_{\delta\to 0^+} V(\rho,\delta,e,z)$ exists by monotonicity and boundedness. Moreover, $V^0(\rho,e,z)$ is still nonincreasing in $\rho$, and we find that $\mathfrak{V}(e,z) = \lim_{\rho\to 0^+} V^0(\rho,e,z)$ exists as well.

To show that $\mathfrak{V}$ is the Mosco limit, we first establish the liminf estimate assuming the weak convergence $z_\delta \rightharpoonup z$ in $\mathrm{W}^{1,r}(\Omega)$. Then, for each $\rho > 0$, there exists $\delta_\rho$ such that $z_\delta \geq \max\{0,z-\rho\}$, where we use the embedding $\mathrm{W}^{1,r}(\Omega) \subset \mathrm{C}(\overline{\Omega})$. Thus, $\mathcal{V}_\delta(e,z_\delta) \geq \mathcal{V}_\delta(e,\max\{0,z-\rho\})$, and we obtain $\liminf_{\delta\to 0^+} \mathcal{V}_\delta(e,z_\delta) \geq V^0(\rho,e,z)$. Taking the limit $\rho \to 0^+$ we obtain the desired liminf estimate. To obtain recovery sequences, we use that by the definition of the double limit we may choose a continuous function $g : [0, \delta_*] \to [0, \rho_*]$ with $g(0) = 0$ such that $V(g(\delta), \delta, e, z) \to \mathfrak{V}(e, z)$. Hence, $z_\delta = \max\{0, z - g(\delta)\}$ provides the desired strongly converging recovery sequence.

Now we establish the structural assumption (17). Starting from $z_j \rightharpoonup z$ as given there we let

$$v = \liminf_{j\to\infty} \mathcal{V}_{\delta_j}(e, z_j) \geq \mathfrak{V}(e, z) \quad \text{and} \quad \gamma = \liminf_{j\to\infty} \mathcal{G}(z_j) \geq \mathcal{G}(z),$$

which gives $\liminf_{j\to\infty} \mathcal{I}_{\delta_j}(t_j, z_j) \geq \mathfrak{I}(t, z)$. The stability of $z_j$ implies

$$\mathcal{I}_{\delta_j}(t_j, z_j) \leq \mathcal{I}_{\delta_j}(t_j, z^\varepsilon) + \mathcal{D}(z_j, z^\varepsilon), \text{ where } z^\varepsilon = \max\{0, z - \varepsilon\}.$$

Doing the $\limsup_{j\to\infty}$ first and the $\lim_{\varepsilon\to 0^+}$ afterwards gives $\limsup_{j\to\infty} \mathcal{I}_{\delta_j}(t_j, z_j) \leq \mathfrak{I}(t, z)$, and we conclude $\mathcal{I}_{\delta_j}(t_j, z_j) \to \mathfrak{I}(t, z)$.

In particular this implies the convergence $\mathcal{G}(z_j) \to \gamma = \mathcal{G}(z)$. Using the strict convexity (4d), we conclude $z_j \to z$, see [Vis84].

To establish the stability of $z$, we take a general test function $\widetilde{z}$ with $\mathcal{D}(z, \widetilde{z}) < \infty$, since otherwise nothing is to be shown. Let $(\widetilde{z}_j)_{j\in\mathbb{N}}$ be a recovery sequence for $\widetilde{z}$, i.e. $\widetilde{z}_j \to \widetilde{z}$ and $\mathcal{I}_{\delta_j}(t_j, \widetilde{z}_j) \to \mathfrak{I}(t, \widetilde{z})$. Then, the stability of $z_j$ implies

$$\mathcal{I}_{\delta_j}(t_j, z_j) \leq \mathcal{I}_{\delta_j}(t_j, \widehat{z}_j) + \mathcal{D}(z_j, \widehat{z}_j) \quad \text{where } \widehat{z}_j = \max\{0, \widetilde{z}_j - \|z - z_j\|_{\mathrm{L}^\infty}\}.$$

Note that $\widehat{z}_j \to \widetilde{z}$ and $\mathcal{I}_{\delta_j}(t_j, \widehat{z}_j) \leq \mathcal{I}_{\delta_j}(t_j, \widetilde{z}_j)$. Thus, $(\widehat{z}_j)_{j\in\mathbb{N}}$ is a recovery sequence as well. Passing to the limit $j \to \infty$ we find $\mathfrak{I}(t, z) \leq \mathcal{I}(t, \widetilde{z}) + \mathcal{D}(z, \widetilde{z})$, giving $z \in \mathfrak{S}(t)$. ■

The importance of the structural assumption lies in the fact that it implies that $\mathfrak{I}$ is a simultaneous $\Gamma$-limit.

