Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Strong stationary solutions to equilibrium problems with equilibrium constraints with applications to an electricity spot market model

René Henrion[1], Jiří Outrata [2], Thomas Surowiec[3]

submitted: January 26, 2009

[1] Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: henrion@wias-berlin.de

[2] Institute of
Information Theory and
Automation Theory
18208 Praha 8
Czech Republic
E-Mail: outrata@utia.cas.cz

[3] Humboldt University Berlin
Unter den Linden 6
10099 Berlin
Germany
E-Mail: surowiec@math.hu-berlin.de

No. 1396
Berlin 2008

2000 *Mathematics Subject Classification.* 90C30, 49J53.

Key words and phrases. Equilibrium problems with equilibrium constraints, EPEC, strong stationary solutions, electricity spot market.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

In this paper, we consider the characterization of strong stationary solutions to equilibrium problems with equilibrium constraints (EPECs). Assuming that the underlying generalized equation satisfies strong regularity in the sense of Robinson, an explicit multiplier-based stationarity condition can be derived. This is applied then to an equilibrium model arising from ISO-regulated electricity spot markets.

1 Introduction

This paper deals with the characterization of strong stationary solutions to so-called equilibrium problems with equilibrium constraints. In its first part, a general result on the characterization of strong stationary points to such problems is proved which in the second part is applied to a concrete model of an electricity spot market.

In optimization problems with smooth objectives it is tempting to describe the local behavior of the constraint set by the respective Frèchet (regular) normal cone (cf. Definition in Section 2), because then the resulting optimality conditions will be as sharp as possible. Unfortunately, in most cases this is not possible due to the fact that in the needed calculus rules one has opposite inclusions (e.g., [15], Th. 6.14). Nevertheless, under some additional conditions, the (wrong) inclusions become equalities and then we arrive at workable sharp optimality conditions. Such a situation has been encountered in the context of mathematical programs with complementarity constraints (MPCCs), where under various additional conditions the limiting normal cone (cf. Definition in Section 2 below) could be replaced by the regular one ([5],[4]). In the first part of this paper (Section 3), we examine another situation of this sort, related to a parameterized equilibrium, governed by a variational inequality with a polyhedral constraint set. In this case under Robinson's strong regularity, the behavior of this equilibrium constraint can also be suitably described by the respective regular normal cone, and we speak then in accordance with [16] about strong stationary points. This result could then be applied to characterize solutions to the following hierarchical model:

Consider a two-level game with n players on the upper level, called *leaders*. Each of them aims to minimize his objective f_i ($i = 1, \ldots, n$) by using a strategy x_i from his set of *admissible strategies* ω_i. The value of f_i depends, however, not only on the vector $x = (x_1, \ldots, x_n)$ of decisions of all leaders but also on the response z of players on a lower level called *followers*. We consider a setting where z is uniquely determined by the generalized equation

$$0 \in F(x, z) + N_{\Xi}(z),$$

where F is a continuously differentiable operator and Ξ is a closed convex set.

Under an *equilibrium problem with equilibrium constraints* (EPEC) we understand now the n-tuple of mutually coupled optimization problems

$$\min_{x_i \in \omega_i, z \in \Xi} \{ f_i(x_{-i}, x_i, z) \, | 0 \in F(x_{-i}, x_i, z) + N_{\Xi}(z) \} \quad (i = 1, \ldots, n), \tag{1}$$

where

$$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

represents the vector of decisions taken by the remaining leaders. In order to simplify the notation, we adopt throughout the paper the convention $x = (x_{-i}, x_i)$ for all $i = 1, \ldots, n$.

Clearly, each of the problems (1) is a standard *mathematical problem with equilibrium constraints* (MPEC) in the sense of [11] or [12] in variables (x_i, z) and parameterized by x_{-i}. As a (local) solution to (1) we declare any vector $(\bar{x}, \bar{z})$ such that for all $i = 1, \ldots, n$ the pair $(\bar{x}_i, \bar{z})$ belongs to the set of (local) solutions to the MPEC

$$\min_{x_i \in \omega_i, z \in \Xi} \{ f_i(\bar{x}_{-i}, x_i, z) \, | 0 \in F(\bar{x}_{-i}, x_i, z) + N_{\Xi}(z) \}.$$

It is well known that an EPEC may very well not possess any solution at all. Nevertheless, it provides a useful modeling framework for a number of problems associated in particular with oligopolistic markets. Such a problem, concerning oligopolistic competition in a regulated electricity spot market is thoroughly analyzed in the second part of the paper. Section 4 presents the simplified model of an ISO-regulated electricity spot market. In Section 5 its basic structural properties required for the application of the abstract result are compiled. Section 6 then provides the explicit translation of the abstract strong stationarity conditions to the concrete model. Finally, the results are illustrated for a small two-settlements example 6.3.

2 Concepts and tools from Variational Analysis

We start by defining the main objects necessary for our analysis. For a closed set $C \subseteq \mathbb{R}^n$ and a point $\bar{x} \in C$, the *Contingent* or *Bouligand cone* to C at $\bar{x} \in C$ is defined

$$T_C(\bar{x}) := \operatorname*{Lim\,sup}_{\tau \searrow 0} \tau^{-1}(C - \bar{x}) = \{v \in \mathbb{R}^n \,|\, \exists \tau_k \searrow 0, \exists v_k \to v : \forall k, \bar{x} + \tau_k v_k \in C\}$$

Here, 'Lim sup' in the definition above is the upper limit of sets in the sense of Kuratowski-Painlevé, cf. [15]. Accordingly, we define the *Fréchet normal cone* to C at $\bar{x} \in C$ by

$$\widehat{N}_C(\bar{x}) := [T_C(\bar{x})]^\circ,$$

where '$[\,\cdot\,]^\circ$' denotes the negative dual operation. Then the *limiting or Mordukhovich normal cone* to C at $\bar{x} \in C$ is derived from the Fréchet normal cone in the following manner:

$$N_C(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C}} \widehat{N}_C(x).$$

Throughout this text we work with generalized equations of the following form

$$0 \in F(x, z) + N_C(z), \tag{2}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is at least continuously differentiable and C is a nonempty closed and convex subset of $\mathbb{R}^m$. Correspondingly, we define the solution set mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ via (2):

$$S(x) := \{z \in \mathbb{R}^m \,|\, 0 \in F(x, z) + N_C(z)\} \tag{3}$$

For some reference point $(\bar{x}, \bar{z})$, where $\bar{z} \in S(\bar{z})$, we define the multifunction $\Sigma : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ via a local partial linearization of (2):

$$\Sigma(\xi) := \{z \in \mathbb{R}^m \,|\, \xi \in F(\bar{x}, \bar{z}) + \nabla_z F(\bar{x}, \bar{z})(z - \bar{z}) + N_C(z)\}$$

Suppose now that there exist neighborhoods U of $0 \in \mathbb{R}^m$ and V of $\bar{z}$ such that the map $\xi \mapsto \Sigma(\xi) \cap V$ is single-valued and Lipschitz on U with modulus κ. Then (2) is called *strongly regular* at $(\bar{x}, \bar{z})$, with Lipschitz modulus κ. In particular, we know from Robinson ([13], Theorem 2.1), that if (2) is strongly regular at $(\bar{x}, \bar{z})$, then for any $\varepsilon > 0$ there exist neighborhoods U_ε of $\bar{x}$ and V_ε of $\bar{z}$ such that the map $x \mapsto \sigma(x) := S(x) \cap V_\varepsilon$ is single-valued and Lipschitz on U_ε with Lipschitz modulus $(\kappa + \varepsilon)L$, where L is the uniform Lipschitz modulus of $F(\cdot, z)$ on U_ε for all $z \in V_\varepsilon$.

3 Strong Stationarity of EPECs

In this section, we commence by discussing the solution map $S(x)$ associated with the generalized equation

$$0 \in \Theta(x, w) + N_\Gamma(w), \tag{4}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $\Theta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable and $\Gamma \subseteq \mathbb{R}^m$ is a polyhedron.

Proposition 3.1. *Consider a reference point $(\bar{x}, \bar{w}) \in \operatorname{gph} S$ and assume that the generalized equation (4) is strongly regular at $(\bar{x}, \bar{w})$. Then*

$$\widehat{N}_{\operatorname{gph} S}(\bar{x}, \bar{w}) = \left\{ \left[\begin{array}{c} -\nabla_x^T \Theta(\bar{x}, \bar{w}) v^* \\ u^* - \nabla_w^T \Theta(\bar{x}, \bar{w}) v^* \end{array} \right] \middle| u^* \in K^\circ, v^* \in K \right\}.$$

Here, $K = T_\Gamma(\bar{w}) \cap \{\Theta(\bar{x}, \bar{w})\}^\perp$ is the critical cone to Γ corresponding to $(\bar{w}, \Theta(\bar{x}, \bar{w}))$.

