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Regression methods for stochastic control problems and their convergence analysis

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Abstract

In this paper we develop several regression algorithms for solving general stochastic optimal control problems via Monte Carlo. This type of algorithms is particularly useful for problems with a high-dimensional state space and complex dependence structure of the underlying Markov process with respect to some control. The main idea behind the algorithms is to simulate a set of trajectories under some reference measure and to use the Bellman principle combined with fast methods for approximating conditional expectations and functional optimization. Theoretical properties of the presented algorithms are investigated and the convergence to the optimal solution is proved under some assumptions. Finally, the presented methods are applied in a numerical example of a high-dimensional controlled Bermudan basket option in a financial market with a large investor.

1 Introduction

Modeling of optimal control is one of the most challenging areas in applied stochastics, particularly in finance. As typical real-world control problems, for example dynamic optimization problems in finance, are too complex to be treated analytically, effective generic computational algorithms are called for. Since the appearance of the ground-breaking articles Carriere (1996), Longstaff and Schwartz (2001), and Tsitsiklis and Van Roy (1999), regression based Monte Carlo methods emerged as an indispensable tool for solving high-dimensional stopping problems in the context of American style derivatives. From a mathematical point of view any optimal stopping problem can be seen as a particular case of a more general stochastic control problem. Optimal stochastic control problems appear in a natural way in many application areas. For instance in mathematical finance, problems such as portfolio optimization under market imperfections, optimal portfolio liquidation, super hedging, etc., do all come down to problems of stochastic optimal control. In fact, an active interplay between stochastic control and financial mathematics has been emerged in the last decades: While stochastic control has been a powerful tool for studying problems in finance on the one hand side, financial applications have been stimulating the development of new methods for optimal stopping and optimal control on the other hand, see, for example, besides the works mentioned above, Rogers (2002), Broadie and Glasserman (2004), Haugh and Kogan (2004), Ibáñez (2004), Meinshausen and Hambly (2004), Belomestny et al. (2006), Bender and Schoenmakers (2006), Belomestny et al. (2007), Kolodko and Schoenmakers (2006), Rogers (2007), and Carmona and Touzi (2008), and many others.

As a canonical general approach for solving an optimal control problem one may consider all possible future evolutions of the process at each time that a control choice is to be made. This method is well developed and may be effective in some special cases, but for more general problems such as optimal control of a diffusion in high dimensions, this approach is impractical. Other recently developed methods for control problems include the Markov chain approximation method of Monoyios (2004), a maturity randomization approach of Bouchard, Karoui and Touzi (2005) and a Malliavin based Monte-Carlo approach of Hansen (2005) (see also Bouchard, Ekeland and Touzi (2004)). However, all these methods are tailored to some specific problems and it is not clear how to generalize them. In this paper we propose a generic Monte Carlo approach combined with fast approximation methods and methods of functional optimization which is applicable to any discrete-time controlled Markov processes. The main idea is to simulate a set of trajectories under some reference measure and then apply a dynamic programming formulation (Bellman principle) to compute recursively estimates for the optimal control process and the optimal stopping rule, where the fast approximation methods allow for computing conditional expectations without nested simulations. In particular we propose several regression procedures and prove for these procedures convergence of the value function estimations under some additional assumptions. Moreover, we present an example of a high-dimensional Bermudan basket option where the dynamics of the underlying are influenced by a large investor, and illustrate the numerical performance of the regression algorithms at this example.

The outline of the paper is as follows. In Section 2 the basic stochastic setup is presented, some notations are introduced and the main problem is formulated. In Section 3 we introduce two kinds of regression methods for stochastic control problems: local regression methods and global regression methods, which are discussed in Sections 3.1 and 3.1 respectively. The convergence analysis of the regression algorithms is done in Section 4. A method of constructing upper bounds is discussed in Section 5. Finally, the numerical example is studied in Section 6.

2 Basic setup

For our framework we adopt the discrete time setup as in Rogers (2007). On a filtered measurable probability space $(\Omega, \mathcal{F})$, with $\mathcal{F} := (\mathcal{F}_r)_{r=0,1,...,T}$, $T \in \mathbb{N}_+$, we consider an adapted control process $\mathbf{a} : \Omega \times \{0, ..., T-1\} \to A$, control for short, where $(A, \mathcal{B})$ is a measurable state space. We assume a given set of admissible controls which is denoted by $\mathcal{A}$. Given a control $\mathbf{a} = (a_0, a_1, ..., a_{T-1}) \in \mathcal{A}$, we consider a *controlled Markov* process X valued in some measurable space $(S, \mathcal{S})$ and defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P}^{\mathbf{a}})$ with $X_0 = x_0$ a.s. and transition kernel of the following type,

$$\mathrm{P}^{\mathbf{a}}(X_{r+1} \in dy \mid X_r = x) = P^{a_r}(x, dy), \quad 0 \leq r < T.$$

So, it is assumed that the distribution of X_{r+1} conditional on $\mathcal{F}_r$ is governed by a (one-step) transition kernel $P^{a_r}(X_r, dy)$ which is in turn controlled by a_r. In this setting we may consider the general optimal control problem

$$Y_0^* := \sup_{\mathbf{a}\in\mathcal{A}} \mathrm{E}^{\mathbf{a}}\left[\sum_{r=0}^{T-1} f_r(X_r, a_r)\right], \tag{2.1}$$

for given functions f_r, $r = 0, \ldots, T-1$. The optimization problem (2.1) contains the standard optimal stopping problem

$$Y_0^* := \sup_{\tau} \mathrm{E}\left[g_\tau(X_\tau)\right],$$

as a special case. Indeed, take $\mathrm{P}^{\mathbf{a}}$ independent of $\mathbf{a}$, $f_r(x,a) = g_r(x)a$, and $\mathcal{A} = \mathcal{A}^{\text{stop}} = \left\{\mathbf{a} = \left(\mathbf{1}_{\{\tau=0\}}, \ldots, \mathbf{1}_{\{\tau=T\}}\right)\right\}$ with τ being $\mathcal{F}$-stopping time taking values in the set $\{0, \ldots, T\}$. Multiple stopping problems may be considered in a similar way by choosing a suitable $\mathcal{A}$. In this article, however, we choose $\mathcal{A}$ to be the set of *all* adapted controls (as in Rogers (2007)), while keeping the standard optimal stopping problem as a special case. This leads to our central goal of solving the optimal control problem

$$Y_0^* = \sup_{\mathbf{a}\in\mathcal{A},\, \tau\in\mathfrak{T}} \mathrm{E}^{\mathbf{a}}\left[\sum_{r=0}^{\tau-1} f_r(X_r, a_r) + g_\tau(X_\tau)\right] \tag{2.2}$$

for a given set of measurable functions $f_r : S \times A \to \mathbb{R}$, $g_r : S \to \mathbb{R}$. For technical reasons f_r and g are assumed to be bounded from below. To exclude trivialities we further assume that

$$\sup_{\mathbf{a}\in\mathcal{A}} \mathrm{E}^{\mathbf{a}}\left[\sum_{r=0}^{T-1} |f_r(X_r, a_r)|\right] < \infty, \quad \sup_{\mathbf{a}\in\mathcal{A}} \mathrm{E}^{\mathbf{a}}[|g_i(X_i)|] < \infty, \quad i = 0, \ldots, T.$$

The supremum in (2.2) is taken over $\mathbf{a} \in \mathcal{A}$ and all $\mathcal{F}$-stopping times with values in a subset $\mathfrak{T} \subset \{0, \ldots, T\}$.

The optimal control problem (2.2) with $\mathfrak{T} = \{0, \ldots, T\}$ will be the main object of our study. Consider the process

$$Y_r^* = \sup_{\mathbf{a}\in\mathcal{A}_r,\, \tau\in\mathfrak{T}_r} \mathrm{E}^{\mathbf{a}}\left[\sum_{s=r}^{\tau-1} f_s(X_s, a_s) + g_\tau(X_\tau) \middle| \mathcal{F}_r\right], \quad 0 \le r \le T, \tag{2.3}$$

with $\mathfrak{T}_r := \{r, \ldots, T\}$ and $\mathcal{A}_r$ being the set of all adapted controls $\mathbf{a} : \Omega \times \{r, \ldots, T-1\} \to A$. Then there exists a vector $h^* = (h_0^*, \ldots, h_T^*)$ of measurable functions on S, such that $Y_j^* = h_j^*(X_j)$ and h^* satisfies

$$\begin{aligned} h_r^*(x) &= \max\left[g_r(x), \left(\mathcal{L}h^*\right)_r(x)\right], \quad 0 \le r < T, \\ h_T^*(x) &= g_T(x), \end{aligned} \tag{2.4}$$

where $\mathcal{L} : h \to \mathcal{L}h$ is a Bellman-type operator defined by

$$(\mathcal{L}h)_r(x) := \sup_{a \in A} [f_r(x,a) + P^a h_{r+1}(x)]$$

and

$$P^a h_{r+1}(x) := \int P^a(x, dy) h_{r+1}(y).$$

We now assume that there exists a reference measure P^* equivalent to $\mathrm{P}^{\mathbf{a}}$, such that

$$P^a(x, dy) = \varphi(x, y, a) P^*(x, dy), \quad a \in A,$$

with $P^*(x, dy) := \mathrm{P}^*(X_{r+1} \in dy \mid X_r = x)$ and the function $\varphi(x, y, a)$ satisfying $\varphi \geq 0$ and $\int P^*(x, dy)\varphi(x, y, a) \equiv 1$. Then for any nonnegative measurable function $G : S^{T+1} \to \mathbb{R}_+$ it holds

$$\mathrm{E}^{\mathbf{a}}[G(X)|\mathcal{F}_j] = \mathrm{E}^*[G(X)\Lambda_{j,T}(\mathbf{a}, X)|\mathcal{F}_j], \tag{2.5}$$

where

$$\Lambda_{j,r}(\mathbf{a},y) := \prod_{l=j}^{r-1} \varphi(y_l, y_{l+1}, a_l), \quad r = j+1, \ldots, T, \quad y \in S^{T+1}.$$

If G depends on $X_0, \ldots, X_r$ only, we have for $0 \leq j \leq r$,

$$\mathrm{E}^{\mathbf{a}}[G(X)|\mathcal{F}_j] = \mathrm{E}^*[G(X)\Lambda_{j,r}(\mathbf{a}, X)|\mathcal{F}_j].$$

In particular, if G depends only on X_{j+1} it holds

$$\mathrm{E}^{\mathbf{a}}[G(X_{j+1})|\mathcal{F}_j] = \mathrm{E}^*[G(X_{j+1})\varphi(X_j, X_{j+1}, a_j)|\mathcal{F}_j]. \tag{2.6}$$

3 Regression methods for control problems

The solution Y_0^* of the optimal control problem (2.2) can in principle be computed backwardly via the dynamic programming principle (2.4). However, if the space S is high-dimensional, an analytic computation of the conditional expectation

$$C_r(x,a) := \mathrm{E}^a[h_r(X_{r+1})|X_r = x] = \mathrm{E}^*\left[\varphi(X_r, X_{r+1}, a) h_{r+1}(X_{r+1}) \mid X_r = x\right],$$

where henceforth for notational convenience $h := h^*$, is usually difficult, even if h_{r+1} is explicitly known. On the other hand, a straightforward backward construction of h using (2.4), by Monte Carlo simulation (under P^*) would lead to nested simulations where the degree of nesting increases with the number of exercise dates. In the context of optimal stopping problems, much research was focused on the development of fast methods to approximate C_r. We will show that these methods can be extended to a more general setting of optimal control problems.