**Proposition 4.5 (Simultaneous $\Gamma$-limit)** *Let the assumptions of Section 2.1 and* (17) *hold. Then, the functional $\mathfrak{I}$ is the simultaneous $\Gamma$-limit of the family $(\mathcal{J}_\delta)_{\delta>0}$.*

**Proof:** Let $e_1, e_2 \in \mathbb{E}$ be given and let $(z_\delta^j)_{\delta>0}$, $j = 1, 2$, be associated recovery sequences for $\mathfrak{I}(e_j, z)$. We define $\widetilde{z}_\delta(x) = \min\{z_\delta^1(x), z_\delta^2(x)\}$ and obtain $\widetilde{z}_\delta \to z$, because of $z_\delta^j \to z$. Moreover, the monotonicity of $W(e, \cdot)$ implies $\mathcal{V}_\delta(e_j, \widetilde{z}_\delta) \leq \mathcal{V}_\delta(e_j, z_\delta^j)$. Thus, we conclude,

$$\mathfrak{V}(e_j, z) \leq \liminf_{\delta\to 0^+} \mathcal{V}_\delta(e_j, \widetilde{z}_\delta) \leq \limsup_{\delta\to 0^+} \mathcal{V}_\delta(e_j, \widetilde{z}_\delta) \leq \limsup_{\delta\to 0^+} \mathcal{V}_\delta(e_j, z_\delta^j) = \mathfrak{V}(e_j, z).$$

Thus, $(\widetilde{z}_\delta)_{\delta>0}$ is a simultaneous recovery sequence. ■

Now, we are able to take profit from the abstract results on parametrized $\Gamma$-convergence of Section 3. In particular, we are able to deduce convexity and differentiability of $\mathcal{V}(\cdot, z)$.

**Proposition 4.6** *Let the assumptions of Section 2.1 and* (17) *hold. Then, $\mathfrak{V}(\cdot, z) : \mathbb{E} \to \mathbb{R}$ is convex for all $z \in \mathcal{Z}$.*

*If additionally $W$ satisfies the differentiability condition* (10)*, then $\mathfrak{V}$ is $\beta$-differentiability in the sense of* (15)*, and for all $e \in \mathbb{E}$ we have*

$$\left.\begin{array}{c} z_\delta \to z \text{ in } \mathcal{Z} \\ \mathcal{V}_\delta(e, z_\delta) \to \mathfrak{V}(e, z) \end{array}\right\} \implies \mathrm{D}_e\mathcal{V}_\delta(e, z_\delta) \rightharpoonup \mathrm{D}_e\mathfrak{V}(e, z) \text{ in } \mathbb{E}^*.$$

The proof of this result is a direct combination of Propositions 3.3, 3.5, 3.6, Lemma 4.1, and Propositions 4.2 and 4.5.

# 5 Proof of Theorem 2.3

Our main Theorem 2.3 provides the convergence of the energetic solutions $z_\delta : [0,T] \to \mathcal{Z}$ for the rate-independent systems $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$ for $\delta \to 0^+$ to energetic solutions $z : [0,T] \to \mathcal{Z}$ of the limit problem $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$, which represents the complete-damage problem. It is stated under the additional assumption " $r > d$".

Here we will provide a more general proof avoiding the explicit use of the embedding $\mathrm{W}^{1,r}(\Omega) \subset \mathrm{C}(\overline{\Omega})$ and replacing it with the *structural assumption* (17), which is satisfied in the case $r > d$, as is shown in Part B of Proposition 4.4.

For the convenience of the reader we provide an almost complete proof, where some details are cited from previous works. We follow the six steps as introduced in [Mie05].

*Step 1. A priori estimates.*

The solutions $z_\delta : [0,T] \to \mathcal{Z}$ are stable. Hence, we have

$$\mathcal{G}(z_\delta(t)) \le \mathcal{I}_\delta(t, z(t)) \le \mathcal{I}_\delta(t, 0) + \mathcal{D}(z_\delta(t), 0) \le C.$$

Together with $z(t,x) \in [0,1]$ we obtain a uniform bound $C > 0$ such that $\|z_\delta(t)\|_{\mathrm{W}^{1,r}} \le C$ for all $t \in [0,T]$ and $\delta > 0$. Moreover, the total dissipation $\mathrm{Diss}_{\mathcal{D}}(z_\delta, [0,T])$ is bounded independently of $\delta > 0$. Thus,

$$\exists\, C > 0\ \forall\, \delta > 0 : \|z_\delta\|_{\mathrm{L}^\infty([0,T],\mathrm{W}^{1,p}(\Omega))} + \|z_\delta\|_{\mathrm{BV}([0,T],\mathrm{L}^1(\Omega))} \le C.$$