Proof. Because Γ is a polyhedron, the strong regularity assumption implies that the Lipschitz localization of S, denoted by $\sigma(x)$, is directionally differentiable at $\bar{x}$ for each $h \in \mathbb{R}^n$ and one has $\sigma'(\bar{x}; h) = v$, where v is the unique solution of the generalized equation

$$0 \in \nabla_x \Theta(\bar{x}, \bar{w}) h + \nabla_w \Theta(\bar{x}, \bar{w}) v + N_K(v)$$

(see e.g., Theorem 6.3 [12]). We define

$$\Phi(h, v) := \left[\begin{array}{c} v \\ -\nabla_x \Theta(\bar{x}, \bar{w}) h - \nabla_w \Theta(\bar{x}, \bar{w}) v \end{array} \right]; \quad \Omega := \operatorname{gph} N_K$$

and calculate first the contingent cone to $\operatorname{gph} S$:

$$\begin{aligned} T_{\operatorname{gph} S}(\bar{x}, \bar{w}) &= \{(h, v) \in \mathbb{R}^n \times \mathbb{R}^m | \\ &\qquad \exists (h_i, v_i) \to (h, v), \tau_i \searrow 0 : \bar{w} + \tau_i v_i = \sigma(\bar{x} + \tau_i h_i) \forall i\} \\ &= \{(h, v) \in \mathbb{R}^n \times \mathbb{R}^m \,| v = \sigma'(\bar{x}; h)\}, \end{aligned}$$

where the last equality follows from the Lipschitz continuity of σ. Hence,

$$\begin{aligned} T_{\operatorname{gph} S}(\bar{x}, \bar{w}) &= \{(h, v) \in \mathbb{R}^n \times \mathbb{R}^m \,| 0 \in \nabla_x \Theta(\bar{x}, \bar{w}) h + \nabla_w \Theta(\bar{x}, \bar{w}) v + N_K(v)\} \\ &= \left\{ (h, v) \in \mathbb{R}^n \times \mathbb{R}^m \middle| \left[\begin{array}{c} v \\ -\nabla_x \Theta(\bar{x}, \bar{w}) h - \nabla_y \Theta(\bar{x}, \bar{w}) v \end{array} \right] \in \operatorname{gph} N_K \right\} \\ &= \Phi^{-1}(\Omega). \end{aligned}$$

Then by definition, $\widehat{N}_{\operatorname{gph} S}(\bar{x}, \bar{w}) = [\Phi^{-1}(\Omega)]^\circ$. Moreover, given that K is a convex cone, it is easy to see

$$\Omega = \{(v, u) \in K \times K^\circ \,| \langle v, u \rangle = 0\}.$$

Clearly, $\Phi^{-1}(K \times K^\circ) \supset \Phi^{-1}(\Omega)$. Consequently, by linearity of Φ,

$$[\Phi^{-1}(\Omega)]^\circ \supset [\Phi^{-1}(K \times K^\circ)]^\circ = \widehat{N}_{\Phi^{-1}(K \times K^\circ)}(0,0). \tag{5}$$

We claim that

$$\widehat{N}_{\Phi^{-1}(K_1 \times K_2)}(0,0) = \nabla^T \Phi(0,0)(K_1^\circ \times K_2^\circ) \tag{6}$$

for arbitrary polyhedral cones $K_1, K_2 \subseteq \mathbb{R}^m$. Indeed, Theorem 6.14 [15] implies

$$\widehat{N}_{\Phi^{-1}(K_1 \times K_2)}(0,0) \supset \nabla^T \Phi(0,0) \widehat{N}_{K_1 \times K_2}(\Phi(0,0)) = \nabla^T \Phi(0,0)(K_1^\circ \times K_2^\circ).$$

On the other hand, the multifunction

$$M(p) := \{(a,b) \,|\, \Phi\,(a,b) + p \in K_1 \times K_2\}$$

is calm at $(0,0,0)$ due to the polyhedrality of K_1, K_2 and linearity of Φ. It follows that we can invoke [6] (Th. 4.1) which yields the inclusion

$$\widehat{N}_{\Phi^{-1}(K_1 \times K_2)} = N_{\Phi^{-1}(K_1 \times K_2)}(0,0) \subset \nabla^T(0,0) N_{K_1 \times K_2}(\Phi(0,0)) = \nabla^T \Phi(0,0)(K_1^\circ \times K_2^\circ),$$

whence (6). Next we prove the reverse inclusion to (5). We observe that both sets $K \times \{0\}$ and $\{0\} \times K^\circ$ are subsets of Ω. Further, taking into account (6) with appropriate settings for K_1, K_2, one has

$$\begin{aligned}
[\Phi^{-1}(\Omega)]^\circ &\subset [\Phi^{-1}(K \times \{0\})]^\circ \cap [\Phi^{-1}(\{0\} \times K^\circ)]^\circ \\
&= \widehat{N}_{\Phi^{-1}(K \times \{0\})}(0,0) \cap \widehat{N}_{\Phi^{-1}(\{0\} \times K^\circ)}(0,0) \\
&= \nabla^T \Phi(0,0)(K^\circ \times \mathbb{R}^m) \cap \nabla^T \Phi(0,0)(\mathbb{R}^m \times K) \\
&= \nabla^T \Phi(0,0)(K^\circ \times K) = \widehat{N}_{\Phi^{-1}(K \times K^\circ)}(0,0),
\end{aligned}$$

and thus (5) becomes an equality. Because

$$\nabla^T \Phi(0,0) = \begin{bmatrix} 0 & -\nabla_x^T \Theta(\bar{x}, \bar{w}) \\ I & -\nabla_w^T \Theta(\bar{x}, \bar{w}) \end{bmatrix},$$

the asserted formula of our proposition holds. $\square$

Since in our later application we are going to deal with generalized equations defined on possibly nonpolyhedral convex sets, we consider now the following extension of (4):

$$0 \in F\,(x,z) + N_\Xi(z), \tag{7}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^l$ and $F : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^l$ is continuously differentiable and

$$\Xi := \left\{ z \in \mathbb{R}^l \,|\, A(z) \leq 0 \right\},$$

such that for $i = 1, \ldots, p$, each function $A_i : \mathbb{R}^l \to \mathbb{R}$ is convex and twice continuously differentiable. We are going to rewrite (7) in the form of the polyhedral case (4) in order to apply Proposition 3.1 to this more general setting: Consider a reference pair $(\bar{x}, \bar{z})$ and assume that $\nabla A(\bar{z})$ is surjective. Then, there is a point $\tilde{z}$, such that $A(\tilde{z}) < 0$, which enables us to rewrite (7) in the "enhanced" form:

$$0 \in \begin{bmatrix} \mathcal{L}(x, z, \lambda) \\ -A(z) \end{bmatrix} + N_{\mathbb{R}^l \times \mathbb{R}^p_+}(z, \lambda), \tag{8}$$

where $\mathcal{L}(x, z, \lambda)$ is the Lagrangian

$$\mathcal{L}(x, z, \lambda) = F(x, z) + \nabla^T A(z)\lambda,$$

and λ is a vector of Lagrange multipliers associated with the constraint mapping A. We introduce the enhanced solution set mapping as follows

$$S^e(x) := \left\{(z, \lambda) \in \mathbb{R}^l \times \mathbb{R}^p \,|(8) \text{ is fulfilled}\right\}. \tag{9}$$

In this setting, we will work with the reference point $(\bar{x}, \bar{z}, \bar{\lambda})$, where $\bar{\lambda} \in \mathbb{R}^p_+$ is uniquely defined by the equality $\mathcal{L}(\bar{x}, \bar{z}, \bar{\lambda}) = 0$ due to the surjectivity of $\nabla A(\bar{z})$. In what follows, we will employ the following index sets and scalars:

$$\begin{array}{rclrcl} I & := & \{i \in \{1, \ldots, p\} \,|A_i(\bar{z}) = 0\} & L & := & \{i \in \{1, \ldots, p\} \,|A_i(\bar{z}) < 0\} \\ I_+ & := & \left\{i \in I \,\middle|\, \bar{\lambda}_i > 0\right\} & I_0 & := & \left\{i \in I \,\middle|\, \bar{\lambda}_i = 0\right\} \\ a_+ & := & \#I_+ & a_0 & := & \#I_0. \end{array} \tag{10}$$

Clearly, (8) is of the form (4), with

$$w := (z, \lambda)\,, \quad \Theta\,(x, w) := \begin{bmatrix} \mathcal{L}(x, z, \lambda) \\ -A(z) \end{bmatrix}, \quad \Gamma := \mathbb{R}^l \times \mathbb{R}^p_+.$$

On the basis of Proposition 3.1 we arrive now at the following statement (using lower index sets to denote corresponding subvectors):

Proposition 3.2. *Consider a reference point $(\bar{x}, \bar{z}, \bar{\lambda}) \in \operatorname{gph} S^e$ and assume that (8) is strongly regular at $(\bar{x}, \bar{z}, \bar{\lambda})$. Then,*

$$\begin{array}{c} \widehat{N}_{\operatorname{gph} S^e}(\bar{x}, \bar{z}, \bar{\lambda}) = \\ \{(a, b, c) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p \mid \\ \exists (v, u, u') \in \mathbb{R}^l \times (\mathbb{R}^{a_+} \times \mathbb{R}^{a_0}_+ \times \{0\}) \times (\{0\} \times \mathbb{R}^{a_0}_- \times \mathbb{R}^{p-a_0-a_+}) : \end{array}$$
$$\begin{array}{rcl} a & = & -\nabla^T_x F(\bar{x}, \bar{z})v \\ b & = & -[\nabla^T_z F(\bar{x}, \bar{z}) + \sum_{i=1}^p \bar{\lambda}_i \nabla^2 A_i(\bar{z})]v + \nabla^T A_{I_+}(\bar{z})u_{I_+} + \nabla^T A_{I_0}(\bar{z})u_{I_0} \\ c_{I_+} & = & -\nabla A_{I_+}(\bar{z})v \\ c_{I_0} & = & u'_{I_0} - \nabla A_{I_0}(\bar{z})v \\ c_L & = & u'_L - \nabla A_L(\bar{z})v\} \end{array}$$

Proof. It suffices to compute

$$\begin{aligned}
K &= T_{\mathbb{R}^l\times\mathbb{R}^p_+}(\bar z,\bar\lambda)\cap\begin{bmatrix}\mathcal{L}(\bar x,\bar z,\bar\lambda)\\ -A(\bar z)\end{bmatrix}^\perp\\
&= \left\{(v,u)\in\mathbb{R}^l\times\mathbb{R}^p\,|u_{L\cup I_0}\geq 0\right\}\cap\begin{bmatrix}0\\ -A(\bar z)\end{bmatrix}^\perp\\
&= \left\{(v,u)\in\mathbb{R}^l\times\mathbb{R}^p\,\middle|-A(\bar z)^Tu=0,\ u_{L\cup I_0}\geq 0\right\}\\
&= \left\{(v,u)\in\mathbb{R}^l\times\mathbb{R}^p\,|u_L=0,u_{I_0}\geq 0\right\}
\end{aligned}$$

and

$$K^\circ=\left\{(v',u')\in\mathbb{R}^m\times\mathbb{R}^p\,\middle|v'=0,u'_{I_+}=0,u'_{I_0}\leq 0\right\}$$

and apply Proposition 3.1 with

$$\begin{aligned}
\nabla_x\Theta(\bar x,\bar w) &= \begin{bmatrix}\nabla_x F(\bar x,\bar z)\\ 0\end{bmatrix}\\
\nabla_w\Theta(\bar x,\bar w) &= \begin{bmatrix}\nabla_z F(\bar x,\bar z)+\sum_{i=1}^p\bar\lambda_i\nabla^2A_i(\bar z) & \nabla^TA(\bar z)\\ -\nabla A(\bar z) & 0\end{bmatrix}.
\end{aligned}$$