From now on we assume that $S \subset \mathbb{R}^d$ for some $d > 0$. Suppose that h_{r+1} is estimated by $\widehat{h}_{r+1}$ and that we want to approximate h_r via (2.4) and (2.5). Define

$$\begin{aligned}\widehat{h}_r(x) &:= \max\left[g_r(x), \sup_{a\in A}\left[f_r(x,a) + P^a\widehat{h}_{r+1}(x)\right]\right] \\ &= \max\left[g_r(x), \sup_{a\in A}\left\{f_r(x,a) + \mathrm{E}^*\left[\varphi(X_r, X_{r+1}, a)\widehat{h}_{r+1}(X_{r+1}) \mid X_r = x\right]\right\}\right].\end{aligned}$$

Let

$$\left(\left(X_r^{(1)}, X_{r+1}^{(1)}\right), \ldots, \left(X_r^{(M)}, X_{r+1}^{(M)}\right)\right)$$

be a Monte Carlo sample from the joint distribution of (X_r, X_{r+1}) under P^* and suppose that, based on this Monte Carlo sample and the approximation $\widehat{h}_{r+1}$ of h_{r+1}, an estimate $\widehat{C}_{r,M}(x,a)$ of the conditional expectation $C_r(x,a)$ is constructed for all $x \in S$ and $a \in A$. In this paper we consider a class of estimation methods with $\widehat{C}_{r,M}$ being of the form

$$\widehat{C}_{r,M}(x,a) = \sum_{m=1}^{M} w_{m,M}(x, \mathbf{X}_r^M)\varphi(x, X_{r+1}^{(m)}, a)\widehat{h}_{r+1}(X_{r+1}^{(m)}), \tag{3.1}$$

where

$$w_{m,M}\left(x, \mathbf{X}_r^M\right) = w_{m,M}\left(x, X_r^{(1)}, \ldots, X_r^{(M)}\right)$$

are some coefficients which are to be specified by the method under consideration. It turns out that this class of approximation methods is very general and contains local and global regression methods. We discuss these two types of method in the next sections.

3.1 Algorithms based on local estimators

By introducing

$$d_r(x,a) := \int_S \varphi(x,y,a)h_{r+1}(y)p_r(x,y)\,dy, \quad p_r(x) := \int_S p_r(x,y)dy,$$

with $p_r(x,y)$ being the joint density of (X_r, X_{r+1}) under P^*, we may write

$$C_r(x,a) = d_r(a,x)/p_r(x).$$

So it is natural to estimate C_r as a ratio of estimates for p_r and d_r, respectively. With this goal in mind we consider, for a given Borel measurable kernel function $\Phi_M(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$, the following estimators

$$\begin{aligned}p_{r,M}(x) &:= M^{-1}\sum_{m=1}^{M}\Phi_M(x, X_r^{(m)}), \\ \widehat{d}_{r,M}(x,a) &:= M^{-1}\sum_{m=1}^{M}\Phi_M(x, X_r^{(m)})\varphi(x, X_{r+1}^{(m)}, a)\widehat{h}_{r+1}(X_{r+1}^{(m)}),\end{aligned}$$

where $x \in \mathbb{R}^d$ and $a \in A$. Then we estimate C_r by

$$\widehat{C}_{r,M}(x,a) := \frac{\widehat{d}_{r,M}(x,a)}{p_{r,M}(x)} \tag{3.2}$$
$$=: \sum_{m=1}^{M} w_{m,M}(x, \mathbf{X}_r^M)\varphi(x, X_{r+1}^{(m)}, a)\widehat{h}_{r+1}(X_{r+1}^{(m)})$$

with weight coefficients defined by

$$w_{m,M}(x,\mathbf{y}) := w_{m,M}(x, y_1, y_2, ...) := \frac{\Phi_M(x, y_m)}{\sum_{m'=1}^{M} \Phi_M(x, y_{m'})}.$$

If $p_{r,M} = 0$ we set $\widehat{C}_{r,M} = 0$. It is important to note that $w_{m,M}$ sum up to one. The name "local" comes from the fact that in most cases the function $\Phi_M(x,y)$ converges (in some sense) to a delta function $\delta(x-y)$ as $M \to \infty$. The class of local estimators is rather large and contains well known examples such as the Nadaraya-Watson and the k-nearest neighbors regression estimators. In recent years, local estimators have become popular in applied financial mathematics, mainly in the context of hedging and Greek estimation (see, e.g. Elie, Fermanian and Touzi (2009)).

Example 3.1. *Let K be a measurable function on $\mathbb{R}^d$. Take*

$$\Phi_M(x,y) = \delta_M^{-d} K((x-y)/\delta_M),$$

where $\{\delta_M\}$ is a sequence of positive numbers tending to zero. Then (3.2) yields the well-known Nadaraya-Watson regression estimator

$$\widehat{C}_{r,M}(x,a) = \frac{\sum_{m=1}^{M} K((x - X_r^{(m)})/\delta_M)\varphi(x, X_{r+1}^{(m)}, a)\widehat{h}_{r+1}(X_{r+1}^{(m)})}{\sum_{m=1}^{M} K((x - X_r^{(m)})/\delta_M)}. \tag{3.3}$$

Example 3.2. *We can modify the estimator in Example 3.1 by specifying an increasing sequence (k_M) of natural numbers with $k_M \leq M$ and by reducing the number of summands in* (3.3) *to k_M in the following way. Consider the first k_M nearest neighbors of x, say $X_r^{(m_1)}, \ldots, X_r^{(m_{k_M})}$ in the Monte Carlo sample $X_r^{(1)}, \ldots, X_r^{(M)}$, and define $R_M := \left\| x - X_r^{(m_{k_M})} \right\|_2$ to obtain the k_M-nearest neighbors regression estimator*

$$\widehat{C}_{r,M}(x,a) = \frac{\sum_{n=1}^{k_M} \varphi(x, X_{r+1}^{(m_n)}, a)\widehat{h}_{r+1}(X_{r+1}^{(m_n)})K((x - X_r^{(m_n)})/R_M)}{\sum_{n=1}^{k_M} K((x - X_r^{(m_n)})/R_M)}. \tag{3.4}$$

Finally, after estimating $C_r(x,a)$ by $\widehat{C}_{r,M}(x,a)$, we construct

$$\widehat{a}_{r,M}(x) := \arg\sup_{a \in A} [f_r(x,a) + \widehat{C}_{r,M}(x,a)], \quad x \in S, \tag{3.5}$$

and estimate h_r by

$$\widehat{h}_{r,M}(x) := \max\{g_r(x), f_r(x, \widehat{a}_{r,M}(x)) + \widehat{C}_{r,M}(x, \widehat{a}_{r,M}(x))\}. \tag{3.6}$$

Starting with $\widehat{h}_{T,M}(x) = g_T(x)$ and working backwardly, we so obtain estimates for all h_r, $r = 0, \ldots, T-1$.

Remark 3.3. Local estimators have in some respects nice theoretical properties, for example, almost sure convergence to C_r under rather weak smoothness assumptions. Basically only local smoothness is required for this. A disadvantage of local estimators is their numerical complexity in general. For instance, if we want to compute the Nadaraya-Watson estimator $\widehat{C}_{r,M}(x,a)$ at M points in $\mathbb{R}^d$, it will require M^2 operations. In the case of the k_M-nearest neighbors estimator this number can be reduced to $M \log M$ using fast search algorithms.

3.2 Global regression estimators

As an alternative to local regression methods we now consider algorithms based on global regression. From a practical point of view global regression estimators are easier to implement in an efficient way than local estimators. The convergence analysis of global estimators is, however, more delicate and usually requires rather strong assumptions on C_r and the underlying Markov process X_r. For the standard Bermudan stopping problem ($f_r \equiv 0$, $\varphi \equiv 1$) we refer to Clément, Lamberton and Protter (2002), Egloff (2005) and Egloff, Kohler and Todorovic (2007). The global regression procedures in the next two sections are in some sense a generalization of the methods of Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), respectively, to optimal control problems.

3.2.1 Algorithms based on continuation functions

For a given Monte Carlo sample $(X_r^{(1)}, \ldots, X_r^{(M)})$, $r = 0, \ldots, L$, under the measure P^* and a system of basis functions $\psi := [\psi_1, \ldots, \psi_K]^\top$ we consider for each $a \in A$ the minimization problem

$$\widehat{\beta}_r(a) := \underset{\beta \in \mathbb{R}^K}{\arg\min} \sum_{m=1}^{M} \left(\psi^\top (X_r^{(m)}) \beta - Y^{(m)}(a) \right)^2, \tag{3.7}$$

where

$$Y^{(m)}(a) := \varphi(X_r^{(m)}, X_{r+1}^{(m)}, a) \widehat{h}_{r+1}(X_{r+1}^{(m)})$$

and an estimate $\widehat{h}_{r+1}$ of h_{r+1} is assumed to be already constructed. The solution of (3.7) is explicitly given by

$$\widehat{\beta}_r(a) = (F^\top F)^{-1} F^\top Y(a) =: F^\dagger Y(a), \tag{3.8}$$

where $F = (F_{mk}) = (\psi_k(X_r^{(m)}))$ is a $M \times K$ design matrix and $Y(a) := (Y^{(m)}(a))_{m=1,\ldots,M}$. Note that the design matrix F does not depend on a. We next consider

$$\widehat{a}_{r,M}(x) = \underset{a \in A}{\arg\max} \{ f_r(x,a) + \widehat{C}_{r,M}(x,a) \}, \tag{3.9}$$

where

$$\widehat{C}_{r,M}(x,a) = \psi^\top(x)\widehat{\beta}_r(a) = \psi^\top(x)F^\dagger Y(a) \qquad (3.10)$$
$$= \sum_{m=1}^{M} w_{m,M}(x, \mathbf{X}_r^M)\varphi(x, X_{r+1}^{(m)}, a)\widehat{h}_{r+1,M}(X_{r+1}^{(m)})$$

with coefficients $w_{m,M}$ given by

$$w_{m,M}(x, \mathbf{X}_r^M) = \psi^\top(x)\left((F^\top F)(X_r^{(\cdot)})\right)^{-1}\psi(X_r^{(m)}). \qquad (3.11)$$

In order to solve (3.9) one may, for instance, construct an approximation procedure for finding the a roots of the stationary point equation

$$\frac{\partial}{\partial a}f_r(x,a) + \sum_{k=1}^{K}\psi_k(x)F^\dagger\frac{\partial}{\partial a}Y(a) = 0.$$

We proceed with a second regression problem

$$\widetilde{\beta}_r = \underset{\beta\in\mathbb{R}^K}{\arg\min}\sum_{m=1}^{M}\left(\varphi(\widetilde{X}_r^{(m)}, \widetilde{X}_{r+1}^{(m)}, \widehat{a}_{r,M}(\widetilde{X}_r^{(m)}))\widehat{h}_{r+1}(\widetilde{X}_{r+1}^{(m)}) - \psi^\top(\widetilde{X}_r^{(m)})\beta\right)^2$$

based on a new set of paths

$$(\widetilde{X}_1^{(m)}, \ldots, \widetilde{X}_T^{(m)}), \quad m = 1, \ldots, M$$

under P^* to end up with

$$\widehat{h}_{r,M}(x) = \max\left[g(x), f_r(x, \widehat{a}_{r,M}(x)) + \psi^\top(x)\widetilde{\beta}_r\right]. \qquad (3.12)$$

The second regression is needed to avoid the multiple vector-matrix multiplication in (3.8) when computing $\widehat{h}_{r,M}(X_r^{(m)})$, $m = 1, \ldots, M$.