*Step 2. Selection of subsequences*

By Helly's selection principle (in its Banach-space version) we extract a subsequence $(\delta_j)_{j\in\mathbb{N}}$ with $\delta_j \to 0^+$ such that for all $t$ we have

$$\mathrm{Diss}_{\mathcal{D}}(z_{\delta_j}, [0,t]) \to \Delta(t), \quad z_{\delta_j}(t) \rightharpoonup z(t) \text{ in } \mathcal{Z},$$

where $\delta : [0,T] \to \mathbb{R}$ is nondecreasing and $z$ lies in $\mathrm{L}^\infty([0,T], \mathrm{W}^{1,p}(\Omega)) \cap \mathrm{BV}([0,T], \mathrm{L}^1(\Omega))$ with $\mathrm{Diss}_{\mathcal{D}}(z, [0,t]) \le \Delta(t)$. Using the structural assumption (17) and Part (A) of Proposition 4.4 we further conclude $z_{\delta_j}(t) \to z(t)$ and $\mathcal{I}_{\delta_j}(t, z_{\delta_j}(t)) \to \mathfrak{I}(t, z(t))$, which means that (i) and (iii) are established.

*Step 3. Stability of the limit process*

The desired stability (S) for energetic solutions means $z(t) \in \mathfrak{S}(t)$ for all $t \in [0,T]$, but this is a direct consequence of Part A of Proposition 4.4.

*Step 4. Upper energy estimate*

For each $\delta > 0$ we have the energy balance

$$\mathcal{I}_\delta(t, z_\delta(t)) + \mathrm{Diss}_{\mathcal{D}}(z_\delta, [0,t]) = \mathcal{I}_\delta(0, \delta) + \int_0^t \partial_s \mathcal{I}(s, z_\delta(s))\,\mathrm{d}s.$$

Using the formula (2ii) and $\partial_s \mathfrak{I}(s,z) = \langle \mathrm{D}_e \mathfrak{V}(e_{\mathrm{D}}(t), z), \mathbf{e}(\dot{u}_{\mathrm{D}}(t))\rangle$ we are now able to pass to the limit $\delta_j \to 0^+$ and obtain

$$\mathfrak{I}(t, z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0,T]) \overset{\text{Step 2}}{\le} \mathfrak{I}(t, z(t)) + \Delta(t) = \mathfrak{I}(0, z(t)) + \int_0^t \partial_s \mathfrak{I}(s, z(s))\,\mathrm{d}s,$$

where we used Proposition 4.6, which also implies (iv).

*Step 5. Lower energy estimate*

The lower estimate $\mathfrak{I}(t, z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0, T]) \geq \mathfrak{I}(0, z(t)) + \int_0^t \partial_s \mathfrak{I}(s, z(s))\,\mathrm{d}s$ is a direct consequence of the stability, see e.g. [Mie05, Prop. 5.7]. Thus, we conclude the energy equality (E) and have established $\mathrm{Diss}_{\mathcal{D}}(z, [0, T]) = \Delta(t)$, which provides (ii).

*Step 6. Improved convergence*

Since the convergences (i)–(iv) in Theorem 2.3 are already established in the previous steps, the convergence proof is finished.

It remains to establish the general existence result for arbitrary initial conditions $z_0 \in \mathfrak{S}(0)$. However, it is standard to apply the existence theory developed in [Mie05, Sect. 5] directly to the limit problem $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$. This concludes the proof of Theorem 2.3.

# 6 Discussion of generalizations

## 6.1 Sufficient conditions for the structural assumption

The reason for introducing the structural condition (17) is that we conjecture its validity also in the case $r \in [1, d]$. To support this conjecture, we highlight an interesting observation from [Tho09], which applies to the uni-directional case, where $\mathcal{D}(z, \widehat{z}) < \infty$ if and only if $\widehat{z} \leq z$. For $z \in \mathcal{S}_\delta(t)$ we find the estimate

$$\begin{aligned}\mathcal{G}(z) &= \mathcal{I}_\delta(t, z) - \mathcal{V}_\delta(e_{\mathrm{D}}(t), z) \;\leq\; \mathcal{I}_\delta(t, \widehat{z}) + \mathcal{D}(z, \widehat{z}) - \mathcal{V}_\delta(e_{\mathrm{D}}(t), z)\\ &= \mathcal{G}(\widehat{z}) + \mathcal{D}(z, \widehat{z}) + \mathcal{V}_\delta(e_{\mathrm{D}}(t), \widehat{z}) - \mathcal{V}_\delta(e_{\mathrm{D}}(t), z) \;\leq\; \mathcal{G}(\widehat{z}) + \mathcal{D}(z, \widehat{z}),\end{aligned}$$

