Weierstraß-Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Local existence, uniqueness, and smooth dependence for nonsmooth quasilinear parabolic problems

Jens A. Griepentrog

Weierstraß-Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39, 10117 Berlin

Lutz Recke

Institut für Mathematik der Humboldt-Universität Berlin
Unter den Linden 6, 10099 Berlin

submitted: February 26, 2009

No. 1409
Berlin 2009

2000 *Mathematics Subject Classification.* 35K60, 35R05, 35K50, 35K90.

Key words and phrases. Sobolev–Morrey spaces, maximal regularity, Implicit Function Theorem, sets with Lipschitz boundary, mixed boundary conditions.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We prove local existence, uniqueness, Hölder regularity in space and time, and smooth dependence in Hölder spaces for a general class of quasilinear parabolic initial boundary value problems with nonsmooth data. As a result the gap between low smoothness of the data, which is typical for many applications, and high smoothness of the solutions, which is necessary for the applicability of differential calculus to abstract formulations of the initial boundary value problems, has been closed. The theory works for any space dimension, and the nonlinearities are allowed to be nonlocal and to have any growth. The main tools are new maximal regularity results [19, 20] in Sobolev–Morrey spaces for linear parabolic initial boundary value problems with nonsmooth data, linearization techniques and the Implicit Function Theorem.

1 Introduction

This paper concerns initial boundary value problems for quasilinear second order parabolic equations in divergence form with nonsmooth data and for weakly coupled systems of such equations. Here *nonsmooth data* means that the domain can be nonsmooth (but has to be a set with Lipschitz boundary), that the coefficients of the equations and the boundary conditions may be discontinuous with respect to the space and time variables (but have to be smooth with respect to the unknown function u), and that the boundary conditions can change type (mixed boundary conditions, where the Dirichlet and the Neumann boundary parts can touch). The coefficients may be local or nonlocal functions of u, they can have any growth with respect to u, and the space dimension can be arbitrary. Typical applications are transport processes of charged particles in semiconductor heterostructures, phase separation processes of nonlocally interacting particles, chemotactic aggregation in heterogeneous environments as well as optimal control by means of of quasilinear elliptic and parabolic PDEs with nonsmooth data.

The main results are Theorem 3.2 about regularity and smooth dependence and, following from that, Theorem 3.3 about local in time existence and uniqueness and

Theorem 3.5 about HÖLDER continuity of the first time derivative of the solution. Here regularity and smooth dependence means that the solutions are HÖLDER continuous in space and time and depend smoothly on the data in parabolic HÖLDER space norms over the space-time cylinder. So the door is open to apply the powerful theorems of differential calculus (principle of linearized stability, analytic bifurcation theory, existence and persistence of smooth invariant manifolds, enabling a reduction of the study of the long-time dynamics to finite dimensions) to those initial boundary value problems. In particular, Theorem 3.2 shows how to apply the classical NEWTON iteration procedure with quadratic convergence rate in parabolic HÖLDER space norms.

Remark that here in the introduction we formulate the results in the language of HÖLDER spaces, which is satisfactory for most of the applications. But it turns out that the proofs cannot be done by working in HÖLDER spaces or in SOBOLEV spaces because the linearized diffential operators do not have the maximal regularity property between such spaces in the case of general nonsmooth data and arbitrary space dimension.

We work in parabolic SOBOLEV–MORREY–CAMPANATO spaces. Those spaces for functions are known (but much less used than SOBOLEV spaces) since 40 years, but for functionals almost unknown and not used. Concerning the delicate questions about embedding theorems, traces on LIPSCHITZ hypersurfaces and behavior under LIPSCHITZ transformations and pointwise multiplication, which appear necessarily in the future analysis, there existed only a few results, and those mainly under unrealistic high smoothness assumptions on the data. In [19] a general theory was developed for parabolic SOBOLEV–MORREY–CAMPANATO spaces on domains with LIPSCHITZ boundary and LIPSCHITZ hypersurfaces as well for functions as for functionals. In [20] it was shown that a general class of linear second order parabolic differential operators has the maximal regularity property between such spaces. Now, in the present paper we show that these maximal regularity properties together with linearization techniques and the Implicit Function Theorem give local existence, uniqueness, regularity and smooth dependence for a general class of quasilinear parabolic initial boundary value problems with nonsmooth data.

Remark that the authors together with K. GRÖGER realized the same programme for quasilinear elliptic boundary value problems with nonsmooth data: Applications of differential calculus to nonlinear problems [23] via maximal regularity in SOBOLEV–MORREY–CAMPANATO spaces for linear problems [18] after investigation of the needed properties of the spaces [17, 22].

Let us close this introduction by some remarks concerning the so far existing literature about quasilinear parabolic initial boundary value problems.

As far as we know up to now there did not exist any results about smoothness (at least continuous differentiability) of the data-to-solution map for quasilinear parabolic initial boundary value problems with nonsmooth data.

What concerns local existence, uniqueness and continuous dependence for quasilinear parabolic initial boundary value problems, we learned a lot from H. AMANN's work [2, 3, 4, 5, 6]. There the main tool is maximal L^p regularity of the corresponding linear operators. The smoothness assumptions on the data are slightly, but, from the point of view of applications, essentially stronger than ours: The leading order coefficients of the elliptic differential operator have to be continuous in space and time, and the DIRICHLET and NEUMANN boundary parts in the mixed boundary conditions are not allowed to touch. On the other hand, H. AMANN's assumptions on the possibly nonlocal coefficient functions are weaker than ours: He includes time delay, we do not. Remark that [3, Theorem 4.1] gives GATEAUX differentialbility of the data-to-solution map on FRÉCHET spaces of coefficient functions, which is a first step to smoothness.

What concerns nonsmoothness of the data, the assumptions in [10, 24, 25] for local existence and uniqueness for quasilinear parabolic initial boundary value problems are as weak as ours. In particular, some domains, which are not LIPSCHITZ domains in the commonly used sense (like two crossing three-dimensional cuboids) are allowed as well as nonlinear ROBIN or NEUMANN boundary conditions. Further, in [10, 24, 25] as well as in our paper the concept of GRÖGER's regular sets, see [21], is used, which enables to handle mixed boundary value problems with touching DIRICHLET and NEUMANN boundary parts. In [10, 24, 25] the assumptions concerning the space dimension (they suppose $n \leq 3$) and the allowed discontinuities in the leading order coefficients are slightly more restrictive than ours (we suppose only L^∞ in space and time), but general enough for most applications.

The idea, to use maximal regularity properties together with linearization techniques and the Implicit Function Theorem in order to get local existence, uniqueness, solution regularity and smooth dependence for quasilinear parabolic problems, is known in the case of problems with sufficiently smooth data, see, for instance, [8, 11, 13, 26].

What concerns strongly coupled systems, it is known that HÖLDER regularity of the solutions cannot be expected in the case $n \geq 3$, in general. Similarly, it turns out that one cannot expect smooth dependence in the case of nonsmooth data, in general, if the equations contain terms which are not affine with respect to the spatial gradient of the solution. Therefore we consider only equations and boundary conditions, which are affine with respect to the spatial gradient. For equations, which are nonlinear with respect to the spatial gradient, even the question of uniqueness is much more difficult, see, for instance, [1, 14, 15, 16].

2 Notation and setting

Let us introduce some notation. Throughout this text we assume $S = (t_0, t_1)$ to be a bounded open interval in $\mathbb{R}$. For $r > 0$ we define the set of subintervals

$$\mathcal{S}_r = \big\{ S \cap (t - r^2, t) : t \in S \big\}.$$

The symbol $|\ |$ is used for both the absolute value and the maximum norm in $\mathbb{R}^n$. We denote by $Q_r(x) = \big\{\xi \in \mathbb{R}^n : |\xi - x| < r\big\}$ the open cube with center $x \in \mathbb{R}^n$ and radius $r > 0$. For subsets Y of $\mathbb{R}^n$ we write Y°, $\overline{Y}$ and ∂Y for the topological interior, the closure, and the boundary of Y, respectively. For $r > 0$ and subsets $Y \subset \mathbb{R}^n$ we use the corresponding calligraphic letter to introduce the set

$$\mathcal{Y}_r = \big\{ Y \cap Q_r(y) : y \in Y \big\}$$

of intersections. Let λ^n be the n-dimensional LEBESGUE measure on the σ-algebra of LEBESGUE measurable subsets of $\mathbb{R}^n$.

2.1 Function spaces and regular sets

Let $X \subset \mathbb{R}^n$ be some bounded open set. The following definition goes back to CAMPANATO [7] and DA PRATO [9]: For $\omega \in [0, n+2]$ the MORREY space $L_2^\omega(S; L^2(X))$ consists of all $u \in L^2(S; L^2(X))$ such that

$$[u]^2_{L_2^\omega(S;L^2(X))} = \sup_{r>0} \sup_{(I,Y)\in \mathcal{S}_r \times \mathcal{X}_r} r^{-\omega} \int_I \int_Y |u(s)|^2 \, d\lambda^n \, ds$$

remains finite. The norm of $u \in L_2^\omega(S; L^2(X))$ is defined by

$$\|u\|^2_{L_2^\omega(S;L^2(X))} = \|u\|^2_{L^2(S;L^2(X))} + [u]^2_{L_2^\omega(S;L^2(X))}.$$

Let $H_0^1(X) \subset H \subset H^1(X)$ be some closed subspace equipped with the usual scalar product of $H^1(X)$. For $\omega \in [0, n+2]$ we introduce the SOBOLEV–MORREY space

$$L_2^\omega(S; H) = \big\{ u \in L^2(S; H) : u \in L_2^\omega(S; L^2(X)),\, |\nabla u| \in L_2^\omega(S; L^2(X)) \big\},$$

and we define the norm of $u \in L_2^\omega(S; H)$ by

$$\|u\|^2_{L_2^\omega(S;H)} = \|u\|^2_{L_2^\omega(S;L^2(X))} + \||\nabla u|\|^2_{L_2^\omega(S;L^2(X))}.$$

Note that the spaces $L_2^\omega(S; L^2(X))$ are usually denoted by $L^{2,\omega}(S \times X)$. Apart from these, later on we use further MORREY-type function spaces. Hence, we have decided to use a different but integrated naming scheme. The set $L^\infty(S \times X)$ of bounded measurable functions is a space of multipliers for $L_2^\omega(S; L^2(X))$.

Analogously, we consider function spaces on LIPSCHITZ hypersurfaces in $\mathbb{R}^n$. Here, a subset M of $\mathbb{R}^n$ is called LIPSCHITZ *hypersurface* in $\mathbb{R}^n$ if for each point $x \in M$ there exist a neighborhood U of x and a LIPSCHITZ transformation Φ from U onto the cube $Q_1(0)$ such that $\Phi[U \cap M] = \{y \in \mathbb{R}^n : |y| < 1,\, y_n = 0\}$ and $\Phi(x) = 0$.

Let M be a compact LIPSCHITZ hypersurface in $\mathbb{R}^n$. By λ_M we denote the $(n-1)$-dimensional LEBESGUE measure on the σ-algebra $\mathfrak{L}_M$ of LEBESGUE measurable subsets of M, see [12]. For $\varkappa \in [0, n+1]$ and relatively open subsets K of M we define the MORREY space $L_2^{\varkappa}(S; L^2(K))$ as the set of all $u \in L^2(S; L^2(K))$ such that

$$[u]^2_{L_2^{\varkappa}(S;L^2(K))} = \sup_{r>0} \sup_{(I,\Gamma)\in\mathcal{S}_r\times\mathcal{K}_r} r^{-\varkappa} \int_I \int_\Gamma |u(s)|^2 \, d\lambda_M \, ds$$

remains finite, and we introduce the norm of $u \in L_2^{\varkappa}(S; L^2(K))$ by

$$\|u\|^2_{L_2^{\varkappa}(S;L^2(K))} = \|u\|^2_{L^2(S;L^2(K))} + [u]^2_{L_2^{\varkappa}(S;L^2(K))}.$$

The set $L^\infty(S \times K)$ is a space of multipliers for $L_2^{\varkappa}(S; L^2(K))$.

For our investigations on global regularity we use a terminology of regular sets $G \subset \mathbb{R}^n$, which is equivalent to the version introduced by K. GRÖGER, see [21]. Being the natural generalization of sets with LIPSCHITZ boundary it allows the proper functional analytic description of elliptic and parabolic problems with mixed boundary conditions in nonsmooth domains. For $x \in \mathbb{R}^n$ and $r > 0$ we introduce the halfcubes

$$\begin{aligned} Q_r^-(x) &= \{\xi \in \mathbb{R}^n : |\xi - x| < r,\, \xi_n - x_n < 0\}, \\ Q_r^+(x) &= \{\xi \in \mathbb{R}^n : |\xi - x| < r,\, \xi_n - x_n \le 0\}, \\ Q_r^\pm(x) &= \{\xi \in Q_r^+(x) : \xi_1 - x_1 > 0 \text{ or } \xi_n - x_n < 0\}. \end{aligned}$$

A bounded set $G \subset \mathbb{R}^n$ is called *regular* if for each $x \in \partial G$ we find some neighborhood U of x in $\mathbb{R}^n$ and a LIPSCHITZ transformation Φ from U onto $Q_1(0)$ such that $T[U \cap G] \in \{Q_1^-(0), Q_1^+(0), Q_1^\pm(0)\}$ and $\Phi(x) = 0$.

Regular sets $G = X \cup \Gamma$ are to be understood as the union of some open set $X \subset \mathbb{R}^n$ with LIPSCHITZ boundary and some relatively open NEUMANN part Γ of the boundary ∂G. From now on we keep in mind this notation.

We define function spaces associated with relatively open subsets Y of regular sets $G \subset \mathbb{R}^n$. By $H_0^1(Y)$ we denote the closure of

$$C_0^\infty(Y) = \{u|Y^\circ : u \in C_0^\infty(\mathbb{R}^n),\ \operatorname{supp}(u) \cap (\overline{Y} \setminus Y) = \varnothing\}$$

in the space $H^1(Y^\circ)$, and we write $H^{-1}(Y)$ for the dual space of $H_0^1(Y)$. Let $I \subset \mathbb{R}$ be an open subinterval of S. If $Z_G : H_0^1(Y) \to H_0^1(G)$ is the zero extension map, then we define $\mathcal{Z}_{S,G} : L^2(I; H^1_0(Y)) \to L^2(S; H^1_0(G))$ by

$$(\mathcal{Z}_{S,G}u)(s) = \begin{cases} Z_G u(s) & \text{if } s \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } u \in L^2(I; H^1_0(Y)).$$

Note that $\mathcal{Z}_{S,Y}$ is a linear isometry from $L^2(I; H^1_0(Y))$ into $L^2(S; H^1_0(G))$.

In the same spirit as the well-established MORREY spaces of functions, in [19] we have constructed a new scale of SOBOLEV–MORREY spaces of functionals as subspaces of $L^2(S; H^{-1}(G))$.