3.2.2 Algorithms based on backward construction of stopping time and control

In this section we present an algorithm where, instead of regressing continuation functions, the control and stopping times are backwardly constructed on a set of simulated trajectories. This method relies on the following consistency theorem proved in Appendix.

Theorem 3.4. *The optimal stopping time $\tau^*(r)$ and the optimal control $\mathbf{a}^*(r)$ solving the problem*

$$Y_r^* = \sup_{\mathbf{a}\in\mathcal{A}_r,\, \tau\in\mathfrak{T}_r} \mathrm{E}^{\mathbf{a}}\left[\sum_{s=r}^{\tau-1} f_s(X_s, a_s) + g_\tau(X_\tau)\,\middle|\, \mathcal{F}_r\right],$$

satisfy the following consistency relations

$$\tau^*(r) > r \Rightarrow \tau^*(r) = \tau^*(r+1) \text{ and } a_j^*(r) = a_j^*(r+1)$$

for all j such that $r + 1 \le j < \tau^*(r+1)$.

Note that $a_j^*(r)$ is only defined for $r \le j < \tau^*(r)$, i.e. the control $\mathbf{a}^*(r)$ is not defined if $\tau^*(r) = r$. Given a sample $(X_0^{(m)}, \ldots, X_T^{(m)})$, $m = 1, ..., M$, we construct estimates $\tau^{(m)}(r)$ and $a_j^{(m)}(r)$, $r \le j < \tau^{(m)}(r)$ for stopping times and control processes respectively in the following way. At the terminal time we set

$$\tau^{(m)}(T) = T, \quad m = 1, ..., M.$$

Let $\tau^{(m)}(r+1)$, $a_j^{(m)}(r+1)$, $r+1 \le j < \tau(r+1)$ be constructed for $m = 1, \ldots, M$, at time $r+1$, $0 \le r < T$. Let $\psi := [\psi_1, \ldots, \psi_K]^\top$ be a system of basis functions. For any $a \in A$ consider the least squares regression problem

$$\widehat{\beta}(a) := \underset{\beta \in \mathbb{R}^K}{\arg\min} \sum_{m=1}^{M} \left(\psi^\top(X_r^{(m)})\beta - Y^{(m)}(a)\right)^2, \tag{3.13}$$

where

$$Y^{(m)}(a) = \varphi(X_r^{(m)}, X_{r+1}^{(m)}, a) Z_{r+1}^{(m)}$$

with

$$\begin{aligned} Z_{r+1}^{(m)} := &\sum_{l=r+1}^{\tau^{(m)}(r+1)-1} \Lambda_{r+1,l}(\mathbf{a}^{(m)}(r+1), X^{(m)}) f_l(X_l^{(m)}, a_l^{(m)}(r+1)) \\ &+ \Lambda_{r+1,\tau^{(m)}(r+1)}(\mathbf{a}^{(m)}(r+1), X^{(m)}) g(X_{\tau^{(m)}(r+1)}^{(m)}). \end{aligned}$$

The solution of (3.13) is given by (3.8) and we can define an estimate $\widehat{C}_{r,M}(x,a) = \psi^\top(x)\widehat{\beta}(a)$ and then $\widehat{a}_{r,M}(x)$ as a solution of (3.9). Now simulate a new set of trajectories

$$(\widetilde{X}_0^{(m)}, \ldots, \widetilde{X}_T^{(m)}), \quad m = 1, \ldots, M,$$

under P^* and define

$$\widetilde{\beta}_r := \underset{\beta \in \mathbb{R}^K}{\arg\min} \sum_{m=1}^{M} \left(\psi^\top(\widetilde{X}_r^{(m)})\beta - \varphi(\widetilde{X}_r^{(m)}, \widetilde{X}_{r+1}^{(m)}, \widehat{a}_{r,M}(\widetilde{X}_r^{(m)})) Z_{r+1}^{(m)}\right)^2.$$

Put $\widetilde{C}_{r,M}(x) = \psi^\top(x)\widetilde{\beta}_r$. By setting for $m = 1, \ldots, M$,

$$\tau^{(m)}(r) = r, \quad \text{if} \quad f_r(X_r^{(m)}, \widehat{a}_{r,M}(X_r^{(m)})) + \widetilde{C}_{r,M}(X_r^{(m)})) \ < g(X_r^{(m)}),$$

and

$$\begin{aligned} &\tau^{(m)}(r) = \tau^{(m)}(r+1), \quad a_r^{(m)}(r) = \widehat{a}_{r,M}(X_r^{(m)}), \\ &a_j^{(m)}(r) = a_j^{(m)}(r+1), \quad r+1 \le j < \tau^{(m)}(r+1), \end{aligned}$$

otherwise, we so end up with a sequence of estimates

$$\widetilde{C}_{r,M}(x) := \sum_{k=1}^{K} \widetilde{\beta}_{r,k}\psi_k(x), \quad r = 0, \ldots, T-1, \tag{3.14}$$

and a sequence of functions $\widehat{a}_{r,M}$, $r = 0, \ldots, T-1$. Based on (3.14) one may use the (generally suboptimal) stopping rule

$$\tau_M := \inf\{0 \leq r \leq T : g(X_r) \geq f_r(X_r, \widehat{a}_{r,M}(X_r)) + \widetilde{C}_{r,M}(X_r)\} \tag{3.15}$$

and the (generally suboptimal) control process

$$\mathbf{a}_M(X) = (\widehat{a}_{0,M}(X_0), \widehat{a}_{1,M}(X_1), \ldots, \widehat{a}_{T-1,M}(X_{T-1})) \tag{3.16}$$

to construct a lower approximation for Y_0^* via a next Monte Carlo simulation.

4 Convergence analysis of regression methods

The issues of convergence for regression algorithms in the context of pricing Bermudan options have been already studied in several papers. Clément, Lamberton and Protter (2002) were first who proved the convergence of the Longstaff-Schwartz algorithm. Glasserman and Yu (2005) have shown that the number of Monte Carlo paths has to be exponential in the number of basis functions used for regression in order to ensure the consistency of the price estimate. Recently, Egloff, Kohler and Todorovic (2007) have derived rates of convergence for continuation values estimates by the so called dynamic look-ahead algorithm (see also Egloff (2005)) that "interpolates" between Longstaff-Schwartz and Tsitsiklis-Roy algorithms. In the case of general control problems the issue of convergence is more delicate because along with the convergence of regression estimates $C_{r,M}(x, a)$ we also need the convergence of control estimates $a_{r,M}$. The latter convergence can be ensured only if the first one is uniform on the set of all possible controls. This type of convergence can be proved only under some additional assumptions.

Generally, a convergence analysis can be divided into two parts. In the first part one considers *local convergence*, that is the convergence of the one step estimate

$$h_{r,M}(x) := \max\left[g_r(x), \sup_{a\in A}\left[f_r(x, a) + C_{r,M}(x, a)\right]\right],$$

based on the "pseudo" estimator

$$C_{r,M}(x, a) := \sum_{m=1}^{M} w_{m,M}(x, \mathbf{X}_r^M)\varphi(x, X_{r+1}^{(m)}, a)h_{r+1}(X_{r+1}^{(m)}), \tag{4.1}$$

i.e. (3.1) with $\widehat{h}_{r+1}$ replaced by the exact solution h_{r+1}. It turns out that the local convergence relies exclusively on the sort of regression estimate under consideration and can be established via standard results from the theory of empirical processes and regression analysis as we will see. The second part deals with the *global convergence*. In practice, one starts from $r = T$ and proceeds backwardly where at each step the previously constructed estimate $\widehat{h}_{r+1}$ is used instead of h_{r+1}. The aim of the global convergence analysis is to prove the convergence of $\widehat{h}_{r,M}$ to h_r in a suitable sense, taking into account all errors from the previous steps. The next theorem provides conditions for the global convergence, assuming that $C_{r,M}$ is known to converge to C_r in a certain sense. In fact, the prove of Theorem 4.2 is quite generic as it involves only general properties of the weights in (3.1).

Theorem 4.1. *Suppose that starting with $\widehat{h}_{T,M} = h_T^*(x) = g_T(x)$, at each backward step $\widehat{h}_{r,M}$ is constructed from $\widehat{h}_{r+1,M}$ via (3.6) or (3.12) using a new independent sample of M trajectories. Suppose further that the function φ is uniformly bounded, that is $|\varphi| \leq A_\varphi$ for some constant A_φ. If*

$$\begin{aligned} &\mathrm{E}\left\{\int_{\mathbb{R}^d} \|C_{r,M}(x,\cdot) - C_r(x,\cdot)\|_A^q \, p_r(x)\, dx\right\}^{1/q} \\ &\qquad = \mathrm{E}\left\{\int_{\mathbb{R}^d} \left[\sup_{a\in A} |C_{r,M}(x,a) - C_r(x,a)|\right]^q \, p_r(x)\, dx\right\}^{1/q} \\ &\qquad\qquad = O(\varepsilon_M), \quad r = 0,\ldots,T-1, \quad M \to \infty \end{aligned} \tag{4.2}$$

with some $q \geq 1$ and some sequence ε_M tending to 0, then it holds

$$\mathrm{E}\left\|\widehat{h}_{r,M} - h_r\right\|_{L_q(p_r)} = O\left(\lambda_{q,M}^{T-r}\varepsilon_M\right), \quad 0 \leq r \leq T,$$

with

$$\lambda_{q,M} = \sup_{0\leq r\leq T} \sum_{m=1}^{M} \|w_{m,M}(\cdot,\cdot)\|_{L_q(p_r \otimes_{l=1}^{M} p_r)}\,. \tag{4.3}$$

Corollary 4.2. *If $q = 1$ and all weights $w_{m,M}$ in (3.1) are nonnegative and sum up to 1 (e.g. in the case (3.2) if $\Phi_M \geq 0$), then $\lambda_{q,M} \leq 1$ and*

$$\mathrm{E}\left\|\widehat{h}_{r,M} - h_r\right\|_{L_1(p_r)} = O\left(\varepsilon_M\right), \quad 0 \leq r \leq T.$$

Thus, in the case of nonnegative weights and $q = 1$ the "global" convergence rates coincide with the rates of a particular regression estimator.

