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Global spatial regularity for time dependent elasto-plasticity and related problems

Dorothee Knees[1]

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[1] Weierstrass Institute for
Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin, Germany
E-Mail: knees@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We study the global spatial regularity of solutions of generalized elasto-plastic models with linear hardening on smooth domains. Under natural smoothness assumptions on the data and the boundary we obtain $u \in L^\infty((0,T);H^{\frac{3}{2}-\delta}(\Omega))$ for the displacements and $z \in L^\infty((0,T);H^{\frac{1}{2}-\delta}(\Omega))$ for the internal variables. The key step in the proof is a reflection argument which gives the regularity result in directions normal to the boundary on the basis of tangential regularity results.

Contents

1 Introduction

This paper is devoted to the study of global spatial regularity properties of solutions to elasto-plastic models in a geometrically linear framework. The model class under consideration comprises rate independent elasto-plasticity with kinematic hardening combined with a von Mises flow rule or a Tresca flow rule, as well as elasto-visco-plastic models which include Cosserat effects.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which represents an elasto-(visco)-plastic body and let $S = (0, T)$ be a time interval. The behavior of the body under the influence of external loadings is characterized by the (generalized) displacements $u : S \times \Omega \to \mathbb{R}^m$ and a vector of internal variables $z : S \times \Omega \to \mathbb{R}^n$, which represent the plastic strains and further hardening variables. The time evolution under the influence of external forces is determined through the quasi-static balance of forces (1.1) and an evolution law for the internal variable (1.2). The resulting model consists of a system of linear elliptic partial differential equations for u which is coupled with an evolution inclusion for z:

$$\operatorname{div}\big(C(x)\nabla u(t,x) + B(x)z(t,x)\big) + f(t,x) = 0 \qquad \text{for } (t,x) \in S \times \Omega, \tag{1.1}$$

$$\partial_t z(t,x) \in g(-(B^\top(x)\nabla u(t,x) + L(x)z(t,x))) \qquad \text{for } (t,x) \in S \times \Omega, \tag{1.2}$$

$$z(0,x) = z_0(x) \quad \text{for } x \in \Omega \tag{1.3}$$

together with boundary conditions for u. The underlying stored elastic energy is given by

$$\mathcal{E}(u,z) = \tfrac{1}{2}\int_\Omega \langle \left(\begin{smallmatrix} C & B \\ B^\top & L \end{smallmatrix}\right) \left(\begin{smallmatrix} \nabla u \\ z \end{smallmatrix}\right), \left(\begin{smallmatrix} \nabla u \\ z \end{smallmatrix}\right)\rangle \, \mathrm{d}x$$

with a symmetric coefficient tensor $A = \left(\begin{smallmatrix} C & B \\ B^\top & L \end{smallmatrix}\right) \in L^\infty(\Omega; \mathrm{Lin}(\mathbb{R}^{m\times d} \times \mathbb{R}^n, \mathbb{R}^{m\times d} \times \mathbb{R}^n))$. Moreover, $g : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a monotone multivalued constitutive function. If $\mathcal{E}$ is positive semi-definite and if $0 \in g(0)$, then the system (1.1)–(1.3) belongs to the class of models of monotone type introduced in [Alb98], which is a generalization of the class of generalized standard materials. With the choice $g = \partial\chi_K$, where $\partial\chi_K$ is the subdifferential of the characteristic function χ_K related to the convex set $K \subset \mathbb{R}^n$, equations (1.1)–(1.3) describe classical rate-independent elasto-plasticity. In this case, the set K is the set of admissible generalized stresses. We give examples for (1.1)–(1.3) in Section 2.2 and a more precise definition of the model in Section 4.

If the elastic energy $\mathcal{E}$ is coercive, i.e. if

$$\mathcal{E}(u,z) \geq \tfrac{\alpha}{2}\big(\|u\|^2_{H^1(\Omega)} + \|z\|^2_{L^2(\Omega)} \big)$$

for all $u \in H^1_0(\Omega)$ and $z \in L^2(\Omega)$ and some constant $\alpha > 0$, then classical results guarantee the existence of a unique pair $(u,z) \in W^{1,1}(S; H^1(\Omega)) \times W^{1,1}(S; L^2(\Omega))$ which solves (1.1)–(1.3), see e.g. [DL72, Joh78, Bré73, HHLN88, HR99, AC04] and the references therein.

The main result of our paper is Theorem 4.1, where we prove the following global spatial regularity for (u,z) provided that $\partial\Omega$ is smooth, that $\mathcal{E}$ is coercive, that the type of the boundary conditions does not change and that the data and coefficients have some natural smoothness properties: For all $\delta > 0$ it holds

$$u \in L^\infty(S; H^{\frac{3}{2}-\delta}(\Omega)), \tag{1.4}$$

$$z \in L^\infty(S; H^{\frac{1}{2}-\delta}(\Omega)), \tag{1.5}$$

where $H^s(\Omega)$ stands for Sobolev-Slobodeckij spaces, see e.g. [Tri83]. This regularity result to our knowledge is new and was announced in the paper [Kne08], where we studied a model problem on a cube. Moreover, as an extension of a result by Alber and Nesenenko [AN08] to our slightly more general system (1.1)–(1.3), we derive the following local and tangential regularity properties, where ∂_{tang} denotes derivatives tangential to the boundary:

$$\begin{aligned} &u \in L^\infty(S; H^2_{\text{loc}}(\Omega)), \quad \partial_{\text{tang}} u \in L^\infty(S; H^1(\Omega)), \\ &z \in L^\infty(S; H^1_{\text{loc}}(\Omega)), \quad \partial_{\text{tang}} z \in L^\infty(S; L^2(\Omega)). \end{aligned} \tag{1.6}$$

The intrinsic difficulty of proving spatial regularity for time-dependent plasticity problems stems from the fact that the flow rule (1.2) is nonsmooth and has no regularizing terms. Hence, spatial regularity has to be maintained during the evolution by careful estimates. Let $\mathcal{Q} \subset H^1(\Omega) \times L^2(\Omega) \ni (u(t), z(t))$ denote the state space. The main problem is that the data to solution map is not Lipschitz as a mapping from $W^{1,1}(S; \mathcal{Q}^*) \to W^{1,1}(S; \mathcal{Q})$, but only as a mapping from $W^{1,1}(S; \mathcal{Q}^*) \to L^\infty(S; \mathcal{Q})$, see Theorem 2.3. This stability estimate is the basis for proving the local and tangential results in (1.6). Since a similar Lipschitz estimate is not available for the rates, we cannot derive a spatial regularity result of the type $\partial_t z \in L^\infty(S; H^1_{\text{loc}}(\Omega))$. Indeed, the example in Section 5.3.5 shows that the latter regularity in general is not valid in spite of smooth data. Since terms of the form $\partial_{\text{tang}} z \in L^\infty(S; L^2(\Omega))$ enter as data when we prove the regularity in normal direction, we cannot apply the aforementioned Lipschitz estimate any more since it would require $\partial_{\text{tang}} z \in W^{1,1}(S; L^2(\Omega))$. In this situation we only have a weaker Hölder estimate with exponent $\frac{1}{2}$ for the solution to data map, see Theorem 2.3. This explains, why in the normal direction we obtain a "half" spatial derivative, only.

The proof of (1.6) is carried out with a difference quotient technique using inner variations and the Lipschitz properties of the data to solution map. These estimates are given in Section 2.1, while the local and tangential regularity results are proved in Section 3.

The essential new idea in this paper is to apply a reflection argument in order to obtain higher differentiability properties for ∇u and z also for directions, which are perpendicular to the boundary. After localizing system (1.1)–(1.3) to a half cube by the usual techniques, we reflect the problem to the full cube using an even extension for the internal variable z and an odd extension for the displacements modified by the value of u on the boundary. We show that the newly defined functions satisfy a problem of the type (1.1)–(1.3) on the full cube with coefficients depending smoothly on the space variable. The right hand side of the extended problem contains tangential derivatives of ∇u and z. Using the tangential results (1.6) and the Hölder property of the data to solution map, we obtain the additional "half" spatial derivative.

It is an unsolved problem, whether our final result (1.4)–(1.5) is optimal or whether one should expect $u \in L^\infty(S; H^2(\Omega))$. This would coincide nicely with the local and tangential properties in (1.6) and also with results for solutions of linear elliptic equations on smooth domains. We show in Section 5.3.1 that the reflection argument applied to stationary elliptic systems (without a coupling to the evolution law) gives a full additional derivative. Thus, in the stationary case our reflection argument is equivalent to the arguments usually applied for elliptic systems, see e.g. [Neč67], and does not intrinsically lead to suboptimal differentiability properties.

Concerning the optimality of our result, we discuss in Section 5.3.4 the case where u is scalar, i.e. $m = 1$. Under strong coupling assumptions between the coefficient matrices C, B, L and the function g, we obtain indeed the full spatial regularity $u \in L^\infty(S; H^2(\Omega))$. Here, we use a reflection argument, which takes into account the explicit structure of the coefficient matrix $A = \left(\begin{smallmatrix} C & B \\ B^\top & L \end{smallmatrix}\right)$.

Let us give a short discussion of regularity results in the literature for systems of the type (1.4)–(1.5). Recently, the question of global spatial regularity attracted much attention. We mention here the contributions by Alber/Nesenenko [AN08] and by Frehse/Löbach [FL08b]. In [AN08] the authors obtain for a model similar to (1.4)–(1.5) the global result $u \in L^\infty(S; H^{1+\frac{1}{3}-\delta}(\Omega))$ and $z \in L^\infty(S; H^{\frac{1}{3}-\delta}(\Omega))$ by first proving the local and tangential result (1.6). They show that this already implies that $u \in L^\infty(S; H^{1+\frac{1}{4}-\delta}(\Omega))$, and similarly for z. By an iteration procedure they improve then the differentiability from $\frac{1}{4}$ to $\frac{1}{3}$. In the paper [FL08b] the authors study regularity properties of rate independent elasto-plastic models with a von Mises flow rule and linear kinematic or isotropic hardening. They show Hölder regularity of the stresses up to the boundary, derive the spatial regularity $\nabla\sigma \in L^\infty(S; L^{1+\delta}(\Omega))$ for the stress σ and prove several additional integrability properties. The investigations take a stress based version of (1.4)–(1.5) as a starting point.

Local regularity properties for the model in (1.4)–(1.5) and variants of it, having e.g. only a positive semi-definite elastic energy, were investigated by several authors [BF96, FL08a, Shi99, Ser92, Dem09, Dem08, NC08]. Here, one typically finds that the stress $\sigma = C\nabla u + Bz$ belongs to $L^\infty(S; H^1_{\text{loc}}(\Omega))$. Similar results are valid for u and z provided that the elastic energy $\mathcal{E}$ is coercive.

Further global results are available for time discretized versions of (1.4)–(1.5), see for example [Rep96, KN08] and the references therein. Here one obtains $\sigma(t_k) \in H^1(\Omega)$ globally for smooth domains and smooth data at every temporal discretization point t_k. However, up to now it is to our knowledge an open question whether a uniform estimate of the form $\sup_{\text{time step } \triangle t>0,\, k\triangle t\le T} \|\sigma(k\triangle t)\|_{H^1(\Omega)} \le c$ is valid. This estimate would allow to carry over the result from the discretized model to the continuous one. Finally, for the stationary Hencky model of perfect plasticity we have the global result $\sigma \in H^{\frac{1}{2}-\delta}(\Omega)$, $\delta > 0$, for domains with Lipschitz boundary and with changing boundary conditions, [Kne06].

2 Abstract existence results and stability estimates

In this section we recall abstract existence results and stability estimates for problems of the type (1.1)–(1.3). The results are based on classical existence theorems by Brézis [Bré73] for evolution equations with maximal monotone operators. We also refer to [AC04, HR99] and the references therein for the discussion of particular elastic-plastic and visco-plastic models.

2.1 Existence result and stability estimates

By $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ we denote the state spaces which is composed of the real, separable Hilbert spaces $\mathcal{U}$ and $\mathcal{Z}$. We identify $\mathcal{Z}^*$ with $\mathcal{Z}$ but distinguish between $\mathcal{U}$ and the dual space $\mathcal{U}^*$. For $u \in \mathcal{U}$ and $z \in \mathcal{Z}$ the stored energy is given by the following quadratic functional

$$\mathcal{E}(u, z) = \tfrac{1}{2}\langle \mathcal{A}(u, z), (u, z)\rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the dual pairing in $\mathcal{Q}^* \times \mathcal{Q}$. It is assumed that $\mathcal{A} \in \mathrm{Lin}(\mathcal{Q}, \mathcal{Q}^*)$ is a linear, bounded and self adjoint operator and that there exists a constant $\alpha > 0$ such that

$$\mathcal{E}(u, z) \geq \tfrac{\alpha}{2}(\|u\|^2_{\mathcal{U}} + \|z\|^2_{\mathcal{Z}}) \tag{2.1}$$

for all $(u, z) \in \mathcal{Q}$.

Let furthermore $\mathcal{G} : \mathcal{Z} \to \mathcal{P}(\mathcal{Z})$ be a maximal monotone operator with $0 \in \mathcal{G}(0)$. The problem under consideration is:

$$\begin{aligned} &\text{Find } u : S \to \mathcal{U},\ z : S \to \mathcal{Z} \text{ such that for a.e. } t \in S \\ &D_u\mathcal{E}(u(t), z(t)) = \ell_1(t) \\ &\qquad \partial_t z(t) \in \mathcal{G}(-D_z\mathcal{E}(u(t), z(t)) + \ell_2(t)) \\ &\qquad z(0) = z_0. \end{aligned} \tag{2.2}$$

Here, $z_0 \in \mathcal{Z}$ and $\ell = (\ell_1, \ell_2) : S \to \mathcal{Q}^*$ are given data.

We call the data z_0 and ℓ compatible if there exists $u_0 \in \mathcal{U}$ with $D_u\mathcal{E}(u_0, z_0) = \ell_1(0)$ and with $-D_z\mathcal{E}(u_0, z_0) + \ell_2(0) \in D(\mathcal{G})$, where $D(\mathcal{G})$ denotes the domain of $\mathcal{G}$.

Theorem 2.1. *Under the above assumptions there exists for every compatible data $\ell \in W^{2,1}(S; \mathcal{Q}^*)$ and $z_0 \in \mathcal{Z}$ a unique pair $(u, z) \in W^{1,\infty}(S; \mathcal{Q})$ which solves* (2.2)*.*

If $\mathcal{G}$ is the subdifferential of the indicator function $\chi_{\mathcal{K}}$ of the convex set $\mathcal{K} \subset \mathcal{Z}$, weaker assumptions on the smoothness of the data are sufficient to obtain existence of solutions.

Theorem 2.2. *Let $\mathcal{G} = \partial\chi_{\mathcal{K}}$, where $\mathcal{K} \subset \mathcal{Z}$ is convex, closed and with $0 \in \mathcal{K}$. Then for every compatible data $\ell \in W^{1,1}(S; \mathcal{Q}^*)$ and $z_0 \in \mathcal{Z}$ there exists a unique pair $(u, z) \in W^{1,1}(S; \mathcal{Q})$ solving* (2.2)*.*

In order to fix the notation, we give here a short sketch of the proofs of Theorems 2.1 and 2.2.

Proof of Theorems 2.1 and 2.2. The linear operator $\mathcal{A}$ is split as follows

$$\mathcal{A}(u,z) = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix}$$

with bounded operators $\mathcal{A}_{11} \in \mathrm{Lin}(\mathcal{U},\mathcal{U}^*)$, $\mathcal{A}_{12} \in \mathrm{Lin}(\mathcal{Z},\mathcal{U}^*)$, $\mathcal{A}_{21} = \mathcal{A}_{12}^* \in \mathrm{Lin}(\mathcal{U},\mathcal{Z})$ and $\mathcal{A}_{22} \in \mathrm{Lin}(\mathcal{Z},\mathcal{Z})$. Due to the assumptions on $\mathcal{A}$, the operators $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are self adjoint and positive definite and hence invertible. By $\mathcal{L} : \mathcal{Z} \to \mathcal{Z}$ we denote the Schur complement operator associated with $\mathcal{A}$, i.e. $\mathcal{L} = \mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12}$. The assumptions on $\mathcal{A}$ imply that $\mathcal{L}$ is a linear, bounded, self adjoint operator with $\langle \mathcal{L}z, z\rangle \geq \alpha \left\|z\right\|_{\mathcal{Z}}^2$ for all $z \in \mathcal{Z}$. The constant α is the same as in (2.1). Problem (2.2) is equivalent to the following reduced version:

$$\begin{aligned} &\text{Find } z : S \to \mathcal{Z} \text{ with} \\ &\partial_t z(t) \in \mathcal{G}(-\mathcal{L}z(t) + F(t)), \quad z(0) = z_0 \end{aligned} \tag{2.3}$$

with $F(t) = \ell_2(t) - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\ell_1(t)$. From this, the function u can be calculated via $u = \mathcal{A}_{11}^{-1}(\ell_1 - \mathcal{A}_{12}z)$.