□

Remark 3.1. *Due to the fact that both variables u'_L and v are free, the component c_L in the statement of Proposition 3.2 becomes inconsequential.*

Remark 3.2. *To ensure the strong regularity in the statement of Proposition 3.2, we usually require that $\nabla_z\mathcal{L}\left(\bar x,\bar z,\bar\lambda\right)$ is positive definite on $\ker\left(\nabla A_{I_+}(\bar z)\right)$, cf. [13] , Theorem 4.1 or [12], Theorem 6.3.*

Based on the structure provided by Proposition 3.2, we can next compute $\widehat N_{\operatorname{gph}S}(\bar x,\bar z)$ for the solution map associated now with the generalized equation (7)

Proposition 3.3.

$$\widehat N_{\operatorname{gph}S}(\bar x,\bar z)=$$

$$\left\{\begin{bmatrix}\nabla_x^TF(\bar x,\bar z)v_1^*\\ \left[\nabla_z^TF(\bar x,\bar z)+\sum_{i=1}^p\lambda_i\nabla^2A_i(\bar z)\right]v_1^*+\nabla^TA_{I_+}(\bar z)v_2^*+\nabla^TA_{I_0}(\bar z)v_3^*\end{bmatrix}\ \middle|\ v^*\in M\times\mathbb{R}^{a_+}\times\mathbb{R}^{a_0}_+\right\},$$

where

$$\begin{aligned}
v^* &= (v_1^*,v_2^*,v_3^*)\in\mathbb{R}^l\times\mathbb{R}^{a_+}\times\mathbb{R}^{a_0}_+\\
M &= \left\{r\in\mathbb{R}^l\,|\nabla A_i(\bar z)r=0\quad(i\in I_+),\ \nabla A_i(\bar z)r\geq 0\quad(i\in I_0)\right\}.
\end{aligned}$$

Proof. Let $\bar\lambda$ be the unique multiplier vector associated with the pair $(\bar x, \bar z)$. We claim that

$$\widehat{N}_{\operatorname{gph} S}(\bar x, \bar z) = \left\{ (a,b) \in \mathbb{R}^n \times \mathbb{R}^l \,\middle|\, (a,b,0) \in \widehat{N}_{\operatorname{gph} S^e}(\bar x, \bar z, \bar\lambda) \right\}.$$

Indeed, by [15], Theorem 6.11 one has $(a,b) \in \widehat{N}_{\operatorname{gph} S}(\bar x, \bar z)$ if and only if there is a smooth function h that achieves its local maximum relative to $\operatorname{gph} S$ at $(\bar x, \bar z)$ and $\nabla h(\bar x, \bar z) = (a,b)$. Then clearly $(\bar x, \bar z, \bar\lambda)$ is a local maximum of the function $\widetilde{h}$ on $\operatorname{gph} S^e$, where

$$\widetilde{h}(x, z, \lambda) = h(x, z) \text{ for all } \lambda.$$

Consequently, $(a,b,0) \in \widehat{N}_{\operatorname{gph} S^e}(\bar x, \bar z, \bar\lambda)$. The reverse inclusion follows directly from the definition of the Fréchet normal cone. Now, the asserted formula follows immediately from Proposition 3.2. $\square$

Proposition 3.3 enables us to provide conditions each solution of the EPEC (1) without constraints ω_i on the leaders strategies has necessarily to satisfy. Points which are feasible w.r.t. to these conditions shall be called *strongly stationary.*

Theorem 3.1. *Let $(\bar x, \bar z)$ be a solution to the EPEC (1) without constraints ω_i on the leaders strategies. Suppose that $\Xi = \{ z \in \mathbb{R}^l \,|\, A(z) \le 0 \}$, where $A : \mathbb{R}^l \to \mathbb{R}^p$ is componentwise convex and twice continuously differentiable. Moreover, assume the following:*

1. *$\nabla A(\bar z)$ is surjective*
2. *(8) is strongly regular at $(\bar x, \bar z, \bar\lambda)$, where $\bar\lambda$ is the unique solution of $\nabla^T A(\bar z)\bar\lambda = -F(\bar x, \bar z)$.*

Then for all $i = 1, \dots, n$, there exist $v^i = (v_1^i, v_2^i, v_3^i) \in M^i \times \mathbb{R}^{a_+} \times \mathbb{R}_+^{a_0}$ such that:

$$0 = \nabla_{x_i} f_i(\bar x, \bar z) + \nabla_{x_i}^T F(\bar x, \bar z) v_1^i \tag{11}$$

$$\begin{aligned} 0 = {} & \nabla_z f_i(\bar x, \bar z) + \left[\nabla_z^T F(\bar x, \bar z) + \sum_{k=1}^{p} \bar\lambda_k \nabla^2 A_k(\bar z) \right] v_1^i + \\ & \nabla^T A_{I_+}(\bar z) v_2^i + \nabla^T A_{I_0}(\bar z) v_3^i \end{aligned} \tag{12}$$

where $M^i := \{ r^i \in \mathbb{R}^m \,|\, \nabla A_k(\bar z) r^i = 0 \ (k \in I_+),\ \nabla A_k(\bar z) r^i \ge 0 \ (k \in I_0\ ) \}$.

Proof. Clearly, $(\bar x_i, \bar z)$ is a locally optimal solution to the MPEC:

$$\min \{ f_i(\bar x_{-i}, x_i, z) \,|\, (\bar x_{-i}, x_i, z) \in \operatorname{gph} S \}.$$

Denote $S(\bar x_{-i}, x_i)$ by $S_{\bar x_i}(x_i)$. By Theorem 6.12 [15] we have

$$0 \in \nabla_{x_i, z} f_i(\bar x_{-i}, \bar x_i, \bar z) + \widehat{N}_{\operatorname{gph} S_{\bar x_i}}(\bar x_i, \bar z).$$

By assumption, (8) is strongly regular at $(\bar{x}, \bar{z}, \bar{\lambda})$, which certainly implies the strong regularity of the more restricted generalized equation (with fixed $\bar{x}_{-i}$)

$$0 \in \begin{bmatrix} \mathcal{L}(\bar{x}_{-i}, x_i, z, \lambda) \\ -A(z) \end{bmatrix} + N_{\mathbb{R}^l \times \mathbb{R}^p_+}(z, \lambda)$$

at $(\bar{x}_i, \bar{z}, \bar{\lambda})$ for all $i = 1, \ldots, n$. Hence, by substituting the solution set mapping $S^e(\bar{x}_{-i}, x_i)$ into the formula of Proposition 3.2 and applying Proposition 3.3 to $\widehat{N}_{\operatorname{gph} S_{\bar{x}_i}}(\bar{x}, \bar{z})$, we obtain (11) and (12). □

Remark 3.3. *It is certainly possible for either index set I_+ or I_0 to be empty. In either case, we do not set the cardinaltities of these sets, a_+ and a_0 respectively, to zero, which would result in either v_2^i or v_3^i being the zero vector. Instead, this admittedly light abuse of notation should be understood to mean that either the component vector v_2^i or v_3^i does not exist. Doing so allows us to conveniently state Theorem 3.1 without having to rewrite (11) and (12) for these two extreme cases.*

4 A Model of Oligopolistic Competition in Electricity Spot Markets

In the following we consider a simplified model for competition in electricity spot markets inspired by work in [3], [9] and [10] and one that has recently been investigated in [8]. We assume that the network of interest is represented by a *connected* oriented graph with m edges (transmission lines) and $N > 1$ nodes. Throughout this paper, $B \in \mathbb{R}^{N \times m}$ is used to represent the incidence matrix of the electricity network, with components denoted b_{ij}, where

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ enters node } i \\ -1 & \text{if edge } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we assume that at each node, electricity is both in demand and generated and that generator i exists only at node i, $i = 1, \ldots, N$. As noted in [3], it is reasonable to use a quadratic term to represent the amount of electricity lost due to transmission. We define such a term via the mapping $L : \mathbb{R}^m \to \mathbb{R}^N$, where

$$L(y) = \left(\tfrac{1}{2} \textstyle\sum_{j=1}^m |b_{ij}| \rho_j y_j^2 \right)_{i=1,\ldots,N} \tag{13}$$

Here, $\rho_j \geq 0$ is the loss coefficient of line j, for all $j = 1, \ldots, m$. Let $i, k \in \{1, \ldots, N\}$, $i \neq k$ be two nodes connected by edge $j \in \{1, \ldots, m\}$. If $\rho_j = 0$, then we interpret this to mean that generators i and k are reasonably geographically close and thus the loss of electricity due to transmission is considered negligible. As will be seen later, setting all $\rho_j = 0$ is also useful for obtaining valuable qualitative information about certain solutions.

Given these considerations, we model the satisfaction of demand as such:

$$q + By \geq d + L(y) \tag{14}$$

Here, the parameter $d \in \mathbb{R}^N$ represents the vector of demands at each node, $q \in \mathbb{R}^N$ is the vector of electricity generated at each respective node, and $y \in \mathbb{R}^m$ denotes the oriented flow vector of electricity along the edges of the graph.

Of course, production has to be nonnegative, i.e., $q \geq 0$ and typically there is also some upper for q, which we will disregard in this paper as only the notational, but not the mathematical, complexity would increase. Moreover, transmission lines have bounded capacities.