for all $\widehat{z} \leq z$. Thus, if we define the set

$$\mathbf{S} = \{\, z \in \mathcal{Z} \mid \mathcal{G}(z) \leq \mathcal{G}(\widehat{z}) + \mathcal{D}(z, \widehat{z}) \text{ for all } \widehat{z} \leq z \,\},$$

we conclude that

$$\forall\, \delta > 0\ \forall\, t \in [0, T]: \quad \mathcal{S}_\delta(t) \subset \mathbf{S} \text{ and } \mathfrak{S}(t) \subset \mathbf{S}.$$

**Conjecture.** *Under the assumptions of Section 2.1 the set* $\mathbf{S}$ *is compact in* $\mathcal{Z}$ *with respect to the strong topology for all* $r \geq 1$.

The argument in favor of the validity of the conjecture derives from the variational inequality defining the elements $z \in \mathbf{S}$. Roughly it provides a one-sided estimate of the weak $r$-Laplacian and there is hope that the results in [Mur81] can be adjusted to prove the conjecture.

Clearly, the validity of the conjecture implies that the structural condition (17) holds.

### 6.2 Generalized energetic solutions

In the case that $W$ does not satisfy the uniform differentiability property (10), we are not able to show the differentiability of $\mathfrak{V}(\cdot, z)$. However, we still have convexity, which implies together with the bounds $0 \leq \mathfrak{V}(e, z) \leq C(1+\|e\|^p)$ that of all $(e, z) \in \mathbb{E} \times \mathcal{Z}$ the (convex) subdifferential $\partial_e^{\text{sub}}\mathfrak{V}(e, z)$ and the directional derivatives $\delta_e\mathfrak{V}(e, z; \widehat{e})$ exist:

$$\begin{aligned} \partial_e^{\text{sub}}\mathfrak{V}(e, z) &= \{\, \eta \in \mathbb{E}^* \mid \forall \widetilde{e}: \ \mathfrak{V}(\widetilde{e}, z) \geq \mathfrak{V}(e, z) + \langle \eta, \widetilde{e}-e\rangle \,\}, \\ \delta_e\mathfrak{V}(e, z; \widehat{e}) &= \lim_{h\to 0^+} \frac{1}{h}\Big(\mathfrak{V}(e+h\widehat{e}, z) - \mathfrak{V}(e, z)\Big) \ = \sup\{\, \langle \sigma, \widehat{e}\rangle \mid \sigma \in \partial_e^{\text{sub}}\mathfrak{V}(e, z) \,\}. \end{aligned} \tag{18}$$

Using $\mathbf{e}_{\text{D}} \in \mathrm{C}^1([0, T]; \mathbb{E})$ we find that the left and right partial derivatives $\partial_t^{\pm}\mathfrak{I}(t, z) = \lim_{h\to 0^+} \frac{\pm 1}{h}\big(\mathfrak{I}(t\pm h, z) - \mathfrak{I}(t, z)\big)$ with respect to $t$ of $\mathfrak{I}$ exist. We have the relations

$$\partial_t^{-}\mathfrak{I}(t, z) = -\delta_e\mathfrak{V}(t, \mathbf{e}_{\text{D}}(t); -\dot{\mathbf{e}}_{\text{D}}(t)) \leq \delta_e\mathfrak{V}(t, \mathbf{e}_{\text{D}}(t); \dot{\mathbf{e}}_{\text{D}}(t)) = \partial_t^{+}\mathfrak{I}(t, z).$$

The Clarke differential of $t \mapsto \mathfrak{I}(t, z)$ is given by $\partial_t^{\text{Cl}}\mathfrak{I}(t, z) = [\partial_t^{-}\mathfrak{I}(t, z), \partial_t^{+}\mathfrak{I}(t, z)]$.

**Definition 6.1** *Let $z : [0, T] \to \mathcal{Z}$ satisfy (S) in (7) for all $t \in [0, T]$. Then, $z$ is called a* generalized energetic solution *of the rate-independent system $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$, if there exists $p \in \mathrm{L}^1([0, T])$ such that $p(\tau) \in \partial_\tau^{\mathrm{Cl}}\mathfrak{I}(\tau, z(\tau))$ a.e. in $[0, T]$ and for all $t \in [0, T]$ we have*

$$\mathfrak{I}(t, z(t)) + \text{Diss}_{\mathcal{D}}(z, [0, t]) = \mathfrak{I}(0, z(0)) + \int_0^t p(\tau)\, \mathrm{d}\tau. \tag{19}$$

Now a slight generalization of the abstract existence theory for rate-independent systems gives the following. Note that we construct the generalized energetic solutions for $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$ directly, without reference to the solutions $z_\delta$ for $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$.