To localize a functional $f \in L^2(S; H^{-1}(G))$ we define the assignment $f \mapsto \mathcal{L}_{I,Y} f$ from $L^2(S; H^{-1}(G))$ into $L^2(I; H^{-1}(Y))$ as the adjoint operator to the zero extension map $\mathcal{Z}_{S,G} : L^2(I; H^1_0(Y)) \to L^2(S; H^1_0(G))$:

$$\langle \mathcal{L}_{I,Y} f, w \rangle_{L^2(I;H^1_0(Y))} = \langle f, \mathcal{Z}_{S,G} w \rangle_{L^2(S;H^1_0(G))} \quad \text{for } w \in L^2(I; H^1_0(Y)).$$

As usual, we denote by $\langle\ ,\ \rangle$ and $(\ |\ )$ dual pairings and scalar products, respectively. Using the isometric property of $\mathcal{Z}_{S,G}$, we get

$$\|\mathcal{L}_{I,Y} f\|_{L^2(I;H^{-1}(Y))} \le \|f\|_{L^2(S;H^{-1}(G))} \quad \text{for all } f \in L^2(S; H^{-1}(G)).$$

For $\omega \in [0, n+2]$ we define the SOBOLEV–MORREY space $L^\omega_2(S; H^{-1}(G))$ as the set of all elements $f \in L^2(S; H^{-1}(G))$ for which

$$[f]^2_{L^\omega_2(S;H^{-1}(G))} = \sup_{r>0} \sup_{(I,Y)\in \mathcal{S}_r \times \mathcal{G}_r} r^{-\omega} \int_I \|(\mathcal{L}_{I,Y} f)(s)\|^2_{H^{-1}(Y)}\, ds$$

has a finite value. We introduce the norm of $f \in L^\omega_2(S; H^{-1}(G))$ by

$$\|f\|^2_{L^\omega_2(S;H^{-1}(G))} = \|f\|^2_{L^2(S;H^{-1}(G))} + [f]^2_{L^\omega_2(S;H^{-1}(G))}.$$

The assignment $(g, g_0, g_\Gamma) \mapsto \Psi(g, g_0, g_\Gamma)$ defined by

$$\begin{aligned} \big\langle \Psi(g, g_0, g_\Gamma), \varphi \big\rangle &= \int_S \int_X g(s) \cdot \nabla\varphi(s)\, d\lambda^n\, ds \\ &\quad + \int_S \int_X g_0(s)\, \varphi(s)\, d\lambda^n\, ds + \int_S \int_\Gamma g_\Gamma(s)\, \varphi(s)\, d\lambda_\Gamma\, ds \end{aligned} \tag{2.1}$$

for $\varphi \in L^2(S; H^1_0(G))$, generates a linear continuous operator

$$\Psi : [L^\omega_2(S; L^2(X))]^n \times L^{\omega-2}_2(S; L^2(X)) \times L^{\omega-1}_2(S; L^2(\Gamma)) \to L^\omega_2(S; H^{-1}(G)), \tag{2.2}$$

and its norm depends on n and G, only, see [19, Theorem 5.6].

Based upon the preceeding definitions, in [19] we have constructed new function classes suitable for the regularity theory of second order parabolic boundary value problems with nonsmooth data, see [20]. Here, we present a version being adequate for systems of equations with $m \in \mathbb{N}$ unknowns. In particular, for the modelling of instationary drift-diffusion problems we are interested in dealing with nonsmooth capacity-like coefficients $a^1, \ldots, a^m \in L^\infty(X)$, which are supposed to be δ-definite with respect to X and $\delta \in (0,1]$, that means, we assume that

$$\delta \le \operatorname*{ess\,inf}_{x\in X} a^\alpha(x), \quad \operatorname*{ess\,sup}_{x\in X} a^\alpha(x) \le \tfrac{1}{\delta} \quad \text{for all } \alpha \in \{1,\ldots,m\}. \tag{2.3}$$

We consider the linear continuous operator $\mathcal{E} = \big(\mathcal{E}^1, \ldots, \mathcal{E}^m\big)$ from $[L^2(S;H^1(X))]^m$ into $[L^2(S;H^{-1}(G))]^m$ being defined componentwise by

$$\langle \mathcal{E}^\alpha v, \varphi\rangle = \int_S \int_X a^\alpha v(s)\varphi(s)\, d\lambda^n\, ds, \quad \alpha \in \{1,\ldots,m\}, \tag{2.4}$$

for $\varphi \in L^2(S;H^1_0(G))$. For $\omega \in [0, n+2]$ and every $\alpha \in \{1,\ldots,m\}$ we introduce the Sobolev–Morrey space $W^\omega_{\mathcal{E}^\alpha}(S;H^1(X))$ as the set of all functions $v \in L^\omega_2(S;H^1(X))$, such that the weak time derivative $(\mathcal{E}^\alpha v)'$ of $\mathcal{E}^\alpha v \in L^2(S;H^{-1}(G))$ exists and belongs to $L^\omega_2(S;H^{-1}(G))$. We define the norm of $v \in W^\omega_{\mathcal{E}^\alpha}(S;H^1(X))$ by

$$\|v\|^2_{W^\omega_{\mathcal{E}^\alpha}(S;H^1(X))} = \|v\|^2_{L^\omega_2(S;H^1(X))} + \|(\mathcal{E}v)'\|^2_{L^\omega_2(S;H^{-1}(G))},$$

and consider the following closed subspaces:

$$\begin{aligned} W^\omega_{\mathcal{E}^\alpha}(S;H^1_0(G)) &= \big\{v \in W^\omega_{\mathcal{E}^\alpha}(S;H^1(X)) : v \in L^\omega_2(S;H^1_0(G))\big\}, \\ W^\omega_{0\mathcal{E}^\alpha}(S;H^1_0(G)) &= \big\{v \in W^\omega_{\mathcal{E}^\alpha}(S;H^1_0(G)) : v(t_0) = 0\big\}. \end{aligned}$$

Finally, as a natural generalization, we set

$$\begin{aligned} W^\omega_{\mathcal{E}}(S;H^1(X)) &= W^\omega_{\mathcal{E}^1}(S;H^1(X)) \times \cdots \times W^\omega_{\mathcal{E}^m}(S;H^1(X)), \\ W^\omega_{\mathcal{E}}(S;H^1_0(G)) &= W^\omega_{\mathcal{E}^1}(S;H^1_0(G)) \times \cdots \times W^\omega_{\mathcal{E}^m}(S;H^1_0(G)), \\ W^\omega_{0\mathcal{E}}(S;H^1_0(G)) &= W^\omega_{0\mathcal{E}^1}(S;H^1_0(G)) \times \cdots \times W^\omega_{0\mathcal{E}^m}(S;H^1_0(G)), \end{aligned}$$

and equip these spaces with the maximum norm of the components, respectively. For $\omega = 0$ we drop superscripted indices.

Note that for $\omega \in (n, n+2]$ and $\beta = (\omega - n)/2$ the space $W^\omega_{\mathcal{E}^\alpha}(S;H^1(X))$ is continuously embedded into the space $C^{0,\beta/2}(\overline{S};C(\overline{X})) \cap C(\overline{S};C^{0,\beta}(\overline{X}))$ of functions, which are Hölder continuous is space and time. This embedding is completely continuous, whenever $0 < \beta < (\omega - n)/2$.

2.2 Formulation of the problem

Let $G = X \cup \Gamma$ be a regular set, U an open subset in $[C(\overline{S};C(\overline{X}))]^m$, Λ a Banach space, and V an open subset in Λ. We look for solutions

$$(u,\lambda) = \big(u^1, \ldots, u^m, \lambda\big) \in \big(U \cap W_{\mathcal{E}}(S;H^1(X))\big) \times V$$

of weakly coupled systems of quasilinear operator equations

$$(\mathcal{E}^\alpha u^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u,\lambda), u^\alpha) = \mathcal{F}^\alpha(u,\lambda), \quad \alpha \in \{1,\dots,m\}, \tag{2.5}$$

of order $m \in \mathbb{N}$, where $\lambda \in V$ plays the role of a control parameter, which includes, for instance, inhomogeneities of the problem. The linear continuous operator

$$\mathcal{E} = (\mathcal{E}^1,\dots,\mathcal{E}^m) : [L^2(S;H^1(X))]^m \to [L^2(S;H^{-1}(G))]^m$$

was already defined in (2.4) via δ-definite leading order coefficients $a^1,\dots,a^m \in L^\infty(X)$, see (2.3). The bilinear continuous map

$$\mathcal{B} : L^\infty(S \times X;\mathbb{R}^{n\times n}) \times L^2(S;H^1(X)) \to L^2(S;H^{-1}(G))$$

is given by

$$\langle \mathcal{B}(A,v),\varphi\rangle = \int_S \int_X A(s)\nabla v(s)\cdot\nabla\varphi(s)\,d\lambda^n\,ds \tag{2.6}$$

for $\varphi \in L^2(S;H^1_0(G))$.

Concerning the nonlinearities we suppose that

$$\mathcal{A}^\alpha \in C^1\big(U \times V; L^\infty(S\times X;\mathbb{R}^{n\times n})\big), \tag{2.7}$$
$$\mathcal{F}^\alpha \in C^1\big(U \times V; L^{\omega_0}_2(S;H^{-1}(G))\big) \tag{2.8}$$

are Volterra operators with respect to u, where $\omega_0 > n$ is a common Morrey exponent for all $\alpha \in \{1,\dots,m\}$. Note, that for all $(u,\lambda) \in U \times V$ the linear continuous operators

$$\tfrac{\partial\mathcal{A}^\alpha}{\partial u}(u,\lambda) : [C(\overline{S};C(\overline{X}))]^m \to L^\infty(S\times X;\mathbb{R}^{n\times n})), \tag{2.9}$$
$$\tfrac{\partial\mathcal{F}^\alpha}{\partial u}(u,\lambda) : [C(\overline{S};C(\overline{X}))]^m \to L^{\omega_0}_2(S;H^{-1}(G)), \tag{2.10}$$

have the Volterra property, too.

2.3 Homogenization and linearization of the problem

Let W be an open subset in $W^{\omega_0}_{\mathcal{E}}(S;H^1(X))$ containing regular inhomogeneities we are interested in, and let U_h be a neighborhood of 0 in $[C(\overline{S};C(\overline{X}))]^m$ such that $\{u_h + w : u_h \in U_h,\, w \in W\} \subset U$. We define nonlinear Volterra operators

$$\mathcal{A}^\alpha_h \in C^1\big(U_h \times W \times V; L^\infty(S\times X;\mathbb{R}^{n\times n})\big),$$
$$\mathcal{F}^\alpha_h \in C^1\big(U_h \times W \times V; L^{\omega_0}_2(S;H^{-1}(G))\big)$$

by setting

$$\mathcal{A}_h^\alpha(u_h, w, \lambda) = \mathcal{A}^\alpha(u_h + w, \lambda),$$
$$\mathcal{F}_h^\alpha(u_h, w, \lambda) = \mathcal{F}^\alpha(u_h + w, \lambda) - \mathcal{B}(\mathcal{A}^\alpha(u_h + w, \lambda), w^\alpha) - (\mathcal{E}^\alpha w^\alpha)',$$

for $(u_h, w, \lambda) \in U_h \times W \times V$ and $\alpha \in \{1, \dots, m\}$. These have the same mapping properties as $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$, respectively, where the old control parameter $\lambda \in V$ has to be replaced by the new control parameter $(w, \lambda) \in W \times V$. Moreover, if

$$(u_h, w, \lambda) \in \big(U_h \cap W_{0\mathcal{E}}(S; H_0^1(G))\big) \times W \times V$$

is a solution to the system

$$(\mathcal{E}^\alpha u_h^\alpha)' + \mathcal{B}(\mathcal{A}_h^\alpha(u_h, w, \lambda), u_h^\alpha) = \mathcal{F}_h^\alpha(u_h, w, \lambda), \quad \alpha \in \{1, \dots, m\},$$

then the pair $(u, \lambda) = (u_h + w, \lambda) \in \big(U \cap W_{\mathcal{E}}(S; H^1(X))\big) \times V$ solves problem (2.5). Hence, for given inhomogeneities $w \in W$, we can restrict ourselves to look for homogeneous solutions $(u, \lambda) \in \big(U \cap W_{0\mathcal{E}}(S; H_0^1(G))\big) \times V$ of problem (2.5).

Besides of the nonlinear operator equation (2.5) we also consider solutions $v \in U \cap W_{0\mathcal{E}}(S; H_0^1(G))$ of its linearization

$$(\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u_0, \lambda_0), v^\alpha) = \tfrac{\partial \mathcal{F}^\alpha}{\partial u}(u_0, \lambda_0)\, v - \mathcal{B}\big(\tfrac{\partial \mathcal{A}^\alpha}{\partial u}(u_0, \lambda_0)\, v, u_0^\alpha\big), \quad \alpha \in \{1, \dots, m\}, \tag{2.11}$$

at $(u_0, \lambda_0) \in \big(U \cap W_{0\mathcal{E}}(S; H_0^1(G))\big) \times V$. In addition to that, we investigate the linear operator equations, determining the sequence of Newton iterations $u_{k+1} \in U \cap W_{0\mathcal{E}}(S; H_0^1(G))$ for given u_k by

$$(\mathcal{E}^\alpha u_{k+1}^\alpha)' + \mathcal{B}\big(\mathcal{A}^\alpha(u_k, \lambda_0), u_{k+1}^\alpha\big) + \mathcal{B}\big(\tfrac{\partial \mathcal{A}^\alpha}{\partial u}(u_k, \lambda_0)(u_{k+1} - u_k), u_k^\alpha\big) = \mathcal{F}^\alpha(u_k, \lambda_0) + \tfrac{\partial \mathcal{F}^\alpha}{\partial u}(u_k, \lambda_0)(u_{k+1} - u_k), \quad \alpha \in \{1, \dots, m\}. \tag{2.12}$$

3 Abstract quasilinear parabolic problems

The following maximal regularity result for linear parabolic boundary value problems in Sobolev–Morrey spaces will serve as the main tool of our considerations. It generalizes the results [20, Theorem 6.8 and Theorem 7.5] to weakly coupled systems of linear parabolic equations including nonlocal operators.

Analogously to the notion of δ-definiteness with respect to X and $\delta \in (0, 1]$ of scalar coefficient functions $a \in L^\infty(X)$, see (2.3), a matrix valued coefficient function $A \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ is called δ-definite with respect to S, X, and $\delta \in (0, 1]$ if

$$\delta\, \|\xi\|^2 \le \operatorname*{ess\,inf}_{s\in S,\, x\in X} A(s, x)\xi \cdot \xi, \quad \operatorname*{ess\,sup}_{s\in S,\, x\in X} A(s, x)\xi \cdot \xi \le \tfrac{1}{\delta}\|\xi\|^2, \tag{3.1}$$

holds true for all $\xi \in \mathbb{R}^n$.

Theorem 3.1. *Suppose that $S = (t_0, t_\ell)$ is an open interval. Further, assume that*

$$\mathcal{N}^\alpha : [C(\overline{S}; C(\overline{X}))]^m \to L_2^{\omega_0}(S; H^{-1}(G)), \quad \alpha \in \{1, \ldots, m\}, \tag{3.2}$$

are linear continuous VOLTERRA *operators having the* MORREY *exponent $\omega_0 > n$ in common. Further, assume that for all $\alpha \in \{1, \ldots, m\}$ the leading coefficients $a^\alpha \in L^\infty(X)$ and $A^\alpha \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ are δ-definite with respect to S, X, and $\delta \in (0, 1]$. Then there exists $\bar\omega \in (n, \omega_0]$ such that for every $\omega \in (n, \bar\omega)$ the assignment*

$$v \mapsto \big((\mathcal{E}^1 v^1)' + \mathcal{B}(A^1, v^1) + \mathcal{N}^1 v, \ldots, (\mathcal{E}^m v^m)' + \mathcal{B}(A^m, v^m) + \mathcal{N}^m v\big) \tag{3.3}$$

generates a linear isomorphism from $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ onto $[L^\omega_2(S; H^{-1}(G))]^m$.

Proof. 1. In view of the maximal regularity result in [20, Theorem 6.8], there exists some exponent $\bar\omega = \bar\omega(\delta, G) \in (n, \omega_0]$ such the linear parabolic operator

$$v \mapsto \big((\mathcal{E}^1 v^1)' + \mathcal{B}(A^1, v^1), \ldots, (\mathcal{E}^m v^m)' + \mathcal{B}(A^m, v^m)\big)$$

is an isomorphism from $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ onto $[L^\omega_2(S; H^{-1}(G))]^m$ for every $\omega \in (n, \bar\omega)$. Due to the completely continuous embedding of $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ into $[C(\overline{S}; C(\overline{X}))]^m$, see [19, Theorem 6.9], the operator $\mathcal{N}^\alpha$ maps $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ completely continuous into $L^\omega_2(S; H^{-1}(G))$ for $\alpha \in \{1, \ldots, m\}$. Hence, the map defined in (3.3) is a FREDHOLM operator of index zero from $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ into $[L^\omega_2(S; H^{-1}(G))]^m$. For the assertion of the theorem, it suffices to prove the injectivity of this map.