In our model, each of the competing generators bids a quadratic cost function to an independent system operator (ISO):

$$c_i(\alpha_i, \beta_i, q_i) = \alpha_i q_i + \beta_i q_i^2 \quad (i = 1, \dots, N)$$

However, the bidded cost coefficients α_i and β_i may in actuality differ from the true cost coefficients, as seen in the "truecost function:

$$C_i(q_i) = \gamma_i q_i + \delta_i q_i^2 \quad (i = 1, \dots, N)$$

Given all bid functions $c_i(\alpha_i, \beta_i, q_i)$, the ISO determines generation and flow such that the demand is met in each node of the network and that the overall costs (according to bidding) are minimized:

$$\min_{q,y} \left\{ \sum_{i=1}^{N} c_i(\alpha_i, \beta_i, q_i) \mid (q, y) \in G \right\}, \tag{15}$$

where

$$G := \left\{ (q, y) \in \mathbb{R}^{N+m} \mid q + By \geq d + L(y),\ q \geq 0,\ -\hat{y} \leq y \leq \hat{y} \right\}.$$

Note that the vector (α, β) appears as a perturbation parameter in the ISO problem and is therefore not considered a decision variable on this level. We will refer to (15) as either the ISO problem or the dispatch problem throughout the text. Given the inherent convexity of (15), for any feasible $(\bar{\alpha}, \bar{\beta})$, we know that the corresponding optimal solution $(\bar{q}, \bar{y})$ to (15) is characterized as a solution of the following generalized equation, which arises from the KKT conditions of (15):

$$0 \in \begin{pmatrix} \alpha + 2[\operatorname{diag}\beta]q \\ 0 \end{pmatrix} + N_G(q, y). \tag{16}$$

We use $[\operatorname{diag}\beta]$ to denote the diagonal matrix with entries β_i along the diagonal. Moreover, given an optimal generation vector q, we may determine generator i's profit function by first calculating the clearing price $\pi(q_i)_i = \alpha_i + 2\beta_i q_i$, i.e. the derivative of its bidding cost function at $\bar{q}_i$ and therefore the lowest marginal price at which transactions may occur in the market, and then subtracting the corresponding true cost coefficients:

$$f_i(\alpha_i, \beta_i, q, y) := (\alpha_i - \gamma_i)q_i + (2\beta_i - \delta_i)q_i^2$$

Therefore, by fixing the decisions of all other competitors, generator i solves the following mathematical program with equilibrium constraints (MPEC):

$$\max_{\substack{(\alpha_i,\beta_i)\in\mathbb{R}^2\\(q,y)\in G}} \left\{ (\alpha_i - \gamma_i)q_i + (2\beta_i - \delta_i)q_i^2 \,\middle|\, 0 \in \begin{pmatrix} \theta(\alpha_i, \beta_i, q) \\ 0 \end{pmatrix} + N_G(q, y) \right\} \tag{17}$$

where

$$\theta(\alpha_i, \beta_i, q) := (\bar{\alpha}_{-i}, \alpha_i) + 2[\operatorname{diag}(\bar{\beta}_{-i}, \beta_i)]q$$

Due to the fact that each competitor i solves the MPEC (17) , the coupled system

$$\min_{\substack{(\alpha_i,\beta_i)\in\mathbb{R}^2\\(q,y)\in G}} \left\{ (\gamma_i - \alpha_i)q_i + (\delta_i - 2\beta_i)q_i^2 \,\middle|\, 0 \in \begin{pmatrix} \alpha + 2[\operatorname{diag}\beta]q \\ 0 \end{pmatrix} + N_G(q, y) \right\} \tag{18}$$

for $i = 1, \ldots, N$ (upon passing from maximization to minimization) represents an equilibrium problem with equilibrium constraints (EPEC). Note, that the expression $N_G(q, y)$ implicitly entails the feasibility condition $(q, y) \in G$. Introducing the notation

$$\begin{aligned} F(\alpha, \beta, q, y) \;&:\; = \begin{pmatrix} \alpha + 2[\operatorname{diag}\beta]q \\ 0 \end{pmatrix}, \\ f_i(\alpha, \beta, q, y) \;&:\; = (\gamma_i - \alpha_i)q_i + (\delta_i - 2\beta_i)q_i^2 \quad (i = 1, \ldots, N), \end{aligned}$$

we may rewrite (18) in the compact form

$$\min_{\substack{(\alpha_i,\beta_i)\in\mathbb{R}^2\\(q,y)\in G}} \{ f_i(\alpha, \beta, q, y) \,|\, 0 \in F(\alpha, \beta, q, y) + N_G(q, y) \} \quad (i = 1, \ldots, N). \tag{19}$$

fitting to the general class of EPECs defined by (1). As noted earlier, we are primarily concerned with *non-cooperative* solutions $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y})$ to (19), the definition of which was provided in the introduction for a general class of EPECs.

Using the terminology from [2], this formulation is known as being *multioptimistic*. As one may suspect, certain problems could arise in terms of the well-posedness of such models. However, as will be seen momentarily, near solutions of particular interest the solution map $S(\alpha, \beta)$ is always a single-valued and locally Lipschitz function, thus possible well-posedness issues are circumvented.

5 Structural Properties

In this section we compile some structural properties of the spot market model which are required in order to apply the results from Section 3. Before doing so, we introduce a restricted class of solutions to (18) we are interested in, as those additionally satisfying the relations

$$\boxed{\begin{array}{cc} \bar{\alpha}_i, \bar{\beta}_i > 0 & i = 1, \ldots, N \\ -\hat{y}_j < y_j < \hat{y}_j & j = 1, \ldots, m \\ \bar{q}_i > 0 & i = 1, \ldots, l \\ \bar{q}_i = 0 & i = l+1, \ldots, N \end{array}} \tag{20}$$

where $1 \leq l \leq N$. First, considering strictly positive bidding coefficients only, prevents us from the analysis of certain pathologies associated with zero coefficients (of course, negative bidding coefficients do not make sense at all economically).

Second, we exclude solutions at which the electricity flow y reaches its upper bound, a phenomonon referred to as *congestion*. Congestion at a solution would have important consequences for the structural properties of our model. In particular, given cycles are present in the network and assuming transmission losses to be zero, the strong regularity assumption needed in the application of Theorem 3.1 would be lost. Therefore, we assume in this paper that no congestion occurs at a solution and leave the analysis of this more complicated case to an upcoming paper [7].

Finally, splitting the vector of generated electricity into strictly positive and zero components allows us to generalize the results found in [8] by including the possibility that some generators may overbid and thus suffer the consequences of not producing any electricity. Figure 1 illustrates the regions of activity in the space of bidding coefficients (projected onto the linear ones for graphical reasons) for an example of three competitors In particular, we are interested in circumstances under which an EPEC solution with a reduced set of competitors exists. It is such cases that introduce the nonsmooth character to the EPEC. Without loss of generality, we assume that the first l generators are active (clearly, the case $l = 0$ is of no economic interest).

Now, recall the feasible set G defined for the ISO-Problem (15). Evidently, close to a solution satisfying (20), G can be described as

$$G = \left\{ (q, y) \in \mathbb{R}^{N+m} \, | H(q, y) \leq 0 \right\},$$

where $H : \mathbb{R}^{N+m} \to \mathbb{R}^{2N-l}$ is a twice continuously differentiable mapping defined by

$$H(q, y) := \begin{pmatrix} d + L(y) - q - By \\ -q^{(2)} \end{pmatrix}. \tag{21}$$

Here, we use the partition $q = (q^{(1)}, q^{(2)}) \in \mathbb{R}^l \times \mathbb{R}^{n-l}$.

Next, we will need the following auxiliary statements:

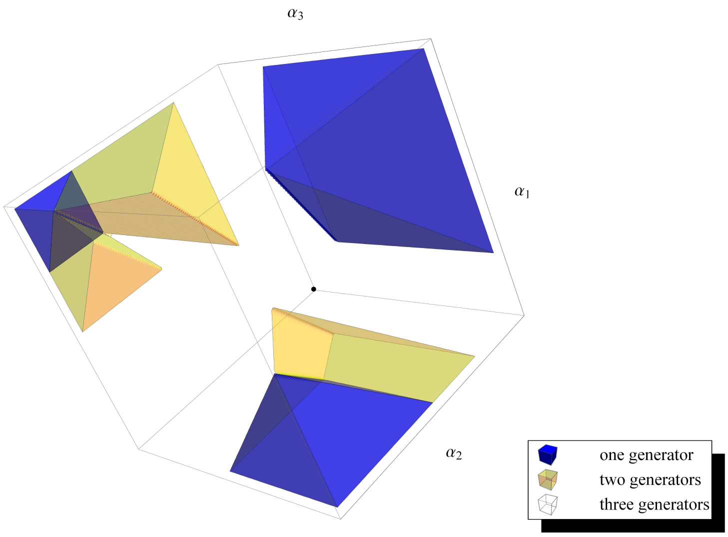

Figure 1: Phase diagram of active producers in the space of bidding coefficients

Lemma 5.1. *Let B be any (N, m)-incidence matrix of some oriented connected graph. Then the following properties hold*

1. $\ker B^T = \mathbb{R}(1, \ldots, 1)^T$
2. *For any integer k such that $1 \leq k \leq N$, each $(N-k, m)$-submatrix of B has rank $N - k$.*
3. $\forall \varepsilon > 0 \quad \exists \Delta > 0 \quad \forall \rho_j \in [0, \Delta) \quad \forall y \in [-\hat{y}, \hat{y}] : \quad \|\nabla L(y)\| < \varepsilon.$
4. $\exists \Delta > 0 \quad \forall \rho_j \in [0, \Delta) \quad \forall y \in [-\hat{y}, \hat{y}]$:

$$\nabla L^T(y) z = B^T z \quad \text{and} \quad \exists i : z_i = 0 \Longrightarrow z = 0.$$

Proof. For 1., see Biggs ([1], Prop 4.3). For 2., assume that the rank of some $(N-k, m)$-submatrix of B is smaller than $N - k$. Then, succesively joining the k left out rows to this submatrix - and thus reconstructing B - can increase the rank at most by $k - 1$, because the last row is already a linear combination of all the remaining $N-1$ rows (see 1.). Whence a contradiction with $\operatorname{rank} B = N-1$ (see 1.). 3. is an immediate consequence of (13). Concerning 4., it follows from 3. that for small enough transmission losses $\nabla L(y)$ can be considered arbitrarily small for all y in the indicated compact range. As a consequence of 1., one has $\operatorname{rank}(\nabla L(y) - B) \geq N - 1$ for small losses. If this rank strictly increases, then the dimension of the corresponding kernel strictly decreases, hence $\ker(\nabla^T L(y) - B^T) = \{0\}$. Otherwise, this rank remains $N - 1$, hence the corresponding kernel keeps having dimension one. Now, by 1. and a continuity argument there exists some $v \neq 0$, which can be chosen arbitrarliy close to $(1, \dots, 1)^T$ such that $\ker(\nabla^T L(y) - B^T) = \mathbb{R}v$. In either case, the asserted implication in 4. follows. $\square$

The following lemma provides first important properties for the constraint mapping H defined in (21).