In terms of the new variable $y(t) = \mathcal{L}^{\frac{1}{2}}z(t) - \mathcal{L}^{-\frac{1}{2}}F(t)$, the mapping $\widetilde{\mathcal{G}} : \mathcal{Z} \to \mathcal{P}(\mathcal{Z})$ with $\widetilde{\mathcal{G}}(\zeta) = \mathcal{L}^{\frac{1}{2}}\mathcal{G}(\mathcal{L}^{\frac{1}{2}}\zeta)$ and the data $f(t) = -\mathcal{L}^{-\frac{1}{2}}\partial_t F(t)$, problem (2.3) can equivalently be written as:

$$\begin{aligned} &\text{Find } y : S \to \mathcal{Z} \text{ with} \\ &\partial_t y(t) \in f(t) + \widetilde{\mathcal{G}}(-y(t)), \quad y(0) = y_0 = \mathcal{L}^{\frac{1}{2}}z_0 - \mathcal{L}^{-\frac{1}{2}}F(0). \end{aligned} \tag{2.4}$$

Note that the operator $\widetilde{\mathcal{G}}$ is maximal monotone with respect to the standard scalar product in $\mathcal{Z}$.

Theorem 3.4 and Proposition 3.3 in [Bré73] applied to (2.4) provide the existence result in case of an arbitrary maximal monotone mapping $\mathcal{G}$, while Proposition 3.4 from [Bré73] gives the result for the case $\mathcal{G} = \partial\chi_{\mathcal{K}}$. □

In the next Theorem we recall stability estimates which are the basis for our regularity results.

Theorem 2.3. *Assume* (2.1) *and let* $\mathcal{G} : \mathcal{Z} \to \mathcal{P}(\mathcal{Z})$ *be a monotone operator.*

(a) *There exists a constant* $\kappa_1 > 0$ *such that for all* $u^i \in L^\infty(S;\mathcal{U})$ *and* $z^i \in W^{1,1}(S;\mathcal{Z})$, $i \in \{1,2\}$, *which are solutions to problem* (2.2) *with data* $z_0^i \in \mathcal{Z}$ *and* $\ell^i = (\ell_1^i, \ell_2^i) \in L^\infty(S;\mathcal{Q}^*)$, *it holds*

$$\begin{aligned} &\left\|u^1 - u^2\right\|_{L^\infty(S;\mathcal{U})}^2 + \left\|z^1 - z^2\right\|_{L^\infty(S;\mathcal{Z})}^2 \\ &\quad \leq \kappa_1\big(\left\|z_0^1 - z_0^2\right\|_{\mathcal{Z}}^2 + \left\|z^1 - z^2\right\|_{W^{1,1}(S;\mathcal{Z})}\left\|\ell^1 - \ell^2\right\|_{L^\infty(S;\mathcal{Q}^*)} + \left\|\ell_1^1 - \ell_1^2\right\|_{L^\infty(S;\mathcal{U}^*)}^2\big) \end{aligned} \tag{2.5}$$

(b) There exists a constant $\kappa_2 > 0$ such that for all $u^i \in W^{1,1}(S;\mathcal{U})$ and $z^i \in W^{1,1}(S;\mathcal{Z})$, which are solutions to problem (2.2) with respect to the data $z_0^i \in \mathcal{Z}$ and $\ell^i \in W^{1,1}(S;\mathcal{Q}^*)$, it holds

$$\|u^1 - u^2\|_{L^\infty(S;\mathcal{U})} + \|z^1 - z^2\|_{L^\infty(S;\mathcal{Z})} \le \kappa_2\big(\|z_0^1 - z_0^2\|_{\mathcal{Z}} + \|\ell^1 - \ell^2\|_{W^{1,1}(S;\mathcal{Q}^*)} \big). \tag{2.6}$$

Proof. Assumption (2.1) implies that there exists $\kappa > 0$ such that

$$\|u^1 - u^2\|_{L^\infty(S;\mathcal{U})} \le \kappa\big(\|z^1 - z^2\|_{L^\infty(S;\mathcal{Z})} + \|\ell_1^1 - \ell_1^2\|_{L^\infty(S;\mathcal{U}^*)} \big). \tag{2.7}$$

Let $\mathcal{L}$ be the operator and F^i, $i \in \{1,2\}$, be the functions defined in the proof of Theorems 2.1 and 2.2. Since $\mathcal{G}$ is monotone and since $\mathcal{L}$ is self adjoint, the solutions z^i of (2.3) satisfy for almost every $t \in S$

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle z^1(t) - z^2(t), \mathcal{L}(z^1(t) - z^2(t))\rangle \le \langle \partial_t(z^1(t) - z^2(t)), F^1(t) - F^2(t)\rangle. \tag{2.8}$$

Integrating this estimate with respect to t and applying Hölder's inequality leads to

$$\|z^1 - z^2\|^2_{L^\infty(S;\mathcal{Z})} \le c\big(\|z_0^1 - z_0^2\|^2 + \|z^1 - z^2\|_{W^{1,1}(S;\mathcal{Z})}\,\|F^1 - F^2\|_{L^\infty(S;\mathcal{Z})} \big).$$

Combining the last estimate with (2.7) results in (2.5).

If $\ell \in W^{1,1}(S;\mathcal{Q}^*)$, then integrating (2.8) with respect to t, partial integration and Young's inequality result in the estimate

$$\begin{aligned} \|z^1(t) - z^2(t)\|^2_{\mathcal{Z}} \le c\Big(&\|z_0^1 - z_0^2\|^2_{\mathcal{Z}} + \int_0^t \|z^1(s) - z^2(s)\|^2_{\mathcal{Z}}\,\mathrm{d}s \\ &+ \|F^1 - F^2\|_{W^{1,1}(S;\mathcal{Z})}\big(\|F^1 - F^2\|_{W^{1,1}(S;\mathcal{Z})} + \|z^1 - z^2\|_{L^\infty(S;\mathcal{Z})} \big)\Big). \end{aligned} \tag{2.9}$$

Applying the Gronwall inequality and Young's inequality to the previous estimate leads in combination with (2.7) to estimate (2.6). □

2.2 Examples

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $m, n \in \mathbb{N}$. By $\mathrm{End}(\mathbb{R}^s)$ we denote the endomorphisms from $\mathbb{R}^s$ to $\mathbb{R}^s$. Choose $A \in L^\infty(\Omega; \mathrm{End}(\mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^n))$ and assume that A is symmetric, i.e. $\langle A(x)\left(\begin{smallmatrix} u_1 \\ F_1 \\ z_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} u_2 \\ F_2 \\ z_2 \end{smallmatrix}\right)\rangle = \langle A(x)\left(\begin{smallmatrix} u_2 \\ F_2 \\ z_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} u_1 \\ F_1 \\ z_1 \end{smallmatrix}\right)\rangle$ for a.e. $x \in \Omega$ and every $u_i \in \mathbb{R}^m$, $F_i \in \mathbb{R}^{m\times d}$, $z_i \in \mathbb{R}^n$ and $i \in \{1,2\}$. Here, $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^n$. For $u \in H^1(\Omega, \mathbb{R}^m)$ and $z \in L^2(\Omega, \mathbb{R}^n)$ we define

$$\mathcal{E}(u, z) = \frac{1}{2}\int_\Omega \langle A(x)\begin{pmatrix} u(x) \\ \nabla u(x) \\ z(x) \end{pmatrix}, \begin{pmatrix} u(x) \\ \nabla u(x) \\ z(x) \end{pmatrix}\rangle\,\mathrm{d}x. \tag{2.10}$$

We put $\mathcal{Z} = L^2(\Omega, \mathbb{R}^n)$ and assume that there exists a closed subspace $\mathcal{U} \subset H^1(\Omega, \mathbb{R}^m)$ and a constant $\alpha > 0$ such that for all $u \in \mathcal{U}$ and $z \in \mathcal{Z}$ it holds

$$\mathcal{E}(u, z) \ge \tfrac{\alpha}{2}\big(\|u\|^2_{H^1(\Omega)} + \|z\|^2_{L^2(\Omega)} \big). \tag{2.11}$$

Estimate (2.11) typically follows from a Poincaré/Friedrichs inequality or Korn's inequality and we will give examples for the choice of A in Sections 2.2.1 and 2.2.2.

Furthermore, let $g : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a maximal monotone mapping with $0 \in g(0)$. In particular, the choice $g = \partial\chi_K$ is admissible, where $K \subset \mathbb{R}^n$ is closed, convex and with $0 \in K$ and where χ_K denotes the characteristic function associated with K. We define

$$\mathcal{G} : \mathcal{Z} \to \mathcal{P}(\mathcal{Z}), \quad \mathcal{G}(z) = \{\, \eta \in L^2(\Omega, \mathbb{R}^n)\,;\, \eta(x) \in g(z(x)) \text{ a.e. in } \Omega \,\}, \tag{2.12}$$

which is a maximal monotone mapping with respect to $\mathcal{Z}$. In this setting, Theorems 2.1 and 2.2 provide the existence of a unique pair $(u, z) \in W^{1,1}(S; \mathcal{U} \times \mathcal{Z})$ satisfying (2.2) with $\mathcal{E}$ from (2.10) and $\mathcal{G}$ from (2.12).

In the sequel we use the following notation: For matrices $T, S \in \mathbb{R}^{m\times d}$ the inner product is denoted by $S : T = \mathrm{tr}(T^\top S)$ with the corresponding norm $|T| = \sqrt{T : T}$. Moreover, $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{d\times d}$.

2.2.1 Elasto-(visco)-plasticity with linear hardening

For setting up an elasto-plastic model with linear hardening we choose $m = d$ and define $\mathcal{U} = \{\, u \in H^1(\Omega, \mathbb{R}^d)\,;\, u\big|_{\Gamma_D} = 0 \,\}$ to be the space of admissible displacements. Here, $\Gamma_D \subset \partial\Omega$ is a nonempty open set and denotes the Dirichlet boundary. Let furthermore $\varepsilon : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be defined through $\varepsilon(F) = \frac{1}{2}(F + F^\top)$ for $F \in \mathbb{R}^{d\times d}$ and let $C \in L^\infty(\Omega, \mathrm{End}(\mathbb{R}^{d\times d}_{\mathrm{sym}}))$. C corresponds to the elasticity tensor. It is assumed that C is self adjoint and that there exists a constant $\alpha > 0$ such that $C(x)F : F \geq \alpha\,|F|^2$ for all $F \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and a.e. $x \in \Omega$. Let moreover $L \in L^\infty(\Omega; \mathrm{End}(\mathbb{R}^n))$ be self adjoint and uniformly positive definite and choose $B \in L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^{d\times d}_{\mathrm{sym}}))$. B maps the vector z of internal variables onto the plastic strain. We define $A \in L^\infty(\Omega, \mathrm{End}(\mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^n))$ via the relation

$$\langle A(x) \left(\begin{smallmatrix} u_1 \\ F_1 \\ z_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} u_2 \\ F_2 \\ z_2 \end{smallmatrix}\right) \rangle = \langle \begin{pmatrix} \varepsilon^* C(x)\varepsilon & -\varepsilon^* C(x) B(x) \\ -B^*(x) C(x)\varepsilon & B^*(x)C(x)B(x) + L(x) \end{pmatrix} \begin{pmatrix} F_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} F_2 \\ z_2 \end{pmatrix} \rangle \tag{2.13}$$

for all $u_i \in \mathbb{R}^d$, $F_i \in \mathbb{R}^{d\times d}$, $z_i \in \mathbb{R}^n$ and almost every $x \in \Omega$. Thus, for $u \in H^1(\Omega, \mathbb{R}^d)$ and $z \in L^2(\Omega, \mathbb{R}^n)$ the stored energy reads

$$\mathcal{E}_{\mathrm{H}}(u, z) = \frac{1}{2} \int_\Omega C(\varepsilon(\nabla u) - Bz) : (\varepsilon(\nabla u) - Bz) + (Lz) \cdot z \,\mathrm{d}x. \tag{2.14}$$

Since C and L are assumed to be positive definite, it follows with Korn's inequality that estimate (2.11) is satisfied for all $u \in \mathcal{U}$ and $z \in \mathcal{Z}$. Problem (2.2) formulated with $\mathcal{E}_{\mathrm{H}}$ from (2.14) and $\mathcal{G}$ from (2.12) constitutes an elastic-(visco)-plastic model with linear hardening and takes the form: Find $(u, z) \in W^{1,1}(S; \mathcal{U}) \times W^{1,1}(S; \mathcal{Z})$ such that for a.e. $t \in S$ and every $v \in \mathcal{U}$

$$\begin{aligned} \int_\Omega C(\varepsilon(\nabla u(t)) - Bz(t)) : \varepsilon(\nabla v) \,\mathrm{d}x &= \langle \ell_1(t), v \rangle_{(\mathcal{U}^*, \mathcal{U})}, \\ \partial_t z(t) \in g\big(-(-B^\top C(\varepsilon(\nabla u(t)) - Bz(t)) &+ Lz(t)) + \ell_2(t) \big). \end{aligned} \tag{2.15}$$

Theorems 2.1 and 2.2 provide the existence of solutions.

This setting comprises linear kinematic hardening while pure isotropic hardening is excluded in our analysis. In the pure isotropic case, the matrix L is positive semidefinite, only. We refer to [HR99, Joh78] for an existence proof for the case with isotropic hardening. Models of the type (2.15) with positive definite L are investigated in [AN08] with respect to regularity questions.