**Theorem 6.2** *For all stable $z^0 \in \mathcal{Z}$ there exists a generalized energetic solution for $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$.*

**Proof:** The existence theory follows the usual steps in the abstract theory for rate-independent processes (cf. [Mie05, FrM06]) via incremental minimization, uniform a priori estimates and Helly's selection principle. This part and the proof of the stability of the limit process work as in [BMR09].

For the upper energy estimate we obtain, by setting $A(t) = \mathfrak{I}(t, z(t)) + \text{Diss}_{\mathcal{D}}(z, [0, t])$,

$$A(s) - A(r) \leq \int_r^s p^{\max}(t)\, \mathrm{d}t \text{ with } p^{\max}(t) = \max \partial_t^{\text{Cl}}\mathfrak{I}(t, z(t)).$$

With a slight generalization of [Mie05, Prop. 5.7] we see that stability of the limit process $z$ implies the lower bound $A(s) - A(r) \geq \int_r^s p^{\min}(t)\, \mathrm{d}t$ with $p^{\min}(t) = \min \partial_t^{\text{Cl}}\mathfrak{I}(t, z(t))$.

Thus, we conclude that $A$ is absolutely continuous and satisfies $p^{\min}(t) \leq A'(t) \leq p^{\max}(t)$. Hence, setting $p(t) = A'(t)$ the proof is complete. ∎

In the following example we show that the notion of generalized energetic solution, which involves the weakened energy balance (19) with the Clarke differential, is really necessary in cases where the one-sided partial derivatives satisfy $\partial_t^- \mathfrak{I}(t,z) < \partial_t^+ \mathfrak{I}(t,z)$ at some points. In particular, it is not possible to make an a priori choice like $p(t) = \max\{\partial_t^{\mathrm{Cl}} \mathfrak{I}(t,z(t))\}$, which worked in [KZM09, MiR08], since there $\partial_t^- \mathfrak{I}(t,z) \geq \partial_t^+ \mathfrak{I}(t,z)$ holds.

**Example 6.3** *This example has a smooth energy $\mathcal{I}_\delta$ such that $\partial_t \mathcal{I}_\delta$ exists, while in the limit $\mathfrak{I}$ is only Lipschitz in $t$. We let $\mathcal{Z} = \mathbb{R}$ and $\mathcal{D}(z, \widetilde{z}) = |\widetilde{z}-z|$. The energy functional reads*

$$\mathcal{I}_\delta(t,z) = H_\delta\big(z-\alpha(t)\big) \quad \text{and } \mathfrak{I}(t,z) = 2|z-\alpha(t)|,$$

*where $\alpha \in \mathrm{C}^1([0,T])$ is given and $H_\delta(u) = 2u^2/\sqrt{\delta^2+u^2}$. For the partial derivatives with respect to time we have*

$$\partial_t \mathcal{I}_\delta(t,z) = -H_\delta'(z-\alpha(t))\dot{\alpha}(t) \text{ and } \partial_t^{Cl} \mathfrak{I}(t,z) = -2\,\mathrm{Sign}(z-\alpha(t))|\dot{\alpha}(t)|.$$

*Since $\mathcal{I}_\delta(t,\cdot)$ is smooth and strictly convex, the energetic solutions for $(\mathbb{R}, \mathcal{I}_\delta, \mathcal{D})$ are exactly the solutions of the doubly nonlinear equation (cf. [MiT04])*

$$0 \in \mathrm{Sign}(\dot{z}(t)) + H_\delta'(z(t)-\alpha(t)).$$

*For $\delta > 0$ the system is smooth, while for $\delta = 0$ we have $H_0(u) = 2|u|$ and set $\mathfrak{I}(t,z) = H_0(z-\alpha(t))$.*

*Consider the special case $\alpha(t) = t$ and $z_\delta(0) = 0$. If $\beta_\delta$ is the unique solution of $H_\delta'(\beta_\delta) = 1$, then the unique energetic solution is $z_\delta(t) = \max\{0, t-\beta_\delta\}$. Using $0 < \beta_\delta \to 0$ we find the limit solution $z(t) = t = \lim_{\delta\to 0} z_\delta(t)$. It is a generalized energetic solution in the sense of Definition 6.1 by using $p(t) = 1 \in [-2,2] = \partial_t^{Cl}\mathfrak{I}(t,t)$.*

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