Lemma 5.2. *Let $(\bar\alpha, \bar\beta, \bar q, \bar y)$ be a solution to (19) satisfying (20), $1 \leq l \leq N$, and $\rho_j \geq 0$ sufficiently small for $j = 1, \dots, m$, then the following properties hold:*

1. *$\nabla H(\bar q, \bar y)$ is surjective*
2. *$H(\bar q, \bar y) = 0$*
3. *Strict complementarity holds for the demand-satisfaction constraints (14).*

Moreover, if $l = N$, then 1., 2., and 3. hold without any requirements on the transmission losses ρ_j.

Proof. We begin with the case $l = N$, as the corresponding arguments used to prove 1., 2., and 3. for $l < N$ mirror those used for this simpler case. Given

$$\nabla H(\bar q, \bar y) = \begin{pmatrix} -I & | & \nabla L(\bar y) - B \end{pmatrix},$$

1. clearly holds without any requirements on the loss coefficients ρ_j. This allows us to write the first order optimality conditions to (15) at point $(\bar\alpha, \bar\beta, \bar q, \bar y)$:

$$\begin{aligned} \exists \lambda &\in \mathbb{R}^N_+ : \\ 0 = \begin{pmatrix} \bar\alpha + 2[\operatorname{diag} \bar\beta]\bar q \\ 0 \end{pmatrix} &+ \nabla^T H(\bar q, \bar y)\lambda && (22) \\ H(\bar q, \bar y) \bullet \lambda &= 0 && (23) \end{aligned}$$

where '$\bullet$' to denotes the Hadamard (component-wise) product. Then, referring to (20), we discern the following relation from (22):

$$0 < \bar\alpha_i + 2\bar\beta_i \bar q_i = \lambda_i, \quad i = 1, \dots, N \tag{24}$$

Now, (24) implies 3. and - along with (23) - also 2.

We now proceed under the assumption that $1 \leq l < N$. Let I and I' denote the identity matrices on $\mathbb{R}^{l \times l}$ and $\mathbb{R}^{N-l \times N-l}$ respectively and reconsider $\nabla H(\bar q, \bar y)$:

$$\nabla H(\bar q, \bar y) = \begin{pmatrix} -I & 0 & \nabla L^1(\bar y) - B^1 \\ 0 & -I' & \nabla L^2(\bar y) - B^2 \\ 0 & -I' & 0 \end{pmatrix},$$

where upper indices represent submatrices made up of the first l and last $N-l$ rows respectively. After elementary row-reductions, the Jacobian transforms into:

$$\left(\begin{array}{ccc} -I & 0 & \nabla L^1(\bar{y}) - B^1 \\ 0 & -I' & \nabla L^2(\bar{y}) - B^2 \\ 0 & 0 & -(\nabla L^2(\bar{y}) - B^2) \end{array} \right)$$

Clearly the surjectivity of $\nabla H(\bar{q}, \bar{y})$ hinges on the linear independence of the rows from $-(\nabla L^2(\bar{y}) - B^2)$. As per Lemma 5.1 (statement 2.), B^2 has rank $N-l$. Then, for small transmission losses, $-(\nabla L^2(\bar{y}) - B^2)$ has rank $N-l$ too by virtue of Lemma 5.1 (statement 3.). This proves 1.

As a consequence, we may write the first order optimality conditions to (15) at point $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y})$:

$$\begin{aligned} \exists(\lambda, \mu) &\in \mathbb{R}^N_+ \times \mathbb{R}^{N-l}_+ : \\ 0 = \left(\begin{array}{c} \bar{\alpha} + 2[\text{diag } \bar{\beta}]\bar{q} \\ 0 \end{array} \right) &+ \nabla^T H(\bar{q}, \bar{y}) \left(\begin{array}{c} \lambda \\ \mu \end{array} \right) \qquad (25) \\ H(\bar{q}, \bar{y}) \bullet (\lambda, \mu) &= 0 \qquad (26) \end{aligned}$$

We obtain the following set of relations from (25) taking into account (20):

$$\begin{aligned} 0 &< \bar{\alpha}_i + 2\bar{\beta}_i \bar{q}_i = \lambda_i, \quad i = 1, \ldots, l \qquad (27) \\ 0 &< \bar{\alpha}_i = \lambda_i + \mu_{i-l}, \quad i = l+1, \ldots, N \qquad (28) \\ 0 &= \left(\nabla^T L(\bar{y}) - B^T \right) \lambda \qquad (29) \end{aligned}$$

Assume that $\lambda_i = 0$ for some $i \in \{1, \ldots, N\}$. Then, (29) combined with Lemma 5.1 (statement 4.) yields the contradiction $\lambda = 0$ to (27), because $l \geq 1$. Hence, $\lambda_i > 0$ for all $i \in \{1, \ldots, N\}$. This proves 3. and, along with (26), also 2. follows (recall that $\bar{q}_i = 0$ for $i = l+1, \ldots, N$). $\square$

We next turn to strong regularity and refer back to Section 2 for the definition of this concept. As was discussed earlier, we need this property in order to derive explicit strong stationarity conditions for EPECs via Theorem 3.1. Moreover, the single-valuedness of the solution set mapping $S(\alpha, \beta)$ of the generalized equation (16) - which comes as a consequence of strong regularity - is integral in the argument justifying the well-posedness of the spot market EPEC. Begin by defining the enhanced generalized equation arising from the already established KKT conditions of (15):

$$0 \in \left[\begin{array}{c} \mathcal{L}(\alpha, \beta, q, y, \lambda, \mu) \\ -H(q, y) \end{array} \right] + N_{\mathbb{R}^{N+m} \times \mathbb{R}^{2N-l}_+}(q, y, \lambda, \mu) \qquad (30)$$

Here,

$$\mathcal{L}(\alpha, \beta, q, y, \lambda, \mu) = \left(\begin{array}{c} \alpha + 2[\text{diag } \beta]q \\ 0 \end{array} \right) + \nabla^T H(q, y) \left(\begin{array}{c} \lambda \\ \mu \end{array} \right).$$

Note that, by virtue of statement 1. in Lemma 5.2, the multipliers (λ, μ) in (30) are uniquely defined for any (α, β, q, y) fixed in a neighbourhood of a solution to (19) satisfying (20).

Proposition 5.1. *Let $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y})$ be a solution to (19) such that (20) holds and assume for all $j = 1, \dots, m$ that $\rho_j > 0$ sufficiently small such that in view of Lemma 5.2 $\nabla H(\bar{q}, \bar{y})$ is surjective. Then (30) is strongly regular at $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}, \bar{\lambda}, \bar{\mu})$, where $(\bar{\lambda}, \bar{\mu})$ is the uniquely defined Lagrange multiplier.*

Proof. Begin by noting that

$$\mathcal{L}(\alpha, \beta, q, y, \lambda, \mu) = \begin{pmatrix} \alpha + 2[\text{diag } \beta]q \\ 0 \end{pmatrix} + \begin{pmatrix} -\lambda^{(1)} \\ -\lambda^{(2)} - \mu \\ (\nabla L(y) - B)^T \lambda \end{pmatrix},$$

where the Lagrange multipliers λ and μ are defined as in Lemma 5.2 with

$$\lambda^{(1)} = (\lambda_1, \dots, \lambda_l), \quad \lambda^{(2)} = (\lambda_{l+1}, \dots, \lambda_N).$$

Because $\nabla H(\bar{q}, \bar{y})$ is surjective and $H(\bar{q}, \bar{y}) = 0$ according to Lemma 5.2, we know via Theorem 4.1 in [13] that (30) is strongly regular, if the partial Jacobian $\nabla_{q,y}\mathcal{L}$ is positive definite on $\ker(\nabla H(\bar{q}, \bar{y}))$. Calculating $\nabla_{q,y}\mathcal{L}(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}, \bar{\lambda}, \bar{\mu})$ we see:

$$\nabla_{q,y}\mathcal{L}(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}, \bar{\lambda}, \bar{\mu}) = \begin{pmatrix} 2[\text{diag } \bar{\beta}] & 0 \\ 0 & A \end{pmatrix}$$

with

$$A = \begin{pmatrix} \rho_1 \sum_{i=1}^N \bar{\lambda}_i |b_{i1}| & & 0 \\ & \ddots & \\ 0 & & \rho_m \sum_{i=1}^N \bar{\lambda}_i |b_{im}| \end{pmatrix}$$

Note that for all $j = 1, \dots, m$, there must exist $k \in \{1, \dots, N\}$ such that $b_{kj} \neq 0$, otherwise edge j would not join any nodes of the graph. Moreover, $\bar{\lambda}_i > 0$ for all $i \in \{1, \dots, N\}$ via Lemma 5.2 (statement 3.). Since finally $\rho_j > 0$ for all j by assumption and $\bar{\beta}_i > 0$ for all i due to (20), we see that the partial Jacobian is in fact positive definite regardless of the structure of $\ker(\nabla H(\bar{q}, \bar{y}))$. □

Corollary 5.1. *Given the assumptions of Proposition 5.1, the solution mapping $S(\alpha, \beta)$ to the generalized equation (16) is locally single-valued and Lipschitzian around $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y})$.*

Proof. From the strong regularity of (30) proved in Proposition 5.1 it follows that the solution set mapping S^e to (30) is locally single-valued and Lipschitzian around the point $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}, \bar{\lambda}, \bar{\mu})$. Now, the assertion follows upon observing that

$$S(\alpha, \beta) = \{(q, y) \,|\, \exists(\lambda, \mu) : (q, y, \lambda, \mu) \in S^e(\alpha, \beta)\}.$$

□

Therefore, as mentioned at the end of Section 4, we can use Corollary 5.1 to argue that the EPEC (19) is well-posed near the solutions we are considering.