4.1 Convergence of local regression estimators

In this section we analyze the convergence of local regression estimators of the form (3.2). Define two sets of functions

$$\begin{aligned} \mathcal{F}_M &:= \{\Phi_M(x,\cdot) : x \in \mathbb{R}^d\}, \\ \mathcal{F}_{\varphi,M} &:= \{\varphi(x,\cdot,a)\Phi_M(x,\cdot) : x \in \mathbb{R}^d,\, a \in A\}. \end{aligned}$$

Assume that for some constant $A_h > 0$,

$$\mathrm{P}\left(|h_r(X_r)| < A_h\right) = 1, \quad r = 0,\ldots,T, \tag{4.4}$$

and that the function φ is uniformly bounded, i.e. there exists a constant A_φ such that

$$\sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}\sup_{a\in A}\varphi(x,y,a) < A_\varphi. \tag{4.5}$$

Theorem 4.3. *Let $\mathcal{F}_M$ and $\mathcal{F}_{\varphi,M}$ be measurable uniformly bounded Vapnik-Červonenkis (VC) classes of functions (see Appendix), such that (7.11) is fulfilled for some $\nu > 0$ and $A > 0$, simultaneously for all M. Furthermore, let σ_M and U_M be two sequences of positive real numbers such that*

$$U_M \geq \sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}|\Phi_M(x,y)|, \tag{4.6}$$

$$\sigma^2_{r,M} \geq \sup_{x\in\mathbb{R}^d}\mathrm{E}[\Phi^2_M(x,X_r)], \tag{4.7}$$

and the following relations hold as $M\to\infty$,

(i) $0 < \sigma_{r,M} < U_M/2$,

(ii) $(U_M/\sigma_{r,M})\sqrt{\log(U_M/\sigma_{r,M})} \leq \sqrt{M}$,

(iii) $\gamma_M := M^{-1/2}\sigma_{r,M}\sqrt{\log(U_M/\sigma_{r,M})} = o(1)$,

(iv) $\log\gamma_M = O(\log(\sigma_{r,M}/U_M))$,

(v) $\|p_r - \mathrm{E}\,p_{r,M}\|_{\mathbb{R}^d} \to 0$,

(vi) $\|d_r - \mathrm{E}\,d_{r,M}\|_{\mathbb{R}^d\times A} \to 0$.

Let $\mathcal{D}$ be a fixed bounded domain such that

$$p_{\min} = p_{\min}(\mathcal{D}) := \min_r \inf_{x\in\mathcal{D}} p_r(x) > 0.$$

Define a truncated version of $C_{r,M}$ (depending on $\mathcal{D}$) as

$$C^{\mathcal{D}}_{r,M}(x,a) := \begin{cases} C_{r,M}(x,a), & |p_{r,M}(x)| > p_{\min}/2 \text{ and } x\in\mathcal{D}, \\ 0, & \text{otherwise.}\end{cases}$$

Then it holds

$$\mathrm{E}\,\|C^{\mathcal{D}}_{r,M} - C_r\|_{\mathcal{D}\times A} \leq \frac{\widetilde{C}_{\max}}{\widetilde{p}_{\min}}\left(L_0\gamma_M + \|p_r - \mathrm{E}\,p_{r,M}\|_{\mathbb{R}^d} + \|d_r - \mathrm{E}\,d_{r,M}\|_{\mathbb{R}^d}\right)$$

with $\widetilde{C}_{\max} := \max(C_{\max}(\mathcal{D}),1)$, where $C_{\max}(\mathcal{D}) = \max_r \sup_{(x,a)\in\mathcal{D}\times A} C_r(x,a)$, $\widetilde{p}_{\min} := 2\min(p_{\min},1)$, and with L_0 depending only on the VC characteristics of the classes $\mathcal{F}_M$ and $\mathcal{F}_{\varphi,M}$.

The proof of Theorem 4.3 is given in the Appendix. This result can be used to prove the condition (4.2) needed for the global convergence. Let us fix some $R > 0$ and consider the ball $\mathcal{B}_R := \mathcal{B}(x_0, R) := \{x : |x - x_0| \leq R\}$ with some fixed $x_0 \in \mathbb{R}^d$. For a fixed $q \geq 1$ we then have

$$\mathrm{E}\left\{\int_{\mathbb{R}^d} \left\|C_{r,M}^{\mathcal{B}_R}(x,\cdot) - C_r(x,\cdot)\right\|_A^q p_r(x)\,dx\right\}^{1/q} \leq$$
$$\mathrm{E}\left\|C_{r,M}^{\mathcal{B}_R} - C_r\right\|_{\mathcal{B}_R\times A} + \left\{\int_{\mathbb{R}^d\setminus\mathcal{B}_R} \|C_r(x,\cdot)\|_A^q\, p_r(x)dx\right\}^{1/q}.$$

So, if R_M is an increasing sequence of positive numbers such that both

$$\mathcal{E}_{1,M} := \frac{\widetilde{C}_{\max}(\mathcal{B}_{R_M})}{\widetilde{p}_{\min}(\mathcal{B}_{R_M})}(L_0\gamma_M + \|p_r - \mathrm{E}\,p_{r,M}\|_{\mathbb{R}^d}$$
$$+ \|d_r - \mathrm{E}\,d_{r,M}\|_{\mathbb{R}^d\times A}) \to 0,$$

and

$$\mathcal{E}_{2,M} \quad:=\quad \left(\int_{\mathbb{R}^d\setminus\mathcal{B}_{R_M}} \|C_r(x,\cdot)\|_A^q\, p_r(x)dx\right)^{1/q} \to 0, \quad M \to \infty,$$

then by Theorem 4.3 it holds

$$\mathrm{E}\left\{\int_{\mathbb{R}^d} \left\|C_{r,M}^{\mathcal{B}_{R_M}}(x,\cdot) - C_r(x,\cdot)\right\|_A^q p_r(x)\,dx\right\}^{1/q} \leq \mathcal{E}_{1,M} + \mathcal{E}_{2,M} \to 0.$$

Kernel type estimators. Let us consider the application of Theorem 4.3 to a kernel type regression estimator (3.3). Let K be a bounded square integrable function on $\mathbb{R}^d$. In Dudley (1999) sufficient conditions are given that ensure that the set

$$\mathcal{F} = \left\{K\left(\frac{x-\cdot}{\delta}\right) : x \in \mathbb{R}^d,\, \delta \in \mathbb{R}\setminus\{0\}\right\} \tag{4.8}$$

is a uniformly bounded VC class, i.e. it satisfies (7.11) with some A and ν and all probability measures P. In particular it is shown that (4.8) is a bounded VC class if $K(x) = f(p(x))$ for some polynomial p and a bounded real function f of bounded variation. Obviously, the standard Gaussian kernel falls into this category. Another example is the case where K is a pyramid, or $K = \mathbf{1}_{[-1,1]^d}$. For constituting new VC classes from given ones the following lemma may be useful.

Lemma 4.4. *If $\mathcal{F}$ is a uniformly bounded VC class, then for any bounded measurable function h the class of functions $h\mathcal{F} := \{h\cdot f : f \in \mathcal{F}\}$ is again a uniformly bounded VC class. In particular, if h is a constant then the VC characteristics of $h\mathcal{F}$ are equal to the VC characteristics of $\mathcal{F}$. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are uniformly bounded VC classes then the function classes $\mathcal{F} \pm \mathcal{G} := \{f \pm g : f \in \mathcal{F},\, g \in \mathcal{G}\}$ and $\mathcal{F}\cdot\mathcal{G}$ $:= \{f\cdot g : f \in \mathcal{F},\, g \in \mathcal{G}\}$ are uniformly bounded VC classes.*

As can be easily seen from the above lemma the class

$$\mathcal{F}_\varphi := \left\{ \varphi(x,\cdot,a) K\left(\frac{x-\cdot}{\delta}\right) : \, x \in \mathbb{R}^d, \, \delta \in \mathbb{R}\setminus\{0\}, \, a \in A \right\}$$

is a uniformly bounded VC class, provided that the function classes (4.8) and

$$\{\varphi(x,\cdot,a) : x \in \mathbb{R}^d, \, a \in A\}$$

are uniformly bounded VC classes. In this case the classes $\mathcal{F}_M$ and $\mathcal{F}_{\varphi,M}$ with

$$\Phi_M(x,\cdot) = \delta_M^{-d} K\left(\frac{x-\cdot}{\delta_M}\right), \quad x \in \mathbb{R}^d, \quad M = 1, 2, \ldots$$

satisfy the conditions of Theorem 4.3. With regard to (4.6) and (4.7), we may take $U_M = \delta_M^{-d} \|K\|_\infty$ and

$$\sigma_{r,M}^2 = \sup_{x \in \mathbb{R}^d} \delta_M^{-d} \int_{\mathbb{R}^d} K^2(u) p_r(x - u\delta_M)\, du \le \delta_M^{-d} \|K\|_2^2 \|p_r\|_\infty,$$

respectively. Note that under this choice of $\sigma_{r,M}$ and U_M the relation (i) of Theorem 4.3 is satisfied. In order to make the conditions (ii)-(iv) hold we additionally suppose that the bandwidths δ_M satisfy for $M \to \infty$,

$$\delta_M \to 0, \qquad \frac{M\delta_M^d}{|\log \delta_M|} \to \infty, \quad \log \frac{M\delta_M^d}{|\log \delta_M|} = O(\log \delta_M). \tag{4.9}$$

Turn now to the conditions (v)-(vi). It can be easily shown that if functions $d_r(x,a)$ and $p_r(x)$ have continuous derivatives in x of order s and these derivatives are uniformly bounded on $\mathbb{R}^d \times A$ and $\mathbb{R}^d$ respectively, then

$$\|p_r - \mathrm{E}\, p_{r,M}\|_{\mathbb{R}^d} = O(\delta_M^s), \quad \|d_r - \mathrm{E}\, d_{r,M}\|_{\mathbb{R}^d \times A} = O(\delta_M^s), \quad M \to \infty,$$

provided that

$$\int_{\mathbb{R}^d} \|x\|^s K(x)\, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} x_j^l K(x)\, dx = 0$$

for $j = 1, \ldots, d$, $l = 1, \ldots, s-1$. Hence, according to Theorem 4.3

$$\mathrm{E}\, \|C_{r,M}^{\mathcal{D}} - C_r\|_{\mathcal{D}\times A} \le \frac{\widetilde{C}_{\max}}{\widetilde{p}_{\min}} \left(D_0 \sqrt{|\log \delta_M| / M\delta_M^d} + D_1 \delta_M^s \right), \quad M \to \infty,$$

where D_0 and D_1 are positive constants independent of the region $\mathcal{D}$.

4.2 Convergence of global regression estimators

Fix some $r > 0$ and consider the one step regression problem

$$\widehat{\beta}(a) := \arg\min_{\beta \in \mathbb{R}^K} \sum_{m=1}^{M} \left(\boldsymbol{\psi}_K^\top (X_r^{(m)}) \beta - Y^{(m)}(a) \right)^2,$$

where

$$Y^{(m)}(a) := \varphi(X_r^{(m)}, X_{r+1}^{(m)}, a) h_{r+1}(X_{r+1}^{(m)}), \quad m = 1, ..., M,$$

and $\boldsymbol{\psi}_K(x) := [\psi_1(x), \ldots, \psi_K(x)]^\top$ with $\{\psi_i(x) : i = 1, 2, ..\}$ being a set of basis functions. Consider the matrix $\Gamma^{M,K}$ with elements

$$\Gamma_{l,k}^{M,K} := \frac{1}{M} \sum_{m=1}^{M} \psi_l \left(X_r^{(m)} \right) \psi_k \left(X_r^{(m)} \right), \quad 1 \le l, k \le K, \tag{4.10}$$

and the matrix $\Gamma^K = (\Gamma_{l,k}^K)_{1 \le l,k \le K}$ with elements

$$\Gamma_{l,k}^K := \mathrm{E}\, \Gamma_{l,k}^{M,K} = \int_{\mathbb{R}^d} \psi_l(z) \psi_k(z) p_r(z)\, dz.$$

In the sequel we assume that the smallest eigenvalue of the matrix Γ^K is bounded from below by $\lambda_{\min} > 0$ for all K and $r > 0$. Let us define a truncated version $C_{r,M}^{\mathfrak{T}}(x, a)$ of the standard least squares regression estimator $C_{r,M}(x, a) = \boldsymbol{\psi}_K^\top(x) \widehat{\beta}$ as follows. If the smallest eigenvalue $\lambda_{\min}^{M,K}$ of $\Gamma^{M,K}$ fulfills $\lambda_{\min}^{M,K} \ge \lambda_{\min}/2$, we set $C_{r,M}^{\mathfrak{T}}(x, a) = C_{r,M}(x, a)$ and otherwise $C_{r,M}^{\mathfrak{T}}(x, a) = 0$. The following theorem holds.