2. Suppose that $v \in W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ is a solution to the system of homogeneous initial boundary value problems

$$(\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(A^\alpha, v^\alpha) + \mathcal{N}^\alpha v = 0, \quad \alpha \in \{1, \ldots, m\}. \tag{3.4}$$

For fixed $t_1 \in S$ we consider the subinterval $S_1 = (t_0, t_1)$ of S, the restriction $v_1 \in W^\omega_{0\mathcal{E}}(S_1; H^1_0(G))$ of v to S_1 and the restrictions $(\mathcal{N}^\alpha v)|S_1 \in L^\omega_2(S_1; H^{-1}(G))$ of $\mathcal{N}^\alpha v \in L^\omega_2(S; H^{-1}(G))$ to S_1. Due to [20, Remark 6.2] we get estimates

$$\|v_1\|_{W^\omega_{0\mathcal{E}}(S_1; H^1_0(G))} \le c_1 \sup_{1\le\alpha\le m} \|(\mathcal{N}^\alpha v)|S_1\|_{L^\omega_2(S_1; H^{-1}(G))}, \tag{3.5}$$

where the constant $c_1 > 0$ may depend on S but not on t_1. To estimate the right hand side of (3.5) we arbitrarily choose $t_1^* \in S$ with $t_1^* > t_1$ and some cut-off function $\vartheta \in C^\infty(\mathbb{R})$ with

$$0 \le \vartheta \le 1, \quad \vartheta(s) = 1 \quad \text{for all } s \le t_1, \quad \vartheta(s) = 0 \quad \text{for all } t \ge t_1^*.$$

The VOLTERRA property of the maps $\mathcal{N}^\alpha : [C(\overline{S}; C(\overline{X}))]^m \to L^\omega_2(S; H^{-1}(G))$ and the definition of the norm in $L^\omega_2(S; H^{-1}(G))$ for all $\alpha \in \{1, \ldots, m\}$ yield that

$$\begin{aligned}\|(\mathcal{N}^\alpha v)|S_1\|_{L^\omega_2(S_1; H^{-1}(G))} &= \|(\mathcal{N}^\alpha(\vartheta v))|S_1\|_{L^\omega_2(S_1; H^{-1}(G))} \\ &\le \|\mathcal{N}^\alpha(\vartheta v)\|_{L^\omega_2(S; H^{-1}(G))} \le c_2 \|\vartheta v\|_{[C(\overline{S}; C(\overline{X}))]^m},\end{aligned} \tag{3.6}$$

where

$$c_2 = \sup\left\{\|\mathcal{N}^\alpha w\|_{L_2^\omega(S;H^{-1}(G))} : \|w\|_{[C(\overline{S};C(\overline{X}))]^m} \le 1,\, \alpha \in \{1,\dots,m\}\right\}$$

is the maximum of the operator norms of $\mathcal{N}^1,\dots,\mathcal{N}^m$. In view of the continuity of the embedding from $W^\omega_{0\mathcal{E}}(S;H^1_0(G))$ into the HÖLDER space $[C^{0,\beta}(\overline{S};C(\overline{X}))]^m$ for $\beta = (\omega-n)/4$, see [19, Theorem 3.4, Theorem 6.8], for all $s \in [t_1,t_1^*]$ we get

$$\begin{aligned}\|v(s)\|_{[C(\overline{X})]^m} &\le \|v(s)-v(t_1)\|_{[C(\overline{X})]^m} + \|v(t_1)\|_{[C(\overline{X})]^m}\\ &\le (t_1^*-t_1)^\beta \|v\|_{[C^{0,\beta}(\overline{S};C(\overline{X}))]^m} + \|v_1\|_{[C(\overline{S_1};C(\overline{X}))]^m},\end{aligned}$$

and, hence,

$$\|\vartheta v\|_{[C(\overline{S};C(\overline{X}))]^m} \le (t_1^*-t_1)^\beta \|v\|_{[C^{0,\beta}(\overline{S};C(\overline{X}))]^m} + \|v_1\|_{[C(\overline{S_1};C(\overline{X}))]^m}.$$

Together with (3.6), for all $\alpha \in \{1,\dots,m\}$ this leads to

$$\|(\mathcal{N}^\alpha v)|S_1\|_{L_2^\omega(S_1;H^{-1}(G))} \le c_2 (t_1^*-t_1)^\beta \|v\|_{[C^{0,\beta}(\overline{S};C(\overline{X}))]^m} + c_2\|v_1\|_{[C(\overline{S_1};C(\overline{X}))]^m}.$$

Since $t_1^* \in S$, $t_1^* > t_1$ was arbitrarily fixed at the beginning, we arrive at

$$\|(\mathcal{N}^\alpha v)|S_1\|_{L_2^\omega(S_1;H^{-1}(G))} \le c_2\|v_1\|_{[C(\overline{S_1};C(\overline{X}))]^m} \quad \text{for all } \alpha \in \{1,\dots,m\}. \tag{3.7}$$

To estimate the left hand side of (3.5) we consider the shifted interval $S_0 = (t_1+t_0-t_\ell, t_1)$ which contains S_1 and has the same length than S, and we define the zero extension $v_0 \in W^\omega_{0\mathcal{E}}(S_0;H^1_0(G))$ of v_1 to S_0 by

$$v_0(s) = \begin{cases} v(s) & \text{if } s \in S_1,\\ 0 & \text{if } s \in S_0 \setminus S_1.\end{cases}$$

Using the continuity of the embedding from $W^\omega_{0\mathcal{E}}(S_0;H^1_0(G))$ into the HÖLDER space $[C^{0,\beta}(\overline{S_0};C(\overline{X}))]^m$ for $\beta = (\omega-n)/4$, and the definition of the norms in the corresponding MORREY and HÖLDER spaces, the above construction yields

$$\|v_1\|_{[C^{0,\beta}(\overline{S_1};C(\overline{X}))]^m} \le \|v_0\|_{[C^{0,\beta}(\overline{S_0};C(\overline{X}))]^m} \le c_3\|v_0\|_{W^\omega_{0\mathcal{E}}(S_0;H^1_0(G))} \le c_3\|v_1\|_{W^\omega_{0\mathcal{E}}(S_1;H^1_0(G))},$$

where the constant $c_3 > 0$ may depend on S but not on t_1. Together with (3.5) and (3.7) this leads to the key estimate

$$\|v_1\|_{[C^{0,\beta}(\overline{S_1};C(\overline{X}))]^m} \le c_4\|v_1\|_{[C(\overline{S_1};C(\overline{X}))]^m}, \tag{3.8}$$

where the constant $c_4 = c_1c_2c_3 \ge 0$ does not depend on t_1.

3. Because $t_1 \in S$ was arbitrarily fixed at the beginning we may choose

$$t_k = t_0 + \tfrac{k}{\ell}(t_\ell - t_0) \quad \text{for } k \in \{1, \dots, \ell\},$$

where $\ell \in \mathbb{N}$, $\ell > 1$ is large enough to satisfy the condition

$$2c_4(t_\ell - t_0)^\beta < \ell^\beta. \tag{3.9}$$

Furthermore, for $k \in \{1, \dots, \ell\}$ we introduce the intervals $S_k = (t_{k-1}, t_k)$ and the restrictions $v_k \in W^\omega_{0\mathcal{E}}(S_k; H^1_0(G))$ of v to S_k.

We prove that for every $k \in \{1, \dots, \ell - 1\}$ from $v(t_{k-1}) = 0$ it follows that $v(s) = 0$ for all $s \in \overline{S_k}$. To do so, we proceed by induction: Starting from $k = 1$ and using (3.8), condition (3.9) ensures that for all $s \in \overline{S_1}$ we have

$$\|v(s) - v(t_0)\|_{[C(\overline{X})]^m} \le (s - t_0)^\beta \|v_1\|_{[C^{0,\beta}(\overline{S_1}; C(\overline{X}))]^m} \le \tfrac{1}{2}\|v_1\|_{[C(\overline{S_1}; C(\overline{X}))]^m}.$$

Since $v(t_0) = 0$ this leads to $v(s) = 0$ for all $s \in \overline{S_1}$.

Assuming that $v(t_{k-1}) = 0$ holds true for some $k \in \{1, \dots, \ell - 1\}$, we apply (3.8) and (3.9) to $v_k \in W^\omega_{0\mathcal{E}}(S_k; H^1_0(G))$ to get

$$\|v(s) - v(t_{k-1})\|_{[C(\overline{X})]^m} \le (s - t_{k-1})^\beta \|v_k\|_{[C^{0,\beta}(\overline{S_k}; C(\overline{X}))]^m} \le \tfrac{1}{2}\|v_k\|_{[C(\overline{S_k}; C(\overline{X}))]^m}$$

for all $s \in \overline{S_k}$. Therefore, $v(t_{k-1}) = 0$ yields $v(s) = 0$ for all $s \in \overline{S_k}$.

Hence, we have proved, that $v = 0$ is the unique solution of the homogeneous problem (3.4) in the space $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$. Due to Step 1, the proof is finished. □

3.1 Regularity and smooth dependence

We continue our considerations with the existence, uniqueness and regularity of solutions to the nonlinear problem (2.5) in the neighborhood of a known solution $(u_0, \lambda_0) \in \big(U \cap W_{0\mathcal{E}}(S; H^1_0(G))\big) \times V$. Moreover, we prove that the solution u depends smoothly on the parameters λ of the problem.

Theorem 3.2. *Let $(u_0, \lambda_0) \in \big(U \cap W_{0\mathcal{E}}(S; H^1_0(G))\big) \times V$ be a solution to (2.5) and suppose that there exists some constant $\varepsilon \in (0, 1]$ such that $a^\alpha \in L^\infty(X)$ and $\mathcal{A}^\alpha(u_0, \lambda_0) \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ are ε-definite with respect to S and X for all $\alpha \in \{1, \dots, m\}$. Then we find a parameter $\omega \in (n, \omega_0]$ and a neighborhood U_0 of u_0 in $[C(\overline{S}; C(\overline{X}))]^m$ with $U_0 \subset U$ such that the following holds true:*

1. There exists a neighborhood V_0 of λ_0 in Λ with $V_0 \subset V$ and a solution map $\Phi \in C^1\big(V_0; W^\omega_{0\mathcal{E}}(S; H^1_0(G))\big)$ such that $(u, \lambda) \in U_0 \times V_0$ is a solution to (2.5) if and only if $u = \Phi(\lambda)$. In particular, for each solution $(u, \lambda) \in U_0 \times V_0$ to (2.5) we get $u \in W^\omega_{0\mathcal{E}}(S; H^1_0(G))$.

2. If for all $\alpha \in \{1, \dots, m\}$ the maps

$$u \in U \mapsto \tfrac{\partial \mathcal{A}^\alpha}{\partial u}(u, \lambda_0) \in \mathscr{L}\big([C(\overline{S}; C(\overline{X}))]^m; L^\infty(S \times X; \mathbb{R}^{n\times n})\big),$$
$$u \in U \mapsto \tfrac{\partial \mathcal{F}^\alpha}{\partial u}(u, \lambda_0) \in \mathscr{L}\big([C(\overline{S}; C(\overline{X}))]^m; L_2^{\omega_0}(S; H^{-1}(G))\big),$$

are locally LIPSCHITZ *continuous, then for each $u_1 \in U_0 \cap W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ the equations* (2.12) *define a sequence of* NEWTON *iterations $u_k \in U_0$ for $k \in \mathbb{N}$, $k \geq 2$, which converges to u_0 in $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ for $k \to \infty$.*

Proof. 1. Let us prove the first assertion. Because of (3.1) there exists some $\delta \in (0, 1]$ such that $a^\alpha \in L^\infty(X)$ and $\mathcal{A}^\alpha(u, \lambda) \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ are δ-definite with respect to S and X for all u, which are close to u_0 in $[C(\overline{S}; C(\overline{X}))]^m$, all λ, which are close to λ_0 in Λ, and all $\alpha \in \{1, \dots, m\}$. Hence, Theorem 3.1 yields that there exist an exponent $\omega \in (n, \omega_0]$ and neighborhoods U_0 of u_0 in $[C(\overline{S}; C(\overline{X}))]^m$ with $U_0 \subset U$ and V_0 of λ_0 in Λ with $V_0 \subset V$ such that for all solutions $(u, \lambda) \in U_0 \times V_0$ we get $u \in W^\omega_{0\mathcal{E}}(S; H^1_0(G))$, in particular, $u_0 \in W^\omega_{0\mathcal{E}}(S; H^1_0(G))$. Hence, close to the solution (u_0, λ_0) it is equivalent to look for solutions $(u, \lambda) \in \big(U_0 \cap W^\omega_{0\mathcal{E}}(S; H^1_0(G))\big) \times V_0$ of problem (2.5). To do so, we will apply the classical Implicit Function Theorem.

Note that $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ is continuously embedded into $[C(\overline{S}; C(\overline{X}))]^m$ for $\omega > n$. Therefore, $U_0 \cap W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ is open in $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$. Moreover, since

$$\mathcal{B} : L^\infty(S \times X; \mathbb{R}^{n\times n}) \times L^\omega_2(S; H^1_0(G)) \to L^\omega_2(S; H^{-1}(G))$$

is a bilinear continuous map, see (2.1), (2.2), for every $\alpha \in \{1, \dots, m\}$ the operator

$$(u, \lambda) \mapsto \mathcal{P}^\alpha(u, \lambda) = (\mathcal{E}^\alpha u^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u, \lambda), u^\alpha) - \mathcal{F}^\alpha(u, \lambda)$$

is a C^1-map from $\big(U_0 \cap W^\omega_{0\mathcal{E}}(S; H^1_0(G))\big) \times V_0$ into $L^\omega_2(S; H^{-1}(G))$. Its partial derivative with respect to u at the solution (u_0, λ_0) is the linear continuous map from $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ into $L^\omega_2(S; H^{-1}(G))$ given by

$$\tfrac{\partial \mathcal{P}^\alpha}{\partial u}(u_0, \lambda_0)\, v = (\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u_0, \lambda_0), v^\alpha) + \mathcal{B}\big(\tfrac{\partial \mathcal{A}^\alpha}{\partial u}(u_0, \lambda_0)\, v, u_0^\alpha\big) - \tfrac{\partial \mathcal{F}^\alpha}{\partial u}(u_0, \lambda_0)\, v$$

for $v \in W^\omega_{0\mathcal{E}}(S; H^1_0(G))$; it corresponds to the linearization (2.11). Applying Theorem 3.1, the derivative

$$v \mapsto \big(\tfrac{\partial \mathcal{P}^1}{\partial u}(u_0, \lambda_0), \dots, \tfrac{\partial \mathcal{P}^m}{\partial u}(u_0, \lambda_0)\big), \tag{3.10}$$

generates a linear isomorphism from $W^\omega_{0\mathcal{E}}(S; H^1_0(G))$ onto $[L^\omega_2(S; H^{-1}(G))]^m$, because the map $\mathcal{N}^\alpha$ defined by

$$\mathcal{N}^\alpha v = \mathcal{B}\big(\tfrac{\partial \mathcal{A}^\alpha}{\partial u}(u_0, \lambda_0)\, v, u_0^\alpha\big) - \tfrac{\partial \mathcal{F}^\alpha}{\partial u}(u_0, \lambda_0)\, v \quad \text{for } v \in [C(\overline{S}; C(\overline{X}))]^m,$$

is a linear continuous Volterra operator from $[C(\overline{S};C(\overline{X}))]^m$ into $L_2^\omega(S;H^{-1}(G))$ for every $\alpha \in \{1,\dots,m\}$, see (2.1), (2.2), (2.6), (2.9) and (2.10). Hence, the Implicit Function Theorem, see [27, Theorem 4.B], works for the first assertion.