An essential assumption in the strong regularity result of Proposition 5.1 was the presence of strictly positive losses. While this is certainly the case in reality, it also makes sense to consider a simplified lossless model facilitating an explicit use of stationarity conditions in order to derive qualitative information about the underlying EPEC. Evidently, strong regularity for the lossless case cannot be obtained via Proposition 5.1. Fortunately, there is a simple remedy by appropriately reducing the model in a way as to satisfy strong regularity. More precisely, observe that in the lossless case the demand satisfaction relation (14) reduces to

$$q + By \geq d. \tag{31}$$

Since by (20) we disregard solutions at which congestion might occur, the demand satisfaction relation my be further reduced to its summarized version $1^T q \geq 1^T d$, where $1^T = (1, \ldots, 1)$. In other words it suffices to require that the total generation of energy meets the total demand. Indeed, the reduced version follows from (31) upon left multiplication with 1^T and using statement 1. of Lemma 5.1. Conversely, given the relation $1^T q \geq 1^T d$, there always exists some appropriate flow vector, such that (31) holds true. This is a consequence of the well-known Gale-Hoffman inequalities (see, e.g., [14]) for the special case of no constraints on y (which is the case here due to the absence of congestion around solutions investigated). Note, that an analogous statement would be false in the presence of losses. On the other hand, the objective in the ISO problem (15) does not depend on the flow y. Therefore, the flow becomes meaningless in the lossless ISO problem and so for the whole EPEC. Once some $\bar{q}$ is fixed, one may recover a feasible flow $\bar{y}$ afterwards. Summarizing, in the lossless case we are allowed to remove the y-variables from the EPEC (18) upon replacing the feasible set G in the ISO problem by a reduced one:

$$\min \left\{ \sum_{i=1}^{N} \alpha_i q_i + \beta_i q_i^2 \,\middle|\, q \in \tilde{G} \right\}, \quad \tilde{G} := \{q \in \mathbb{R}^n \,|\, q \geq 0,\ 1^T q \geq 1^T d\}. \tag{32}$$

Then, the lossless EPEC becomes

$$\min_{\substack{(\alpha_i, \beta_i) \in \mathbb{R}^2 \\ q \in \tilde{G}}} \left\{ \tilde{f}_i(\alpha, \beta, q) \,\middle|\, 0 \in \tilde{F}(\alpha, \beta, q) + N_{\tilde{G}}(q) \right\} \quad (i = 1, \ldots, N). \tag{33}$$

Here $\tilde{f}_i(\alpha, \beta, q) := f_i(\alpha, \beta, q, y)$ and $\tilde{F}(\alpha, \beta, q) := \alpha + 2[\operatorname{diag} \beta]q$ are the same functions as in the original EPEC (19) but without the formal dependence on y of the latter.

The following proposition states strong regularity of the generalized equation induced from the KKT conditions of (32). This is in the same spirit as in Proposition 5.1 with respect to the KKT conditions of (15). Again we use a local description of $\tilde{G}$. This allows to reduce the non-negativity constraints to those components of q belonging to non-active generators. The demand satisfaction inequality has to be included too in this local description because at a solution it is always satisfied as an equality (as a consequence of statement 2. in Lemma 5.2). Therefore, around some $\bar{q}$ as in (20), $\tilde{G}$ may be locally described by

$$h(q) \leq 0, \quad h(q) := \left(1^T d - 1^T q, -q^{(2)}\right)^T \tag{34}$$

and $q^{(2)} := (q_{l+1}, \ldots, q_N)$.

Proposition 5.2. *Assume $\rho_j = 0$ for $j = 1, \ldots, m$ and let $(\bar{\alpha}, \bar{\beta}, \bar{q})$ be a solution to (33) satisfying (20). Then, the generalized equation*

$$0 \in \begin{bmatrix} \mathcal{L}(\alpha, \beta, q, \eta, \xi) \\ 1^T q - 1^T d \\ q^{(2)} \end{bmatrix} + N_{\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}_+^{N-l}}(q, \eta, \xi), \tag{35}$$

is strongly regular at $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{\eta}, \bar{\xi})$. Here,

$$\mathcal{L}(\alpha, \beta, q, \eta, \xi) := \alpha + 2[\text{diag } \beta]q + \nabla^T h(\bar{q})(\eta, \xi)$$

and $(\bar{\eta}, \bar{\xi})$ is the uniquely defined Lagrange multiplier associated with $(\bar{\alpha}, \bar{\beta}, \bar{q})$ in (32).

Proof. Following the same argument as in the proof of Proposition 5.1, we end up with

$$\nabla_q \mathcal{L}(\bar{\alpha}, \bar{\beta}, \bar{q}) = 2[\text{diag } \bar{\beta}],$$

which is always positive definite, regardless of $\ker(\nabla h(\bar{q}))$. Furthermore, the Jacobian of the feasible set mapping to (32), $\nabla h(\bar{q})$, has the structure:

$$\begin{pmatrix} -1 & \ldots & -1 \\ 0 & | & -I_{N-l} \end{pmatrix}$$

This matrix is always surjective which by the way justifies the uniqueness of the multiplier stated in the assertion of this proposition. Therefore, strong regularity follows as in Proposition 5.1. $\square$

Propositions 5.1 and 5.2 tell us among other that the lower-level solutions can indeed by determined as (locally Lipschitzian) functions of the bidding coefficients. The lossless model has the big advantage of even providing an explicit formula for this function. We present it here for the special case of $l = N$ (all generators active).

Proposition 5.3. *In the setting of Proposition 5.2, if $l = N$, then*

$$\bar{q}_i = \frac{1^T d + \frac{1}{2}\sum_{k=1}^N \frac{\bar{\alpha}_k}{\bar{\beta}_k}}{\sum_{k=1}^N \frac{\bar{\beta}_i}{\bar{\beta}_k}} - \frac{\bar{\alpha}_i}{2\bar{\beta}_i} \quad (i = 1, \ldots, N)$$

Proof. Writing the KKT conditions to (32) at $\bar{q}$ for the special case $l = N$ we see that there exists some $\eta \geq 0$ such that

$$
\begin{aligned}
0 &= \bar{\alpha} + 2[\text{diag } \bar{\beta}]\bar{q} + \nabla^T h(\bar{q})\eta && (36)\\
0 &= \eta \cdot h(\bar{q}) = \eta \cdot (1^T d - 1^T q) && (37)
\end{aligned}
$$

Therefore $\nabla h(\bar{q}) = (-1, \ldots, -1)$ along with (36) implies

$$
\bar{\alpha}_i + 2\bar{\beta}_i \bar{q}_i = \eta \quad i = 1, \ldots, N.
$$

Thus $\eta > 0$ due to (20) and $1^T q = 1^T d$ via (37). Then, $\bar{q}_i = (\eta - \bar{\alpha}_i)/\,2\bar{\beta}_i$ for all i and

$$
\sum_{k=1}^{N} \left(\frac{\eta - \bar{\alpha}_k}{2\bar{\beta}_k} \right) = 1^T d.
$$

Now, resolving for η and substituting in the formula obtained for the $\bar{q}_i$ yields the asserted formula. □

6 Characterizations of Solutions to the Spot Market EPEC using Strong Stationarity Conditions

6.1 Stationarity Conditions in general form

Given the previous sections' results we are now able to provide explicit characterizations of solutions to the spot market EPEC (18) via the S-Stationarity conditions guaranteed by Theorem 3.1. For the mapping A figuring in Theorem 3.1 we put $A := H$ with H as in (21) for the general EPEC (18) and $A := h$ with h as in (34) in the lossless case. Specializing the index sets from (10) to the data from (18), we get

$$
\begin{aligned}
I &= \{1, \ldots, 2N - l\}\\
I_+ &= \{1, \ldots, N\} \cup \{k \in \{N+1, \ldots, 2N - l\} \,|\, \bar{\mu}_{k-N} > 0\}\\
I_0 &= \{k \in \{N+1, \ldots, 2N - l\} \,|\, \bar{\mu}_{k-N} = 0\}
\end{aligned}
$$

for the general EPEC and

$$
\begin{aligned}
I &= \{1, \ldots, N - l + 1\}\\
I_+ &= \{1\} \cup \big\{k \in \{2, \ldots, N - l + 1\} \,\big|\, \bar{\xi}_{k-1} > 0\big\}\\
I_0 &= \big\{k \in \{2, \ldots, N - l + 1\} \,\big|\, \bar{\xi}_{k-1} = 0\big\}
\end{aligned}
$$

for the lossless EPEC. Here, the multipliers $\bar{\lambda}, \bar{\mu}, \bar{\eta}, \bar{\xi}$ refer to those figuring in Propositions 5.1 and 5.2, respectively. We note that in both cases the set I of active indices is complete as a consequence of Lemma 5.2 (statement 2.). Similarly, the form of I_+ and I_0 already exploits the strict complementarity shown in 5.2 (statement 3.). As in (10) we put $a_+ := \#I_+$ and $a_0 := \#I_0$. The association of the index sets with the respective case should be evident from the context in the following.

Now, we are in a position to apply Theorem 3.1 to the two versions of EPEC (18):

Corollary 6.1. *Let $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y})$ be a solution to (19) satisfying (20) and assume that for $j = 1, \ldots, m$, $\rho_j > 0$ sufficiently small such that $\nabla H(\bar{q}, \bar{y})$ is surjective. Then for $i = 1, \ldots, N$, there exist multipliers $v^i = (v_1^i, v_2^i, v_3^i) \in M^i \times \mathbb{R}^{a_+} \times \mathbb{R}_+^{a_0}$, with*

$$M^i := \left\{ z \in \mathbb{R}^{N+m} \,|\nabla H_k(\bar{q}, \bar{y})z = 0, k \in I_+,\ \nabla H_k(\bar{q}, \bar{y})z \geq 0, k \in I_0 \right\}$$

such that

$$\begin{aligned} 0 &= \nabla_{\alpha_i, \beta_i} f_i(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}) + \nabla^T_{\alpha_i, \beta_i} F(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}) v_1^i &\quad (38)\\ 0 &= \nabla_{q,y} f_i(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}) + \nabla^T_{q,y} F(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}) v_1^i + \sum_{k=1}^{N} \bar{\lambda}_k \nabla^2 H_k(\bar{q}, \bar{y}) v_1^i + \\ &\quad + \nabla^T H_{I_+}(\bar{q}, \bar{y}) v_2^i + \nabla^T H_{I_0}(\bar{q}, \bar{y}) v_3^i &\quad (39) \end{aligned}$$

Here, as before, $\bar{\lambda}$ refers to (a part of) the uniquely defined Lagrange multiplier in the ISO problem (15).