Theorem 4.5. *Suppose that conditions (4.4) and (4.5) are fulfilled and let $\{\psi_k, k = 1, 2, \ldots\}$ be a system of basis functions on $\mathbb{R}^d$ which are uniformly bounded, that is there exists a constant $A_\psi > 0$ such that $\max_k \|\psi_k\|_\infty < A_\psi$. Let further the families of functions*

$$\left\{ \varphi(x, \cdot, a) : x \in \mathbb{R}^d, a \in A \right\} \quad \text{and} \quad \left\{ \psi_k(\cdot) : k = 1, 2, ... \right\}$$

be bounded VC classes. Then it holds

$$\begin{aligned} \mathrm{E} \left(\int \sup_{a \in A} \left| C_{r,M}^{\mathfrak{T}}(x, a) - C_r(x, a) \right|^2 p_r(x) dx \right)^{1/2} & \\ \le 2 C_{\max} K^2 \exp\left[-B_0 M / K^2 \right] + B_1 \frac{K^2}{\sqrt{M}} + & \\ \left(\int_{\mathbb{R}} \sup_{a \in A} \left| \Delta_r(x, a) \right|^2 p_r(x)\, dx \right)^{1/2}, & \end{aligned} \tag{4.11}$$

where B_0 and B_1 are some positive constants, $C_{\max} := \max_r \sup_{(x,a) \in \mathbb{R}^d \times A} C_r(x, a)$ and

$$\Delta_r(x, a) = \mathrm{E}\left[\boldsymbol{\psi}_K^\top(x) \left(\Gamma^K \right)^{-1} \boldsymbol{\psi}_K(X_r^{(1)}) C_r(X_r^{(1)}, a) \right] - C_r(x, a).$$

Corollary 4.6. *Suppose that*

$$C_r(x,a) = \sum_{k=1}^{\infty} \beta_k(a)\psi_k(x), \tag{4.12}$$

where the convergence takes place both pointwise and in $L_2(p_r)$ sense. Then (4.11) becomes

$$\begin{aligned} &\mathrm{E}\left(\int \sup_{a\in A}\left|C^{\mathfrak{T}}_{r,M}(x,a) - C_r(x,a)\right|^2 p_r(x)dx\right)^{1/2} \\ &\leq 2C_{\max}K^2 \exp\left[-B_0M/K^2\right] + B_1\frac{K^2}{\sqrt{M}} + \gamma_K \end{aligned} \tag{4.13}$$

with

$$\begin{aligned} \gamma_K := &\left(\mathrm{E}\sup_{a\in A}\left|\sum_{k=K+1}^{\infty} \beta_k(a)\psi_k(X_r)\right|^2\right)^{1/2} \\ &\leq \left(\sup_{a\in A}\sum_{k,k'=K+1}^{\infty} |\beta_k(a)\beta_{k'}(a)|\,\Gamma^{1/2}_{kk}\Gamma^{1/2}_{k'k'}\right)^{1/2}. \end{aligned} \tag{4.14}$$

Corollary 4.7. *We can represent the truncated estimator $C^{\mathfrak{T}}_{r,M}(x,a)$ in the form*

$$C^{\mathfrak{T}}_{r,M}(x,a) := \sum_{m=1}^{M} \widetilde{w}_{m,M}(x,\mathbf{X}^M_r)\varphi(X^{(m)}_r, X^{(m)}_{r+1}, a)h_{r+1}(X^{(m)}_{r+1})$$

with $\widetilde{w}_{m,M}(x,\mathbf{X}^M_r) := M^{-1}\boldsymbol{\psi}^{\top}_K(x)\left(\Gamma^{M,K}\right)^{-1}\boldsymbol{\psi}_K(X^{(m)}_r)$ if $\lambda^{M,K}_{\min} \geq \lambda_{\min}/2$ and 0 otherwise. A straightforward calculations lead to the bound

$$\|\widetilde{w}_{m,M}(\cdot,\cdot)\|_{L_2(p_r\otimes^M_{l=1}p_r)} = \left(\mathrm{E}\left|\widetilde{w}_{m,M}(X_r,\mathbf{X}^M_r)\right|^2\right)^{1/2} \leq B_4K^{1/2}M^{-1}$$

and hence we obtain $\lambda_{2,M} = O(\sqrt{K})$ with $\lambda_{2,M}$ being defined in (4.3)*.*

Corollary 4.8. *Suppose that $K^2/M = o(\log^{-1}(M))$ as $M \to 0$, then*

$$\mathrm{E}\left\|\widehat{h}_{r,M} - h_r\right\|_{L_2(p_r)} = O\left(K^{T/2}(\gamma_K + K^2/\sqrt{M})\right), \quad r = 0,\ldots,T-1,$$

for $M \to \infty$. Moreover, if (4.12) *holds and the coefficients $\{\beta_k(a)\}$ in* (4.12) *fulfill*

$$\sup_a \sum_{k=0}^{\infty} |\beta_k(a)| \exp(\mu k^{\alpha}) < \infty$$

for some positive α and μ, then under the choice $K = ((\log M)/2\mu)^{1/\alpha}$, we get

$$\mathrm{E}\left\|\widehat{h}_{r,M} - h_r\right\|_{L_2(p_r)} \leq A_1\frac{\log^{(T+2)/\alpha}(M)}{\sqrt{M}}, \quad r = 0,\ldots,T-1.$$

5 Dual upper bounds

In order to assess the quality of our estimates we need to construct upper bounds for the value process. To this aim we extend the approach in Rogers (2007) to problem (2.2). In fact, the following theorem is a generalization of Theorem 1 in Rogers (2007).

Theorem 5.1. *Let Y_r^* be the solution of the optimal control problem* (2.3)*, then the following representation holds*

$$Y_r^* = \inf_{h \in \mathcal{H}} \left\{ h_r(X_r) + \mathrm{E}^* \left[\sum_{j=r}^{T-1} W_{r,j} \left(\left(\mathcal{L}h\right)_j (X_j) - h_j(X_j) \right)^+ \right.\right.$$
$$\left.\left. + \max_{r \le i \le T} W_{r,i} \left(g_i(X_i) - h_i(X_i) \right)^+ \middle| \mathcal{F}_r \right] \right\},$$

where $W_{r,j} = \sup_{\mathbf{a} \in \mathcal{A}} \left[\Lambda_{r,j}(\mathbf{a}, X) \right]$ and $\mathcal{H}$ is the space of bounded measurable vector functions $h = (h_0, ..., h_T)$ on S^{T+1}.

6 Numerical example

Now we illustrate our algorithms by pricing a Bermudan basket call option in a model, where asset prices can be influenced by an investor holding large amounts of shares of the asset. In our model the large investor can increase the expected value of future asset prices, hence the future option pay-off, by borrowing assets (and return them later on).

Let X_r, $r = 0, \dots, T$ be a discrete time Markov process. Consider a Bermudan call option on a basket of d assets with the payoff

$$g(X_r) := \left(\frac{1}{d} \sum_{i=1}^{d} X_r^{(i)} - K \right)^+, \quad K > 0$$

which can be exercised at times $r = 1, \dots, T$. We assume that the large investor borrows $a_r \times 100\%$ $(0 \le a_r \le 1)$ of each asset at time r and pays to his lender the so called lending fee which is proportional to a_r:

$$\alpha a_r \sum_{k=1}^{d} X_r^{(k)}, \quad \alpha > 0. \tag{6.1}$$

Furthermore, the dynamic of X_{r+1} given X_r depends on a_r via

$$X_{r+1}^{(i)} = X_r^{(i)} \exp\left(-\frac{\sigma^2}{2} \delta_r + \sigma \sqrt{\delta_r} \zeta_{r,i} \right) \gamma(a_r), \quad X_0^{(i)} = x_0, \quad i = 1, ..., d,$$

where $\zeta_{r,i}$ are i.i.d. standard Gaussian random variables, $\gamma : [0,1] \to \mathbb{R}_+$ is some function, and δ_r is a time scaling parameter. The transition kernel of the process X is given by

$$\mathrm{P}^{a_r}(x, dy) \quad = \quad \frac{y_1^{-1} \cdot \ldots \cdot y_d^{-1}}{\sigma^d \sqrt{2\pi\delta_r^d}} \exp\left(-\frac{\sum_{j=1}^d (\ln \frac{y_j}{x_j} + \sigma^2\delta_r/2 - \ln\gamma(a_r))^2}{2\sigma^2\delta_r} \right) dy.$$

In our particular example we take $\gamma(a) = \exp(a/20)$ and choose as a reference measure the one corresponding to $a = 0$. Hence

$$\mathrm{P}^a(x, dy) = \varphi(x, y; a)\, \mathrm{P}^*(x, dy)$$

with

$$\varphi(x, y; a) = \exp\left(\frac{\sum_{j=1}^d \ln(y_j/x_j) + d\sigma^2\delta_r/2}{\sigma^2\delta_r} \ln\gamma(a) - \frac{d\ln^2\gamma(a)}{2\sigma^2\delta_r} \right).$$

The value of the controlled Bermudan option contract in this situation is given by (2.2) with $g_r \equiv g$ and $f_r(x, a) = -\alpha a \left(\sum_{k=1}^d x_k \right)$.

We now study a numerical example with $d = 5$, $T = 3$, $\delta_r \equiv 1$, $x_0 = 100$, $K = 90$, $\sigma = 0.2$ where we shall construct lower bounds for the option price using local regression and global regression methods. First, using the k-nearest neighbor estimator (3.4) and the corresponding estimator (3.5), based on M paths of the process X, we construct a suboptimal stopping time and a suboptimal control. Then averaging over a new independent set of 50000 trajectories, we get a lower bound denoted by $Y_{0,M}^{knn,low}$. This lower bound is shown in Table 1 for different M and different numbers of nearest neighbors used to construct (3.4). Similarly, a suboptimal stopping time (3.15) and a suboptimal control (3.16) lead to a lower bound denoted by $Y_{0,M}^{gr,low}$. In Table 2 the values of $Y_{0,M}^{gr,low}$ are presented in dependence on the set of basis functions used for the least squares approximation.

Furthermore, we construct upper bounds $Y_{0,M}^{knn,up}$ and $Y_{0,M}^{gr,up}$ for the option price based on the dual representation in Theorem 5.1, using approximative value functions (3.6) and (3.12), respectively. To get these upper bounds we simulate 50 ("outer") trajectories where on each trajectory the conditional expectations in $(\mathcal{L}h)_r$ are estimated using 10000 independent ("inner") trajectories.