2. Finally, we prove the second assertion of the theorem. Remembering (2.12), the sequence of Newton iterations is defined by

$$\begin{aligned}(\mathcal{E}^\alpha u_{k+1}^\alpha)' + \mathcal{B}\big(\mathcal{A}^\alpha(u_k,\lambda_0),u_{k+1}^\alpha\big) + \mathcal{B}\big(\tfrac{\partial\mathcal{A}^\alpha}{\partial u}(u_k,\lambda_0)\,u_{k+1},u_k^\alpha\big) - \tfrac{\partial\mathcal{F}^\alpha}{\partial u}(u_k,\lambda_0)\,u_{k+1}\\ = \mathcal{B}\big(\tfrac{\partial\mathcal{A}^\alpha}{\partial u}(u_k,\lambda_0)\,u_k,u_k^\alpha\big) + \mathcal{F}^\alpha(u_k,\lambda_0) - \tfrac{\partial\mathcal{F}^\alpha}{\partial u}(u_k,\lambda_0)\,u_k, \quad \alpha\in\{1,\dots,m\}. \quad (3.11)\end{aligned}$$

Starting the iteration with $k=1$ and $u_1 \in U \cap W_{0\mathcal{E}}^\omega(S;H_0^1(G))$, the right hand side of (3.11) belongs to $L_2^\omega(S;H^{-1}(G))$. Since we have

$$\tfrac{\partial\mathcal{P}^\alpha}{\partial u}(u,\lambda_0)\,v = (\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u,\lambda_0),v^\alpha) + \mathcal{B}\big(\tfrac{\partial\mathcal{A}^\alpha}{\partial u}(u,\lambda_0)\,v,u^\alpha\big) - \tfrac{\partial\mathcal{F}^\alpha}{\partial u}(u,\lambda_0)\,v$$

for all $v \in W_{0\mathcal{E}}^\omega(S;H_0^1(G))$ and $\alpha\in\{1,\dots,m\}$, the derivative

$$v \mapsto \big(\tfrac{\partial\mathcal{P}^1}{\partial u}(u,\lambda_0),\dots,\tfrac{\partial\mathcal{P}^m}{\partial u}(u,\lambda_0)\big), \quad (3.12)$$

is close to the isomorphism defined by (3.10) with respect to the operator norm in the space $\mathscr{L}\big(W_{0\mathcal{E}}^\omega(S;H_0^1(G));[L_2^\omega(S;H^{-1}(G))]^m\big)$ and, therefore, an isomorphism from $W_{0\mathcal{E}}^\omega(S;H_0^1(G))$ onto $[L_2^\omega(S;H^{-1}(G))]^m$, too, whenever u is sufficiently close to u_0 in $W_{0\mathcal{E}}^\omega(S;H_0^1(G))$. Hence, if u_1 is sufficiently close to u_0 in $W_{0\mathcal{E}}^\omega(S;H_0^1(G))$, then the new iteration u_2 is uniquely defined by (3.11), belongs to $W_{0\mathcal{E}}^\omega(S;H_0^1(G))$ and is close to u_0 in $W_{0\mathcal{E}}^\omega(S;H_0^1(G))$. Now, the classical Newton iteration procedure, see [27, Proposition 5.1], works for problem (2.5), since the norm of the map (3.12) in $\mathscr{L}\big(W_{0\mathcal{E}}^\omega(S;H_0^1(G));[L_2^\omega(S;H^{-1}(G))]^m\big)$ depends even Lipschitz continuously on u in a neighborhood of u_0 in $W_{0\mathcal{E}}^\omega(S;H_0^1(G))$. □

3.2 Local existence and uniqueness

One cannot expect that solutions $(u,\lambda) \in \big(U \cap W_{0\mathcal{E}}(S;H_0^1(G))\big) \times V$ to problem (2.5) exist on arbitrarily long time intervals $S=(t_0,t_1)$ without imposing further structural or growth conditions on the nonlinear operators $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$. Setting $U_\tau = \{u|S_\tau : u \in U\}$, our next assertion deals with the fact, that in the case $(0,\lambda) \in U \times V$ we can always find a solution $(u_\tau,\lambda) \in \big(U_\tau \cap W_{0\mathcal{E}}(S_\tau;H_0^1(G))\big) \times V$ to the problem

$$(\mathcal{E}_\tau^\alpha u_\tau^\alpha)' + \mathcal{B}_\tau\big(\mathcal{A}_\tau^\alpha(u_\tau,\lambda),u_\tau^\alpha\big) = \mathcal{F}_\tau^\alpha(u_\tau,\lambda), \quad \alpha\in\{1,\dots,m\}, \quad (3.13)$$

restricted to the subinterval $S_\tau = (t_0,t_0+\tau)$ of S, whenever we choose $\tau \in (0,t_1-t_0]$ small enough. For $\alpha\in\{1,\dots,m\}$ and leading order coefficients $a^\alpha \in L^\infty(X)$ and $A_\tau^\alpha \in L^\infty(S_\tau\times X;\mathbb{R}^{n\times n})$, the linear continuous operator

$$\mathcal{E}_\tau^\alpha : L^2(S_\tau;H^1(X)) \to L^2(S_\tau;H^{-1}(G))$$

as well as the bilinear continuous map

$$\mathcal{B}_\tau : L^\infty(S_\tau \times X; \mathbb{R}^{n\times n}) \times L^2(S_\tau; H^1(X)) \to L^2(S_\tau; H^{-1}(G))$$

are defined analogously to (2.4) and (2.6). Furthermore, using the VOLTERRA property of $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$ with respect to u, both the nonlinear operators

$$\mathcal{A}^\alpha_\tau \in C^1\big(U_\tau \times V; L^\infty(S_\tau \times X; \mathbb{R}^{n\times n})\big),$$
$$\mathcal{F}^\alpha_\tau \in C^1\big(U_\tau \times V; L_2^{\omega_0}(S_\tau; H^{-1}(G))\big),$$

are uniquely defined by the identities

$$\mathcal{A}^\alpha_\tau(u|S_\tau, \lambda) = \mathcal{A}^\alpha(u,\lambda)|S_\tau, \quad \mathcal{F}^\alpha_\tau(u|S_\tau, \lambda) = \mathcal{F}^\alpha(u,\lambda)|S_\tau,$$

for $(u,\lambda) \in U \times V$ and $\alpha \in \{1,\dots,m\}$.

Theorem 3.3. *Assume that $(0,\lambda)$ belongs to $U \times V$, and let $\varepsilon \in (0,1]$ be a constant such that $a^\alpha \in L^\infty(X)$ and $\mathcal{A}^\alpha(0,\lambda) \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ are ε-definite with respect to S and X for all $\alpha \in \{1,\dots,m\}$.*

Then we find a parameter $\omega \in (n, \omega_0]$ and some $\tau \in (0, t_1 - t_0]$ such that there exists a solution $(u_\tau, \lambda) \in \big(U_\tau \cap W^\omega_{0\mathcal{E}}(S_\tau; H^1_0(G))\big) \times V$ to (3.13) on the subinterval $S_\tau = (t_0, t_0 + \tau)$ of S.

Proof. 1. Because $\mathcal{F}^\alpha(0,\lambda) \in L_2^{\omega_0}(S; H^{-1}(G))$ holds true for every $\alpha \in \{1,\dots,m\}$, the maximal regularity result in [20, Theorem 6.8] yields some $\bar\omega = \bar\omega(\varepsilon, G) \in (n, \omega_0]$ such that the solution $w \in W_{0\mathcal{E}}(S; H^1_0(G))$ of the linear auxiliary problem

$$(\mathcal{E}^\alpha w^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(0,\lambda), w^\alpha) = \mathcal{F}^\alpha(0,\lambda), \quad \alpha \in \{1,\dots,m\}, \tag{3.14}$$

belongs to $W^{\bar\omega}_{0\mathcal{E}}(S; H^1_0(G))$.

2. We choose two neighborhoods U_λ and V_λ of 0 in $[C(\overline{S}; C(\overline{X}))]^m$ such that the inclusion $\big\{u + v : (u,v) \in U_\lambda \times V_\lambda\big\} \subset U$ holds true. Now, we look for solutions $(u,v) \in \big(U_\lambda \cap W_{0\mathcal{E}}(S; H^1_0(G))\big) \times V_\lambda$ of the nonlinear auxiliary problem

$$(\mathcal{E}^\alpha u^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha_\lambda(u,v), u^\alpha) = \mathcal{F}^\alpha_\lambda(u,v), \quad \alpha \in \{1,\dots,m\}, \tag{3.15}$$

where the VOLTERRA operators $\mathcal{A}^\alpha_\lambda \in C^1\big(U_\lambda \times V_\lambda; L^\infty(S \times X; \mathbb{R}^{n\times n})\big)$ and $\mathcal{F}^\alpha_\lambda \in C^1\big(U_\lambda \times V_\lambda; L_2^{\bar\omega}(S; H^{-1}(G))\big)$ are defined by

$$\mathcal{A}^\alpha_\lambda(u,v) = \mathcal{A}^\alpha(u+v,\lambda), \tag{3.16}$$
$$\mathcal{F}^\alpha_\lambda(u,v) = \big(\mathcal{F}^\alpha(u+v,\lambda) - \mathcal{F}^\alpha(0,\lambda)\big) - \mathcal{B}\big(\mathcal{A}^\alpha(u+v,\lambda) - \mathcal{A}^\alpha(0,\lambda), w^\alpha\big), \tag{3.17}$$

for $(u,v) \in U_\lambda \times V_\lambda$ and $\alpha \in \{1,\dots,m\}$. Because of $\mathcal{A}^\alpha_\lambda(0,0) = \mathcal{A}^\alpha(0,\lambda)$ and $\mathcal{F}^\alpha_\lambda(0,0) = 0$, the pair $(u,v) = (0,0) \in U_\lambda \times V_\lambda$ is a solution of (3.15).

In view of Theorem 3.2 we find a parameter $\omega \in (n, \bar{\omega}]$, two neighborhoods U_0 and V_0 of 0 in $[C(\overline{S}; C(\overline{X}))]^m$ with $U_0 \times V_0 \subset U_\lambda \times V_\lambda$ and a solution map $\Phi \in C^1\big(V_0; W^{\omega}_{0\mathcal{E}}(S; H^1_0(G))\big)$ such that $(u, v) \in U_0 \times V_0$ is a solution of (3.15) if and only if $u = \Phi(v)$. In particular, for each solution $(u, v) \in U_0 \times V_0$ of (3.15) we get $u \in W^{\omega}_{0\mathcal{E}}(S; H^1_0(G))$.

3. Next, we make use of the fact that the solution $w \in W^{\bar{\omega}}_{0\mathcal{E}}(S; H^1_0(G))$ of (3.14) is small in the norm of $[C(\overline{S_t}; C(\overline{X}))]^m$ on subintervals $S_t = (t_0, t_0 + t)$ of S, whenever $t \in (0, t_1 - t_0]$ is small enough: Indeed, due to the continuous embedding of $W^{\bar{\omega}}_{0\mathcal{E}}(S; H^1_0(G))$ into $[C^{0,\beta}(\overline{S}; C(\overline{X}))]^m$, see [19, Theorem 3.4, Theorem 6.8], for all $t \in (0, t_1 - t_0]$ and $s \in \overline{S_t}$ we get

$$\|w(s) - w(t_0)\|_{[C(\overline{X})]^m} \le (s - t_0)^\beta \|w\|_{[C^{0,\beta}(\overline{S}; C(\overline{X}))]^m} \le c_1 t^\beta \|w\|_{W^{\bar{\omega}}_{0\mathcal{E}}(S; H^1_0(G))},$$

where $\beta = (\bar{\omega} - n)/4$ and the constant $c_1 > 0$ does not depend on $\tau > 0$. Since $w(t_0) = 0$ holds true, we can find some τ, $t \in (0, t_1 - t_0]$ with $\tau < t$ and a cut-off function $\vartheta \in C^\infty(\mathbb{R})$ with

$$0 \le \vartheta \le 1, \quad \vartheta(s) = 1 \quad \text{for all } s \le t_0 + \tau, \quad \vartheta(s) = 0 \quad \text{for all } s \ge t_0 + t,$$

such that $v = \vartheta w \in [C(\overline{S}; C(\overline{X}))]^m$ belongs to V_0. Now, we apply the result of Step 2 to get a solution $(u, v) \in U_0 \times V_0$ of (3.15) with $u = \Phi(v) \in W^{\omega}_{0\mathcal{E}}(S; H^1_0(G))$. Because $w \in W^{\bar{\omega}}_{0\mathcal{E}}(S; H^1_0(G))$ solves problem (3.14), by (3.16) and (3.17) we arrive at the identity

$$(\mathcal{E}^\alpha(u^\alpha + w^\alpha))' + \mathcal{B}(\mathcal{A}^\alpha(u + v, \lambda), u^\alpha + w^\alpha) = \mathcal{F}^\alpha(u + v, \lambda), \quad \alpha \in \{1, \ldots, m\}.$$

Note, that $u_\tau = (u + v)|S_\tau = (u + w)|S_\tau$ belongs to $W^{\omega}_{0\mathcal{E}}(S_\tau; H^1_0(G))$. Hence, restricting the functionals on both sides of the last identity to the subinterval S_τ, the Volterra property of the maps $\mathcal{E}^\alpha$, $\mathcal{A}^\alpha$, $\mathcal{B}$, $\mathcal{F}^\alpha$ and the definition of their restrictions $\mathcal{E}^\alpha_\tau$, $\mathcal{A}^\alpha_\tau$, $\mathcal{B}_\tau$, $\mathcal{F}^\alpha_\tau$ to S_τ yield the fact, that $(u_\tau, \lambda) \in \big(U_\tau \cap W^{\omega}_{0\mathcal{E}}(S_\tau; H^1_0(G))\big) \times V$ is a solution of problem (3.13). □

3.3 Additional temporal regularity

In this subsection we formulate assumptions on the nonlinearities (2.7) and (2.8), which ensure an additional temporal regularity of solutions to (2.5). To do so, we consider C^∞-isotopies $T : \overline{\Sigma} \times \overline{S} \to \overline{S}$, where $\Sigma = (-\sigma_1, \sigma_1)$ and $S = (t_0, t_1)$ are open intervals. Introducing the notation

$$T_\sigma(t) = T(\sigma, t) \quad \text{for } \sigma \in \overline{\Sigma} \text{ and } t \in \overline{S},$$

we assume that both the families $\{T_\sigma\}_{\sigma\in\Sigma}$ and $\{T_\sigma^{-1}\}_{\sigma\in\Sigma}$ of monotone diffeomorphisms from $\overline{S}$ onto itself have uniformly bounded derivatives of arbitrary order. Moreover, we suppose that $T_0 : \overline{S} \to \overline{S}$ is the identity.

In the following, for $\sigma \in \Sigma$ we consider maps, which assign functions $w : \overline{S} \to H$ with values in a HILBERT space H to its temporal transformation

$$t \in \overline{S} \mapsto w(T_\sigma(t)) \in H.$$

As a simple consequence of the change of variables formula and the uniform properties of the above families, these maps generate linear isomorphisms

$$\begin{aligned} &\mathcal{T}_\sigma^0 \text{ from } L^2(S;L^2(X)) \text{ onto itself}, &&\mathcal{T}_\sigma^\Gamma \text{ from } L^2(S;L^2(\Gamma)) \text{ onto itself},\\ &\mathcal{T}_\sigma \text{ from } L^2(S;H_0^1(G)) \text{ onto itself}, &&\mathcal{M}_\sigma \text{ from } L^2(S;L^2(X;\mathbb{R}^{n\times n})) \text{ onto itself},\end{aligned}$$

as well as their adjoint operators,

$$\begin{aligned} &\mathcal{T}_0^\sigma \text{ from } L^2(S;L^2(X)) \text{ onto itself}, &&\mathcal{T}_\Gamma^\sigma \text{ from } L^2(S;L^2(\Gamma)) \text{ onto itself},\\ &\mathcal{T}^\sigma \text{ from } L^2(S;H^{-1}(G)) \text{ onto itself}, &&\mathcal{M}^\sigma \text{ from } L^2(S;L^2(X;\mathbb{R}^{n\times n})) \text{ onto itself}.\end{aligned}$$

Obviously, $\mathcal{T}_\sigma^0$ maps $C(\overline{S};C(\overline{X}))$ isomorphically onto itself. Moreover, we get

Lemma 3.4. *For $\omega \in [0, n+2]$, $\varkappa \in [0, n+1]$, and $\sigma \in \Sigma$ the following holds true:*

1. $\mathcal{T}_\sigma^0$ and $\mathcal{T}_0^\sigma$ map $L_2^\omega(S;L^2(X))$ isomorphically onto itself.