Similarly, let $(\bar{\alpha}, \bar{\beta}, \bar{q})$ be a solution to the lossless EPEC (33) satisfying (20). Then, there exist multipliers $w^i = (w_1^i, w_2^i, w_3^i) \in \widetilde{M^i} \times \mathbb{R}^{a_+} \times \mathbb{R}_+^{a_0}$ for $i = 1, \ldots, N$ with

$$\widetilde{M^i} := \left\{ \tilde{z} \in \mathbb{R}^{N} \,|\nabla h_k(\bar{q})\tilde{z} = 0, k \in I_+,\ \nabla h_k(\bar{q})\tilde{z} \geq 0, k \in I_0 \right\}$$

such that

$$\begin{aligned} 0 &= \nabla_{\alpha_i, \beta_i} \tilde{f}_i(\bar{\alpha}, \bar{\beta}, \bar{q}) + \nabla^T_{\alpha_i, \beta_i} \tilde{F}(\bar{\alpha}, \bar{\beta}, \bar{q}) w_1^i &\quad (40)\\ 0 &= \nabla_q \tilde{f}_i(\bar{\alpha}, \bar{\beta}, \bar{q}) + \nabla^T_q \tilde{F}(\bar{\alpha}, \bar{\beta}, \bar{q}) w_1^i + \nabla^T h_{I_+}(\bar{q}) w_2^i + \nabla^T h_{I_0}(\bar{q}) w_3^i. &\quad (41) \end{aligned}$$

Proof. Due to Proposition 5.1 and Proposition 5.2, (30) and (35) are strongly regular at $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}, \bar{\lambda}, \bar{\mu})$ and $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{\eta}, \bar{\xi})$ respectively. Then given the surjectivity of $\nabla H(\bar{q}, \bar{y})$ (see Lemma 5.2) and $\nabla h(\bar{q})$ (see Proof of Prop. (5.2)), Theorem 3.1 implies that the stationarity conditions (11) and (12) hold for the first case, yielding (38) and (39), and the second case as well, yielding (40) and (41). Here, it has to be noted that the summation in (39) can be restricted from $N + m$ to N because $\nabla^2 H_k(\bar{q}, \bar{y}) = 0$ for $k = N + 1, \ldots, N + m$. Therefore, only the first part of the multiplier $(\bar{\lambda}, \bar{\mu})$ from the ISO problem occurs in (39). □

6.2 Stationarity Conditions in a special case

In this section we provide explicit formulae for the stationarity conditions derived in Corollary 6.1. In order not to reduce the complexity, we restrict the analysis to a particular case, namely $I_+ = \{1, \ldots, N\}$ and $I_0 = \{N+1, \ldots, 2N-l\}$ for the general EPEC (19). The meaning of this choice of index sets is the following: putting all multipliers related with non-active generators equal to zero, implies that a corresponding EPEC solution is at the 'phase-boundary' between the region where all N generators are active and the region where just the first l generators are active and the remaining $N-l$ generators become inactive. In Figure 1, this correpsonds to the boundary between the empty space and one of the shaded areas (depending on whether $l=1$ or $l=2$).

Let $(\bar\alpha, \bar\beta, \bar q, \bar y)$ be a solution to (19) satisfying (20) and assume that for $j = 1, \ldots, m$, $\rho_j > 0$ sufficiently small such that $\nabla H(\bar q, \bar y)$ is surjective. Furthermore, assume that $I_+ = \{1, \ldots, N\}$ and $I_0 = \{N+1, \ldots, 2N-l\}$. Then by Corollary 6.1, we know that for $i = 1, \ldots, N$, there exist multipliers $v^i = (v_1^i, v_2^i, v_3^i) \in M^i \times \mathbb{R}^{a_+} \times \mathbb{R}_+^{a_0}$, with M^i as defined in Corollary (6.1), such that (38) and (39) hold. Fix some $i \in \{1, \ldots, N\}$, then we may rewrite (38) as follows:

$$(0,0) = (-\bar q_i, -2\bar q_i^2) + (v_{1,i}^i, 2\bar q_i v_{1,i}^i).$$

Given $\bar q_i > 0$ for $i \in \{1, \ldots, l\}$ and $\bar q_i = 0$ for $i \in \{l+1, \ldots, N\}$, we obtain the relations

$$v_{1i}^i = \bar q_i \quad (i \in \{1, \ldots, l\}), \qquad v_{1i}^i = 0 \quad (i \in \{l+1, \ldots, N\}) \tag{42}$$

Next, denoting by e_i the ith standard unit vector in $\mathbb{R}^{N+m}$, we observe that

$$\begin{aligned} \nabla_{q,y} f_i(\bar\alpha, \bar\beta, \bar q, \bar y) &= (\gamma_i - \bar\alpha_i + 2(\delta_i - 2\bar\beta_i)\bar q_i)e_i \\ \nabla_{q,y}^T F(\bar\alpha, \bar\beta, \bar q, \bar y) &= \begin{pmatrix} 2[\mathrm{diag}\ \bar\beta] & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and, with I_1 and I_2 being the identity matrices in $\mathbb{R}^{l\times l}$ and $\mathbb{R}^{(N-l)\times(N-l)}$, respectively,

$$\begin{aligned} \nabla^T H_{I_+}(\bar q, \bar y) &= \begin{pmatrix} -I_1 & 0 \\ 0 & -I_2 \\ \multicolumn{2}{c}{(\nabla L(\bar y) - B)^T} \end{pmatrix}, \quad \nabla^T H_{I_0}(\bar q, \bar y) = \begin{pmatrix} 0 \\ -I_2 \\ 0 \end{pmatrix} \\ \nabla^2 H_k(\bar q, \bar y) &= \begin{pmatrix} 0 & 0 \\ 0 & \nabla^2 L_k(\bar y) \end{pmatrix} \quad \text{if } k \in \{1, \ldots, N\}. \end{aligned}$$

Using these data we read now all components $j = 1, \ldots, N+m$ of the equation system (39) for all generators $i = 1, \ldots, N$. To do so, it is necessary to make case distinction according to the partition of components. For components $j = 1, \ldots, N$ we get:

$$\begin{aligned} j \in \{1, \ldots, l\} \setminus \{i\} &\implies 2\bar\beta_j v_{1,j}^i = v_{2,j}^i \\ j \in \{l+1, \ldots, N\} \setminus \{i\} &\implies 2\bar\beta_j v_{1,j}^i = v_{2,j}^i + v_{3,j-l}^i \\ j \in \{1, \ldots, l\},\, j = i &\implies \gamma_i - \bar\alpha_i + 2(\delta_i - \bar\beta_i)\bar q_i = v_{2,i}^i \\ j \in \{l+1, \ldots, N\},\, j = i &\implies \gamma_i - \bar\alpha_i = v_{2,i}^i + v_{3,i-l}^i \end{aligned} \tag{43}$$

Here, we already exploited (42) and the fact that $\bar{q}_i = 0$ for $i \in \{l+1, \ldots, N\}$. For components $j = N+1, \ldots, N+m$ we may write the whole block of equations as

$$\sum_{k=1}^{N} \bar{\lambda}_k \nabla^2 L_k(\bar{y}) v^i_{1,[3]} + (\nabla L(\bar{y}) - B)^T v^i_2 = 0, \tag{44}$$

where we make use of the partition $v^i_1 = (v^i_{1,[1]}, v^i_{1,[2]}, v^i_{1,[3]}) \in \mathbb{R}^l \times \mathbb{R}^{N-l} \times \mathbb{R}^m$. With this same notation, and exploiting the formulae for ∇H_{I_+} and ∇H_{I_0} presented above, we may summarize the relation $v^i_1 \in M^i$ (see statement of Corollary 6.1) as

$$v^i_{1,[1]} = (\nabla L^1(\bar{y}) - B_1) v^i_{1,[3]}, \quad v^i_{1,[2]} = (\nabla L^2(\bar{y}) - B_2) v^i_{1,[3]}, \quad v^i_{1,[2]} \leq 0. \tag{45}$$

Finally, we recall that $v^i_3 \geq 0$ (see statement of Corollary 6.1). Therefore, this inequality along with (42), (43), (44) and (45) yields a complete set of relations for the strong stationarity conditions of the EPEC (18).

We now address the lossless case in the setting $I_+ := \{1\}$ and $I_0 := \{2, \ldots, N-l+1\}$ which has the same interpretation as given in the beginning of this section for the general EPEC. The same (but now slightly simplified) analysis as above now leads to the following set of relations for the multipliers w^i_j in Corollary 6.1:

$$w^i_{1,i} = \bar{q}_i \quad (i \in \{1, \ldots, l\}), \quad w^i_{1,i} = 0 \quad (i \in \{l+1, \ldots, N\}) \tag{46}$$

$$\left.\begin{array}{rcl} j \in \{1, \ldots, l\} \setminus \{i\} & \Longrightarrow & 2\bar{\beta}_j w^i_{1,j} = w^i_2 \\ j \in \{l+1, \ldots, N\} \setminus \{i\} & \Longrightarrow & 2\bar{\beta}_j w^i_{1,j} = w^i_2 + w^i_{3,j-l} \\ j \in \{1, \ldots, l\},\, j = i & \Longrightarrow & \gamma_i - \bar{\alpha}_i + 2(\delta_i - \bar{\beta}_i)\bar{q}_i = w^i_2 \\ j \in \{l+1, \ldots, N\},\, j = i & \Longrightarrow & \gamma_i - \bar{\alpha}_i = w^i_2 + w^i_{3,i-l} \end{array}\right. \tag{47}$$

$$\textstyle\sum_{j=1}^{N} w^i_{1,j} = 0, \quad w^i_{1,j} \leq 0 \quad (j = l+1, \ldots, N), \quad w^i_3 \geq 0. \tag{48}$$