2.2.2 Elasto-plasticity coupled with Cosserat micropolar effects

In [NC05] an elastic-plastic model was introduced which incorporates Cosserat micropolar effects. This model is analyzed in [NC05, NC08] with respect to existence and local regularity and in [KN08] with respect to global regularity of a time discretized version. Let $\Omega \subset \mathbb{R}^d$, $d = 3$, be bounded with Lipschitz boundary. In this model, not only the displacements u but also linearized micro-rotations Q are taken into account. These micro-rotations are represented with skew-symmetric tensors which are identified with vectors in $\mathbb{R}^{\frac{d(d-1)}{2}}$. Consequently we choose $m = d + d(d-1)/2$. The generalized displacements are now given by the pair $(u, Q) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{skew}} \cong \mathbb{R}^m$. The internal variable z is identified with the plastic strain tensor $z = \varepsilon_p \in \mathbb{R}^{d\times d}_{\text{sym, dev}} \cong \mathbb{R}^n$ with a suitable $n \in \mathbb{N}$. The set $\mathbb{R}^{d\times d}_{\text{sym, dev}}$ consists of the symmetric matrices with zero trace. The coefficient function $A \in L^\infty(\Omega, \text{End}(\mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^n))$ is defined through the relation

$$\begin{aligned}\langle A(x)\begin{pmatrix}\scriptstyle (u_1,Q_1)\\ \scriptstyle (F_1^u,F_1^Q)\\ \scriptstyle \varepsilon_{p,1}\end{pmatrix}, \begin{pmatrix}\scriptstyle (u_2,Q_2)\\ \scriptstyle (F_2^u,F_2^Q)\\ \scriptstyle \varepsilon_{p,2}\end{pmatrix}\rangle&\\ = 2\mu\big(\varepsilon(F_1^u) - \varepsilon_{p,1}\big) : \big(\varepsilon(F_2^u) - \varepsilon_{p,2}\big) &+ 2\mu_c\big(\text{skew}\,(F_1^u - Q_1)\big) : \big(\text{skew}\,(F_2^u - Q_2)\big)\\ + \lambda(\text{tr}\,F_1^u)(\text{tr}\,F_2^u) &+ 2\gamma(F_1^Q : F_2^Q)\end{aligned} \tag{2.16}$$

for every $u_i \in \mathbb{R}^d$, $Q_i \in \mathbb{R}^{d\times d}_{\text{skew}}$, $F_i^u \in \mathbb{R}^{d\times d}$, $F_i^Q \in \mathbb{R}^{d\times d(d-1)/2}$ and $z_i = \varepsilon_{p,i} \in \mathbb{R}^{d\times d}_{\text{sym, dev}}$. The operator ε is the same as in the previous section. Here, $\lambda, \mu > 0$ are the Lamé constants, $\mu_c > 0$ is the Cosserat couple modulus and $\gamma > 0$ depends on the Lamé constants and an internal length parameter. For $u \in H^1(\Omega, \mathbb{R}^d)$, $Q \in H^1(\Omega, \mathbb{R}^{d\times d}_{\text{skew}})$ and $\varepsilon_p \in L^2(\Omega, \mathbb{R}^{d\times d}_{\text{sym,dev}})$ the stored energy reads

$$\mathcal{E}_{\text{C}}((u,Q),\varepsilon_p) = \int_\Omega \mu\,|\varepsilon(\nabla u) - \varepsilon_p|^2 + \mu_c\,|\text{skew}\,(\nabla u - Q)|^2 + \frac{\lambda}{2}\,|\text{tr}\,\nabla u|^2 + \gamma\,|\nabla Q|^2\,\mathrm{d}x. \tag{2.17}$$

Let $\mathcal{U} = H_0^1(\Omega, \mathbb{R}^d) \times H_0^1(\Omega, \mathbb{R}^{d\times d}_{\text{skew}})$ and $\mathcal{Z} = L^2(\Omega, \mathbb{R}^{d\times d}_{\text{sym, dev}})$. On the basis of the div/curl inequality, see e.g. [GR86], and the Poincaré inequality it follows for C^1-smooth domains that there exists a constant $\alpha > 0$ such that for all $(u, Q) \in \mathcal{U}$ and $\varepsilon_p \in \mathcal{Z}$ we have

$$\mathcal{E}_{\text{C}}((u,Q),\varepsilon_p) \geq \frac{\alpha}{2}\big(\,\|u\|^2_{H^1(\Omega)} + \|Q\|^2_{H^1(\Omega)} + \|\varepsilon_p\|^2_{L^2(\Omega)}\,\big) \tag{2.18}$$

and therefore $\mathcal{E}_{\text{C}}$ satisfies the assumption (2.11). We refer to [NC05] for a proof of inequality (2.18). Problem (2.2) formulated with $\mathcal{E}_{\text{C}}$ from (2.17) and with a maximal monotone operator $\mathcal{G}$ defined as in (2.12) describes elasto-plastic material behavior which is coupled with Cosserat micropolar effects. Note that $D_u\mathcal{E}(u,z)$ in (2.2) has to be interpreted as $D_{(u,Q)}\mathcal{E}_{\mathrm{C}}(u,Q,\varepsilon_p)$. The existence of solutions was first investigated in [NC05] and is also a consequence of Theorems 2.1 and 2.2.

3 Regularity for model problems on a cube

3.1 Local regularity

The starting point of our global regularity analysis is to study the local regularity properties on cubes of solutions of systems, which consist of the principal part of the systems described in Section 2.2. These properties are derived with a difference quotient technique which is based on inner variations. The results in part (b) of the regularity Theorem 3.1 here below are a straightforward extension of the results from [AN08] for energies of the form described in (2.14) to our more general setting. In part (a) of Theorem 3.1 we discuss the local regularity properties for data which have less temporal regularity.

For $r > 0$ let $C_r = (-r,r)^d$ be a cube with side length $2r$. Let $m,n \in \mathbb{N}$. We choose $\mathcal{U} = H^1_0(C_r,\mathbb{R}^m)$ and $\mathcal{Z} = L^2(C_r,\mathbb{R}^n)$. The coefficient function A shall satisfy

A1 $A \in C^{0,1}(\overline{C_r},\mathrm{End}(\mathbb{R}^{m\times d}\times\mathbb{R}^n))$ is symmetric and there exists a constant $\alpha > 0$ such that for all $u \in \mathcal{U}$ and $z \in \mathcal{Z}$ we have $\mathcal{E}(u,z) \geq \frac{\alpha}{2}\big(\|u\|^2_{H^1(C_r)} + \|z\|^2_{L^2(C_r)}\big)$.

Here, $\mathcal{E}(u,z) = \int_{C_r}\langle A(x)\left(\begin{smallmatrix}\nabla u\\ z\end{smallmatrix}\right),\left(\begin{smallmatrix}\nabla u\\ z\end{smallmatrix}\right)\rangle\,\mathrm{d}x$. It is assumed that the term ℓ_1 in (2.2) can be written as

$$\langle \ell_1(t), v\rangle_{(\mathcal{U}^*,\mathcal{U})} = \int_{C_r} f(t)\cdot v + H(t) : \nabla v\,\mathrm{d}x, \tag{3.1}$$

with suitable $f \in L^\infty(S;L^2(C_r))$ and $H \in L^\infty(S;L^2(C_r))$. We study the spatial regularity of functions $u \in L^\infty(S;\mathcal{U})$ and $z \in W^{1,1}(S;\mathcal{Z})$ which satisfy for a.e. $t \in S$ and every $v \in \mathcal{U}$ the relations

$$\begin{aligned}\int_{C_r}\langle A\begin{pmatrix}\nabla u(t)\\ z(t)\end{pmatrix},\left(\begin{smallmatrix}\nabla v\\ 0\end{smallmatrix}\right)\rangle\,\mathrm{d}x &= \int_{C_r} f(t)\cdot v + H(t):\nabla v\,\mathrm{d}x,\\ \partial_t z(t) &\in \mathcal{G}(-D_z\mathcal{E}(u(t),z(t)) + \ell_2(t))\\ z(0) &= z_0.\end{aligned} \tag{3.2}$$

In terms of the projection operators $\mathbb{P}_{m\times d} : \mathbb{R}^{m\times d}\times\mathbb{R}^n \to \mathbb{R}^{m\times d}$, $(F,z)\mapsto F$ and $\mathbb{P}_n : \mathbb{R}^{m\times d}\times\mathbb{R}^n \to \mathbb{R}^n$, $(F,z)\mapsto z$, problem (3.2) can equivalently be written as

$$\begin{aligned}\int_{C_r}\mathbb{P}_{m\times d}[A\begin{pmatrix}\nabla u(t)\\ z(t)\end{pmatrix}]:\nabla v\,\mathrm{d}x &= \int_{C_r} f(t)\cdot v + H(t):\nabla v\,\mathrm{d}x,\\ \partial_t z(t) &\in \mathcal{G}(-\mathbb{P}_n[A\begin{pmatrix}\nabla u(t)\\ z(t)\end{pmatrix}] + \ell_2(t)).\end{aligned} \tag{3.3}$$

In the sequel we use the following spaces defined for domains $\Omega_2 \subset \Omega_1$ and $i \in \{1, \ldots, d\}$:

$$\mathcal{F}_i(\Omega_1, \Omega_2) = \{ v \in L^2(\Omega_1) \, ; \, \partial_{x_i}(v\big|_{\Omega_2}) \in L^2(\Omega_2) \} \tag{3.4}$$

with $\|v\|_{\mathcal{F}_i(\Omega_1,\Omega_2)} = \|v\|_{L^2(\Omega_1)} + \|\partial_{x_i} v\|_{L^2(\Omega_2)}$. Moreover, finite differences are denoted by

$$\triangle_{he} u(x) := u(x + he) - u(x)$$

for $h \in \mathbb{R}$ and $e \in \mathbb{R}^d \backslash \{0\}$. By $\{e_1, \ldots, e_d\}$ we denote the standard basis in $\mathbb{R}^d$. The spatial regularity of solutions is discussed under different assumptions on the temporal smoothness of the data. In particular, the cases **A2** and **A3** here below are considered:

A2 There exists $\rho \in (0, r)$ and $i \in \{1, \ldots, d\}$ such that $z_0 \in \mathcal{F}_i(C_r, C_\rho)$, $f \in L^\infty(S; L^2(C_r))$, $H \in L^\infty(S; \mathcal{F}_i(C_r, C_\rho))$ and $\ell_2 \in L^\infty(S; \mathcal{F}_i(C_r, C_\rho))$.

A3 There exists $\rho \in (0, r)$ and $i \in \{1, \ldots, d\}$ such that $z_0 \in \mathcal{F}_i(C_r, C_\rho)$, $f \in W^{1,1}(S; L^2(C_r))$, $H \in W^{1,1}(S; \mathcal{F}_i(C_r, C_\rho))$ and $\ell_2 \in W^{1,1}(S; \mathcal{F}_i(C_r, C_\rho))$.

Theorem 3.1 (Local regularity on cubes)**.** *Let condition* **A1** *be satisfied.*

(a) Let the pair $(u, z) \in L^\infty(S; \mathcal{U}) \times W^{1,1}(S; \mathcal{Z})$ solve (3.2) with data according to assumption **A2***. Then there exists $h_0 > 0$ such that*

$$\sup_{0<h<h_0} h^{-\frac{1}{2}} \|\triangle_{he_i} \nabla u\|_{L^\infty(S; L^2(C_{\rho/2}))} < \infty,$$
$$\sup_{0<h<h_0} h^{-\frac{1}{2}} \|\triangle_{he_i} z\|_{L^\infty(S; L^2(C_{\rho/2}))} < \infty.$$

(b) Let the pair $(u, z) \in W^{1,1}(S; \mathcal{U} \times \mathcal{Z})$ satisfy (3.2) with data according to **A3***. Then*

$$\nabla u \in L^\infty(S; \mathcal{F}_i(C_r, C_{\frac{\rho}{2}})), \quad z \in L^\infty(S; \mathcal{F}_i(C_r, C_{\frac{\rho}{2}})).$$

If the assumptions of part (b) of Theorem 3.1 are valid for all $i \in \{1, \ldots, d\}$, then

$$u\big|_{C_{\rho/2}} \in L^\infty(S; H^2(C_{\rho/2})), \quad z\big|_{C_{\rho/2}} \in L^\infty(S; H^1(C_{\rho/2})).$$

If the assumptions of part (a) are satisfied for every $i \in \{1, \ldots, d\}$, then it follows that

$$\text{essup}_{t \in S} \|u(t)\|_{B^{\frac{3}{2}}_{2,\infty}(C_{\rho/2})} < \infty, \qquad \text{essup}_{t \in S} \|z(t)\|_{B^{\frac{1}{2}}_{2,\infty}(C_{\rho/2})} < \infty. \tag{3.5}$$

The spaces $B^s_{p,q}(\Omega)$ are Besov spaces and we refer to [Tri83] for a precise definition. We recall that $v \in B^s_{2,\infty}(\Omega)$ for $s \in (0, 1)$ if and only if $v \in L^2(\Omega)$ and

$$\sup_{1 \le i \le d,\, \widetilde{\Omega} \Subset \Omega,\, 0<h<h_0} h^{-s} \|\triangle_{he_i} v\|_{L^2(\widetilde{\Omega})} < \infty,$$

where $\{e_1, \ldots, e_d\}$ is an arbitrary basis in $\mathbb{R}^d$. Moreover, for every $\delta > 0$ and $s > 0$ with $s \notin \mathbb{N}$ the embeddings $H^s(\Omega) \subset B^s_{2,\infty}(\Omega) \subset H^{s-\delta}(\Omega)$ are continuous. Due to Lemma 3.2 here below we obtain therefore from (3.5) that

$$u\big|_{C_{\rho/2}} \in L^\infty(S; H^{\frac{3}{2}-\delta}(C_{\rho/2})), \qquad z\big|_{C_{\rho/2}} \in L^\infty(S; H^{\frac{1}{2}-\delta}(C_{\rho/2}))$$

for every $\delta > 0$.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a domain and assume that $v : S \to L^2(\Omega)$ is measurable and that there exists $s > 0$ such that $v(t) \in H^s(\Omega)$ for every $t \in S$. Then $v : S \to H^s(\Omega)$ is measurable.*

Proof. Since the space $H^s(\Omega)$ is separable, measurability is equivalent to weak measurability. Let $\eta \in (H^s(\Omega))'$ be arbitrary. Since $L^2(\Omega)$ is dense in $(H^s(\Omega))'$, there exists a sequence $(\eta_n)_n \subset L^2(\Omega)$ with $\eta_n \to \eta$ in $(H^s(\Omega))'$. Obviously, for every $t \in S$ we have $\int_\Omega \eta_n v(t)\,\mathrm{d}x = \langle \eta_n, v(t)\rangle_{H^s(\Omega)} \to \langle \eta, v(t)\rangle_{H^s(\Omega)}$. Due to the measurability of $v : S \to L^2(\Omega)$, the real valued functions $t \mapsto \int_\Omega \eta_n v(t)\,\mathrm{d}x$ are measurable as well, and hence also the limit function $t \mapsto \langle \eta, v(t)\rangle_{H^s(\Omega)}$ is measurable. This proves the weak measurability of $v : S \to H^s(\Omega)$. □

Proof of Theorem 3.1. Let $\rho \in (0, r)$ be given according to the assumptions in Theorem 3.1 and choose $\varphi \in C_0^\infty(C_r)$ with $\varphi(x) = 1$ on $C_{\rho/2}$ and $\operatorname{supp}\varphi \subset C_\rho$. For $h \in \mathbb{R}^d$ we introduce the following family of inner variations $\tau_h : C_r \to \mathbb{R}^d$, $x \to \tau_h(x) = x + \varphi(x)h$.

Let $h_0 = \min\{\operatorname{dist}(\operatorname{supp}\varphi, \partial C_r), \|\varphi\|_{W^{1,\infty}(C_r)})^{-1}\}$. For every $h \in \mathbb{R}^d$ with $|h| < h_0$, the mapping τ_h is a diffeomorphism from C_r onto itself with $\tau_h(x) = x$ for every $x \in \partial C_r$, see e.g. [GH96]. Obviously,

$$\nabla\tau_h(x) = \big(\mathbb{I} + h \otimes \nabla\varphi(x)\big), \quad \det(\nabla\tau_h(x)) = 1 + h \cdot \nabla\varphi(x). \tag{3.6}$$

Let the pair $(u, z) \in L^\infty(S;\mathcal{U}) \times W^{1,1}(S;\mathcal{Z})$ be a solution of problem (3.2) and e_i the vector introduced in assumptions **A2** and **A3**. For $h \in \mathbb{R}e_i$ with $|h| < h_0$ we define the shifted functions $u_h(t, x) := u(t, \tau_h(x))$ and $z_h(t, x) := z(t, \tau_h(x))$. Clearly, the shifted functions have the same temporal and spatial regularity as u and z since the shift τ_h induces linear isomorphisms on $\mathcal{U}$ and $\mathcal{Z}$, respectively.