Note that it can be advantageous to take the number of nearest neighbors k_M in (3.4) depending on x. To illustrate this we plot in Figure 1 the root-mean-square errors of the estimates $\widehat{C}_{2,10000}^{knn}(x, 1)$ and $\widehat{C}_{2,50000}^{knn}(x, 1)$, relative to the "exact" values $C_2(x, 1)$, computed using 10^6 Monte Carlo trajectories, for different numbers of nearest neighbors and for two points $x^{(0)}$ and $x^{(1)}$ with

$$x_k^{(i)} = x_0 \exp(-\frac{\sigma^2}{2}(\delta_0 + \delta_1) + \zeta_i(\sigma\sqrt{\delta_0} + \sigma\sqrt{\delta_1})), \quad k = 1, \ldots, d, \quad i = 0, 1,$$

where $\zeta_0 \equiv 0$ (left figure) and $\zeta_1 \equiv 1.5$ (right figure). Here the best value of k_M for the "centralpoint $x^{(0)}$ is about $0.1 \times M$ and the RMS error does not exceed 5% for

$M = 10000$. However, the error becomes rather large if x lies in the region with a small concentration of the pre-simulated regression points (the optimal k_M is about 10 in the right-hand side figure). Thus, the performance of the k-nearest neighbor estimator can be improved by choosing k_M adaptively depending on x.

As can be seen from our simulation study, global regression estimators provide a smaller gap between lower and upper bounds for the option price than their local regression counterparts. The gap between lower and upper bounds in the case of global regression for the best choice of basis functions does not exceed 4% (relative to the lower estimate), while for the local regression estimator the smallest gap is larger than 15%.

Table 1: Lower and upper bounds obtained via the k-nearest neighbor estimator (3.4) for different numbers of the nearest neighbors.

k	$\widehat{h}_{0,10000}^{knn,low}$(SD)	$\widehat{h}_{0,10000}^{knn,up}$(SD)	$\widehat{h}_{0,50000}^{knn,low}$(SD)	$\widehat{h}_{0,50000}^{knn,up}(SD)$
10	13.94(0.06)	20.94(0.23)	13.82(0.06)	21.22(0.27)
20	14.10(0.06)	18.89(0.20)	14.20(0.06)	18.41(0.16)
50	14.08(0.06)	16.74(0.09)	14.33(0.06)	17.08(0.14)
100	14.13(0.05)	16.59(0.14)	14.19(0.05)	16.68(0.13
500	14.17(0.05)	16.73(0.14)	14.17(0.05)	16.48(0.13)
1000	13.56(0.05)	17.04(0.13)	14.06(0.05)	16.27(0.11)

Table 2: Lower and upper bounds using global regression algorithms with different sets of basis functions.

base functions	$\widehat{h}_{0,200000}^{gr,low}$(SD)	$\widehat{h}_{0,200000}^{gr,up}$(SD)
up to 2nd degree polynomials on $g_r(X_r)$	15.15(0.06)	15.75(0.10)
up to 3th degree polynomials on $g_r(X_r)$	15.10(0.07)	15.62(0.07)
up to 4th degree polynomials on $g_r(X_r)$	15.13(0.07)	15.70(0.09)
1, $X_r^{(1)}, \ldots, X_r^{(5)}$, $g_r(X_r)$	15.01(0.07)	15.76(0.08)
up to 2nd degree polynomials on $X_r^{(1)}, \ldots, X_r^{(5)}$, $g_r(X_r)$	15.09(0.06)	15.55(0.07)

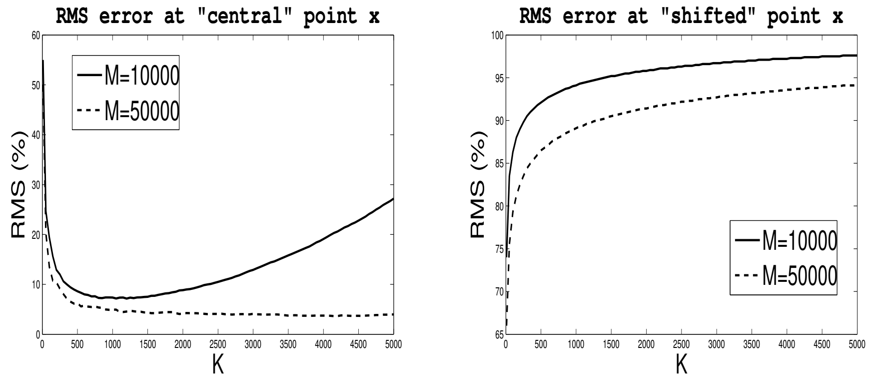

Figure 1: Root-mean-square errors (in %) of the estimators $\widehat{C}^{knn}_{2,10000}(x,0)$ (solid line) and $\widehat{C}^{knn}_{2,50000}(x,0)$ (dashed line) for different numbers k of nearest neighbors at two points $x_0\exp(-\sigma^2)$ (left) and $x_0\exp(-\sigma^2+1.5(\sigma\sqrt{\delta_0}+\sigma\sqrt{\delta_1}))$ (right).

7 Appendix

7.1 Proof of Theorem 3.4

The statement of the theorem holds trivially true for $r=T$. For $r<T$ we have

$$
\begin{aligned}
1_{\{\tau^*(r)>r\}}Y_r^* &= 1_{\{\tau^*(r)>r\}} \sup_{\mathbf{a}\in\mathcal{A}_r,\,\tau\in\mathfrak{T}_r} \mathrm{E}^{\mathbf{a}}\left[\sum_{j=r}^{\tau-1} f_j(X_j,a_j)+g_\tau(X_\tau)\,\middle|\,\mathcal{F}_r\right] \\
&= 1_{\{\tau^*(r)>r\}} \sup_{\tau\in\mathfrak{T}_{r+1}} \mathrm{E}_r^{\mathbf{a}^*(r)}\left[\sum_{j=r}^{\tau-1} f_j(X_j,a_j)+g_\tau(X_\tau)\,\middle|\,\mathcal{F}_r\right] \\
&= 1_{\{\tau^*(r)>r\}} f_r(X_r,a_r^*(r))+ \\
&\quad + 1_{\{\tau^*(r)>r\}} \sup_{\tau\in\mathfrak{T}_{r+1}} \mathrm{E}_r^{a^*(r)}\,\mathrm{E}_{r+1}^{(a^*_{r+1}(r),\ldots)}\left[\sum_{j=r+1}^{\tau-1} f_j(X_j,a_j^*(r))+g_\tau(X_\tau)\,\middle|\,\mathcal{F}_{r+1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq 1_{\{\tau^*(r)>r\}} f_r(X_r,a_r^*(r))+ \\
&1_{\{\tau^*(r)>r\}}\,\mathrm{E}_r^{a^*(r)} \sup_{\mathbf{a}\in\mathcal{A}_{r+1},\,\tau\in\mathfrak{T}_{r+1}} \mathrm{E}^{(a_{r+1}(r),\ldots)}\left[\sum_{j=r+1}^{\tau-1} f_j(X_j,a_j)+g_\tau(X_\tau)\,\middle|\,\mathcal{F}_{r+1}\right] \\
&= 1_{\{\tau^*(r)>r\}} f_r(X_r,a_r^*(r)) + 1_{\{\tau^*(r)>r\}}\,\mathrm{E}^{(a^*(r),a^*_{r+1}(r+1),\ldots)}\times \\
&\times\left[\sum_{j=r+1}^{\tau^*(r+1)-1} f_j(X_j,a_j^*(r+1))+g_{\tau^*(r+1)}(X_{\tau^*(r+1)})\,\middle|\,\mathcal{F}_r\right] \\
&= 1_{\{\tau^*(r)>r\}} f_r(X_r,a_r^*(r)) + 1_{\{\tau^*(r)>r\}}\,\mathrm{E}_r^{a^*(r)}\,Y_{r+1}^* = 1_{\{\tau^*(r)>r\}}Y_r^*,
\end{aligned}
$$

due to the Bellman principle. Hence

$$
\begin{aligned}
1_{\{\tau^*(r)>r\}} Y_r^* &= 1_{\{\tau^*(r)>r\}} f_r(X_r, a_r^*(r)) + 1_{\{\tau^*(r)>r\}} \mathrm{E}_r^{(a^*(r), a_{r+1}^*(r+1), \ldots)} \times \\
&\quad \times \left[\sum_{j=r+1}^{\tau^*(r+1)-1} f_j(X_j, a_j^*(r+1)) + g_{\tau^*(r+1)}(X_{\tau^*(r+1)}) \middle| \mathcal{F}_r \right]
\end{aligned}
$$

from which the consistency relations follow.

7.2 Proof of Theorem 4.1

For $r = T$ the statement is trivial. As induction hypothesis we assume that

$$
\mathrm{E} \left\| \widehat{h}_{r+1,M} - h_{r+1} \right\|_{L_q(p_{r+1})} = O\left(\lambda_{q,M}^{T-r-1} \varepsilon_M \right), \quad M \to \infty. \tag{7.1}
$$

Based on a new sample $(X_r^{(m)}, X_{r+1}^{(m)})$, $m = 1, \ldots, M$, independent of the samples needed for constructing the estimate $\widehat{h}_{r+1,M}$, we define

$$
\begin{aligned}
a_{r,M}(x) &:= \arg\sup_{a \in A} [f_r(x,a) + C_{r,M}(x,a)], \\
\widehat{a}_{r,M}(x) &:= \arg\sup_{a \in A} [f_r(x,a) + \widehat{C}_{r,M}(x,a)],
\end{aligned}
$$

where

$$
\widehat{C}_{r,M}(x,a) := \sum_{m=1}^{M} w_{m,M}(x, \mathbf{X}_r^M) \varphi(x, X_{r+1}^{(m)}, a) \widehat{h}_{r+1,M}(X_{r+1}^{(m)}).
$$

Observe that due to

$$
\begin{aligned}
&- \sup_{a \in A} \left| \widehat{C}_{r,M}(x,a) - C_{r,M}(x,a) \right| \\
&\quad \le f_r(x, \widehat{a}_{r,M}(x)) + \widehat{C}_{r,M}(x, \widehat{a}_{r,M}(x)) - \{ f_r(x, a_{r,M}(x)) + C_{r,M}(x, a_{r,M}(x)) \} \\
&\quad\quad \le \sup_{a \in A} \left| \widehat{C}_{r,M}(x,a) - C_{r,M}(x,a) \right|
\end{aligned}
$$

the inequality

$$
\left| \widehat{h}_{r,M}(x) - h_{r,M}(x) \right| \le \sup_{a \in A} \left| \widehat{C}_{r,M}(x,a) - C_{r,M}(x,a) \right|
$$

holds for all x and a, where

$$
h_{r,M}(x) := \max\{ g_r(x), f_r(x, a_{r,M}(x)) + C_{r,M}(x, a_{r,M}(x)) \}.
$$

Analogously one can show that

$$
|h_r(x) - h_{r,M}(x)| \le \sup_{a \in A} |C_r(x,a) - C_{r,M}(x,a)|. \tag{7.2}
$$

On the other hand we have

$$\widehat{C}_{r,M}(x,a) - C_{r,M}(x,a) = \tag{7.3}$$
$$\sum_{m=1}^{M} w_{m,M}(x, \mathbf{X}_r^M)\varphi(x, X_{r+1}^{(m)}, a)(\widehat{h}_{r+1,M}(X_{r+1}^{(m)}) - h_{r+1}(X_{r+1}^{(m)})),$$

hence

$$\left|\widehat{h}_{r,M}(x) - h_{r,M}(x)\right|$$
$$\leq A_\varphi \sum_{m=1}^{M} |w_{m,M}(x, \mathbf{X}_r^M)| \left|\widehat{h}_{r+1,M}(X_{r+1}^{(m)}) - h_{r+1}(X_{r+1}^{(m)})\right|, \quad x \in \mathbb{R}^d.$$