2. $\mathcal{T}_\sigma^\Gamma$ and $\mathcal{T}_\Gamma^\sigma$ map $L_2^\varkappa(S;L^2(\Gamma))$ isomorphically onto itself.

3. $\mathcal{T}_\sigma$ maps $L_2^\omega(S;H_0^1(G))$ isomorphically onto itself.

4. $\mathcal{T}^\sigma$ maps $L_2^\omega(S;H^{-1}(G))$ isomorphically onto itself.

5. Let $\mathcal{E}^\alpha \in \mathscr{L}\big(L^2(S;H^1(X));L^2(S;H^{-1}(G))\big)$ be defined as in (2.4). Then, $\mathcal{T}_\sigma$ maps $W_{\mathcal{E}^\alpha}^\omega(S;H_0^1(G))$ isomorphically onto itself, and for all $w \in W_{\mathcal{E}^\alpha}^\omega(S;H_0^1(G))$ we have the identity

$$\mathcal{T}^\sigma(\mathcal{E}^\alpha\mathcal{T}_\sigma w)' = (\mathcal{E}^\alpha w)'. \tag{3.18}$$

6. $\mathcal{M}_\sigma$ and $\mathcal{M}^\sigma$ map $L^\infty(S\times X;\mathbb{R}^{n\times n})$ isomorphically onto itself. Furthermore, we get the transformation rule

$$\mathcal{T}^\sigma\mathcal{B}(A,\mathcal{T}_\sigma w) = \mathcal{B}(\mathcal{M}^\sigma A, w) \tag{3.19}$$

for all $A \in L^\infty(S\times X;\mathbb{R}^{n\times n})$ and $w \in L_2^\omega(S;H_0^1(G))$.

Proof. Suppose that $L \geq 1$ is a LIPSCHITZ constant for both the transformations T_σ and T_σ^{-1}. Since the map T_σ and its inverse T_σ^{-1} have the same differential and topological properties, for the above isomorphism results it is enough to prove the continuity of the operator under consideration or the continuity of its inverse.

For the proof of the desired results in MORREY spaces we arbitrarily fix some radius $r > 0$ and corresponding subsets

$$\begin{aligned} S_r &\in \big\{ S \cap (t - r^2, t) : t \in S \big\}, \quad X_r \in \big\{ X \cap Q_r(x) : x \in X \big\}, \\ \Gamma_r &\in \big\{ \Gamma \cap Q_r(x) : x \in \Gamma \big\}, \qquad G_r \in \big\{ G \cap Q_r(x) : x \in G \big\}. \end{aligned}$$

Setting $\delta = Lr$ we can always choose suitable intersections

$$\begin{aligned} S_\delta &\in \big\{ S \cap (t - \delta^2, t) : t \in S \big\}, \quad X_\delta \in \big\{ X \cap Q_\delta(x) : x \in X \big\}, \\ \Gamma_\delta &\in \big\{ \Gamma \cap Q_\delta(x) : x \in \Gamma \big\}, \qquad G_\delta \in \big\{ G \cap Q_\delta(x) : x \in G \big\}, \end{aligned}$$

with $S_r \subset T_\sigma[S_\delta]$, $X_r \subset X_\delta$, $\Gamma_r \subset \Gamma_\delta$, and $G_r \subset G_\delta$. In the following we only derive the essential estimates on these intersections. As usual, the final step to get estimates in MORREY spaces consists of multiplying both sides of the inequality under consideration with radial weights and of taking the suprema over all these radii and intersections on both sides of the inequality.

1. By a change of variables for $u \in L_2^\omega(S; L^2(X))$ and $v \in L_2^\varkappa(S; L^2(\Gamma))$ we get

$$\begin{aligned} \int_{S_r} \int_{X_r} \big| u(T_\sigma^{-1}(s)) \big|^2 \, d\lambda^n \, ds &\le L \int_{S_\delta} \int_{X_\delta} |u(t)|^2 \, d\lambda^n \, dt, \\ \int_{S_r} \int_{\Gamma_r} \big| v(T_\sigma^{-1}(s)) \big|^2 \, d\lambda_\Gamma \, ds &\le L \int_{S_\delta} \int_{\Gamma_\delta} |v(t)|^2 \, d\lambda_\Gamma \, dt, \end{aligned}$$

which yields the continuity of the map $(\mathcal{T}_\sigma^0)^{-1}$ from $L_2^\omega(S; L^2(X))$ into itself and of the map $(\mathcal{T}_\sigma^\Gamma)^{-1}$ from $L_2^\varkappa(S; L^2(\Gamma))$ into itself.

2. For $u \in L_2^\omega(S; L^2(X))$ and $\varphi \in L^2(S; L^2(X))$, which satisfy $\varphi|(S \setminus S_r) = 0$ and $\varphi(s)|(X \setminus X_r) = 0$ for almost all $s \in S$, we obtain

$$\begin{aligned} \int_{S_r} \int_{X_r} (\mathcal{T}_0^\sigma u)(s) \, \varphi(s) \, d\lambda^n \, ds &= \int_S \int_X (\mathcal{T}_0^\sigma u)(s) \, \varphi(s) \, d\lambda^n \, ds \\ &= \int_S \int_X u(t) \, (\mathcal{T}_\sigma^0 \varphi)(t) \, d\lambda^n \, dt = \int_{S_\delta} \int_{X_\delta} u(t) \, (\mathcal{T}_\sigma^0 \varphi)(t) \, d\lambda^n \, dt, \end{aligned}$$

which leads to the estimate

$$\int_{S_r} \int_{X_r} \big| (\mathcal{T}_0^\sigma u)(s) \big|^2 \, d\lambda^n \, ds \le \|\mathcal{T}_\sigma^0\|^2 \int_{S_\delta} \int_{X_\delta} |u(t)|^2 \, d\lambda^n \, ds,$$

where $\|\mathcal{T}_\sigma^0\|$ is the norm of the operator $\mathcal{T}_\sigma^0$ mapping $L^2(S; L^2(X))$ into itself. Consequently, $\mathcal{T}_0^\sigma$ is a linear continuous operator from $L_2^\omega(S; L^2(X))$ into itself.

For all $u \in L_2^{\varkappa}(S; L^2(\Gamma))$ and $\varphi \in L^2(S; L^2(\Gamma))$ satisfying $\varphi|(S \setminus S_r) = 0$ and $\varphi(s)|(\Gamma \setminus \Gamma_r) = 0$ for almost all $s \in S$, we have

$$\begin{aligned}\int_{S_r}\int_{\Gamma_r} (\mathcal{T}_\Gamma^\sigma u)(s)\,\varphi(s)\,d\lambda_\Gamma\,ds &= \int_S\int_\Gamma (\mathcal{T}_\Gamma^\sigma u)(s)\,\varphi(s)\,d\lambda_\Gamma\,ds \\ &= \int_S\int_\Gamma u(t)\,(\mathcal{T}_\sigma^\Gamma \varphi)(t)\,d\lambda_\Gamma\,dt = \int_{S_\delta}\int_{\Gamma_\delta} u(t)\,(\mathcal{T}_\sigma^\Gamma \varphi)(t)\,d\lambda_\Gamma\,dt,\end{aligned}$$

which leads to the estimate

$$\int_{S_r}\int_{\Gamma_r} \big|(\mathcal{T}_\Gamma^\sigma u)(s)\big|^2\,d\lambda_\Gamma\,ds \le \|\mathcal{T}_\sigma^\Gamma\|^2 \int_{S_\delta}\int_{\Gamma_\delta} |u(t)|^2\,d\lambda_\Gamma\,ds,$$

where $\|\mathcal{T}_\sigma^\Gamma\|$ is the norm of the operator $\mathcal{T}_\sigma^\Gamma$ mapping $L^2(S; L^2(\Gamma))$ into itself. Hence, $\mathcal{T}_\Gamma^\sigma$ is a linear continuous operator from $L_2^{\varkappa}(S; L^2(\Gamma))$ into itself.

3. Due to a change of variables for all $u \in L_2^{\omega}(S; H_0^1(G))$ we obtain

$$\int_{S_r}\int_{G_r} \big|\nabla u(T_\sigma^{-1}(s))\big|^2\,d\lambda^n\,ds \le L \int_{S_\delta}\int_{G_\delta} |\nabla u(t)|^2\,d\lambda^n\,dt.$$

Together with Step 1 this proves the continuity of $\mathcal{T}_\sigma^{-1}$ from $L_2^{\omega}(S; H_0^1(G))$ into itself.

4. For all $f \in L_2^{\omega}(S; H^{-1}(G))$ and $v \in L^2(S_r; H_0^1(G_r))$ the properties of the zero extension map $\mathcal{Z}_{S,G}$ and the localization operators ensure that the identity

$$\begin{aligned}&\int_{S_r} \langle (\mathcal{L}_{S_r,G_r}\mathcal{T}^\sigma f)(s), v(s)\rangle_{H_0^1(G_r)}\,ds = \int_S \langle (\mathcal{T}^\sigma f)(s), (\mathcal{Z}_{S,G}v)(s)\rangle_{H_0^1(G)}\,ds \\ &= \int_S \langle f(t), (\mathcal{T}_\sigma \mathcal{Z}_{S,G}v)(t)\rangle_{H_0^1(G)}\,dt = \int_{S_\delta} \langle (\mathcal{L}_{S_\delta,G_\delta} f)(t), (\mathcal{R}_{S_\delta,G_\delta}\mathcal{T}_\sigma\mathcal{Z}_{S,G}v)(t)\rangle_{H_0^1(G_\delta)}\,dt\end{aligned}$$

holds true, which yields the estimate

$$\int_{S_r} \|(\mathcal{L}_{S_r,G_r}\mathcal{T}^\sigma f)(s)\|_{H^{-1}(G_r)}^2\,ds \le \|\mathcal{T}_\sigma\|^2 \int_{S_\delta} \|(\mathcal{L}_{S_\delta,G_\delta} f)(t)\|_{H^{-1}(G_\delta)}^2\,dt,$$

where $\|\mathcal{T}_\sigma\|$ is the norm of the operator $\mathcal{T}_\sigma$ mapping $L^2(S; H_0^1(G))$ into itself. Hence, $\mathcal{T}^\sigma$ is a linear continuous operator from $L_2^{\omega}(S; H^{-1}(G))$ into itself.

5. If $u = \mathcal{T}_\sigma w \in W_{\mathcal{E}^\alpha}^{\omega}(S; H_0^1(G))$, then $w = \mathcal{T}_\sigma^{-1} u$ belongs to $L_2^{\omega}(S; H_0^1(G))$ due to Step 3. Furthermore, for all $\vartheta \in C_0^\infty(S)$ and $\varphi \in H_0^1(G)$ we have

$$\begin{aligned}\int_S \big\langle \tfrac{\partial}{\partial t}(\mathcal{E}^\alpha \mathcal{T}_\sigma w)(t), (\vartheta\varphi)(T_\sigma(t))\big\rangle\,dt &= -\int_S \big\langle (\mathcal{E}^\alpha w)(T_\sigma(t)), \tfrac{\partial \vartheta}{\partial s}(T_\sigma(t))\varphi\big\rangle\,\tfrac{\partial T}{\partial t}(\sigma, t)\,dt \\ &= -\int_S \big\langle (\mathcal{E}^\alpha w)(s), \tfrac{\partial \vartheta}{\partial s}(s)\varphi\big\rangle\,ds.\end{aligned}$$

Hence, Step 4 yields the identity $(\mathcal{E}^\alpha w)' = \mathcal{T}^\sigma(\mathcal{E}^\alpha \mathcal{T}_\sigma w)' \in L_2^\omega(S; H^{-1}(G))$ and, therefore, $w \in W_{\mathcal{E}^\alpha}^\omega(S; H_0^1(G))$ with a corresponding norm estimate. If, additionally, we have $u(t_0) = 0$, this implies $w(t_0) = 0$.

6. Clearly, the operator $\mathcal{M}_\sigma$ maps $L^\infty(S \times X; \mathbb{R}^{n\times n})$ into itself. Using a change of variables, for matrix functions $A \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ and $B \in C(\overline{S} \times \overline{X}; \mathbb{R}^{n\times n}))$ we obtain the estimate

$$\int_S \int_X A : (\mathcal{M}_\sigma B)\, d\lambda^n\, dt \le \|A\|_\infty \, \|(\mathcal{M}_\sigma B)\|_1 \le L\, \|A\|_\infty \, \|B\|_1,$$

Here we have denoted by $\|A\|_\infty$ and $\|B\|_1$ the norms of A and B in $L^\infty(S\times X; \mathbb{R}^{n\times n})$ and $L^1(S \times X; \mathbb{R}^{n\times n})$, respectively. A density argument shows that $\mathcal{M}^\sigma$ is a linear continuous operator from $L^\infty(S \times X; \mathbb{R}^{n\times n})$ into itself.

7. By definition for all $A \in L^\infty(S \times X; \mathbb{R}^{n\times n})$, $w \in L_2^\omega(S; H_0^1(G))$, $\vartheta \in C_0^\infty(S)$ and $v \in C_0^\infty(G)$ we get the identity

$$\begin{aligned}\big\langle \mathcal{B}(A, \mathcal{T}_\sigma w), \mathcal{T}_\sigma(\vartheta v)\big\rangle &= \int_S \int_X A : \mathcal{M}_\sigma(\nabla w \otimes \nabla(\vartheta v))\, d\lambda^n\, dt \\ &= \int_S \int_X (\mathcal{M}^\sigma A) : (\nabla w \otimes \nabla(\vartheta v))\, d\lambda^n\, ds = \big\langle \mathcal{B}(\mathcal{M}^\sigma A, w), \vartheta v\big\rangle.\end{aligned}$$

Using a density argument, we see that (3.19) holds true. □

Theorem 3.5. *Let $(u, \lambda) \in \big(U \cap W_{0\mathcal{E}}(S; H_0^1(G))\big) \times V$ be a solution to (2.5) and assume that there exists a constant $\varepsilon \in (0, 1]$ such that $a^\alpha \in L^\infty(X)$ and $\mathcal{A}^\alpha(u, \lambda) \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ are ε-finite with respect to S and X for all $\alpha \in \{1, \ldots, m\}$.*

Moreover, let W be some neighborhood of u in $[C(\overline{S}; C(\overline{X}))]^m$ such that

$$\big(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m\big) \in U \quad \text{for all } w \in W \text{ and } \sigma \in \Sigma,$$

and suppose that the nonlinear Volterra *operators, defined by*

$$\mathcal{A}_\lambda^\alpha(w, \sigma) = \mathcal{M}^\sigma \mathcal{A}^\alpha(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{3.20}$$

$$\mathcal{F}_\lambda^\alpha(w, \sigma) = \mathcal{T}^\sigma \mathcal{F}^\alpha(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{3.21}$$

for $(w, \sigma) \in W \times \Sigma$ and $\alpha \in \{1, \ldots, m\}$, are continuously differentiable in the sense

$$\mathcal{A}_\lambda^\alpha \in C^1\big(W \times \Sigma; L^\infty(S \times X; \mathbb{R}^{n\times n})\big), \tag{3.22}$$

$$\mathcal{F}_\lambda^\alpha \in C^1\big(W \times \Sigma; L_2^{\omega_0}(S; H^{-1}(G))\big). \tag{3.23}$$

Then, we can find a parameter $\omega \in (n, \omega_0]$ such that both the solution u and the time derivative of the weighted solution $\frac{\partial T}{\partial \sigma}(0)\, u$ belongs to $W_{0\mathcal{E}}^\omega(S; H_0^1(G))$.