6.3 Illustration via a Small Example

In order to illustrate how the explicit stationary conditions can be resolved in the lossless EPEC, we consider a toy example of $N = 2$ generators linked together by one $(m = 1)$ transmission line. We want to identify situations in which - at an EPEC solution - just the first generator is active whereas the second one becomes inactive $(l = 1)$. More precisely, we are interested in an EPEC solution located on the boundary between both generators being active and the second one becoming inactive. This corresponds to the setting $I_+ := \{1\}$ and $I_0 := \{2\}$ (see discussion in previous section). Now, (46), (47) and (48) allow us to extract the following relations:

$$\begin{array}{rcl}
w_{1,1}^1 &=& \bar{q}_1 \\
w_{1,2}^2 &=& 0 \\
\gamma_1 - \bar{\alpha}_1 + 2(\delta_1 - \bar{\beta}_1)\bar{q}_1 &=& w_2^1 \\
2\bar{\beta}_2 w_{1,2}^1 &=& w_2^1 + w_3^1 \\
2\bar{\beta}_1 w_{1,1}^2 &=& w_2^2 \\
\gamma_2 - \bar{\alpha}_2 &=& w_2^2 + w_3^2 \\
w_{1,1}^1 + w_{1,2}^1 &=& 0 \\
w_{1,1}^2 + w_{1,2}^2 &=& 0 \\
w_{1,2}^1, w_{1,2}^2 &\leq& 0 \\
w_3^1, w_3^2 &\geq& 0
\end{array}$$

We recall that at a solution of the lossless EPEC, the total generation equals the total demand (strict complementarity of demand satisfaction): $1^T\bar{q} = 1^T d$. In our special setting with the second generator being inactive, this amounts to $\bar{q}_1 = d_1 + d_2 =: \bar{d}$. Using this information, the previous relations reduce to

$$\begin{array}{rcl}
\gamma_1 - \bar{\alpha}_1 + 2(\delta_1 - \bar{\beta}_1)\bar{d} &=& w_2^1 \\
-2\bar{\beta}_2\bar{d} &=& w_2^1 + w_3^1 \\
\gamma_2 - \bar{\alpha}_2 &=& w_3^2 \\
w_3^1, w_3^2 &\geq& 0.
\end{array}$$

Equating the expressions for w_2^1, we end up at the following two inequalities that the generators bidding coefficients have to satisfy at an EPEC solution with respect to the true cost coefficients:

$$\begin{array}{rcl}
\gamma_1 - \bar{\alpha}_1 + 2(\delta_1 - \bar{\beta}_1 + \bar{\beta}_2)\bar{d} &\leq& 0 \\
\bar{\alpha}_2 &\leq& \gamma_2
\end{array}$$

So, in particular, at such EPEC solution the generator is forced to become inactive while bidding a linear coefficient not larger than the true one.

At this point one should not forget that the lower level ISO problem provides additional information. Indeed, the KKT conditions for (33) yield

$$\bar{\alpha}_1 + 2\bar{\beta}_1\bar{q}_1 = \eta, \quad \bar{\alpha}_2 + 2\bar{\beta}_2\bar{q}_2 = \eta + \xi.$$

Recalling that $\xi = 0$ (due to $I_0 = \{2\}$), $\bar{q}_1 = \bar{d}$ and $\bar{q}_2 = 0$, we may summarize the necessary conditions for EPEC solutions in our special case as:

$$\begin{array}{rcl}
\gamma_1 - \bar{\alpha}_1 + 2(\delta_1 - \bar{\beta}_1 + \bar{\beta}_2)\bar{d} &\leq& 0 \\
\bar{\alpha}_2 &\leq& \gamma_2 \\
\bar{\alpha}_1 - \bar{\alpha}_2 + 2\bar{\beta}_1\bar{d} &=& 0.
\end{array} \tag{49}$$

Obviously, these relations do not uniquely identify the set of stationary solutions (the presence of a continuum of solutions to such EPECs has already been observed in [10]). Nevertheless they provide a useful test set in order to find potential candiates for EPEC solutions or to rule out numerical solutions which are not true EPEC solutions. To provide a numerical example, assume a total demand of $\bar{d}$=1 and cost coefficients

$$\gamma_1 = 1, \delta_1 = 0.25, \gamma_2 = 2, \delta_2 = 1.$$

Then, the constellation

$$\alpha_1 = 1, \beta_1 = 0.5, \alpha_2 = 2, \beta_2 = 0.25$$

of bidding coefficients evidently satisfies the stationarity conditions (49). Though, these stationarity conditions are much sharper than many others derived for MPECs or EPECs, it is not sure whether or not a true EPEC solution has been identified. In this simple example, however, graphical verification is possible. Indeed the plot

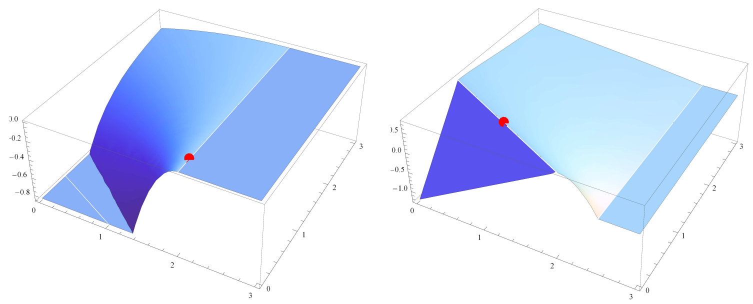

Figure 2: Plot of profit functions for a solution of the two generators EPEC

of the profit functions for the two generators, given the stationary solution of the respective other generator, reveals that the own stationary solution is indeed a global maximum in boeth cases. Hence, this a global EPEC solution.

Finally, we conclude this paper with a small sensitivity result for this type of EPEC solution based on the true cost coefficients.

Proposition 6.1. *Let* $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y})$ *be a solution to the lossless EPEC as defined above, i.e.,* $l = 1$*,* $\rho = 0$*. Then*

$$\gamma_2 \geq \gamma_1 + 2\delta_1 \bar{d}$$

Proof. Notice that (49) can be re-written as the inequality system:

$$\underbrace{\begin{pmatrix} -1 & -2\bar{d} & 0 & 2\bar{d} \\ 0 & 0 & 1 & 0 \\ 1 & 2\bar{d} & -1 & 0 \\ -1 & -2\bar{d} & 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\beta}_1 \\ \bar{\alpha}_2 \\ \bar{\beta}_2 \end{pmatrix} \leq \underbrace{\begin{pmatrix} -(\gamma_1 + 2\delta_1\bar{d}) \\ \gamma_2 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

Since $(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y})$ is a strongly stationary solution, (49) must hold. Furthermore, since $\bar{\alpha}_i, \bar{\beta}_i > 0$, for all $i = 1, 2$; the Farkas Lemma indicates that the following must also hold:

$$\forall u \in \mathbb{R}^4_+ : A^T u \geq 0,\ b^T u \geq 0$$

From $u \in \mathbb{R}^4_+$ and $A^T u \geq 0$ we obtain

$$\left.\begin{array}{rcl} -u_1 + u_3 - u_4 & \geq & 0 \\ -2\bar{d}u_1 + 2\bar{d}u_3 - 2\bar{d}u_4 & \geq & 0 \\ u_2 - u_3 + u_4 & \geq & 0 \\ 2\bar{d}u_1 & \geq & 0 \end{array}\right. \Rightarrow u_2 \geq u_3 - u_4 \geq u_1 \geq 0$$

In particular, $b^T u \geq 0$ for all $u \in \mathbb{R}^4$ such that $u_2 \geq u_1 \geq 0$. Because $b_3 = b_4 = 0$, this implies

$$(b_1, b_2) \in \left\{(w_1, w_2) \in \mathbb{R}^2 \,|w_2 \geq -w_1,\ w_2 \geq 0\right\}$$

Then the statement of the proposition follows by substitution (note that the second relation $w_2 \geq 0$ does not provide any new information). $\square$

Proposition 6.1 indicates that if the market mechanism is set up using an ISO, then electricity producers whose operating costs are lower than their competitors, could bid in such a way so as to force them out of the market. While this might be an expected phenomenon in the qualitative sense, the proposition yields an exact quantitative relationship between the cost coefficients of competitors leading to a monopolistic situation.

References

[1] N. Biggs, Algebraic Graph Theory, Cambridge University Press, Cambrige, 2 edition 1994.

[2] M. Cervinka, Hierarchal Structures in Equilibrium Problems, PhD thesis, Charles University Prague, 2008.

[3] J.F. Escobar and A. Jofre, Oligopolistic competition in electricity spot markets, working paper 2005, available at `http://ssrn.com/abstract=878762`.

[4] M. Flegel, Ch. Kanzow and J. Outrata, Optimality conditions for disjunctive programs with application to mathematical programs with equilibrium constraints, Set-Valued Analysis 15 (2007), 139-162.

[5] M. Fukushima and J.-S. Pang, Some feasibility issues in mathematical programs with equilibrium constraints, SIAM Journal on Optimization 8 (1998), 673-681.

[6] R. Henrion, J. Outrata, and A. Jourani, On the calmness of a class of multifunctions, SIAM Journal on Optimization 13 (2002), 603-618, .

[7] R. Henrion, J. Outrata, and T. Surowiec, Analysis of M-stationarity points to an EPEC modeling competition in an electricity spot market, *In Preparation.*

[8] R. Henrion and W. Römisch, On M-stationary points for a stochastic equilibrium problem under equilibrium constraints in electricity spot market modeling, Applications of Mathematics 52 (2007), 473-494.

[9] X. Hu and D. Ralph, Using EPECs to model bilevel games in restructured electricity markets with locational prices, Operations Research 55 (2007), 809-827.

[10] X. Hu, D. Ralph, E.K. Ralph, P. Bardsley and M.C. Ferris, Electricity generation with looped transmission networks: Bidding to an ISO, Research Paper No. 2004/16, Judge Institute of Management, Cambridge University, 2004.

[11] Z.Q. Luo, J.S. Pang and D. Ralph, Mathematical programs with equilibrium constraints, Cambridge University Press, Cambridge 1996.

[12] J.V. Outrata, M. Kocvara and J. Zowe, Nonsmooth approach to optimization problems with equilibrium constraints, Kluwer Academic Publishers, Dordrecht, 1998.

[13] S.M. Robinson, Strongly regular generalized equations, Mathematics of Operations Research 5 (1980), 43-62.

[14] R.T. Rockafellar, Network Flows and Monotropic Optimization, Wiley-Interscience, 1984.

[15] R.T. Rockafellar and R. J.-B. Wets, Variational Analysis, Springer, Berlin, 1998.

[16] H. Scheel and S. Scholtes, Mathematical programs with equilibrium constraints: Stationarity, optimality and sensitivity, Mathematics of Operations Research 25 (2000), 1-22.