Denote with $\mathcal{G}_{r+1}$ the σ-algebra generated by the samples used from T down to $r+1$. The application of Hölder's and Jensen inequality leads to

$$\begin{aligned}
& \mathrm{E}\left\|\widehat{h}_{r,M} - h_{r,M}\right\|_{L_q(p_r)} \\
\leq\ & A_\varphi \,\mathrm{E} \sum_{m=1}^{M} \mathrm{E}^{\mathcal{G}_{r+1}} \left[\left|\widehat{h}_{r+1,M}(X_{r+1}^{(m)}) - h_{r+1}(X_{r+1}^{(m)})\right| \left\|w_{m,M}(\cdot, X_r^{(\cdot)})\right\|_{L_q(p_r)}\right] \\
\leq\ & A_\varphi \,\mathrm{E}\left\{\left[\mathrm{E}^{\mathcal{G}_{r+1}}\left|\widehat{h}_{r+1,M}(X_{r+1}^{(1)}) - h_{r+1}(X_{r+1}^{(1)})\right|^q\right]^{1/q}\right. \\
& \left.\times \sum_{m=1}^{M}\left[\mathrm{E}^{\mathcal{G}_{r+1}}\left\|w_{m,M}(\cdot, \mathbf{X}_r^M)\right\|_{L_q(p_r)}^{\frac{q}{q-1}}\right]^{1-1/q}\right\} \\
\leq\ & A_\varphi \,\mathrm{E}\left\|\widehat{h}_{r+1,M} - h_{r+1}\right\|_{L_q(p_{r+1})} \sum_{m=1}^{M}\left(\int p_r(x)\,\mathrm{E}\left|w_{m,M}(x, \mathbf{X}_r^M)\right|^q dx\right)^{\frac{1}{q}} \\
=\ & A_\varphi \,\mathrm{E}\left\|\widehat{h}_{r+1,M} - h_{r+1}\right\|_{L_q(p_{r+1})} \sum_{m=1}^{M} \|w_{m,M}(\cdot,\cdot)\|_{L_q(p_r \otimes_{l=1}^M p_r)}.
\end{aligned}$$

The induction assumption (7.1) implies now that

$$\mathrm{E}\left\|\widehat{h}_{r,M} - h_{r,M}\right\|_{L_q(p_r)} = O(\varepsilon_M \lambda_{q,M}^{T-r}).$$

Note that by letting $q \downarrow 1$, the last estimate holds true for $q = 1$ as well. Further we have

$$\mathrm{E}\left\|\widehat{h}_{r,M} - h_r\right\|_{L_q(p_r)} \leq \mathrm{E}\left\|\widehat{h}_{r,M} - h_{r,M}\right\|_{L_q(p_r)} + \mathrm{E}\left\|h_{r,M} - h_r\right\|_{L_q(p_r)}$$

and due to (7.2)

$$\mathrm{E}\left\|h_{r,M} - h_r\right\|_{L_q(p_r)}$$
$$\leq \left\{\int_{\mathbb{R}^d} \|C_r(x,\cdot) - C_{r,M}(x,\cdot)\|_A^q \, p_r(x)\, dx\right\}^{1/q} = O(\varepsilon_M), \quad M \to \infty.$$

7.3 Proof of Theorem 4.3

For any $x \in \mathcal{D}$ we have on the set $\{|p_{r,M}(x)| > p_{\min}/2\}$

$$C^{\mathcal{D}}_{r,M}(x,a) - C_r(x,a) = \frac{d_{r,M}}{p_{r,M}} - \frac{d_r}{p_r} = \frac{d_{r,M} - d_r}{p_{r,M}} + C_r \frac{p_r - p_{r,M}}{p_{r,M}}$$

and so

$$\left\| C^{\mathcal{D}}_{r,M} - C_r \right\|_{\mathcal{D}\times A} \leq 2p_{\min}^{-1} \left\| d_{r,M} - d_r \right\|_{\mathcal{D}\times A} + 2C_{\max} p_{\min}^{-1} \left\| p_r - p_{r,M} \right\|_{\mathcal{D}}.$$

Hence

$$\begin{aligned} \mathrm{E} \left\| C^{\mathcal{D}}_{r,M} - C_r \right\|_{\mathcal{D}\times A} &\leq 2p_{\min}^{-1} \,\mathrm{E} \left\| d_{r,M} - d_r \right\|_{\mathcal{D}\times A} + 2C_{\max} p_{\min}^{-1} \,\mathrm{E} \left\| p_r - p_{r,M} \right\|_{\mathcal{D}} \\ &\quad + C_{\max} \,\mathrm{P} \left(\left\| p_r - p_{r,M} \right\|_{\mathcal{D}} > p_{\min}/2 \right). \end{aligned} \tag{7.4}$$

Since

$$p_{r,M}(x) - \mathrm{E}\, p_{r,M}(x) = \frac{1}{M} \sum_{m=1}^{M} \left(\Phi_M(x, X_r^{(m)}) - \mathrm{E}\, \Phi_M(x, X_r^{(m)}) \right)$$

we immediately get from Theorem 7.1 taking into account conditions (i), (ii), and (iii) in Theorem 4.3,

$$\begin{aligned} \mathrm{E} \left\| p_{r,M} - \mathrm{E}\, p_{r,M} \right\|_{\mathbb{R}^d} &\leq \frac{B}{M} \left[\nu U_M \log \frac{A U_M}{\sigma_{r,M}} + \sqrt{\nu} \sqrt{M \sigma_{r,M}^2 \log \frac{A U_M}{\sigma_{r,M}}} \right] \\ &\leq B_1 \sqrt{\nu} \gamma_M, \quad M \to \infty, \end{aligned}$$

with some universal positive constants constants B and B_1. Similarly,

$$\begin{aligned} &\mathrm{P} \left(\left\| p_{r,M} - \mathrm{E}\, p_{r,M} \right\|_{\mathbb{R}^d} > C_1 \gamma_M \right) \leq \\ &\qquad L \exp \left(- \frac{C_1 \log(1 + C_1/(4L))}{L} \log \frac{U_M}{\sigma_{r,M}} \right), \quad M \to \infty, \end{aligned}$$

for any $C_1 \geq C$, where positive constants C and L only depend on the VC-characteristics A and ν. Due to condition (iv) there exists $W > 0$ such that

$$\left(\frac{\sigma_{r,M}}{U_M} \right)^W \leq \gamma_M.$$

Then for any fixed $C_1 \geq C$ such that

$$\frac{C_1 \log(1 + C_1/(4L))}{L} \geq W$$

we have

$$\begin{aligned} &\mathrm{P} \left(\left\| p_{r,M} - \mathrm{E}\, p_{r,M} \right\|_{\mathbb{R}^d} > C_1 \gamma_M \right) \\ &\qquad \leq L \left(\frac{\sigma_{r,M}}{U_M} \right)^{\frac{C_1 \log(1 + C_1/(4L))}{L}} \leq L \left(\frac{\sigma_{r,M}}{U_M} \right)^W \leq L \gamma_M, \quad M \to \infty. \end{aligned}$$

Due to (iii) we can now find M_0 such that for all $M > M_0$ it holds $C_1\gamma_M \le p_{\min}/4$. Hence

$$\mathrm{P}\left(\|p_r - \mathrm{E}\, p_{r,M}\|_{\mathbb{R}^d} > p_{\min}/4\right) \le L\gamma_M\,, \quad M \to \infty.$$

Since

$$\begin{aligned}\mathrm{P}\left(\|p_r - p_{r,M}\|_{\mathcal{D}} > p_{\min}/2\right) &\le \mathrm{P}\left(\|p_r - \mathrm{E}\, p_{r,M}\|_{\mathbb{R}^d} > p_{\min}/4\right) \\ &\quad + \mathrm{P}\left(\|\mathrm{E}\, p_r - p_{r,M}\|_{\mathbb{R}^d} > p_{\min}/4\right)\end{aligned}$$

and $\|p_r - \mathrm{E}\, p_{r,M}\|_{\mathcal{D}}$ goes to zero for $M \to \infty$, we end up with

$$\mathrm{P}\left(\|p_r - p_{r,M}\|_{\mathcal{D}} > p_{\min}/2\right) \le L\gamma_M, \quad M \to \infty.$$

Similarly,

$$\begin{aligned}\mathrm{E}\,\|p_r - p_{r,M}\|_{\mathcal{D}} &\le \|p_r - \mathrm{E}\, p_{r,M}\|_{\mathbb{R}^d} + \mathrm{E}\,\|\mathrm{E}\, p_{r,M} - p_{r,M}\|_{\mathbb{R}^d} \\ &\le \|p_r - \mathrm{E}\, p_{r,M}\|_{\mathbb{R}^d} + L_1\gamma_M\end{aligned}$$

with $L_1 := B_1\sqrt{\nu}$ only depending on the VC characteristics. Next, by applying Theorem 7.1 to the representation

$$\begin{aligned}d_{r,M}(x) - \mathrm{E}\, d_{r,M}(x) = \frac{1}{M}\sum_{m=1}^{M}\Big(&\Phi_M(x, X_r^{(m)})\varphi(x, X_{r+1}^{(m)}, a)h_{r+1}(X_{r+1}^{(m)}) \\ &- \mathrm{E}\left[\Phi_M(x, X_r^{(m)})\varphi(x, X_{r+1}^{(m)}, a)h_{r+1}(X_{r+1}^{(m)})\right]\Big)\,,\end{aligned}$$

with $\widetilde{U}_M := A_\phi A_h U_M$ and $\widetilde{\sigma}_{r,M} := A_\phi A_h \sigma_{r,M}$, and observing that (i)-(iv) in Theorem 4.3 are trivially fulfilled for the sequences $\widetilde{U}_M$ and $\widetilde{\sigma}_{r,M}$, we obtain in an analogous way the estimate

$$\mathrm{E}\,\|d_r - d_{r,M}\|_{\mathbb{R}^d\times A} \le \mathrm{E}\,\|d_r - \mathrm{E}\, d_{r,M}\|_{\mathbb{R}^d\times A} + L_2\gamma_M$$

with some constant $L_2 > 0$ only depending on the VC characteristics. Taking all together, (7.4) yields

$$\begin{aligned}\mathrm{E}\,\|C_{r,M}^{\mathcal{D}} - C_r\|_{\mathcal{D}\times A} &\le \left(C_{\max}L + 2C_{\max}p_{\min}^{-1}L_1 + 2p_{\min}^{-1}L_2\right)\gamma_M \\ &\quad + 2C_{\max}p_{\min}^{-1}\,\|p_r - \mathrm{E}\, p_{r,M}\|_{\mathbb{R}^d} + 2p_{\min}^{-1}\,\|d_r - \mathrm{E}\, d_{r,M}\|_{\mathbb{R}^d}\,,\end{aligned}$$

from which the statement of the theorem follows with $L_0 := \frac{1}{2}L + L_1 + L_2$.