Proof. 1. Because the temporal transformations T_σ are close to identity for small $\sigma \in \Sigma$, from $u \in W$ it follows that there exists a neighborhood Σ_1 of 0 in $\mathbb{R}$ with $\Sigma_1 \subset \Sigma$, such that the temporally transformed function satisfies

$$w = \big(\mathcal{T}_\sigma^{-1} u^1, \dots, \mathcal{T}_\sigma^{-1} u^m\big) \in W \cap W_{0\mathcal{E}}(S; H_0^1(G)) \quad \text{for every } \sigma \in \Sigma_1.$$

If we apply the adjoint operator $\mathcal{T}^\sigma$ to the functionals on both sides of

$$(\mathcal{E}^\alpha u^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u,\lambda), u^\alpha) = \mathcal{F}^\alpha(u,\lambda), \quad \alpha \in \{1,\dots,m\},$$

then, following Lemma 3.4 and the transformation rules (3.18) and (3.19), for all $\sigma \in \Sigma_2$ and $\alpha \in \{1,\dots,m\}$ we get

$$\begin{aligned}&(\mathcal{E}^\alpha w^\alpha)' + \mathcal{B}\big(\mathcal{M}^\sigma \mathcal{A}^\alpha(\mathcal{T}_\sigma w^1, \dots, \mathcal{T}_\sigma w^m, \lambda), w^\alpha\big)\\ &\quad = \mathcal{T}^\sigma(\mathcal{E}^\alpha u^\alpha)' + \mathcal{T}^\sigma \mathcal{B}(\mathcal{A}^\alpha(u,\lambda), u^\alpha) = \mathcal{T}^\sigma \mathcal{F}^\alpha(u,\lambda) = \mathcal{T}^\sigma \mathcal{F}^\alpha(\mathcal{T}_\sigma w^1, \dots, \mathcal{T}_\sigma w^m, \lambda).\end{aligned}$$

Hence, the pair $(w,\sigma) \in \big(W \cap W_{0\mathcal{E}}(S; H_0^1(G))\big) \times \Sigma$ solves the transformed problem

$$(\mathcal{E}^\alpha w^\alpha)' + \mathcal{B}\big(\mathcal{A}_\lambda^\alpha(w,\sigma), w^\alpha\big) = \mathcal{F}_\lambda^\alpha(w,\sigma), \quad \alpha \in \{1,\dots,m\}. \tag{3.24}$$

Note, that the pair $(w,\sigma) = (u,0)$ is a solution of both the problems (2.5) and (3.24). In view of (3.22) and (3.23) we can apply Theorem 3.2 to find some Morrey exponent $\omega \in (n, \omega_0]$ and a neighborhood W_0 of u in $[C(\overline{S}; C(\overline{X}))]^m$ with $W_0 \subset W$ such that the following holds true: There exists a neighborhood Σ_0 of 0 in $\mathbb{R}$ with $\Sigma_0 \subset \Sigma_1$ and a solution map $\Phi \in C^1\big(\Sigma_0; W_{0\mathcal{E}}^\omega(S; H_0^1(G))\big)$ such that $(w,\sigma) \in W_0 \times \Sigma_0$ is a solution to (3.24) if and only if $w = \Phi(\sigma)$. Because of the above construction this yields $\Phi(\sigma) = \big(\mathcal{T}_\sigma^{-1} u^1, \dots, \mathcal{T}_\sigma^{-1} u^m\big)$ for all $\sigma \in \Sigma_0$.

2. To prove the temporal regularity of the solution we calculate the derivative $\frac{\partial \Phi}{\partial \sigma}(0) \in W_{0\mathcal{E}}^\omega(S; H_0^1(G))$: For every $\alpha \in \{1,\dots,m\}$, $\sigma \in \Sigma_0$, $\vartheta \in C_0^\infty(S)$ and $\varphi \in H_0^1(G)$ we obtain

$$\begin{aligned}\tfrac{\partial}{\partial \sigma} \int_S \big(u^\alpha(T_\sigma^{-1}(s)) \big| \varphi\big)\, \vartheta(s)\, ds &= \int_S \big(u^\alpha(t) \big| \varphi\big)\, \tfrac{\partial}{\partial \sigma}\big(\vartheta(T_\sigma(t)) \tfrac{\partial T}{\partial t}(\sigma,t)\big)\, dt\\ &= \int_S \big(u^\alpha(t) \big| \varphi\big)\, \big(\tfrac{\partial \vartheta}{\partial s}(T_\sigma(t)) \tfrac{\partial T}{\partial \sigma}(\sigma,t) \tfrac{\partial T}{\partial t}(\sigma,t) + \vartheta(T_\sigma(t)) \tfrac{\partial}{\partial \sigma} \tfrac{\partial T}{\partial t}(\sigma,t)\big)\, dt\end{aligned}$$

and, furthermore,

$$\int_S \big(u^\alpha(t) \big| \varphi\big) \tfrac{\partial \vartheta}{\partial s}(T_\sigma(t)) \tfrac{\partial T}{\partial \sigma}(\sigma,t) \tfrac{\partial T}{\partial t}(\sigma,t)\, dt = \int_S \big(u^\alpha(T_\sigma^{-1}(s))\, \tfrac{\partial T}{\partial \sigma}(\sigma, T_\sigma^{-1}(s)) \big| \varphi\big) \tfrac{\partial \vartheta}{\partial s}(s)\, ds.$$

Specifying $\sigma = 0$, from both identities it follows that

$$\int_S \big(\tfrac{\partial \Phi}{\partial \sigma}(0,t) - u(t) \tfrac{\partial}{\partial t} \tfrac{\partial T}{\partial \sigma}(0,t) \big| \varphi\big)\, \vartheta(t)\, dt = \int_S \big(u(t) \tfrac{\partial T}{\partial \sigma}(0,t) \big| \varphi\big)\, \tfrac{\partial \vartheta}{\partial t}(t)\, dt$$

for all $\vartheta \in C_0^\infty(S)$ and $\varphi \in H_0^1(G)$, in other words,

$$\tfrac{\partial}{\partial t}\big(\tfrac{\partial T}{\partial\sigma}(0)\,u\big) = u\,\tfrac{\partial}{\partial t}\tfrac{\partial T}{\partial\sigma}(0) - \tfrac{\partial\Phi}{\partial\sigma}(0) \in W^\omega_{0\mathcal{E}}(S;H_0^1(G)), \tag{3.25}$$

which finishes the proof. □

Example 3.6. Let $S = (t_0, t_1)$ be a bounded open time interval and fix $\sigma_1 > 0$ such that $2\sigma_1(t_1 - t_0) < 1$. Given an open interval $\Sigma = (-\sigma_1, \sigma_1)$ of parameters, we consider the polynomial $T : \overline{\Sigma} \times \overline{S} \to \mathbb{R}$ and the corresponding family of temporal transformations $T_\sigma : \overline{S} \to \mathbb{R}$ defined by

$$T_\sigma(t) = T(\sigma, t) = t + \sigma(t - t_0)(t_1 - t) \quad \text{for } \sigma \in \overline{\Sigma} \text{ and } t \in \overline{S}. \tag{3.26}$$

Since we have the uniform estimate

$$\tfrac{\partial T}{\partial t}(\sigma, t) = 1 + \sigma\big((t_1 - t) - (t - t_0)\big) \in \big(\tfrac{1}{2}, \tfrac{3}{2}\big) \quad \text{for all } \sigma \in \overline{\Sigma} \text{ and } t \in \overline{S},$$

every transformation T_σ is a monotone diffeomorphism from $\overline{S}$ onto itself and all derivatives of the families $\{T_\sigma\}_{\sigma\in\Sigma}$ and $\{T_\sigma^{-1}\}_{\sigma\in\Sigma}$ are uniformly bounded. Obviously, T_0 is the identity. Further partial derivatives being of interest with respect to Theorem 3.5 and (3.25) are given by

$$\tfrac{\partial T}{\partial\sigma}(\sigma, t) = (t - t_0)(t_1 - t), \quad \tfrac{\partial}{\partial t}\tfrac{\partial T}{\partial\sigma}(\sigma, t) = (t_1 - t) - (t - t_0) \quad \text{for } \sigma \in \Sigma \text{ and } t \in \overline{S}.$$

Note, that the function $t \mapsto \frac{\partial T}{\partial\sigma}(\sigma, t)$ degenerates at the endpoints of the interval S. Nevertheless, on compact subintervals of S this function is uniformly bounded from below and from above by positive constants.

Corollary 3.7. *Let $(u, \lambda) \in \big(U \cap W_{0\mathcal{E}}(S; H_0^1(G))\big) \times V$ be a solution of problem (2.5). Suppose that the assumptions of Theorem 3.5 are satisfied with respect to the family $\{T_\sigma\}_{\sigma\in\Sigma}$ given by (3.26).*

Then, there exists an exponent $\omega \in (n, \omega_0]$ such that the solution u as well as the time derivative of the weighted solution $t \mapsto (t - t_0)(t_1 - t)u(t)$ belongs to $W^\omega_{0\mathcal{E}}(S; H_0^1(G))$. In particular, on compact subintervals I of S the time derivative of the restricted solution $u|I \in W^\omega_{\mathcal{E}}(I; H_0^1(G))$ is an element of $W^\omega_{\mathcal{E}}(I; H_0^1(G))$, too.

4 Examples of nonlinear operators

In this section we indicate some classes of nonlinear operators, which are candidates for the leading order coefficient maps $\mathcal{A}^\alpha$ and the right hand sides $\mathcal{F}^\alpha$ occuring in the operator equations in Sections 2 and 3.

4.1 Leading order coefficients

In place of the maps $\mathcal{A}^\alpha$ and $\mathcal{A}^\alpha_\lambda$ of Section 3 we consider superposition operators

$$\mathcal{A}(u,\lambda)(t,x) = A(t,x,u(t,x),\lambda) \quad \text{for almost all } (t,x) \in S \times X. \tag{4.1}$$

$$\mathcal{C}(u,\lambda,\sigma)(t,x) = A(T_\sigma(t),x,u(t,x),\lambda) \quad \text{for almost all } (t,x) \in S \times X. \tag{4.2}$$

Here, $A : S\times X\times\Omega\times V \to \mathbb{R}$ is the function, generating the superposition operators, Ω is an open subset in $\mathbb{R}^m$, and V is an open subset of the Banach space Λ. Introduced in Section 3, we consider the family of diffeomorphisms $T_\sigma : \overline{S} \to \overline{S}$ with uniform properties with respect to $\sigma \in \Sigma = (-\sigma_1, \sigma_1)$. In view of applications to parabolic systems we will consider vector valued functions $u : S \times X \to \mathbb{R}^m$.

We define U as the subset of all $u \in [C(\overline{S}; C(\overline{X}))]^m$, for which we can find a compact set $F \subset \Omega$ such that $u(t,x) \in F$ for all $(t,x) \in S \times X$. Obviously, U is open in $[C(\overline{S}; C(\overline{X}))]^m$. Next, we state conditions on the function A, which ensure that $\mathcal{A} \in C^1\big(U \times V; L^\infty(S \times X)\big)$ and $\mathcal{C} \in C^1\big(U \times V \times \Sigma; L^\infty(S \times X)\big)$:

Theorem 4.1. *Let us formulate the following C^1-Carathéodory conditions on A:*

(C1) *$(\xi,\lambda) \mapsto A(t,x,\xi,\lambda)$ belongs to $C^1(\Omega\times V)$ for almost all $(t,x) \in S\times X$, and $(t,x) \mapsto \frac{\partial A}{\partial \xi}(t,x,\xi,\lambda)$ and $(t,x) \mapsto \frac{\partial A}{\partial \lambda}(t,x,\xi,\lambda)$ are measurable for all $(\xi,\lambda) \in \Omega\times V$.*

(C2) *For all $\lambda \in V$ and compact sets $F \subset \Omega$ there exists a $\varrho > 0$ such that*

$$\big|\tfrac{\partial A}{\partial \xi}(t,x,\xi,\lambda)\big| + \big\|\tfrac{\partial A}{\partial \lambda}(t,x,\xi,\lambda)\big\|_{\Lambda^*} + \big|A(t,x,\xi,\lambda)\big| \le \varrho$$

for almost all $(t,x) \in S \times X$ and all $\xi \in F$.

(C3) *For all $\lambda \in V$, compact sets $F \subset \Omega$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\big|\tfrac{\partial A}{\partial \xi}(t,x,\xi,\lambda) - \tfrac{\partial A}{\partial \xi}(t,x,\eta,\mu)\big| + \big\|\tfrac{\partial A}{\partial \lambda}(t,x,\xi,\lambda) - \tfrac{\partial A}{\partial \lambda}(t,x,\eta,\mu)\big\|_{\Lambda^*} + \big|A(t,x,\xi,\lambda) - A(t,x,\eta,\mu)\big| < \varepsilon,$$

for almost all $(t,x) \in S\times X$ and all $\xi \in F$, $(\eta,\mu) \in F\times V$ with $|\xi-\eta|+\|\lambda-\mu\|_\Lambda < \delta$.

(C4) *For all $\lambda \in V$ and compact sets $F \subset \Omega$ there exists an $L > 0$ such that*

$$\big|\tfrac{\partial A}{\partial \xi}(t,x,\xi,\lambda) - \tfrac{\partial A}{\partial \xi}(t,x,\eta,\lambda)\big| \le L|\xi - \eta|,$$

for almost all $(t,x) \in S \times X$ and all ξ, $\eta \in F$.

(C5) *$t \mapsto A(t,x,\xi,\lambda)$ belongs to $C^1(\overline{S})$ for almost all $x \in X$ and all $(\xi,\lambda) \in \Omega\times V$, and $x \mapsto \frac{\partial A}{\partial s}(t,x,\xi,\lambda)$ is measurable for all $t \in \overline{S}$ and $(\xi,\lambda) \in \Omega \times V$.*

(C6) *For all $\lambda \in V$ and compact sets $F \subset \Omega$ there exists a $\varrho > 0$ such that*

$$\big|\tfrac{\partial A}{\partial s}(t,x,\xi,\lambda)\big| \le \varrho$$

for almost all $x \in X$, all $t \in \overline{S}$ and $\xi \in F$.

(C7) *For all $\lambda \in V$, compact sets $F \subset \Omega$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\begin{aligned}\big|\tfrac{\partial A}{\partial \xi}(t,x,\xi,\lambda) - \tfrac{\partial A}{\partial \xi}(s,x,\eta,\mu)\big| + \big\|\tfrac{\partial A}{\partial \lambda}(t,x,\xi,\lambda) - \tfrac{\partial A}{\partial \lambda}(s,x,\eta,\mu)\big\|_{\Lambda^*} \\ + \big|\tfrac{\partial A}{\partial s}(t,x,\xi,\lambda) - \tfrac{\partial A}{\partial s}(s,x,\eta,\mu)\big| + \big|A(t,x,\xi,\lambda) - A(s,x,\eta,\mu)\big| < \varepsilon\end{aligned}$$

for almost all $x \in X$, all s, $t \in \overline{S}$ and all $\xi \in F$, $(\eta,\mu) \in F \times V$, which satisfy the conditions $|s-t| < \delta$ and $|\xi - \eta| + \|\lambda - \mu\|_\Lambda < \delta$.

1. If conditions (C1), (C2), *and* (C3) *are satisfied, then the operator $\mathcal{A}$ defined by* (4.1) *belongs to $C^1\big(U \times V; L^\infty(S \times X)\big)$. Moreover, we have*

$$\big(\tfrac{\partial \mathcal{A}}{\partial u}(u,\lambda)\, v\big)(t,x) = \tfrac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda)\, v(t,x), \tag{4.3a}$$

$$\big(\tfrac{\partial \mathcal{A}}{\partial \lambda}(u,\lambda)\, \mu\big)(t,x) = \tfrac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda)\, \mu, \tag{4.3b}$$

for almost all $(t,x) \in S \times X$, all $(u,\lambda) \in U \times V$, $v \in [C(\overline{S}; C(\overline{X}))]^m$, and $\mu \in \Lambda$.