Straightforward calculations, which are based on a change of coordinates with τ_h, imply that for almost every t and every $v \in \mathcal{U}$ the shifted functions u_h and z_h satisfy

$$\begin{aligned}
&\int_{C_r} \langle A \left(\begin{smallmatrix}\nabla u_h(t)\\ z_h(t)\end{smallmatrix}\right), \left(\begin{smallmatrix}\nabla v\\ 0\end{smallmatrix}\right)\rangle\,\mathrm{d}x \\
&\qquad= \int_{C_r} \det\nabla\tau_h f_h(t)\cdot v\,\mathrm{d}x + \int_{C_r} (\det\nabla\tau_h)\big(H_h(t)(\nabla\tau_h)^{-\top}\big) : \nabla v\,\mathrm{d}x \\
&\qquad\quad+ \int_{C_r} \langle F_1^h(t), \nabla v\rangle\,\mathrm{d}x \\
&\qquad=: \langle \ell_1^h(t), v\rangle_{(\mathcal{U}^*,\mathcal{U})}.
\end{aligned} \tag{3.7}$$

Here, $f_h = f \circ \tau_h$, $H_h = H \circ \tau_h$, $A_h = A \circ \tau_h$ and

$$\begin{aligned}
\int_{C_r} \langle F_1^h(t), \nabla v\rangle\,\mathrm{d}x &= \int_{C_r} \langle A \left(\begin{smallmatrix}\nabla u_h(t)\\ z_h(t)\end{smallmatrix}\right) - \det\nabla\tau_h A_h \left(\begin{smallmatrix}\nabla u_h(t)(\nabla\tau_h)^{-1}\\ z_h(t)\end{smallmatrix}\right), \left(\begin{smallmatrix}\nabla v\\ 0\end{smallmatrix}\right)\rangle\,\mathrm{d}x \\
&\quad- \int_{C_r} \det\nabla\tau_h \langle A_h \left(\begin{smallmatrix}\nabla u_h(t)(\nabla\tau_h)^{-1}\\ z_h(t)\end{smallmatrix}\right), \left(\begin{smallmatrix}\nabla v(\nabla\tau_h^{-1} - \mathbb{I})\\ 0\end{smallmatrix}\right)\rangle\,\mathrm{d}x.
\end{aligned}$$

Moreover, the following evolution law is satisfied by u_h and z_h

$$\partial_t z_h(t) \in g(-D_z\mathcal{E}(u_h(t), z_h(t)) + F_2^h(t) + \ell_{2,h}(t)), \quad z_h(0) = z_0\circ\tau_h. \tag{3.8}$$

Here, $\ell_{2,h} = \ell_2\circ\tau_h$ and

$$F_2^h(t) = -\mathbb{P}_n\left((A_h - A)\begin{pmatrix}\nabla u_h(t)\\ z_h(t)\end{pmatrix} + A_h\begin{pmatrix}\nabla u_h(t)(\nabla\tau_h^{-1} - \mathbb{I})\\ 0\end{pmatrix}\right),$$

where $\mathbb{P}_n$ is the already introduced projection onto the $\mathbb{R}^n$ component.

From the Lipschitz continuity of the coefficient matrix A and the properties of τ_h, where we use in particular the relations in (3.6), we deduce the estimate

$$\|F_1^h\|_{L^\infty(S;L^2(C_r))} + \|F_2^h\|_{L^\infty(S;L^2(C_r))} \le c\,|h|\,\big(\|u\|_{L^\infty(S;\mathcal{U})} + \|z\|_{L^\infty(S;\mathcal{Z})}\big), \tag{3.9}$$

and if $(u, z) \in W^{1,1}(S;\mathcal{U}\times\mathcal{Z})$, then

$$\|F_1^h\|_{W^{1,1}(S;L^2(C_r))} + \|F_2^h\|_{W^{1,1}(S;L^2(C_r))} \le c\,|h|\,\big(\|u\|_{W^{1,1}(S;\mathcal{U})} + \|z\|_{W^{1,1}(S;\mathcal{Z})}\big). \tag{3.10}$$

In both inequalities, the constant c depends on $\|\varphi\|_{W^{1,\infty}(C_r)}$ and $\|A\|_{W^{1,\infty}(C_r)}$ but is independent of h. By estimates (3.9) and (3.10) we have $\ell_1^h \in L^\infty(S;\mathcal{U}^*)$ in the situation described in part (a) of Theorem 3.1 and $\ell_1^h \in W^{1,1}(S;\mathcal{U}^*)$ if the assumptions of part (b) are valid.

Let first the assumptions of part (a) be satisfied. From the stability estimate (2.5) applied to (3.7) and (3.8) we deduce that there exists a constant $c > 0$ which is independent of h such that

$$\begin{aligned}\|u - u_h\|^2_{L^\infty(S;\mathcal{U})} + \|z - z_h\|^2_{L^\infty(S;\mathcal{Z})} \le c\Big(&\|z_0 - z_{0,h}\|^2_{L^2(C_r)} + \|\ell_1 - \ell_1^h\|^2_{L^\infty(S;\mathcal{U}^*)}\\ &+ 2c(\varphi)\,\|z\|_{W^{1,1}(S;\mathcal{Z})}\big(\|\ell_1 - \ell_1^h\|_{L^\infty(S;\mathcal{U}^*)} + \|\ell_2 - \ell_{2,h} - F_2^h\|_{L^\infty(S;\mathcal{Z})}\big)\Big).\end{aligned} \tag{3.11}$$

In view of **A2** it follows (see e.g. Lemma 4.1 in [KM08]) that

$$\|z_0 - z_{0,h}\|_{L^\infty(S;\mathcal{Z})} \le c\,|h|\,\|z_0\|_{\mathcal{F}_i(C_r,C_\rho)}.$$

The last term in (3.11) can be estimated in the same way. For estimating the terms with ℓ_1 observe that

$$\begin{aligned}\|f(t) - \det\nabla\tau_h f_h(t)\|_{\mathcal{U}^*} &= \sup_{v\in\mathcal{U},\,\|v\|_{\mathcal{U}}=1}\int_{C_r}(f(t) - \det\nabla\tau_h f_h(t))\cdot v\,\mathrm{d}x\\ &= \sup_{v\in\mathcal{U},\,\|v\|_{\mathcal{U}}=1}\int_{C_r} f(t)\cdot(v - v\circ\tau_h^{-1})\,\mathrm{d}x \le c(\varphi)\,|h|\,\|f(t)\|_{L^2(C_\rho)}.\end{aligned}$$

Thus, altogether it follows that there exists a constant $\kappa > 0$ such that for all $h \in \mathbb{R}e_i\backslash\{0\}$ with $|h| < h_0$ we have

$$\begin{aligned}&|h|^{-\frac{1}{2}}\big(\|\triangle_h u\|_{L^\infty(S;H^1(C_{\rho/2}))} + \|\triangle_h z\|_{L^\infty(S;L^2(C_{\rho/2}))}\big)\\ &\quad\le \kappa\Big(\|z_0\|_{\mathcal{F}_i(C_r,C_\rho)} + \|f\|_{L^\infty(S;L^2(C_\rho))} + \|H\|_{L^\infty(S;\mathcal{F}_i(C_r,C_\rho))} + \|\ell_2\|_{L^\infty(S;\mathcal{F}_i(C_r,C_\rho))}\\ &\qquad\qquad + \|u\|_{L^\infty(S;\mathcal{U})} + \|z\|_{W^{1,1}(S;\mathcal{Z})}\Big).\end{aligned}$$

This proves the assertions of Theorem 3.1, part (a).

The results in part (b) follow in the same way by applying stability estimate (2.6) to (3.7) and (3.8). □

3.2 Tangential regularity on a half cube

For $r > 0$ let $K_r = (-r,r)^{d-1} \times (0,r)$ be a half cube with bottom $\Gamma_0 = (-r,r)^{d-1} \times \{0\}$ and let $m, n \in \mathbb{N}$. We choose $\mathcal{Z} = L^2(K_r, \mathbb{R}^n)$ and consider closed subspaces $\mathcal{U}(K_r) \subset H^1(K_r, \mathbb{R}^m)$ allowing for different types of boundary conditions for different components of $u \in \mathcal{U}(K_r)$. In particular, let $D \subset \{1,\ldots,m\}$, D might also be the empty set. Then $\mathcal{U}(K_r) := \{\, u \in H^1(K_r, \mathbb{R}^m)\,;\, u\big|_{\partial K_r \setminus \Gamma_0} = 0,\, u_i\big|_{\Gamma_0} = 0 \text{ for } i \in D \,\}$.

Theorem 3.3. *Assume that the coefficient function A satisfies* **A1** *from Section 3.1 with respect to K_r and $\mathcal{U}(K_r) \times \mathcal{Z}$. Let the pair $(u,z) \in W^{1,1}(S;\mathcal{U}(K_r) \times \mathcal{Z})$ satisfy* (3.2) *on K_r and assume that the data has the following regularity for some $\rho \in (0,r)$:*

$$z_0 \in H^1(K_r), \quad f \in W^{1,1}(S;L^2(K_r)), \quad H \in W^{1,1}(S;L^2(K_r) \cap H^1(K_\rho)), \tag{3.12}$$

$$\ell_2 \in W^{1,1}(S;L^2(K_r) \cap H^1(K_\rho)). \tag{3.13}$$

Then for $1 \le i \le d-1$ we have the tangential regularity

$$\partial_{x_i} \nabla u \in L^\infty(S;L^2(K_{\frac{\rho}{2}})), \qquad \partial_{x_i} z \in L^\infty(S;L^2(K_{\frac{\rho}{2}})). \tag{3.14}$$

This theorem is a straightforward generalization of a recent result by Alber/Nesenenko [AN08], where the case $m = d$ and pure Dirichlet conditions on Γ_0 are considered. The theorem can be derived in the same way as the results in part (b) of Theorem 3.1 and we omit the proof. We just remark that the space $\mathcal{U}$ is invariant with respect to inner variations τ_h which are tangential to Γ_0.

3.3 Global regularity on a half cube

Before we formulate the key result of this paper, Theorem 3.4, we need some further notation. Let again $K_r = (-r,r)^{d-1} \times (0,r)$ be the half cube with bottom $\Gamma_0 = (-r,r)^{d-1} \times \{0\}$ and let $m, n \in \mathbb{N}$. By $R = \mathbb{I} - 2e_d \otimes e_d$ we denote the reflection at the boundary Γ_0. The extended coefficient function A_e is defined via $A_e(x) = A(x)$ for $x \in \overline{K_r}$ and $A_e(x) = A(Rx)$ for $x \in C_r \setminus K_r$.

Theorem 3.4 (Global regularity on a half cube)**.** *Assume that the extended coefficient function A_e satisfies condition* **A1** *from Section 3.1 with respect to the full cube C_r and $H^1_0(C_r) \times L^2(C_r)$. Let the pair $(u,z) \in L^\infty(S;H^1(K_r)) \times W^{1,1}(S;\mathcal{Z})$ satisfy* (3.2) *on K_r for all $v \in H^1_0(K_r)$. Assume furthermore that for all t it holds* $\operatorname{supp} u(t) \subset \overline{K_{\frac{3r}{4}}}$ *and that for $1 \le i \le d-1$ the functions u and z have the tangential regularity*

$$\partial_i \nabla u \in L^\infty(S;L^2(K_r)),\ \partial_i z \in L^\infty(S;L^2(K_r)).$$

For the data we assume that

$$z_0 \in H^1(K_r), \quad f \in L^\infty(S; L^2(K_r)),$$
$$\ell_2,\, H \in L^\infty(S; \cap_{i=1}^{d-1} \mathcal{F}_i(K_r, K_r)) \cap L^\infty(S; H^1(K_{\frac{r}{2}})).$$

Then $\operatorname{essup}_{t\in S} \|u(t)\|_{B^{3/2}_{2,\infty}(K_{r/4})} < \infty$, $\operatorname{essup}_{t\in S} \|z(t)\|_{B^{1/2}_{2,\infty}(K_{r/4})} < \infty$, *and for every* $\delta > 0$ *we have*

$$u \in L^\infty(S; H^{\frac{3}{2}-\delta}(K_{\frac{r}{4}})), \quad z \in L^\infty(S; H^{\frac{1}{2}-\delta}(K_{\frac{r}{4}})).$$

The proof of this theorem relies on a reflection argument which was developed in [Kne08] for periodic problems with constant coefficients and is carried out in the Lemmata 3.5–3.6 here below.

Let $(u, z) \in L^\infty(S; H^1(K_r)) \times W^{1,1}(S; \times\mathcal{Z})$ be a solution to problem (3.2) on K_r as described in Theorem 3.4. Choose a function $\varphi \in C^\infty([0, r])$ with $\varphi(s) = 1$ in a neighborhood of $s = 0$, $\varphi(s) = 0$ for $s \geq \frac{r}{2}$ and $0 \leq \varphi \leq 1$. By γ_0 we denote the trace operator from $H^1(K_r)$ to $H^{\frac{1}{2}}(\Gamma_0)$ and define for $x = (x', x_d) \in K_r$

$$\hat{u}(t, x) := \varphi(x_d)(\gamma_0 u(t))(x'). \tag{3.15}$$

The tangential regularity of u entails the following regularity for $\hat{u}$:

Lemma 3.5. *Under the assumptions of Theorem 3.4 it holds* $\hat{u}$, $\partial_d \hat{u} \in L^\infty(S; H^1(K_r))$ *with* $\operatorname{supp} \hat{u}(t) \subset \overline{K_{\frac{3r}{4}}}$.

Proof. The proof is similar to the proof of Lemma 4.2 from [Kne08] with obvious modifications. □

The following extensions to C_r will be used:

$$u_e(t, x) := \begin{cases} u(t, x) - \hat{u}(t, x) & x \in K_r \\ -u(t, Rx) + \hat{u}(t, Rx) & x \in C_r \backslash K_r \end{cases}. \tag{3.16}$$

For the inner variable we use an even extension:

$$z_e(t, x) := \begin{cases} z(t, x) & x \in K_r \\ z(t, Rx) & x \in C_r \backslash K_r \end{cases} \tag{3.17}$$

and similar for z_0, where the extension is denoted by $z_{0,e}$. The extended functions have the smoothness

$$u_e \in L^\infty(S; H^1_0(C_r)), \quad z_e \in W^{1,1}(S; L^2(C_r)), \quad z_{0,e} \in H^1(C_r).$$

Finally let $\mathcal{E}_e(v, \eta) = \int_{C_r} \frac{1}{2} \langle A_e \left(\begin{smallmatrix} \nabla v \\ \eta \end{smallmatrix}\right), \left(\begin{smallmatrix} \nabla v \\ \eta \end{smallmatrix}\right) \rangle \, dx$.

Lemma 3.6. *Let the assumptions of Theorem 3.4 be satisfied. There exist functions* $f_e \in L^\infty(S; L^2(C_r))$*,* $H_e \in L^\infty(S; \mathcal{F}_d(C_r, C_{\frac{r}{2}}))$ *and* $\ell_{2,e} \in L^\infty(S; \mathcal{F}_d(C_r, C_{\frac{r}{2}}))$ *such that for all* $v \in H^1_0(C_r)$ *we have*

$$\begin{aligned}\int_{C_r} \langle A_e \left(\begin{smallmatrix} \nabla u_e(t) \\ z_e(t) \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x = \int_{C_r} f_e(t) \cdot v \, \mathrm{d}x + \int_{C_r} H_e(t) : \nabla v \, \mathrm{d}x, \\ \partial_t z_e(t) \in \mathcal{G}(-D_z \mathcal{E}_e(u_e(t), z_e(t)) + \ell_{2,e}(t)) \\ z_e(0) = z_{0,e}.\end{aligned} \tag{3.18}$$

Proof. Observe first that for all $v \in H^1_0(C_r)$ it holds with $\widetilde{v}(x) = v(Rx)$

$$\begin{aligned}\int_{C_r} \langle A_e \left(\begin{smallmatrix} \nabla u_e \\ z_e \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x &= \int_{K_r} \langle A \left(\begin{smallmatrix} \nabla u \\ z \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla (v - \widetilde{v}) \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x - \int_{K_r} \langle A \left(\begin{smallmatrix} \nabla \hat{u} \\ 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x \\ &+ \int_{C_r \setminus K_r} \langle A_e \left(\begin{smallmatrix} \nabla u \\ z \end{smallmatrix} \right) \big|_{Rx}, \left(\begin{smallmatrix} \nabla v (R + \mathbb{I}) \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x \\ &+ \int_{C_r \setminus K_r} \langle A_e \left(\begin{smallmatrix} \nabla \hat{u} R - \nabla u (R + \mathbb{I}) \\ 0 \end{smallmatrix} \right) \big|_{Rx}, \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x.\end{aligned}$$

Since the pair (u, z) solves (3.3) and since $v - \widetilde{v} \in H^1_0(K_r)$, we may replace the first term on the right hand side with f and H and obtain after rearranging the terms the following relation:

$$\begin{aligned}\int_{C_r} \langle A_e \left(\begin{smallmatrix} \nabla u_e \\ z_e \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix} \right) \rangle \, \mathrm{d}x &= \int_{K_r} f v \, \mathrm{d}x + \int_{C_R \setminus K_r} (-f \circ R) v \, \mathrm{d}x \\ &- \int_{C_r \setminus K_r} H \circ R (R + \mathbb{I}) : \nabla v \, \mathrm{d}x \\ &+ \int_{C_r \setminus K_r} \left(\mathbb{P}_{m \times d} \big[A \left(\begin{smallmatrix} \nabla u \\ z \end{smallmatrix} \right) \big] \circ R \right) (R + \mathbb{I}) : \nabla v \, \mathrm{d}x \\ &+ \int_{K_r} H : \nabla v \, \mathrm{d}x + \int_{C_r \setminus K_r} H \circ R : \nabla v \, \mathrm{d}x \\ &- \int_{K_r} \mathbb{P}_{m \times d} \big[A \left(\begin{smallmatrix} \nabla \hat{u} \\ 0 \end{smallmatrix} \right) \big] : \nabla v \, \mathrm{d}x \\ &+ \int_{C_r \setminus K_r} \mathbb{P}_{m \times d} \big[A \left(\begin{smallmatrix} \nabla \hat{u} R - \nabla u (R + \mathbb{I}) \\ 0 \end{smallmatrix} \right) \big] \circ R : \nabla v \, \mathrm{d}x.\end{aligned} \tag{3.19}$$

Observe that the regularity assumption on u and z imply that

$$\operatorname{div} \Big(\big(H \circ R - \mathbb{P}_{m \times d} \big[A \left(\begin{smallmatrix} \nabla u \\ z \end{smallmatrix} \right) \big] \circ R \big) (R + \mathbb{I}) \Big) \in L^\infty(S; L^2(C_r \setminus K_r))$$

since due to the factor $R + \mathbb{I}$ the derivative with respect to x_d does not appear. Thus, after applying the Gauss Theorem, the first four integrals on the right hand side in (3.19) can be replaced with the term $\int_{C_r} f_e(t, x) \cdot v(x) \, \mathrm{d}x$, where

$$f_e(t, x) = \begin{cases} f(t, x) & x \in K_r, \\ -f(t, Rx) + 2 \operatorname{div}_{x'} \left(H(t, Rx) - \mathbb{P}_{m \times d} \big[A \left(\begin{smallmatrix} \nabla u(t) \\ z(t) \end{smallmatrix} \right) \big] \circ R \right) & x \in C_r \setminus K_r. \end{cases} \tag{3.20}$$

Here, we use that $\operatorname{div}_{x'} \sigma = \partial_{x_1}\sigma_1 + \ldots + \partial_{x_{d-1}}\sigma_{d-1} = \frac{1}{2}\operatorname{div}(\sigma(R+\mathbb{I}))$ for $\sigma : C_r \to \mathbb{R}^{m\times d}$, and σ_i is the i-th column of σ. Note that $f_e \in L^\infty(S; L^2(C_r))$.

Let

$$\theta_e(t,x) = \begin{cases} \nabla\hat{u}(t,x) & x \in K_r, \\ -\nabla(\hat{u}(t,Rx)) + \big(\nabla(u(t,Rx))\big)(R+\mathbb{I}) & x \in C_r\backslash K_r. \end{cases}$$

From the assumptions on u and from Lemma 3.5 we conclude that $\partial_d(\theta_e\big|_{K_r}) \in L^\infty(S; L^2(K_r))$ and $\partial_d\theta_e\big|_{C_r\backslash K_r} \in L^\infty(S; L^2(C_r\backslash K_r))$. Since the traces on Γ_0 of $\theta_e\big|_{K_r}$ and of $\theta_e\big|_{C_r\backslash K_r}$ coincide, it follows that $\theta_e \in L^\infty(S; \mathcal{F}_d(C_r, C_r))$. Moreover, we define

$$H_e(t,x) = -\mathbb{P}_{m\times d}\big[A_e(x)\left(\begin{smallmatrix}\theta_e(t,x)\\ 0\end{smallmatrix}\right)\big] + \begin{cases} H(t,x) & x \in K_r, \\ H(t,Rx) & x \in C_r\backslash K_r. \end{cases} \tag{3.21}$$

The assumptions on H and the properties of θ_e imply that $H_e \in L^\infty(S; \mathcal{F}_d(C_r, C_{\frac{r}{2}}))$. With these definitions, the right hand side in (3.19) is equal to $\int_{C_r} f_e(t)\cdot v \,\mathrm{d}x + \int_{C_r} H_e(t) : \nabla v \,\mathrm{d}x$, which leads to the first equation in (3.18).

Finally we define

$$\ell_{2,e}(t,x) = -\mathbb{P}_n\big[A_e(x)\left(\begin{smallmatrix}\theta_e(t,x)\\ 0\end{smallmatrix}\right)\big] + \begin{cases} \ell_2(t,x) & x \in K_r, \\ \ell_2(t,Rx) & x \in C_r\backslash K_r. \end{cases}$$

As before we have $\ell_{2,e} \in L^\infty(S; \mathcal{F}_d(C_r, C_{\frac{r}{2}}))$. Moreover, straightforward calculations show that the extended functions satisfy the second relation in (3.18). This finishes the proof of Lemma 3.6. □

Proof of Theorem 3.4. Theorem 3.4 is an immediate consequence of part (a) of Theorem 3.1 and of Lemma 3.6. □

Observe that even with stronger assumptions on the temporal regularity of the data we cannot extend in the proof of Theorems 3.1 and 3.3 the regularity of u from $L^\infty(S; H^2_{\text{loc}}(K_r) \cap H^2_{\text{tang}}(K_r))$ to $W^{1,1}(S; H^2_{\text{loc}}(K_r) \cap H^2_{\text{tang}}(K_r))$. In fact, the example in Section 5.3.5 shows that in spite of arbitrary smooth data, u does not belong to $W^{1,1}(S; H^2(\Omega))$ in general. Thus we cannot expect that the extended data in the proof of Theorem 3.4 (see Lemma 3.6), which contain tangential derivatives of u and z, have the temporal regularity formulated in assumption **A3**. Hence, in order to obtain the global regularity, we can only apply the weak result formulated in part (a) of Theorem 3.1, and not the stronger result stated in part (b) of Theorem 3.1. This explains the loss of a "half" derivative in the normal direction. However, as we point out in Section 5.3.1, for time independent problems our reflection argument gives a full additional spatial derivative.

4 Main regularity theorem

4.1 Basic assumptions and main result

We are now ready to formulate and prove the main regularity theorem for generalized elasto-(visco)-plastic models on smooth domains. In particular we assume the following:

R1 $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^{1,1}$-smooth boundary, see e.g. [Gri85].

For $A \in L^\infty(\Omega, \mathrm{End}(\mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^n))$, $u \in H^1(\Omega, \mathbb{R}^m)$ and $z \in L^2(\Omega, \mathbb{R}^n)$ the energy $\mathcal{E}$ and the corresponding "principal part" $\mathcal{E}_{pp}$ are defined via

$$\mathcal{E}(u,z) = \tfrac{1}{2}\int_\Omega \langle A\begin{pmatrix} u \\ \nabla u \\ z\end{pmatrix}, \begin{pmatrix} u \\ \nabla u \\ z\end{pmatrix}\rangle \,\mathrm{d}x,$$
$$\mathcal{E}_{pp}(u,z) = \tfrac{1}{2}\int_\Omega \langle A\begin{pmatrix} 0 \\ \nabla u \\ z\end{pmatrix}, \begin{pmatrix} 0 \\ \nabla u \\ z\end{pmatrix}\rangle \,\mathrm{d}x.$$

R2 The coefficient function A belongs to $C^{0,1}(\overline{\Omega}, \mathrm{End}(\mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^n))$, is self adjoint and the principle part satisfies $\mathcal{E}_{pp}(v,z) \geq \frac{\alpha}{2}\big(\|v\|^2_{H^1(\Omega)} + \|z\|^2_{L^2(\Omega)}\big)$ for all $v \in H^1_0(\Omega)$ and $z \in L^2(\Omega)$.

Note that **R2** shall be satisfied for $v \in H^1_0(\Omega)$, only, independently of the type of boundary conditions which finally are imposed on the generalized displacements.

R3 $g : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is maximal monotone with $0 \in g(0)$. Moreover, $\mathcal{G} : L^2(\Omega, \mathbb{R}^n) \to \mathcal{P}(L^2(\Omega, \mathbb{R}^n))$ is defined as in (2.12).

We make the following assumptions on the data:

R4 $z_0 \in H^1(\Omega, \mathbb{R}^n)$, $f \in W^{1,1}(S; L^2(\Omega, \mathbb{R}^m))$, $H \in W^{1,1}(S; H^1(\Omega, \mathbb{R}^{m\times d}))$, $\ell_2 \in W^{1,1}(S; H^1(\Omega, \mathbb{R}^n))$ and $u_0 \in W^{1,1}(S; H^{\frac{3}{2}}(\partial\Omega, \mathbb{R}^m))$.

For $D \subset \{1,\dots,m\}$, where $D=\emptyset$ is not excluded, the set of admissible generalized displacements is given by

$$\mathcal{U} = \{\, v \in H^1(\Omega, \mathbb{R}^m)\,;\, v_i\big|_{\partial\Omega} = 0 \text{ for } i \in D\,\}. \tag{4.1}$$

With this choice it is possible to define different types of boundary conditions for the different components of u.

We consider functions $(u,z) \in W^{1,1}(S; H^1(\Omega, \mathbb{R}^m)) \times W^{1,1}(S; L^2(\Omega, \mathbb{R}^n))$ which for all $v \in \mathcal{U}$ and a.e. $t \in S$ satisfy the following relations

$$\begin{aligned} D_u\mathcal{E}(u(t), z(t))[v] = \int_\Omega \langle A\begin{pmatrix} u(t) \\ \nabla u(t) \\ z(t)\end{pmatrix}, \begin{pmatrix} v \\ \nabla v \\ 0\end{pmatrix}\rangle \,\mathrm{d}x &= \int_\Omega f(t)\cdot v + H(t) : \nabla v \,\mathrm{d}x, \\ \partial_t z(t) &\in \mathcal{G}\big(-D_z\mathcal{E}(t, u(t), z(t)) + \ell_2(t)\big), \\ z(0) &= z_0, \\ u_i(t)\big|_{\partial\Omega} &= u_{0,i}(t) \text{ for } i \in D. \end{aligned} \tag{4.2}$$

Theorem 4.1 (Main Regularity Theorem)**.** *Let* **R1**–**R4** *be satisfied and assume that the pair* $(u, z) \in W^{1,1}(S; H^1(\Omega)) \times W^{1,1}(S; L^2(\Omega))$ *satisfies* (4.2) *for all* $v \in \mathcal{U}$ *and almost every* $t \in S$*. Then*

$$\mathrm{essup}_{t\in S} \|u(t)\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega)} < \infty, \quad \mathrm{essup}_{t\in S} \|z(t)\|_{B^{\frac{1}{2}}_{2,\infty}(\Omega)} < \infty,$$

and for every $\delta > 0$ *we have*

$$u \in L^\infty(S; H^{\frac{3}{2}-\delta}(\Omega)), \quad z \in L^\infty(S; H^{\frac{1}{2}-\delta}(\Omega)). \tag{4.3}$$

In addition, step 2 of the proof of Theorem 4.1 shows that the following local result is valid if **R1**–**R4** hold: $\nabla u \in L^\infty(S; H^1_{\mathrm{loc}}(\Omega))$, $\nabla z \in L^\infty(S; L^2_{\mathrm{loc}}(\Omega))$. This regularity is also valid for tangential derivatives at the boundary of Ω. Similar local and tangential results were recently derived by Alber/Nesenenko [AN08] for problems with pure Dirichlet boundary conditions. The optimality of Theorem 4.3 and further examples are discussed in Section 5.

The proof of Theorem 4.1 is carried out in Sections 4.2–4.4. By the usual arguments we may assume for the Dirichlet datum that $u_0 \equiv 0$ and thus $u \in W^{1,1}(S;\mathcal{U})$.

4.2 Step 1: Elimination of the lower order terms

Let the assumptions of Theorem 4.1 be satisfied and let $(u, z) \in W^{1,1}(S;\mathcal{U})\times W^{1,1}(S; L^2(\Omega))$ be a solution to (4.2). Then there exist functions $\widetilde{f} \in W^{1,1}(S; L^2(\Omega))$, $\widetilde{H} \in W^{1,1}(S; H^1(\Omega))$ and $\widetilde{\ell}_2 \in W^{1,1}(S; H^1(\Omega))$ such that for every $v \in \mathcal{U}$ and a.e. $t \in S$ we have

$$D_u\mathcal{E}_{pp}(u(t), z(t)) \equiv \int_\Omega \langle A \begin{pmatrix} 0 \\ \nabla u(t) \\ z(t) \end{pmatrix}, \begin{pmatrix} 0 \\ \nabla v \\ 0 \end{pmatrix}\rangle \,\mathrm{d}x = \int_\Omega \widetilde{f}(t) \cdot v + \widetilde{H}(t) : \nabla v \,\mathrm{d}x,$$

$$\partial_t z(t) \in \mathcal{G}\left(-\mathbb{P}_n\big[A \begin{pmatrix} 0 \\ \nabla u(t) \\ z(t) \end{pmatrix}\big] + \widetilde{\ell}_2(t)\right).$$

Here, $\mathbb{P}_n$ is the projection operator introduced in Section 3.1. The assertion follows immediately by rearranging the terms in (4.2). Thus, from now on we assume that $\mathcal{E}(u, z) = \mathcal{E}_{pp}(u, z)$ and $A \in C^{0,1}(\overline{\Omega}, \mathrm{End}(\mathbb{R}^{m\times d} \times \mathbb{R}^n))$.

4.3 Step 2: Localization of the model and tangential regularity

Assumption **R1** implies that for every $y_0 \in \partial\Omega$ there exists a neighborhood V_{y_0} of y_0 and a $C^{1,1}$-diffeomorphism $\Phi_{y_0} : V_{y_0} \to C_1$ having the properties $\Phi_{y_0}(y_0) = 0$, $\Phi_{y_0}(\partial\Omega\cap V_{y_0}) = \Gamma_0$, $\Phi_{y_0}(\Omega \cap V_{y_0}) = K_1$ and $\Phi_{y_0}(V_{y_0}\backslash\overline{\Omega}) = C_1\backslash K_1$. The diffeomorphism Φ_{y_0} is chosen in such a way that $\det \nabla\Phi_{y_0}$ is constant. This choice is always possible for $C^{1,1}$-smooth domains, see for example [Gri85]. The inverse of Φ_{y_0} is denoted by $\Psi_{y_0} : C_1 \to V_{y_0}$.

Let $A \in C^{0,1}(\overline{\Omega}; \mathrm{End}(\mathbb{R}^{m\times d} \times \mathbb{R}^n))$ be the coefficient function in (4.2). For $x \in \overline{K_1}$, $F_i \in \mathbb{R}^{m\times d}$, $z_i \in \mathbb{R}^n$ we define $A_{\Phi_{y_0}} \in C^{0,1}(\overline{K_1}; \mathrm{End}(\mathbb{R}^{m\times d} \times \mathbb{R}^n))$ via

$$\langle A_{\Phi_{y_0}}(x) \begin{pmatrix} F_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} F_2 \\ z_2 \end{pmatrix}\rangle = \langle A(\Psi_{y_0}(x)) \begin{pmatrix} F_1(\nabla\Psi_{y_0}(x))^{-1} \\ z_1 \end{pmatrix}, \begin{pmatrix} F_2(\nabla\Psi_{y_0}(x))^{-1} \\ z_2 \end{pmatrix}\rangle.$$

Moreover,

$$\mathcal{E}_{\Phi_{y_0}}(v,\zeta) := \frac{1}{2}\int_{K_1} \langle A_{\Phi_{y_0}} \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right), \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right)\rangle \,\mathrm{d}x.$$

Finally we define $\mathcal{W}(K_r) = \{\, v \in H^1(K_r)\,;\, v\big|_{\partial K_r\setminus\Gamma_0} = 0 \,\}$ for $r > 0$.

In the next Lemma, we extend the coercivity assumption on A from $H_0^1(\Omega) \times L^2(\Omega)$ to functions $v \in H^1(\Omega)$, which vanish only on some parts of the boundary.

Lemma 4.2. *Let conditions* **R1** *and* **R2** *be satisfied. For every* $y_0 \in \partial\Omega$ *there exists* $r \in (0,1)$ *such that for all* $v \in \mathcal{W}(K_r)$ *and* $\zeta \in L^2(K_r)$ *it holds*

$$\mathcal{E}_{\Phi_{y_0}}(v,\zeta) \geq \tfrac{\alpha}{4}\big(\, \|\nabla v\|^2_{L^2(K_r)} + \|\zeta\|^2_{L^2(K_r)}\,\big) \tag{4.4}$$

with α *from* **R2**.