2. If, additionally, (C4) *holds true, then $u \mapsto \frac{\partial \mathcal{A}}{\partial u}(u,\lambda)$ is locally* Lipschitz *continuous from U into $\mathscr{L}\big([C(\overline{S}; C(\overline{X}))]^m; L^\infty(S \times X)\big)$ for all $\lambda \in V$.*

3. If conditions (C1), (C2), (C3), *and* (C5), (C6), (C7) *are satisfied, then the operator $\mathcal{C}$ defined by* (4.2) *belongs to $C^1\big(U \times V \times \Sigma; L^\infty(S \times X)\big)$, and we get*

$$\big(\tfrac{\partial \mathcal{C}}{\partial u}(u,\lambda,\sigma)\, v\big)(t,x) = \tfrac{\partial A}{\partial \xi}(T_\sigma(t),x,u(t,x),\lambda)\, v(t,x), \tag{4.4a}$$

$$\big(\tfrac{\partial \mathcal{C}}{\partial \lambda}(u,\lambda,\sigma)\, \mu\big)(t,x) = \tfrac{\partial A}{\partial \lambda}(T_\sigma(t),x,u(t,x),\lambda)\, \mu, \tag{4.4b}$$

$$\tfrac{\partial \mathcal{C}}{\partial \sigma}(u,\lambda,\sigma)(t,x) = \tfrac{\partial A}{\partial s}(T_\sigma(t),x,u(t,x),\lambda)\, \tfrac{\partial T}{\partial \sigma}(\sigma,t), \tag{4.4c}$$

for almost all $(t,x) \in S \times X$ and all $(u,\lambda,\sigma) \in U \times V \times \Sigma$, $v \in [C(\overline{S}; C(\overline{X}))]^m$, and $\mu \in \Lambda$.

Proof. For the sake of simplicity let us denote by $\|\ \|_C$ the norm in $[C(\overline{S}; C(\overline{X}))]^m$.

1. From the definition of the set U and from conditions (C1) and (C2) it follows that $\mathcal{A}$ maps $U \times V$ into $L^\infty(S \times X)$.

In order to prove (4.3a) we fix a pair $(u,\lambda) \in U \times V$ and $\varepsilon > 0$. Again, the definition of U and conditions (C1) and (C2) yield that the operator, which assigns v to

$$(t,x) \mapsto \tfrac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda)\, v(t,x),$$

is a linear continuous map from $[C(\overline{S}; C(\overline{X}))]^m$ in to $L^\infty(S \times X)$. Since U is an open subset of $[C(\overline{S}; C(\overline{X}))]^m$, we can choose a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ with $\|v\|_C < \delta$ we have both $u(t,x) \in F$ and $(u+v)(t,x) \in F$ for almost all $(t,x) \in S \times X$. Taking δ small enough we can assume that this δ corresponds to λ, F and ε with respect to condition (C3). Hence, for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ with $\|v\|_C < \delta$ and almost all $(t,x) \in S \times X$ we obtain

$$\begin{aligned}&\big|A(t,x,(u+v)(t,x),\lambda) - A(t,x,u(t,x),\lambda) - \tfrac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda)\, v(t,x)\big| \\ &\qquad = \big| \textstyle\int_0^1 \big(\tfrac{\partial A}{\partial \xi}(t,x,(u+\tau v)(t,x),\lambda) - \tfrac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda)\big)\, d\tau\, v(t,x)\big| \le \varepsilon \|v\|_C,\end{aligned}$$

which proves (4.3a) and the differentiability of $\mathcal{A}$ in $U \times V$ with respect to u.

To show that (4.3b) holds true, we fix a pair $(u,\lambda) \in U \times V$ and $\varepsilon > 0$. From the definition of the set U and from conditions (C1) and (C2) it follows that the operator, which assigns μ to

$$(t,x) \mapsto \tfrac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda)\, \mu,$$

is a linear continuous map from Λ into $L^\infty(S \times X)$. We choose $F \subset \Omega$ such that $u(t,x) \in F$ for almost all $(t,x) \in S \times X$. Additionally, we take $\delta > 0$ small enough such that $\lambda + \mu \in V$ holds true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ and we suppose that this δ is suitable for λ, F and $\varepsilon > 0$ from condition (C3). Then, for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ and almost all $(t,x) \in S \times X$ we get

$$\begin{aligned}&\big|A(t,x,u(t,x),\lambda+\mu) - A(t,x,u(t,x),\lambda) - \tfrac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda)\, \mu\big| \\ &\qquad = \big| \textstyle\int_0^1 \big(\tfrac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda+\tau\mu) - \tfrac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda)\big)\, d\tau\, \mu\big| \le \varepsilon \|\mu\|_\Lambda,\end{aligned}$$

which leads to (4.3b) and the differentiability of $\mathcal{A}$ in $U \times V$ with respect to λ.

2. In order to prove that $\frac{\partial \mathcal{A}}{\partial u}$ and $\frac{\partial \mathcal{A}}{\partial \lambda}$ are continuous maps, we fix a pair $(u,\lambda) \in U \times V$ and some $\varepsilon > 0$. Because U is open in $[C(\overline{S}; C(\overline{X}))]^m$, we can find a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ with $\|v\|_C < \delta$ we have both $u(t,x) \in F$ and $(u+v)(t,x) \in F$ for almost all $(t,x) \in S \times X$. Taking δ small enough we ensure that this δ corresponds to λ, F and ε with respect to condition (C3) and that $\lambda + \mu \in V$ holds true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$. Hence, for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ and $\mu \in \Lambda$ with $\|v\|_C + \|\mu\|_\Lambda < \delta$, for all $\varphi \in [C(\overline{S}; C(\overline{X}))]^m$ and $\chi \in \Lambda$ and almost all $(t,x) \in S \times X$, condition (C3) yields

$$\begin{aligned}&\big|\big(\tfrac{\partial \mathcal{A}}{\partial u}(u+v,\lambda+\mu)\, \varphi - \tfrac{\partial \mathcal{A}}{\partial u}(u,\lambda)\, \varphi\big)(t,x)\big| \\ &\qquad = \big|\big(\tfrac{\partial A}{\partial \xi}(t,x,(u+v)(t,x),\lambda+\mu) - \tfrac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda)\big)\, \varphi(t,x)\big| \le \varepsilon \|\varphi\|_C,\end{aligned}$$

and

$$\begin{aligned}&\big|\big(\tfrac{\partial \mathcal{A}}{\partial \lambda}(u+v,\lambda+\mu)\, \chi - \tfrac{\partial \mathcal{A}}{\partial \lambda}(u,\lambda)\, \chi\big)(t,x)\big| \\ &\qquad = \big|\big(\tfrac{\partial A}{\partial \lambda}(t,x,(u+v)(t,x),\lambda+\mu) - \tfrac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda)\big)\, \chi\big| \le \varepsilon \|\chi\|_\Lambda,\end{aligned}$$

in other words, $\frac{\partial \mathcal{A}}{\partial u}$ and $\frac{\partial \mathcal{A}}{\partial \lambda}$ are continuous on $U \times V$.

3. Next, we show that $u \mapsto \frac{\partial \mathcal{A}}{\partial u}$ is locally Lipschitz continuous, whenever all the conditions (C1), (C2), (C3), and (C4) are satisfied. To do so, we fix $(u,\lambda) \in U \times V$, and, again, we choose a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ with $\|v\|_C < \delta$ we have both $u(t,x) \in F$ and $(u+v)(t,x) \in F$ for almost all $(t,x) \in S \times X$. Let $L > 0$ be the Lipschitz constant, which corresponds to λ and F in condition (C4). Then, for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ with $\|v\|_C < \delta$, for all $\varphi \in [C(\overline{S}; C(\overline{X}))]^m$ and almost all $(t,x) \in S \times X$, we arrive at

$$\begin{aligned}&\big|\big(\tfrac{\partial \mathcal{A}}{\partial u}(u+v,\lambda)\,\varphi - \tfrac{\partial \mathcal{A}}{\partial u}(u,\lambda)\,\varphi\big)(t,x)\big| \\ &\qquad = \big|\big(\tfrac{\partial \mathcal{A}}{\partial \xi}(t,x,(u+v)(t,x),\lambda) - \tfrac{\partial \mathcal{A}}{\partial \xi}(t,x,u(t,x),\lambda)\big)\,\varphi(t,x)\big| \le L\|v\|_C\|\varphi\|_C,\end{aligned}$$

which leads to the local Lipschitz continuity of $u \mapsto \frac{\partial \mathcal{A}}{\partial u}(u,\lambda)$.

4. Analogously to Step 1, we can use the definition of the set U and conditions (C1), (C2), (C3), (C5), (C6), and (C7) to show that $\mathcal{C}$ maps $U \times V \times \Lambda$ into $L^\infty(S \times X)$, that $\mathcal{C}$ is differentiable with respect to u and λ in $U \times V \times \Sigma$, and that (4.4a) and (4.4b) are the corresponding derivatives.

5. To prove the remaining assertions, we make use of the uniform properties of the family of temporal transformations: We take $M > 0$ such that $\big|\frac{\partial T}{\partial \sigma}(\sigma,t)\big| \le M$ for all $t \in \overline{S}$ and $\sigma \in \Sigma$.

In order to show that $\frac{\partial \mathcal{C}}{\partial u}$ and $\frac{\partial \mathcal{C}}{\partial \lambda}$ are continuous operators, we fix a triple $(u,\lambda,\sigma) \in U \times V \times \Sigma$ and $\varepsilon > 0$. Since U is an open subset of $[C(\overline{S}; C(\overline{X}))]^m$, there exists a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ with $\|v\|_C < \delta$ we have both $u(t,x) \in F$ and $(u+v)(t,x) \in F$ for almost all $(t,x) \in S \times X$. We choose δ small enough such that it corresponds to λ, F and ε with respect to condition (C7) and that $\lambda + \mu \in V$ and $\sigma + \kappa \in \Sigma$ hold true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ and all $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$. Consequently, for all $v \in [C(\overline{S}; C(\overline{X}))]^m$, $\mu \in \Lambda$, and $\kappa \in \mathbb{R}$ with $\|v\|_C + \|\mu\|_\Lambda < \delta$ and $|\kappa| < \frac{\delta}{M}$, for all $\varphi \in [C(\overline{S}; C(\overline{X}))]^m$ and $\chi \in \Lambda$, and almost all $(t,x) \in S \times X$, we obtain

$$\begin{aligned}&\big|\big(\tfrac{\partial \mathcal{C}}{\partial u}(u+v,\lambda+\mu,\sigma+\kappa)\,\varphi - \tfrac{\partial \mathcal{C}}{\partial u}(u,\lambda,\sigma)\,\varphi\big)(t,x)\big| \\ &= \big|\big(\tfrac{\partial \mathcal{A}}{\partial \xi}(T_{\sigma+\kappa}(t),x,(u+v)(t,x),\lambda+\mu) - \tfrac{\partial \mathcal{A}}{\partial \xi}(T_\sigma(t),x,u(t,x),\lambda)\big)\,\varphi(t,x)\big| \le \varepsilon\|\varphi\|_C,\end{aligned}$$

and

$$\begin{aligned}&\big|\big(\tfrac{\partial \mathcal{C}}{\partial \lambda}(u+v,\lambda+\mu,\sigma+\kappa)\,\chi - \tfrac{\partial \mathcal{C}}{\partial \lambda}(u,\lambda,\sigma)\,\chi\big)(t,x)\big| \\ &\qquad = \big|\tfrac{\partial \mathcal{A}}{\partial \lambda}(T_{\sigma+\kappa}(t),x,(u+v)(t,x),\lambda+\mu)\,\chi - \tfrac{\partial \mathcal{A}}{\partial \lambda}(T_\sigma(t),x,u(t,x),\lambda)\,\chi\big| \le \varepsilon\|\chi\|_\Lambda,\end{aligned}$$

which means, $\frac{\partial \mathcal{C}}{\partial u}$ and $\frac{\partial \mathcal{C}}{\partial \lambda}$ are continuous on $U \times V$.

6. For the proof of (4.4c) we fix a triple $(u, \lambda, \sigma) \in U \times V \times \Sigma$ and $\varepsilon > 0$. Because of conditions (C1), (C2), (C5), and (C6) the function

$$(t, x) \mapsto \tfrac{\partial A}{\partial s}(T_\sigma(t), x, u(t, x), \lambda)\, \tfrac{\partial T}{\partial \sigma}(\sigma, t),$$

belongs to $L^\infty(S \times X)$. We choose a compact set $F \subset \Omega$ such that $u(t, x) \in F$ for almost all $(t, x) \in S \times X$ and some bound $\varrho > 0$ corresponding to λ and F with respect to (C6). Furthermore, we can find some $\delta > 0$ such that for all $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$ we have both $\sigma + \kappa \in \Sigma$ and $\bigl|T_{\sigma+\kappa}(t) - T_\sigma(t) - \frac{\partial T}{\partial \sigma}(\sigma, t)\kappa\bigr| < \varepsilon|\kappa|$. Simultaneously, we take δ small enough such it corresponds to λ, F and ε from condition (C7). For all $\xi \in F$ and $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$ and almost all $(t, x) \in S \times X$ this leads to

$$\begin{aligned} &A(T_{\sigma+\kappa}(t), x, \xi, \lambda) - A(T_\sigma(t), x, \xi, \lambda) - \tfrac{\partial A}{\partial s}(T_\sigma(t), x, \xi, \lambda)\, \tfrac{\partial T}{\partial \sigma}(\sigma, t)\kappa \\ &\quad = \int_0^1 \tfrac{\partial A}{\partial s}\bigl((1-\tau)T_\sigma(t) + \tau T_{\sigma+\kappa}(t), x, \xi, \lambda\bigr)\, d\tau \, \bigl(T_{\sigma+\kappa}(t) - T_\sigma(t) - \tfrac{\partial T}{\partial \sigma}(\sigma, t)\kappa\bigr) \\ &\qquad + \int_0^1 \bigl(\tfrac{\partial A}{\partial s}\bigl((1-\tau)T_\sigma(t) + \tau T_{\sigma+\kappa}(t), x, \xi, \lambda\bigr) - \tfrac{\partial A}{\partial s}(T_\sigma(t), x, \xi, \lambda)\bigr)\, d\tau\, \tfrac{\partial T}{\partial \sigma}(\sigma, t)\kappa, \end{aligned}$$

which yields

$$\begin{aligned} &\bigl|A(T_{\sigma+\kappa}(t), x, u(t, x), \lambda) - A(T_\sigma(t), x, u(t, x), \lambda) - \tfrac{\partial A}{\partial s}(T_\sigma(t), x, u(t, x), \lambda)\, \tfrac{\partial T}{\partial \sigma}(\sigma, t)\kappa\bigr| \\ &\qquad \le (\varrho + M)\varepsilon|\kappa| \quad \text{for all } \kappa \in \mathbb{R} \text{ with } |\kappa| < \tfrac{\delta}{M} \text{ and almost all } (t, x) \in S \times X. \end{aligned}$$

This proves (4.4c) and the differentiability of $\mathcal{C}$ in $U \times V \times \Sigma$ with respect to σ.

7. Finally, we show that $\frac{\partial \mathcal{C}}{\partial \sigma}$ is continuous: We fix a triple $(u, \lambda, \sigma) \in U \times V \times \Sigma$ and some $\varepsilon > 0$. Because U is open in $[C(\overline{S}; C(\overline{X}))]^m$, we can find a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in [C(\overline{S}; C(\overline{X}))]^m$ with $\|v\|_C < \delta$ we have $u(t, x) \in F$ and $(u + v)(t, x) \in F$ for almost all $(t, x) \in S \times X$. In view of (C6) we take some bound $\varrho > 0$ depending on λ and F, and we choose δ small enough such that it corresponds to λ, F and ε with respect to condition (C7), that $\lambda + \mu \in V$, $\sigma + \kappa \in \Sigma$ and $\bigl|\frac{\partial T}{\partial \sigma}(\sigma + \kappa, t) - \frac{\partial T}{\partial \sigma}(\sigma, t)\bigr| < \varepsilon$ hold true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ all $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$, and all $t \in \overline{S}$. Consequently, for all $v \in [C(\overline{S}; C(\overline{X}))]^m$, $\mu \in \Lambda$, and $\kappa \in \mathbb{R}$ with $\|v\|_C + \|\mu\|_\Lambda < \delta$ and $|\kappa| < \frac{\delta}{M}$, and almost all $(t, x) \in S \times X$, we get

$$\begin{aligned} &\bigl|\bigl(\tfrac{\partial \mathcal{C}}{\partial \sigma}(u + v, \lambda + \mu, \sigma + \kappa) - \tfrac{\partial \mathcal{C}}{\partial \sigma}(u, \lambda, \sigma)\bigr)(t, x)\bigr| \\ &\quad \le \bigl|\bigl(\tfrac{\partial A}{\partial s}(T_{\sigma+\kappa}(t), x, (u + v)(t, x), \lambda + \mu) - \tfrac{\partial A}{\partial s}(T_\sigma(t), x, u(t, x), \lambda)\bigr)\, \tfrac{\partial T}{\partial \sigma}(\sigma + \kappa, t)\bigr| \\ &\qquad + \bigl|\tfrac{\partial A}{\partial s}(T_\sigma(t), x, u(t, x), \lambda)\bigl(\tfrac{\partial T}{\partial \sigma}(\sigma + \kappa, t) - \tfrac{\partial T}{\partial \sigma}(\sigma, t)\bigr)\bigr| \le (M + \varrho)\varepsilon, \end{aligned}$$

which finishes the proof. □

4.2 Right hand sides

In this subsection we consider operators $\mathcal{F} \in C^1\big(U \times V; L_2^\omega(S; H^{-1}(G))\big)$, which are candidates for the right hand sides $\mathcal{F}^\alpha$ of problem (2.5). Please, remember that the notation $G = X \cup \Gamma$ indicates the decomposition of the regular set $G \subset \mathbb{R}^n$ into its interior $X \subset \mathbb{R}^n$ and its Neumann boundary part $\Gamma \subset \partial G$.