Proof. Assume that conditions **R1** and **R2** are satisfied. By a localization argument similar to the one described in [GH96, Chap. 4.1.3, Legendre-Hadamard condition] and a scaling argument it follows that for all $x_0 \in \overline{\Omega}$, $r > 0$, $v \in H_0^1(C_r)$ and $\zeta \in L^2(C_r)$ it holds

$$\int_{C_r} \langle A(x_0) \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right), \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right)\rangle \,\mathrm{d}x \geq \alpha(\|\nabla v\|^2_{L^2(C_r)} + \|\zeta\|^2_{L^2(C_r)}). \tag{4.5}$$

Here, α is the same constant as in condition **R2** and does not depend on r. Moreover, by using even extensions for $v \in \mathcal{W}(K_r)$ and odd extensions for $\zeta \in L^2(K_r)$ from K_r to C_r, it follows that estimate (4.5) is valid also on $\mathcal{W}(K_r) \times L^2(K_r)$ with the same constant α as in (4.5).

Let now $y_0 \in \partial\Omega$ be arbitrary. For all $r \in (0,1]$, $v \in \mathcal{W}(K_r)$ and $\zeta \in L^2(K_r)$ we have

$$\begin{aligned} 2\mathcal{E}_{\Phi_{y_0}}(v,\zeta) &= \int_{K_r} \langle A(y_0) \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right), \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right)\rangle \,\mathrm{d}x + \int_{K_r} \langle (A_{\Phi_{y_0}}(x) - A(y_0)) \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right), \left(\begin{smallmatrix}\nabla v\\ \zeta\end{smallmatrix}\right)\rangle \,\mathrm{d}x \\ &\geq (\alpha - c_{A,\Phi_{y_0}} \operatorname{diam}(K_r))(\|\nabla v\|^2_{L^2(C_r)} + \|\zeta\|^2_{L^2(C_r)}). \end{aligned}$$

The constant $c_{A,\Phi_{y_0}}$ depends on Φ_{y_0} and on the Lipschitz properties of A, but is independent of r. For small enough r we therefore arrive at (4.4). □

In the sequel we omit the index y_0.

Let $(u,z) \in W^{1,1}(S;\mathcal{U}) \times W^{1,1}(S;L^2(\Omega))$ be given as in Theorem 4.1, choose $y_0 \in \partial\Omega$ and let $r \in (0,1)$ be given according to Lemma 4.2. Let furthermore $\varphi \in C_0^\infty(C_{\frac{3r}{4}})$ with $0 \leq \varphi \leq 1$ and with $\varphi \equiv 1$ on $C_{\frac{r}{2}}$. For $(t,x) \in S \times K_r$ we define

$$u_\Phi(t,x) = \varphi(x)u(t,\Psi(x)), \quad z_\Phi(t,x) = z(t,\Psi(x)).$$

Furthermore, the space

$$\mathcal{U}(K_r) = \{\, w \in H^1(K_r)\,;\, v\big|_{\partial K_r\setminus\Gamma_0} = 0,\, v_i\big|_{\Gamma_0} = 0 \text{ for } i \in D \,\}$$

is defined in the same way as in Section 3.2. Obviously, $\mathcal{U}(K_r) \subset \mathcal{W}(K_r)$ and

$$(u_\Phi, z_\Phi) \in W^{1,1}(S;\mathcal{U}(K_r)) \times W^{1,1}(S;L^2(K_r)).$$

Testing (4.2) with $v\circ\Phi$, where $v \in \mathcal{U}(K_r)$, changing the coordinates using Φ and moving the lower order terms to the right hand side, we arrive at the following relations taking into account that $|\det \nabla\Phi|$ is constant:

For all $v \in \mathcal{U}(K_r)$ and almost every $t \in S$ it holds

$$\begin{aligned}
\int_{K_r} \langle A_\Phi(x) \left(\begin{smallmatrix} \nabla u_\Phi(t) \\ z_\Phi(t) \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix} \right) \rangle \,\mathrm{d}x = \int_{K_r} f_\Phi(t) \cdot v \,\mathrm{d}x + H_\Phi(t) : \nabla v \,\mathrm{d}x, \\
\partial_t z_\Phi(t) \in \mathcal{G}\Big(- \mathbb{P}_n\big[A_\Phi \left(\begin{smallmatrix} \nabla u_\Phi(t) \\ z_\Phi(t) \end{smallmatrix} \right)\big] + \ell_{2,\Phi}(t)\Big), \\
z_\Phi(0) = z_{0,\Phi}.
\end{aligned} \tag{4.6}$$

Here, $z_{0,\Phi} = z_0 \circ \Psi$. Moreover, with $\widetilde{u}(t,x) = u(t, \Psi(x))$ we have

$$\begin{aligned}
f_\Phi(t) &= f(t)\circ\Psi, \\
H_\Phi(t) &= H(t)\circ\Psi\nabla\Psi^{-\top} - \mathbb{P}_{m\times d}\big[A_\Phi \left(\begin{smallmatrix} \nabla((1-\varphi)\widetilde{u}) \\ 0 \end{smallmatrix} \right)\big], \\
\ell_{2,\Phi}(t) &= \ell_2(t)\circ\Psi - \mathbb{P}_n\big[A_\Phi \left(\begin{smallmatrix} \nabla((1-\varphi)\widetilde{u}) \\ 0 \end{smallmatrix} \right)\big].
\end{aligned} \tag{4.7}$$

From assumption **R4** and using that $(1-\varphi)\widetilde{u} = 0$ on $K_{r/2}$, we obtain

$$\begin{aligned}
f_\Phi \in W^{1,1}(S;L^2(K_r)), \quad H_\Phi \in W^{1,1}(S;L^2(K_r)) \cap W^{1,1}(S;H^1(K_{\frac{r}{2}})), \\
\ell_{2,\Phi} \in W^{1,1}(S;L^2(K_r)) \cap W^{1,1}(S;H^1(K_{\frac{r}{2}})).
\end{aligned} \tag{4.8}$$

In view of Lemma 4.2 we are now exactly in the situation described in Section 3.2 on tangential regularity. Theorem 3.3 therefore implies that for $1 \le i \le d-1$ we have

$$\partial_{x_i} \nabla u_\Phi \in L^\infty(S;L^2(K_{\frac{r}{4}})), \quad \partial_{x_i} z_\Phi \in L^\infty(S;L^2(K_{\frac{r}{4}})).$$

Since $y_0 \in \partial\Omega$ was arbitrary and since $\partial\Omega$ can be covered with a finite number of the domains $\Psi_{y_0}(K_{\frac{r}{4}})$, the tangential regularity result is also valid for u and z on the whole domain Ω.

4.4 Step 3: Global regularity

We consider again the localized problem (4.6). Thanks to the second step we have the additional regularity $\partial_i \nabla u_\Phi \in L^\infty(S;L^2(K_r))$ and $\partial_i z_\Phi \in L^\infty(S;L^2(K_r))$ for $1 \le i \le d-1$. Thus, in addition to (4.8) the data in (4.7) satisfy

$$H_\Phi,\ \ell_{2,\Phi} \in L^\infty(S;\cap_{i=1}^{d-1}\mathcal{F}_i(K_r,K_r)).$$

By a reflection argument it follows from Lemma 4.2 that the extended coefficient function $A_{\Phi,e}$, which is defined by $A_{\Phi,e}(x) = A_\Phi(x)$ for $x \in K_r$ and $A_{\Phi,e}(x) = A_\Phi(Rx)$ for $x \in C_r \backslash K_r$, satisfies

$$\int_{C_r} \langle A_{\Phi,e} \left(\begin{smallmatrix} \nabla v \\ \zeta \end{smallmatrix} \right), \left(\begin{smallmatrix} \nabla v \\ \zeta \end{smallmatrix} \right) \rangle \,\mathrm{d}x \ge \kappa\big(\|v\|^2_{H^1(C_r)} + \|\zeta\|^2_{L^2(C_r)} \big)$$

for all $v \in H_0^1(C_r)$, $\zeta \in L^2(C_r)$ and some constant $\kappa > 0$. Theorem 3.4 now guarantees that $\mathrm{essup}_{t\in S} \|u_\Phi(t)\|_{B^{3/2}_{2,\infty}(K_{r/4})} < \infty$, $\mathrm{essup}_{t\in S} \|z_\Phi(t)\|_{B^{1/2}_{2,\infty}(K_{r/4})} < \infty$, and that for every $\delta > 0$ we have $u_\Phi \in L^\infty(S; H^{\frac{3}{2}-\delta}(K_{\frac{r}{4}}))$ and $z_\Phi \in L^\infty(S; H^{\frac{1}{2}-\delta}(K_{\frac{r}{4}}))$. Since $y_0 \in \partial\Omega$ is arbitrary and since $\partial\Omega$ can be covered with a finite number of domains $\Psi_{y_0}(K_{\frac{r}{4}})$, we arrive finally at (4.3) and the proof of Theorem 4.1 is finished.

5 Examples and Discussion

5.1 Elastic-plastic models with linear hardening

Regularity Theorem 4.1 is in particular applicable to classical elastic–plastic models with linear hardening having positive definite hardening coefficients.

Let $m = d$ and let $\mathcal{E}_H$ be the stored energy introduced in Section 2.2.1. If the elasticity tensor $C \in C^{0,1}(\overline{\Omega}; \mathrm{End}(\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ and the hardening coefficients $L \in C^{0,1}(\overline{\Omega}; \mathrm{End}(\mathbb{R}^n))$ are symmetric and uniformly positive definite on Ω, then condition **R2** is satisfied. This is an immediate consequence of the Korn inequality. Assume furthermore that the mapping $g : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ and the data are chosen according to **R3** and **R4**, respectively.

If the pair $(u, z) \in W^{1,1}(S; H^1(\Omega)) \times W^{1,1}(S; L^2(\Omega))$ is a solution to (4.2) with pure Dirichlet or pure Neumann boundary conditions for u, i.e. $\mathcal{U} = H_0^1(\Omega)$ or $\mathcal{U} = H^1(\Omega)$, then the regularity results stated in Theorem 4.1 are valid for u and z.

In particular the results hold for elastic–plastic models with linear kinematic hardening and with von Mises or Tresca flow rule.

5.2 Elastic-plastic models with Cosserat effects

In the case of the elastic-plastic model with Cosserat effects described in Section 2.2.2, the generalized displacements consist of the true displacements $u : \Omega \to \mathbb{R}^d$ and the micro-rotation tensor $Q : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{skew}}$. Moreover, the inner variable is identified with the plastic strains, i.e. $z = \varepsilon_p : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym,\ dev}}$. The corresponding stored energy $\mathcal{E}_{\mathrm{C}}((u, Q), z)$ is defined in (2.17). If the coefficients $\mu, \mu_c, \lambda, \gamma \in C^{0,1}(\overline{\Omega}; \mathbb{R})$ are uniformly positive, then the principal part of $\mathcal{E}_{\mathrm{C}}$, which is given by

$$\mathcal{E}_{\mathrm{C},pp}((u, Q), \varepsilon_p) = \int_\Omega \mu\, |\varepsilon(\nabla u) - \varepsilon_p|^2 + \mu_c\, |\mathrm{skew}\, \nabla u|^2 + \frac{\lambda}{2}\, |\mathrm{tr}\, \nabla u|^2 + \gamma\, |\nabla Q|^2 \,\mathrm{d}x,$$

satisfies condition **R2**. This follows in the same way as the inequality (2.18), see e.g. [NC05]. Thus, if in addition **R1**, **R3** and **R4** are valid, then by Theorem 4.1 for every $\delta > 0$ we have $u \in L^\infty(S; H^{\frac{3}{2}-\delta}(\Omega))$ and $\varepsilon_p \in L^\infty(S; H^{\frac{1}{2}-\delta}(\Omega))$, while the existence proof already provides $Q \in W^{1,1}(S; H^2(\Omega))$, see [NC05]. Note that it is possible to choose $\mathcal{U} = H_0^1(\Omega, \mathbb{R}^d) \times H^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{skew}})$, which means that Dirichlet conditions are prescribed for the displacements and Neumann conditions for the micro-rotation tensor Q.

5.3 Discussion of the optimality of Theorem 4.1

It is not clear whether the result presented in Theorem 4.1 is optimal or whether one should expect that $u \in L^\infty(S; H^2(\Omega))$ and $z \in L^\infty(S; H^1(\Omega))$ in the general framework of Theorem 4.1. The latter regularity would fit well to the local result provided in Theorem 3.1 and to regularity results for elliptic equations. In this section we discuss several aspects and special cases in view of the question of optimality. For simplicity, we omit the lower order terms, so that $A \in C^{0,1}(\overline{\Omega}; \mathrm{End}(\mathbb{R}^{m\times d} \times \mathbb{R}^n))$. Moreover, we use the notation $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ with coefficient matrices $A_{11} \in C^{0,1}(\overline{\Omega}; \mathrm{End}(\mathbb{R}^{m\times d}))$, $A_{12} = A_{21}^* \in C^{0,1}(\overline{\Omega}; \mathrm{Lin}(\mathbb{R}^{m\times d}, \mathbb{R}^n))$ and $A_{22} \in C^{0,1}(\overline{\Omega}; \mathrm{End}(\mathbb{R}^n))$.

5.3.1 Reflection technique and regularity for elliptic systems

Assume first that the maximal monotone operator $\mathcal{G}$ in (4.2) is identically 0 and that f, H are constant in time. Then z and consequently u are constant in time as well and u in fact is the solution of the following linear elliptic system of PDEs of second order

$$\int_\Omega A_{11}\nabla u : \nabla v \,\mathrm{d}x = \int_\Omega f \cdot v + (H - A_{12} z_0) : \nabla v \,\mathrm{d}x$$

for all $v \in \mathcal{U}$ with $\mathcal{U}$ like in (4.1).

It is well known that solutions of such systems belong to $H^2(\Omega)$ provided that **R1** is satisfied and that $f \in L^2(\Omega)$ and $z_0, H \in H^1(\Omega)$, see e.g. [Neč67]. This result follows from tangential regularity results by solving the elliptic equation for the missing second derivative in normal direction.

Alternatively, this result can also be obtained by applying the reflection technique introduced in Section 3.3. This can be seen as follows: Assume that Ω is a half cube and that we are in the situation described in Theorem 3.4 with time independent data and with $\mathcal{G} \equiv 0$. Let u_e and A_e be the extended functions defined Section 3.3. By adapting the proof of Lemma 3.6 to this particular situation, it follows that the extended function u_e satisfies

$$\int_{C_1} \langle A_e \left(\begin{smallmatrix} \nabla u_e \\ z_{0,e} \end{smallmatrix}\right), \left(\begin{smallmatrix} \nabla v \\ 0 \end{smallmatrix}\right) \rangle \,\mathrm{d}x = \int_\Omega f_e \cdot v + H_e : \nabla v \,\mathrm{d}x$$

for all $v \in H_0^1(C_1)$, where f_e and H_e are defined as in (3.20) and (3.21). From the tangential regularity of u we deduce that $f_e \in L^2(C_1)$ and $H_e \in \mathcal{F}_d(C_1, C_{\frac{1}{2}})$. Thus the local results for linear elliptic equations guarantee that $u_e \in H^2(C_{\frac{1}{2}})$ and finally $u \in H^2(K_{\frac{1}{2}})$. This shows that in the stationary case the reflection argument is equivalent to the usual argument for proving global regularity for solutions of linear elliptic systems.

5.3.2 The decoupled case

We consider now the case where $A_{12} = 0$ but with arbitrary $\mathcal{G}$ satisfying **R3**. In this case, the elliptic equation and the evolution equation in (4.2) are completely decoupled. The extended function $\ell_{2,e}$ occurring in the proof of Lemma 3.6 is now given by

$$\ell_{2,e}(t,x) = \begin{cases} \ell_2(t) & x \in K_r \\ \ell_2(t, Rx) & x \in C_r \backslash K_r \end{cases}$$

and belongs to $W^{1,1}(S; H^1(C_r))$ instead of $L^\infty(S; \mathcal{F}_d(C_r, C_\rho))$. Thus part (b) of Theorem 3.1 is applicable and yields $z \in L^\infty(S; H^1(K_r))$. Under the assumptions of Theorem 4.1 it therefore holds in the decoupled case that $u \in W^{1,1}(S; H^2(\Omega))$ and $z \in L^\infty(S; H^1(\Omega))$.

5.3.3 The one dimensional case

Let $d = 1$ and $K_1 = (0, 1)$. Furthermore, let the pair $(u, z) \in W^{1,1}(S; H^1(K_1) \times L^2(K_1))$ be a solution of (4.2). Applying the reflection procedure from Section 3.3 leads to extended functions having the regularity $(u_e, z_e) \in W^{1,1}(S; H^1(C_1) \times L^2(C_1))$, $f_e \in W^{1,1}(S; L^2(C_1))$, $\theta_e, H_e, \ell_{2,e} \in W^{1,1}(S; H^1(C_1))$. Thus part (b) of Theorem 3.1 gives

Theorem 5.1. *Let $d = 1$ and assume that* **R1**–**R4** *are satisfied. Then the solutions u and z have the regularity $u \in L^\infty(S; H^2(\Omega))$, $z \in L^\infty(S; H^1(\Omega))$.*

5.3.4 The case where u is scalar

If the function u is scalar, i.e. $m = 1$, improved regularity results can be obtained provided that certain coupling conditions between the coefficient matrix A and the function g are satisfied. For the proof of the result we apply again a reflection argument. In contrast to the approach presented in Section 3 the model is not reflected perpendicular to the boundary but in a direction which is locally given by $A_{11}(x)\nu(x)$ for $x \in \partial\Omega$. Here, $\nu : \partial\Omega \to \mathbb{R}^d$ denotes the interior unit normal vector. In particular we assume

R1' $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^{2,1}$-smooth boundary and $\partial\Omega = \Gamma_{\mathrm{D}}$.

R2' The coefficient matrix A belongs to $C^{1,1}(\overline{\Omega}; \mathrm{End}(\mathbb{R}^d \times \mathbb{R}^n))$, is self adjoint and satisfies $\mathcal{E}(v, z) \geq \frac{\alpha}{2}(\|v\|^2_{H^1(\Omega)} + \|z\|^2_{L^2(\Omega)})$ for every $v \in H^1_0(\Omega)$ and $z \in L^2(\Omega)$.

R3' $g : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ satisfies **R3**.

R4' $z_0 \in H^1(\Omega, \mathbb{R}^n)$, $f \in W^{1,1}(S; L^2(\Omega))$ and $u_0 \in W^{1,1}(S; H^2(\Omega))$.

In order to formulate the compatibility conditions, we define for $x \in \partial\Omega$ and $A_{11}(x) \in \mathbb{R}^{d\times d}$

$$R_\nu(x) = \mathbb{I} - \frac{2}{\langle A_{11}(x)\nu(x), \nu(x)\rangle} A_{11}(x)\nu(x) \otimes \nu(x) \tag{5.1}$$

with the interior normal vector $\nu : \partial\Omega \to \mathbb{R}^d$. The matrix R_ν locally determines the reflection at $\partial\Omega$. Observe that for all $x \in \partial\Omega$ we have

$$(R_\nu(x))^2 = \mathbb{I} \quad \text{and} \quad R_\nu(x)A_{11}(x)R_\nu(x)^\top = A_{11}(x). \tag{5.2}$$

R5' For every $x_0 \in \partial\Omega$ there exists a neighborhood $W \subset \mathbb{R}^d$ and a mapping $P \in C^{0,1}(\partial\Omega \cap W; \mathrm{End}(\mathbb{R}^n))$ such that the inverse matrix $(P(x))^{-1}$ exists for every $x \in \partial\Omega \cap W$ and such that the following conditions hold for every $x \in \partial\Omega \cap W$:

(a) $R_\nu(x)A_{12}(x)P(x) = A_{12}(x)$,

(b) $P(x)^\top A_{22}(x)P(x) = A_{22}(x)$,

(c) $-P(x)^{-1}g(-P(x)^{-\top}\eta) = g(\eta)$ for all $\eta \in \mathbb{R}^n$,

(d) Compatibility for the initial datum: $(\mathbb{I} + P^{-1})z_0 = 0$ on $\partial\Omega$.

We consider the problem to find $(u, z) \in W^{1,1}(S; H^1(\Omega)) \times W^{1,1}(S; L^2(\Omega))$ which satisfy for a.e. $t \in S$ and every $v \in H^1_0(\Omega)$ the relations

$$\begin{aligned} \int_\Omega \langle A \begin{pmatrix} \nabla u(t) \\ z(t) \end{pmatrix}, \begin{pmatrix} \nabla v \\ 0 \end{pmatrix} \rangle \,\mathrm{d}x &= \int_V f(t)v \,\mathrm{d}x, \\ \partial_t z(t) &\in g(-\mathbb{P}_n[A \begin{pmatrix} \nabla u(t) \\ z(t) \end{pmatrix}]) \\ z(0) = z_0, \quad u(t)\big|_{\partial\Omega} &= u_0(t)\big|_{\partial\Omega}. \end{aligned} \tag{5.3}$$

Theorem 5.2. *Let* **R1'**–**R5'** *be satisfied and assume that* $(u, z) \in W^{1,1}(S; H^1(\Omega) \times L^2(\Omega))$ *solves* (5.3)*. Then* $u \in L^\infty(S; H^2(\Omega))$*,* $z \in L^\infty(S; H^1(\Omega))$ *and* $(\mathbb{I} + P^{-1})z = 0$ *on* $\partial\Omega$*.*

The proof is carried out in the next two lemmata, where we first construct a local diffeomorphism from $\Omega \cap W$ to $W\backslash\Omega$. This diffeomorphism is closely related with R_ν. In the second step we localize and extend problem (5.3) from $\Omega \cap W$ to W and show that the new problem satisfies the smoothness assumptions of part (b) of Theorem 3.1.

Lemma 5.3. *For every* $x_0 \in \partial\Omega$ *exists a neighborhood* V *with* $V_+ = \Omega \cap V$ *and* $V_- = V\backslash\overline{\Omega}$ *and a* $C^{1,1}$*-diffeomorphism* $T : V \to V$ *with the properties* $T(x) = x$ *for all* $x \in \partial\Omega \cap V$*,* $T(V_\pm) = V_\mp$ *and* $\nabla T(x) = R_\nu(x)$ *for all* $x \in \partial\Omega \cap V$*.*

Proof. We define the following mapping

$$\widetilde{T} : \partial\Omega \times \mathbb{R} \to \mathbb{R}^d; \quad (\widetilde{y}, y_d) \mapsto \widetilde{y} + y_d A_{11}(\widetilde{y})\nu(\widetilde{y}).$$

Since $\partial\Omega$ is assumed to be $C^{2,1}$-smooth, the mapping $\widetilde{T}$ belongs to $C^{1,1}(\partial\Omega \times \mathbb{R}^d)$. For $y_d = 0$ we have

$$\nabla\widetilde{T}(\widetilde{y}, 0) = \mathbb{I}_{\mathrm{T}_{\widetilde{y}}\partial\Omega} + A_{11}(\widetilde{y})\nu(\widetilde{y}) \otimes \nu(\widetilde{y}) = \mathbb{I}_{\mathbb{R}^d} + (A_{11}(\widetilde{y}) - \mathbb{I}_{\mathbb{R}^d})\nu(\widetilde{y}) \otimes \nu(\widetilde{y}), \tag{5.4}$$

where $\mathbb{I}_{\mathrm{T}_{\widetilde{y}}\partial\Omega}$ is the restriction of the identity to the tangent space of $\partial\Omega$ in $\widetilde{y}$. Moreover, $\det \nabla\widetilde{T}(\widetilde{y}, 0) = \langle A_{11}(\widetilde{y})\nu(\widetilde{y})), \nu(\widetilde{y})\rangle > 0$ since A_{11} is uniformly positive definite. Thus the inverse of $\nabla\widetilde{T}(\widetilde{y}, 0)$ exists in all points $(\widetilde{y}, 0) \in \partial\Omega \times \mathbb{R}$ and is given by

$$(\nabla\widetilde{T}(\widetilde{y}, 0))^{-1} = \mathbb{I}_{\mathbb{R}^d} - \frac{1}{\langle A_{11}\nu, \nu\rangle}(A_{11} - \mathbb{I}_{\mathbb{R}^d})\nu \otimes \nu.$$

Let now $x_0 \in \partial\Omega$ be arbitrary. By the Implicit Function Theorem there exists a neighborhood $V \subset \mathbb{R}^d$ of x_0 and a neighborhood $\widetilde{V} \subset \partial\Omega \times \mathbb{R}$ of $(x_0, 0)$ such that $\widetilde{T} : \widetilde{V} \to V$ is a $C^{1,1}$-diffeomorphism with $\widetilde{T}(\widetilde{V}_\pm) = V_\pm$. Here, $\widetilde{V}_\pm = \{ (\widetilde{y}, y_d) \in \widetilde{V} \,;\, y_d \gtrless 0 \}$, $V_+ = \Omega \cap V$, $V_- = V \backslash \overline{\Omega}$. Let the reflection at $\partial\Omega$ be given by $R : \partial\Omega \times \mathbb{R} \to \partial\Omega \times \mathbb{R}$, $R(\widetilde{y}, y_d) = (\widetilde{y}, -y_d)$ with $\nabla R(\widetilde{y}, 0) = \mathbb{I}_{\mathbb{R}^d} - 2\nu \times \nu$. The mapping T we are looking for is defined through

$$T : V \to V, \quad T(x) = \widetilde{T}(R(\widetilde{T}^{-1}(x))).$$

By construction, T is a $C^{1,1}$-diffeomorphism with $T(V_\pm) = V_\mp$. Moreover, straightforward calculations show that for every $x \in \partial\Omega$ we have

$$\nabla T(x) = \nabla\widetilde{T}(x, 0)\nabla R(x, 0)(\nabla\widetilde{T}(x, 0))^{-1} = R_\nu(x).$$

This finishes the proof of Lemma 5.3. □

From now on we assume that $u_0 = 0$ and $u(t) \in H_0^1(\Omega)$. Otherwise, the volume term f should be replaced with $\widetilde{f} = f + \operatorname{div} A_{11} u_0 \in W^{1,1}(S; L^2(\Omega))$. Moreover we assume that the set V from Lemma 5.3 is contained in the set W from **R5'**.

The following extended functions will be considered in the sequel: Choose $x_0 \in \partial\Omega$ and let $T : V \to V$ be the corresponding diffeomorphism from Lemma 5.3. Choose $\varphi \in C_0^\infty(V)$ with $\varphi\big|_{B_\delta(x_0)} = 1$ for some $\delta > 0$. The matrix valued function P introduced in condition **R5'** is extended to V in the following way: Let $\widetilde{T}$ be the diffeomorphism defined in the proof of Lemma 5.3. For $x \in V$ we have $\widetilde{T}^{-1}(x) = (\widetilde{y}, y_d) \in \partial\Omega \times \mathbb{R}$. By $\widetilde{T}^{-1}_{\partial\Omega}$ we denote the projection onto the point $\widetilde{y}$, i.e. $\widetilde{T}^{-1}_{\partial\Omega}(x) = \widetilde{y} \in \partial\Omega$. The extension of P is now defined as

$$P_e(x) = P(\widetilde{T}^{-1}_{\partial\Omega}(x)), \quad x \in V.$$

By construction, $P_e \in C^{0,1}(\overline{V}, \operatorname{End}(\mathbb{R}^n))$. Observe that the inverse matrix $(P_e(x))^{-1}$ exists for every $x \in \overline{V}$ and that $(P_e(\cdot))^{-1}$ belongs to $C^{0,1}(\overline{V}, \operatorname{End}(\mathbb{R}^n))$. We define

$$u_e(t, x) = \begin{cases} (\varphi u)(t, x) & (t, x) \in S \times V_+ \\ -(\varphi u)(t, T^{-1}(x)) & (t, x) \in S \times V_- \end{cases},$$

$$z_e(t, x) = \begin{cases} z(t, x) & (t, x) \in S \times V_+ \\ -(P_e^{-1} z)(t, T^{-1}(x)) & (t, x) \in S \times V_- \end{cases}.$$

Obviously, $(u_e, z_e) \in W^{1,1}(S; H_0^1(V) \times L^2(V))$. The coefficient function A is extended as follows

$$A_{11,e} = \begin{cases} A_{11} & \text{on } V_+ \\ (\nabla T A_{11} \nabla T^\top) \circ T^{-1} & \text{on } V_- \end{cases}, \quad A_{22,e} = \begin{cases} A_{22} & \text{on } V_+ \\ (P_e^\top A_{22} P_e) \circ T^{-1} & \text{on } V_- \end{cases},$$

$$A_{12,e} = \begin{cases} A_{12} & \text{on } V_+ \\ (\nabla T A_{12} P_e) \circ T^{-1} & \text{on } V_- \end{cases}, \qquad A_{21,e} = A_{12,e}^\top.$$

Due to the compatibility condition **R5'**, the coefficient matrix $A_e = \begin{pmatrix} A_{11,e} & A_{12,e} \\ A_{21,e} & A_{22,e} \end{pmatrix}$ belongs to $C^{0,1}(\overline{V}, \mathrm{End}(\mathbb{R}^d \times \mathbb{R}^n))$. Moreover, the data is extended as follows

$$f_e = \begin{cases} f & \text{on } V_+ \\ \Big(-f + |\det \nabla T|^{-1} (A_{11}\nabla u + A_{12} z) \cdot \nabla |\det \nabla T|\Big) \circ T^{-1} & \text{on } V_- \end{cases},$$

$$H_e = \begin{cases} A_{11}\nabla((\varphi - 1)u) & \text{on } V_+ \\ -\big(\nabla T A_{11}\nabla((\varphi-1)u)\big)\circ T^{-1} & \text{on } V_- \end{cases},$$

$$\ell_{2,e} = \begin{cases} A_{21}\nabla((\varphi - 1)u) & \text{on } V_+ \\ -(P_e^\top A_{21}\nabla((\varphi-1)u))\circ T^{-1} & \text{on } V_- \end{cases},$$

$$z_{0,e} = \begin{cases} z_0 & \text{on } V_+ \\ -(P_e^{-1} z_0)\circ T^{-1} & \text{on } V_- \end{cases}.$$

Thanks to **R5'**, the extended functions have the regularity $f_e \in W^{1,1}(S; L^2(V))$, $H_e, \ell_{2,e} \in W^{1,1}(S; L^2(V)) \cap W^{1,1}(S; H^1(B_\delta(x_0)))$ and $z_{0,e} \in H^1(V)$. Finally, for $\eta \in \mathbb{R}^n$ we define

$$g_e(x,\eta) = \begin{cases} g(\eta) & \text{on } V_+ \\ -P_e^{-1}\circ T^{-1}\, g(-P_e^{-\top}\circ T^{-1}\,\eta) & \text{on } V_- \end{cases}.$$

Due to condition **R5'**, we have in fact the identity $g_e(x,\eta) = g(\eta)$ for all $x \in V$ and $\eta \in \mathbb{R}^n$.

Lemma 5.4. *Assume that* **R1'-R5'** *are satisfied. For a.e. $t \in S$ and every $v \in H^1_0(V)$ the above defined extended functions $(u_e, z_e) \in W^{1,1}(S; H^1_0(V)) \times W^{1,1}(S; L^2(V))$ satisfy*

$$\begin{aligned} \int_V \langle A_e \begin{pmatrix} \nabla u_e(t) \\ z_e(t) \end{pmatrix}, \begin{pmatrix} \nabla v \\ 0 \end{pmatrix}\rangle \,\mathrm{d}x &= \int_V f_e(t) v + H_e(t)\cdot \nabla v \,\mathrm{d}x, \\ \partial_t z_e(t) &\in g_e(-\mathbb{P}_n[A_e \begin{pmatrix} \nabla u_e(t) \\ z_e(t) \end{pmatrix}] + \ell_{2,e}(t)) \\ z_e(0) &= z_{0,e}. \end{aligned} \tag{5.5}$$

Moreover, the coefficients and the data $z_{0,e}$, f_e, H_e and $\ell_{2,e}$ have the smoothness described in conditions **A1** *and* **A3** *of Section 3.1. Thus, $u \in L^\infty(S; H^2(B_\delta(x_0) \cap \Omega))$ and $z \in L^\infty(S; H^1(B_\delta(x_0) \cap \Omega))$.*

Proof. The last assertion of Lemma 5.4 is an immediate consequence of Theorem 3.1 applied to u_e and z_e and (5.5). We recall that $g_e(x,\eta) = g(\eta)$ for all $x \in V$ and $\eta \in \mathbb{R}^n$. Since $z_e \in L^\infty(S; H^1(B_\delta(x_0))$, the traces of z_e from V_+ and from V_- on $\partial\Omega$ coincide, which entails $(\mathbb{I} + P^{-1})z = 0$ on $\partial\Omega$.