Theorem 4.2. *If $\omega \in [0, n+2]$ and*

$$\begin{aligned} \mathcal{G}^\ell &\in C^1\big(U \times V; L_2^\omega(S; L^2(X))\big),\\ \mathcal{G}^0 &\in C^1\big(U \times V; L_2^{\omega-2}(S; L^2(X))\big),\\ \mathcal{G}^\Gamma &\in C^1\big(U \times V; L_2^{\omega-1}(S; L^2(\Gamma))\big), \end{aligned}$$

are Volterra *operators for $\ell \in \{1, \dots, n\}$, then the following statements hold true:*

1. The map $\mathcal{F}$, defined by

$$\begin{aligned} \big\langle \mathcal{F}(u,\lambda), \varphi \big\rangle = &\int_S \int_X \sum_{\ell=1}^n \mathcal{G}^\ell(u,\lambda)(s)\, \tfrac{\partial \varphi}{\partial x_\ell}(s)\, d\lambda^n\, ds \\ &+ \int_S \int_X \mathcal{G}^0(u,\lambda)(s)\, \varphi(s)\, d\lambda^n\, ds + \int_S \int_\Gamma \mathcal{G}^\Gamma(u,\lambda)(s)\, \varphi(s)\, d\lambda_\Gamma\, ds \end{aligned} \tag{4.5}$$

for $(u,\lambda) \in U \times V$ and $\varphi \in L^2(S; H_0^1(G))$, belongs to $C^1\big(U \times V; L_2^\omega(S; H^{-1}(G))\big)$ and admits the Volterra *property.*

2. If, for certain $\lambda \in V$ and all $\ell \in \{1, \dots, n\}$ the maps

$$u \in U \mapsto \tfrac{\partial \mathcal{G}^\ell}{\partial u}(u,\lambda) \in \mathscr{L}\big([C(\overline{S}; C(\overline{X}))]^m; L_2^\omega(S; L^2(X))\big), \tag{4.6a}$$
$$u \in U \mapsto \tfrac{\partial \mathcal{G}^0}{\partial u}(u,\lambda) \in \mathscr{L}\big([C(\overline{S}; C(\overline{X}))]^m; L_2^{\omega-2}(S; L^2(X))\big), \tag{4.6b}$$
$$u \in U \mapsto \tfrac{\partial \mathcal{G}^\Gamma}{\partial u}(u,\lambda) \in \mathscr{L}\big([C(\overline{S}; C(\overline{X}))]^m; L_2^{\omega-1}(S; L^2(\Gamma))\big), \tag{4.6c}$$

are locally Lipschitz *continuous, then the operator*

$$u \in U \mapsto \tfrac{\partial \mathcal{F}}{\partial u}(u,\lambda) \in \mathscr{L}\big([C(\overline{S}; C(\overline{X}))]^m; L_2^\omega(S; H^{-1}(G))\big), \tag{4.7}$$

is also locally Lipschitz *continuous.*

3. Let W be an open set in $[C(\overline{S}; C(\overline{X}))]^m$ such that $\big(\mathcal{T}_\sigma^0 w^1, \dots, \mathcal{T}_\sigma^0 w^m\big) \in U$ holds true for every $w \in W$ and $\sigma \in \Sigma$, and assume that for some $\lambda \in V$ and all $\ell \in \{1, \dots, n\}$ the assignments

$$(w,\sigma) \mapsto \mathcal{T}_0^\sigma \mathcal{G}^\ell(\mathcal{T}_\sigma^0 w^1, \dots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{4.8a}$$
$$(w,\sigma) \mapsto \mathcal{T}_0^\sigma \mathcal{G}^0(\mathcal{T}_\sigma^0 w^1, \dots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{4.8b}$$
$$(w,\sigma) \mapsto \mathcal{T}_\Gamma^\sigma \mathcal{G}^\Gamma(\mathcal{T}_\sigma^0 w^1, \dots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{4.8c}$$

generate continuously differentiable operators

$$\begin{aligned}\mathcal{G}^{\ell}_{\lambda} &\in C^1\big(W \times \Sigma; L^{\omega}_2(S; L^2(X))\big),\\ \mathcal{G}^{0}_{\lambda} &\in C^1\big(W \times \Sigma; L^{\omega-2}_2(S; L^2(X))\big),\\ \mathcal{G}^{\Gamma}_{\lambda} &\in C^1\big(W \times \Sigma; L^{\omega-1}_2(S; L^2(\Gamma))\big),\end{aligned}$$

respectively. Then the map $(w,\sigma) \mapsto \mathcal{T}^{\sigma}\mathcal{F}(\mathcal{T}^0_{\sigma}w^1, \dots, \mathcal{T}^0_{\sigma}w^m, \lambda)$ *defines a* VOLTERRA *operator* $\mathcal{F}_{\lambda} \in C^1\big(W \times \Sigma; L^{\omega}_2(S; H^{-1}(G))\big)$.

Proof. 1. Due to (2.1), (2.2) the assignment $(g, g_0, g_{\Gamma}) \mapsto \Psi(g, g_0, g_{\Gamma})$, defined by

$$\begin{aligned}\big\langle \Psi(g, g_0, g_{\Gamma}), \varphi\big\rangle &= \int_S \int_X g(s) \cdot \nabla\varphi(s)\, d\lambda^n\, ds\\ &\quad + \int_S \int_X g_0(s)\, \varphi(s)\, d\lambda^n\, ds + \int_S \int_{\Gamma} g_{\Gamma}(s)\, \varphi(s)\, d\lambda_{\Gamma}\, ds \qquad (4.9)\end{aligned}$$

for $\varphi \in L^2(S; H^1_0(G))$, generates a linear continuous operator

$$\Psi : [L^{\omega}_2(S; L^2(X))]^n \times L^{\omega-2}_2(S; L^2(X)) \times L^{\omega-1}_2(S; L^2(\Gamma)) \to L^{\omega}_2(S; H^{-1}(G)),$$

and its norm depends on n and G, only. Since (4.5) holds true, we obtain

$$\mathcal{F}(u,\lambda) = \Psi\big(\mathcal{G}^1(u,\lambda), \dots, \mathcal{G}^n(u,\lambda), \mathcal{G}^0(u,\lambda), \mathcal{G}^{\Gamma}(u,\lambda)\big) \quad \text{for all } (u,\lambda) \in U \times V.$$

Hence, as a superposition of continuously differentiable operators, the map $\mathcal{F}$ belongs to $C^1\big(U \times V; L^{\omega}_2(S; H^{-1}(G))\big)$. Moreover, for all $(u,\lambda) \in U \times V$ and $v \in [C(\overline{S}; C(\overline{X}))]^m$ we get identity

$$\begin{aligned}\tfrac{\partial\mathcal{F}}{\partial u}(u,\lambda)\, v &= \Psi\big(\tfrac{\partial\mathcal{G}^1}{\partial u}(u,\lambda)\, v, \dots, \tfrac{\partial\mathcal{G}^n}{\partial u}(u,\lambda)\, v, \mathcal{G}^0(u,\lambda), \mathcal{G}^{\Gamma}(u,\lambda)\big)\\ &\quad + \Psi\big(\mathcal{G}^1(u,\lambda), \dots, \mathcal{G}^n(u,\lambda), \tfrac{\partial\mathcal{G}^0}{\partial u}(u,\lambda)\, v, \mathcal{G}^{\Gamma}(u,\lambda)\big)\\ &\quad + \Psi\big(\mathcal{G}^1(u,\lambda), \dots, \mathcal{G}^n(u,\lambda), \mathcal{G}^0(u,\lambda), \tfrac{\partial\mathcal{G}^{\Gamma}}{\partial u}(u,\lambda)\, v\big). \qquad (4.10)\end{aligned}$$

2. For $\ell \in \{1, \dots, n\}$ the maps $u \mapsto \frac{\partial\mathcal{G}^{\ell}}{\partial u}(u,\lambda)$, $u \mapsto \frac{\partial\mathcal{G}^0}{\partial u}(u,\lambda)$, and $u \mapsto \frac{\partial\mathcal{G}^{\Gamma}}{\partial u}(u,\lambda)$ are locally LIPSCHITZ continuous in the sense of (4.6) by assumption. Applying the mean value theorem, the maps $u \mapsto \mathcal{G}^{\ell}(u,\lambda)$, $u \mapsto \mathcal{G}^0(u,\lambda)$, and $u \mapsto \mathcal{G}^{\Gamma}(u,\lambda)$ are locally LIPSCHITZ continuous in the sense of (4.6), too. Now, it is an easy consequence of (4.10) that the operator $u \mapsto \frac{\partial\mathcal{F}}{\partial u}(u,\lambda)$ given by (4.7) is locally LIPSCHITZ continuous.

3. Because of (4.8) and (4.9) for all $(w,\sigma) \in W \times \Sigma$ we obtain

$$\mathcal{F}_{\lambda}(w,\sigma) = \Psi\big(\mathcal{G}^1_{\lambda}(w,\sigma), \dots, \mathcal{G}^n_{\lambda}(w,\sigma), \mathcal{G}^0_{\lambda}(w,\sigma), \mathcal{G}^{\Gamma}_{\lambda}(w,\sigma)\big).$$

As a superposition of continuously differentiable maps, the operator $\mathcal{F}_{\lambda}$ belongs to $C^1\big(W \times \Sigma; L^{\omega}_2(S; H^{-1}(G))\big)$. □

Acknowledgement

Jens A. Griepentrog gratefully acknowledges financial support by the DFG Research Center Matheon *Mathematics for key technologies*. Research about the subject of this text has been carried out in the framework of the Matheon project D19: *Local existence, uniqueness, and smooth dependence for quasilinear parabolic problems with nonsmooth data.*

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral: Regularity and nonuniqueness results for parabolic problems arising in some physical models, having natural growth in the gradient. *J. Math. Pures Appl. 90*, 242–269 (2008).

[2] H. Amann: Abstract methods in differential equations. *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 97*, 89–105 (2003).

[3] H. Amann: Quasilinear parabolic problems via maximal regularity. *Adv. Differential Equations 10*, 1081–1110 (2005).

[4] H. Amann: Maximal regularity and quasilinear parabolic boundary value problems. Editors: C.-C. Chen, M. Chipot, C.-S. Liu. In: *Recent Advances in Elliptic and Parabolic Problems*, 1–17. World Scientific, 2005.

[5] H. Amann: Nonlocal quasilinear parabolic equations. *Russian Math. Surveys 60*, 1021–1033 (2005).

[6] H. Amann: Quasilinear parabolic functional evolution equations. Editors: M. Chipot, H. Ninomiya. In: *Recent Advances in Elliptic and Parabolic Issues*, 19–44. World Scientific, 2006.

[7] S. Campanato: Equazioni paraboliche del secondo ordine e spazi $\mathfrak{L}^{(2,\theta)}(\Omega,\delta)$. *Ann. Mat. Pura Appl. 73*, 55–102 (1966).

[8] P. Clement, G. Simonett: Maximal regularity in continuous interpolation spaces and quasilinear parabolic equations. *J. Evolution Equations 1*, 39–67 (2001).

[9] G. Da Prato: Spazi $\mathfrak{L}^{p,\theta}(\Omega,\delta)$ e loro proprieta. *Ann. Mat. Pura Appl. 69*, 383–392 (1965).

[10] J. Elschner, V. Maz'ya, J. Rehberg, G. Schmidt: Solutions for quasilinear nonsmooth evolution systems in L^p. *Arch. Rational Mech. Anal. 171*, 219–262 (2004).

[11] J. Escher, G. Prokert: Analyticity of solutions to nonlinear parabolic equations on manifolds and applications to Stokes problems. *J. Math. Fluid Mech. 8*, 1–35 (2006).

[12] L.C. Evans, R.F. Gariepy: *Measure Theory and Fine Properties of Functions.* Boca Raton, New York, London, Tokyo: CRC Press, 1992.

[13] J. Escher, J. Prüss, G. Simonett: A new approach to regularity of solutions for parabolic equations. Editors: G. Ruiz Goldstein et al., In: *Evolution Equations*, 167–190, *Lecture Notes Pure Appl. Math. 234*. New York: Dekker, 2003.

[14] H. Gajewski, I. V. Skrypnik: On the uniqueness of solutions for nonlinear elliptic-parabolic equations. *J. Evolution Equations 3*, 247–281 (2003).

[15] H. Gajewski, I. V. Skrypnik: On the unique solvability of nonlocal drift-diffusion-type problems. *Nonlinear Anal. 56*, 803–830 (2004).

[16] H. Gajewski, I. V. Skrypnik: Existence and uniqueness results for reaction-diffusion processes of electrically charged species. Editors: H. Brezis, M. Chipot, J. Escher. In: *Progress in Nonlinear Differential Equations and Their Applications 64*, 151–188. Basel: Birkhäuser-Verlag, 2005.

[17] J. A. Griepentrog: Linear elliptic boundary value problems with nonsmooth data: Campanato spaces of functionals. *Math. Nachr. 243*, 19–42 (2002).

[18] J. A. Griepentrog, L. Recke: Linear elliptic boundary value problems with nonsmooth data: Normal solvability on Sobolev–Campanato spaces. *Math. Nachr. 225*, 39–74 (2001).

[19] J. A. Griepentrog: Sobolev–Morrey spaces associated with evolution equations. *Adv. Differential Equations 12*, 781–840 (2007).

[20] J. A. Griepentrog: Maximal regularity for nonsmooth parabolic problems in Sobolev–Morrey spaces. *Adv. Differential Equations 12*, 1031–1078 (2007).

[21] K. Gröger: A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order eliptic differential equations. *Math. Ann. 283*, 679–687 (1989).

[22] K. Gröger, L. Recke: Preduals of Campanato and Sobolev–Campanato spaces: A general approach. *Math. Nachr. 230*, 45–72 (2001).

[23] K. Gröger, L. Recke: Applications of differential calculus to quasilinear elliptic boundary value problems with nonsmooth data. *NoDEA Nonlinear Differential Equations Appl. 13*, 263–285 (2006).

[24] R. Haller-Dintelmann, J. Rehberg: Maximal parabolic regularity for divergence operators including mixed boundary conditions. *WIAS Preprint 1288* (2008).

[25] M. Hieber, J. Rehberg: Quasilinear parabolic systems with mixed boundary conditions on nonsmooth domains. *SIAM J. Math. Anal. 40*, 292–305 (2008).

[26] J. J. Sharples: Linear and quasilinear parabolic equations in Sobolev space. *J. Differential Equations 202*, 111–142 (2004).

[27] E. Zeidler: *Nonlinear Functional Analysis and its Applications. Part I: Fixed-Point Theorems.* Second Corrected Printing. New York, Berlin, Heidelberg: Springer, 1992.