Relation (5.5) can be derived as follows: straightforward calculations show that for $v \in H^1_0(V)$ it holds

$$\begin{aligned} \int_V \langle A_e \begin{pmatrix} \nabla u_e(t) \\ z_e(t) \end{pmatrix}, \begin{pmatrix} \nabla v \\ 0 \end{pmatrix}\rangle \,\mathrm{d}x &= \int_{V_+} (A_{11}\nabla u_e + A_{12} z)\cdot \nabla(v - |\det \nabla T|\, v\circ T)\,\mathrm{d}y \\ &+ \int_{V_+} \big((A_{11}\nabla u_e + A_{12} z)\cdot \nabla |\det \nabla T|\big) v\circ T \,\mathrm{d}y. \end{aligned}$$

Since $|\det \nabla T(y)| = |\det R_\nu(y)| = 1$ for $y \in \partial\Omega$, it follows that $v - |\det \nabla T|\, v\circ T \in H^1_0(V_+)$. Thus, on the basis of (5.3), we arrive at the following relation

$$\int_V \langle A_e \begin{pmatrix} \nabla u_e(t) \\ z_e(t) \end{pmatrix}, \begin{pmatrix} \nabla v \\ 0 \end{pmatrix}\rangle \,\mathrm{d}x = \int_{V_+} f\big(v - |\det \nabla T|\, v\circ T\big) + H_e \cdot \nabla(v - |\det \nabla T|\, v\circ T)\,\mathrm{d}y$$
$$+ \int_{V_-} \big(|\det \nabla T|^{-1} \left(A_{11}\nabla u_e + A_{12} z\right) \cdot \nabla\, |\det \nabla T| \,\big)\circ T^{-1} v\,\mathrm{d}x$$

with H_e from above. After a transformation of coordinates we obtain the first relation in Lemma 5.4. The second relation is an immediate consequence of the definitions of the extended functions in combination with relation (5.3). □

A concrete example, where condition **R5'** is satisfied, is the following: We choose $n = d$ and coefficients $A = \begin{pmatrix} A_{11} & -A_{11} \\ -A_{11} & A_{11}+A_{22} \end{pmatrix}$ with symmetric, positive definite and constant matrices $A_{11},\, A_{22} \in \mathbb{R}^{d\times d}$. The corresponding stored energy reads

$$\mathcal{E}(u,z) = \tfrac{1}{2}\int_\Omega A_{11}(\nabla u - z)\cdot(\nabla u - z) + A_{22} z\cdot z\,\mathrm{d}x.$$

Moreover, $g = \partial\chi_{\mathcal{K}}$ for some convex and closed set $\mathcal{K} \subset \mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ satisfy **R1'** and let $R_\nu := \mathbb{I} - \frac{2}{\langle A_{11}\nu,\nu\rangle} A_{11}\nu\otimes\nu$ for $\nu \in \mathbb{R}^d\backslash\{0\}$. Observe that $R_\nu^{-1} = R_\nu$. With $P_\nu = A_{11}^{-1} R_\nu^{-1} A_{11} = R_\nu^\top$, condition **R5'** reads as follows

(a) $R_\nu A_{22} R_\nu^\top = A_{22}$ for all $\nu \in \mathbb{R}^d\backslash\{0\}$,

(b) $-R_\nu^\top g(-R_\nu \eta) = g(\eta)$ for all $\eta \in \mathbb{R}^d$ and $\nu \in \mathbb{R}^d\backslash\{0\}$.

Proposition 5.5. *The compatibility condition* **R5'** *is satisfied if and only if* $\mathcal{K} = -R_\nu\mathcal{K}$ *for every* $\nu \in \mathbb{R}^d\backslash\{0\}$ *and if there exists* $\alpha > 0$ *such that* $A_{22} = \alpha A_{11}$.

If $\mathcal{K} = \{\,\eta \in \mathbb{R}^d\,;\, \langle B\eta,\eta\rangle \le 1\,\}$ *for some symmetric and positive definite* $B \in \mathbb{R}^{d\times d}$, *then the condition on* $\mathcal{K}$ *is satisfied if and only if there exists a constant* $\beta > 0$ *such that* $B = \beta A_{11}^{-1}$.

Proof. The Proposition follows from Lemma A.1 and Lemma A.2 in the appendix. □

This scalar example shows that if the anisotropy of "Hooke's law" given by A_{11} is strongly correlated with the anisotropy in the hardening coefficients A_{22} and the convex set $\mathcal{K}$, then the displacements $u(t)$ have full H^2 regularity up to the boundary of Ω. The crucial point in the scalar case is the existence of the local diffeomorphism T from Ω to some larger domain having the property (5.2) for $R_\nu = \nabla T$. It is not clear, whether a similar construction is possible for true elasto-plasticity, where $m = d$, or for the general vectorial case with $m > 1$.

An other open question is, whether or not in the case of non matching anisotropies there exist examples with $u(t) \notin H^2(\Omega)$ in spite of smooth data. This will be the subject of further studies.

5.3.5 Example: $\partial_t z \notin L^\infty(S; H^1(\Omega))$

In this section we give an example which shows that in spite of smooth data the rate $\partial_t z$ does not belong to $L^\infty(S; H^1(\Omega))$. This example is inspired by Seregin's paper [Ser99].

Let $0 < R_1 < R_2$. We set $\Omega = B_{R_2}(0) \backslash B_{R_1}(0)$ and choose the following energy for $u, z : \Omega \to \mathbb{R}$:

$$\mathcal{E}(u, z) = \frac{1}{2} \int_\Omega \left|\nabla u - \tfrac{x}{|x|} z\right|^2 + z^2 \,\mathrm{d}x.$$

Moreover, $g(\eta) := \partial\chi_{[-1,1]}(\eta)$ for $\eta \in \mathbb{R}$. We assume that $u(t)\big|_{\partial B_{R_1}} = 0$, $u(t)\big|_{\partial B_{R_2}} = t$, $z_0 = 0$ and that the remaining data $(f,\ H,\ \ell_2)$ vanish. It is easily checked that the assumptions of Theorem 5.2 are satisfied and hence the problem has a unique solution with the regularity $\nabla u, z \in W^{1,1}(S; L^2(\Omega)) \cap L^\infty(S; H^1(\Omega))$. Due to the rotational symmetry of the problem the solution does not depend on the angle and can explicitely be calculated. Introducing polar-coordinates, the solution $u, z : S \times (R_1, R_2) \to \mathbb{R}$ has to satisfy

$$\begin{aligned} \partial_r^2 u + r^{-1}\partial_r u - \partial_r z - r^{-1} z = 0 \quad &\text{in } S \times (R_1, R_2), \\ \partial_t z \in \partial\chi_{[-1,1]}(\partial_r u - 2z) \quad &\text{in } S \times (R_1, R_2), \\ z(0, \cdot) = 0, \quad u(t, R_1) = 0, \quad &u(t, R_2) = t. \end{aligned}$$

For $t \le t_1 := R_1 \ln(R_2/R_1)$ it follows that $u(t, r) = \frac{t \ln(r/R_1)}{\ln(R_2/R_1)}$, $z(t, r) = 0$. In this regime, no plastic strains are present. For $t > t_1$ the plastic variable z starts to grow and there exists $r_*(t)$ such that $z(t, r) > 0$ for $r < r_*$ and $z(r, t) = 0$ for $r > r_*$, i.e. $r_*(t)$ separates the plastic region from the elastic region. The dependence of r_* of t is given implicitly through the relation

$$t(r_*) = R_1 - r_* + r_* \ln \frac{R_2 r_*}{R_1^2}.$$

Simple calculations show that $t(r_*)$ is strictly increasing, and hence $r_*(t) \ge R_1$ is strictly growing, as well. Moreover, for $t \ge t_1$ we have

$$u(t, r) = \begin{cases} b(t) - r + 2r_*(t) \ln r & \text{if } r \le r_*(t) \\ c(t) + r_*(t) \ln r & \text{else} \end{cases}, \quad z(t, r) = \begin{cases} -1 + r_*(t) r^{-1} & \text{if } r \le r_*(t), \\ 0 & \text{else} \end{cases},$$

with functions $b(t) = R_1 - 2r_*(t) \ln R_1$ and $c(t) = t - r_*(t) \ln R_2$. Since $r_*'(t) > 0$ for $t \ge t_1$ it follows that $\partial_t z(t, \cdot) \notin H^1(R_1, R_2)$ for $t > t_1$.

A Proof of Proposition 5.5

Lemma A.1. *Let $A, B \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^d)$ be symmetric with $\det A \neq 0$ and assume that for all $\nu \in \mathbb{R}^d \backslash \{0\}$ we have $R_\nu B R_\nu^\top = B$ with $R_\nu = \mathbb{I} - \frac{2}{\langle A\nu, \nu\rangle} A\nu \otimes \nu$. Then there exists $\alpha \in \mathbb{R}$ such that $B = \alpha A$.*

Proof. Let $\{e_1, \dots, e_d\}$ be an orthonormal system of eigenvectors of A, i.e. $Ae_i = \lambda_i e_i$ for some $\lambda_i \in \mathbb{R}\backslash\{0\}$ and $\langle e_i, e_j\rangle = \delta_{ij}$. Then the set $\{\, e_i \otimes e_j \,;\, i, j \in \{1, \dots, d\}\,\}$ is a basis of $\mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^d)$ which is orthonormal with respect to the inner product defined by $S : T = \mathrm{tr}(T^\top S)$. This means that $(e_i \otimes e_j) : (e_k \otimes e_l) = \delta_{ik}\delta_{jl}$ and $A : (e_i \otimes e_j) = \lambda_i \delta_{ij}$. Thus the identity $R_\nu B R_\nu^\top = B$ is valid for all $\nu \in \mathbb{R}^d\backslash\{0\}$ if and only if

$$(R_\nu B R_\nu^\top) : (e_i \otimes e_j) = B : (e_i \otimes e_j) \tag{A.1}$$

for all $\nu \in \mathbb{R}^d\backslash\{0\}$ and all $i, j \in \{1, \dots, d\}$. Observe that (A.1) is equivalent to

$$2\lambda_i\lambda_j\langle \nu, e_i\rangle\langle \nu, e_j\rangle\langle B\nu, \nu\rangle = \langle A\nu, \nu\rangle\big(\lambda_j\langle \nu, e_j\rangle\langle \nu, Be_i\rangle + \lambda_i\langle \nu, e_i\rangle\langle \nu, Be_j\rangle\big) \tag{A.2}$$

for all $\nu \in \mathbb{R}^d\backslash\{0\}$ and all $i, j \in \{1, \dots, d\}$. With $\nu = e_i \neq e_j$ we obtain from (A.2) the condition $0 = \lambda_i^2\langle e_i, Be_j\rangle$. Since $\lambda_i \neq 0$, it follows that

$$B : (e_i \otimes e_j) = B : (e_j \otimes e_i) = \langle e_i, Be_j\rangle = 0 = A : (e_i \otimes e_j) \tag{A.3}$$

for all $i \neq j$. Assume again that $i \neq j$. With the choice $\nu = a_i e_i + a_j e_j$, where $a_i^2 + a_j^2 = 1$ and $a_i a_j \neq 0$, it follows from (A.2) in combination with (A.3) that

$$a_i^2\big(\langle Be_i, e_i\rangle - \lambda_i c_{ij}\big) + a_j^2\big(\langle Be_j, e_j\rangle - \lambda_j c_{ij}\big) = 0$$

for all these a_i and a_j. Here, $c_{ij} = (2\lambda_i\lambda_j)^{-1}\big(\lambda_j\langle Be_i, e_i\rangle + \lambda_i\langle Be_j, e_j\rangle\big)$. This implies that $\langle Be_i, e_i\rangle - \lambda_i c_{ij} = 0$ for all $i \neq j$ from which we deduce (with $j = 1$) that

$$\langle Be_i, e_i\rangle = \frac{\langle Be_1, e_1\rangle}{\lambda_1}\langle Ae_i, e_i\rangle$$

for all $i \in \{1, \dots, d\}$. Together with (A.3) it follows that $B = \frac{\langle Be_1, e_1\rangle}{\lambda_1} A$. □

Lemma A.2. *Let $A, B \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^d)$ be symmetric with $\det A > 0$ and $\det B > 0$. Assume that for all $\nu \in \mathbb{R}^d\backslash\{0\}$ we have $-R_\nu\mathcal{K} = \mathcal{K}$, where $\mathcal{K} = \{\, \eta \in \mathbb{R}^d \,;\, \langle B\eta, \eta\rangle \leq 1 \,\}$ and $R_\nu = \mathbb{I} - \frac{2}{\langle A\nu, \nu\rangle} A\nu \otimes \nu$. Then there exists $\beta > 0$ such that $B = \beta A^{-1}$.*

Proof. Short calculations show that

$$\begin{aligned} R_\nu^\top B R_\nu &= B + \tfrac{2}{\langle A\nu, \nu\rangle^2}\big(- \langle A\nu, \nu\rangle(BA\nu \otimes \nu + \nu \otimes BA\nu) + 2\langle BA\nu, A\nu\rangle \nu \otimes \nu\big) \\ &=: B + \tfrac{2}{\langle A\nu, \nu\rangle^2}\, T_\nu. \end{aligned}$$

The assumption $-R_\nu\mathcal{K} = \mathcal{K}$ implies that for all $\nu \in \mathbb{R}^d\backslash\{0\}$ and all $\eta \in \mathbb{R}^d$ we have

$$\langle B\eta, \eta\rangle \leq 1 \quad \Leftrightarrow \quad \langle B\eta, \eta\rangle + \tfrac{2}{\langle A\nu, \nu\rangle^2}\langle T_\nu\eta, \eta\rangle \leq 1.$$

Thus, $\langle T_\nu\eta, \eta\rangle = 0$ for all $\eta \in \mathbb{R}^d$. Note that

$$\langle T_\nu\eta, \eta\rangle = 2\langle \nu, \eta\rangle\big(\langle BA\nu, A\nu\rangle\langle \eta, \nu\rangle - \langle A\nu, \nu\rangle\langle BA\nu, \eta\rangle\big).$$

Let $\{e_1, \dots, e_d\}$ be an orthonormal basis of eigenvectors of A with eigenvalues $\lambda_i > 0$. Let furthermore $\nu = e_i + \alpha e_j$ and $\eta = e_i$ for $i \neq j$ and $\alpha \in \mathbb{R}$. From $\langle T_\nu \eta, \eta \rangle = 0$ it follows that for all $\alpha \in \mathbb{R}$ we have

$$0 = \alpha \lambda_i \lambda_j \langle Be_i, e_j \rangle + \alpha^2 \lambda_j \big(\lambda_j \langle Be_j, e_j \rangle - \lambda_i \langle Be_i, Be_i \rangle \big) - \alpha^3 \lambda_j^2 \langle Be_j, e_i \rangle.$$

This implies that $\langle Be_i, e_j \rangle = 0$ for $i \neq j$ and $\lambda_j \langle Be_j, e_j \rangle = \lambda_i \langle Be_i, Be_i \rangle$ for all i, j, from which we conclude that $\langle Be_j, e_j \rangle = \lambda_1 \langle Be_1, e_1 \rangle \lambda_j^{-1} = \lambda_1 \langle Be_1, e_1 \rangle \langle A^{-1} e_j, e_j \rangle$. In the same way as in the proof of the previous Lemma, it follows finally that $B = \lambda_1 \langle Be_1, e_1 \rangle A^{-1}$. ☐

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