7.4 Proof of Theorem 4.5

We have

$$\begin{aligned}\lambda_{\min}^M &= \min_{\|w\|=1} w^\top \Gamma^{M,K} w \\ &\ge \min_{\|w\|=1} w^\top \Gamma^K w + \min_{\|w\|=1} w^\top(\Gamma^{M,K} - \Gamma^K)w \\ &\ge \lambda_{\min} - K\max_{1\le k,l\le K}|\Gamma_{l,k}^{M,K} - \Gamma_{l,k}^K|. \qquad (7.5)\end{aligned}$$

By the uniform boundedness of $\psi_k(x)$ on $\mathbb{R}^d$ it follows that

$$\operatorname{Var}\left[\psi_l\left(X_r\right)\psi_k\left(X_r\right)\right] \leq \mathrm{E}\left[\psi_l^2\left(X_r\right)\psi_k^2\left(X_r\right)\right] \leq A_\psi^4,$$

and so we get by Bernstein's inequality

$$\mathrm{P}(|\Gamma_{l,k}^M - \Gamma_{l,k}| > \delta) \leq 2\exp\left[-\frac{M\delta^2}{2A_\psi^2\left(A_\psi^2 + 2\delta/3\right)}\right]. \tag{7.6}$$

Combining (7.5) and (7.6), we get

$$\begin{aligned}\mathrm{P}(\lambda_{\min}^{M,K} < \lambda_{\min}/2) &\leq \mathrm{P}\left(\max_{1\leq k,l\leq K}\left|\Gamma_{l,k}^{M,K} - \Gamma_{l,k}^K\right| > \lambda_{\min}/2K\right)\\ &\leq 2K^2\exp\left[-\frac{M\lambda_{\min}^2}{8K^2A_\psi^2\left(A_\psi^2+2\delta/3\right)}\right] \leq 2K^2\exp\left[-B_0M/K^2\right]\end{aligned} \tag{7.7}$$

with some constant $B_0 > 0$ independent of K and M. We further have

$$\mathbf{1}_{\{\lambda_{\min}^M \geq \lambda_{\min}/2\}}\left|C_{r,M}^{\mathfrak{T}}(x,a) - C_r(x,a)\right| \leq \mathcal{E}_{r,M}^{(1)} + \mathcal{E}_{r,M}^{(2)} + |\Delta_r(x,a)| \tag{7.8}$$

with

$$\begin{aligned}\mathcal{E}_{r,M}^{(1)} &= \sup_{(x,a)\in A}\left|\frac{1}{M}\sum_{m=1}^M \boldsymbol{\psi}_K^\top(x)\left[\left(\Gamma^{M,K}\right)^{-1} - \left(\Gamma^K\right)^{-1}\right]\boldsymbol{\psi}_K(X_r^{(m)})Y^{(m)}(a)\right|,\\ \mathcal{E}_{r,M}^{(2)} &= \sup_{(x,a)\in A}\left|\frac{1}{M}\sum_{m=1}^M\left(\boldsymbol{\psi}_K^\top(x)\left(\Gamma^K\right)^{-1}\boldsymbol{\psi}_K(X_r^{(m)})Y^{(m)}(a)\right.\right.\\ &\qquad \left.\left.-\,\mathrm{E}\left[\boldsymbol{\psi}_K^\top(x)\left(\Gamma^K\right)^{-1}\boldsymbol{\psi}_K(X_r^{(m)})Y^{(m)}(a)\right]\right)\right|.\end{aligned}$$

The matrix identity $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$ and the multiplicativity of the spectral matrix norm imply on the set $\{\lambda_{\min}^{M,K} \geq \lambda_{\min}/2\}$,

$$\left\|\left(\Gamma^{M,K}\right)^{-1} - \left(\Gamma^K\right)^{-1}\right\|_2 \leq \frac{2}{\lambda_{\min}^2}\left\|\Gamma^K - \Gamma^{M,K}\right\|_2.$$

Hence, it holds on the set $\{\lambda_{\min}^{M,K} \geq \lambda_{\min}/2\}$,

$$\begin{aligned}&\sup_{(x,a)\in A}\left|\boldsymbol{\psi}_K^\top(x)\left[\left(\Gamma^{M,K}\right)^{-1} - \left(\Gamma^K\right)^{-1}\right]\boldsymbol{\psi}_K(X_r^{(m)})Y^{(m)}(a)\right|\\ &\qquad\leq \frac{2A_\varphi A_h}{\lambda_{\min}^2}\left\|\boldsymbol{\psi}_K(x)\right\|_2\left\|\boldsymbol{\psi}_K(X_r^{(m)})\right\|_2\left\|\Gamma^K - \Gamma^{M,K}\right\|_2\\ &\qquad\qquad\leq K^2A_\varphi A_h A_\psi^2\frac{2}{\lambda_{\min}^2}\left\|\Gamma^K - \Gamma^{M,K}\right\|_{\max},\end{aligned}$$

where $\|\cdot\|_{\max}$ denotes the elements-wise maximum. Due to our assumptions it follows from Theorem 7.1 that

$$\mathrm{E}\,\mathcal{E}_{r,M}^{(1)} \leq B_2\frac{K^2}{\sqrt{M}}, \tag{7.9}$$

where the constant B_2 does not depend on K and M. Since

$$\begin{aligned}
\mathcal{E}^{(2)}_{r,M} &\le \sqrt{K}\frac{A_\psi}{\lambda_{\min}}\sup_{a\in A}\left\|\frac{1}{M}\sum_{m=1}^{M}\left(\boldsymbol{\psi}_K(X_r^{(m)})Y^{(m)}(a)-\mathrm{E}\,\boldsymbol{\psi}_K(X_r^{(m)})Y^{(m)}(a)\right)\right\|_2 \\
&\le \frac{KA_\psi}{\lambda_{\min}}\sup_{a\in A}\max_{1\le k\le K}\left|\frac{1}{M}\sum_{m=1}^{M}\left(\psi_k(X_r^{(m)})Y^{(m)}(a)-\mathrm{E}\,\psi_k(X_r^{(m)})Y^{(m)}(a)\right)\right|,
\end{aligned}$$

our assumptions and Theorem 7.1 lead to the following bound

$$\mathrm{E}\,\mathcal{E}^{(2)}_{r,M} \le B_3\frac{K}{\sqrt{M}}, \tag{7.10}$$

where constant B_3 does not depend on K and M. Combining (7.8) with (7.9) and (7.10), we arrive at (4.11).

7.5 Proof of Theorem 5.1

For any $h=(h_0,\ldots,h_T)\in\mathcal{H}$ and $\mathbf{a}\in\mathcal{A}$ let consider a martingale M_r from the Doob decomposition of $h_r(X_r)$:

$$M^{\mathbf{a}}_{r+1}-M^{\mathbf{a}}_r = h_{r+1}(X_{r+1})-\mathrm{E}^{\mathbf{a}}\left[h_{r+1}(X_r)|\mathcal{F}_r\right],$$

with $M^{\mathbf{a}}_0=0$, i.e.,

$$M^{\mathbf{a}}_r=\sum_{j=0}^{r-1}\left(M^{\mathbf{a}}_{j+1}-M^{\mathbf{a}}_j\right)=\sum_{j=0}^{r-1}\left(h_{j+1}(X_j)-P^{a_j}h_{j+1}(X_j)\right).$$

We then have

$$\begin{aligned}
Y_r^* &= \inf_h \sup_{\substack{\mathbf{a}\in\mathcal{A}_r\\ \tau,\,\tau\ge r}} \mathrm{E}^{\mathbf{a}}\left[\sum_{j=r}^{\tau-1} f_j(X_j,a_j)+g_\tau(X_\tau)-\sum_{j=r}^{\tau-1}\left(h_{j+1}(X_j)-P^{a_j}h_{j+1}(X_j)\right)\middle|\,\mathcal{F}_r\right] \\
&\le \inf_h\left\{h_r(X_r)+\sup_{\mathbf{a}\in\mathcal{A}_r}\mathrm{E}^*\left[\sum_{j=r}^{i-1}\Lambda_{r,j}(\mathbf{a},X)\left(f_j(X_j,a_j)+P^{a_j}h_{j+1}(X_j)-h_j(X_j)\right)\right.\right. \\
&\quad \left.\left.+\Lambda_{r,i}(\mathbf{a},X)\left(g_i(X_i)-h_i(X_i)\right)\right|\mathcal{F}_r\right]\Bigg\} \\
&\le \inf_h\left\{h_r(X_r)+\mathrm{E}^*\left[\sum_{j=r}^{T-1}\sup_{\mathbf{a}\in\mathcal{A}_r}\Lambda_{r,j}(\mathbf{a},X)\left(\left(\mathcal{L}h\right)_j(X_j)-h_j(X_j)\right)^+\right.\right. \\
&\quad \left.\left.+\max_{i\ge r}\sup_{\mathbf{a}\in\mathcal{A}_r}\Lambda_{r,i}(\mathbf{a},X)\left(g_i(X_i)-h_i(X_i)\right)^+\middle|\,\mathcal{F}_r\right]\right\}.
\end{aligned}$$

For $h=h^*$ it holds $\max\left[g_i,\left(\mathcal{L}h^*\right)_i\right]=h_i^*$, and $h_T^*(x)=g_T(x)$, so we finally have identity.

7.6 Some results from the theory of empirical processes

For the readers convenience we here recall some definitions and corner stone results from the theory of empirical processes.

Definition A class $\mathcal{F}$ of measurable functions on a measurable space $(S, \mathcal{S})$ is called a Vapnik-Červonenkis class if there exist positive numbers A and ν such that, for any probability measure P on $(S, \mathcal{S})$ and any $0 < \rho < 1$,

$$\mathcal{N}(\mathcal{F}, L_2(\mathrm{P}), \rho \|F\|_{L_2(\mathrm{P})}) \leq \left(\frac{A}{\rho}\right)^{\nu}, \tag{7.11}$$

where $\mathcal{N}(\mathcal{F}, d, \varepsilon)$ denotes the ε-covering number of $\mathcal{F}$ in a metric d, that is the minimal number of spheres with radius ε needed to cover $\mathcal{F}$, and $F := \sup_{f\in\mathcal{F}} |f|$ is the envelope of $\mathcal{F}$ (with here and below sup denoting esssup with respect to P). The following proposition is a key tool for obtaining convergence rates for the local and global type estimators considered in this paper.

Theorem 7.1 (Talagrand (1994), Giné and Guillou (2002))**.** *Let $\mathcal{F}$ be a measurable uniformly bounded VC class of functions. Let* P *be any measure on $(S, \mathcal{S})$, and let $(X_m)_{m=1,2,\ldots}$ be an i.i.d. sequence of S-valued random variables with distribution* P*. Let σ and U be any numbers such that*

$$\sup_{f\in\mathcal{F}} \mathrm{Var}_{\mathrm{P}}(f) \leq \sigma^2, \quad \sup_{f\in\mathcal{F}} \|f\|_\infty \leq U$$

and $0 < \sigma \leq U$. Then, there exist a universal constant B such that

$$\mathrm{E}\left[\sup_{f\in\mathcal{F}} \left|\sum_{m=1}^{M} (f(X_m) - \mathrm{E}\, f(X_1))\right|\right] \leq B\left[\nu U \log\frac{AU}{\sigma} + \sqrt{\nu}\sqrt{M\sigma^2 \log\frac{AU}{\sigma}}\right].$$

If moreover $0 < \sigma < U/2$ and $\sqrt{M}\sigma \geq U\sqrt{\log(U/\sigma)}$, there exist constants L and C which only depend on the VC characteristics of $\mathcal{F}$, such that for all $\lambda \geq C$ and t satisfying

$$C\sqrt{M}\sigma\sqrt{\log\frac{U}{\sigma}} \leq t \leq \lambda\frac{M\sigma^2}{U},$$

it holds

$$\mathrm{P}\left(\sup_{f\in\mathcal{F}} \left|\sum_{m=1}^{M} (f(X_m) - \mathrm{E}\, f(X_1))\right| > t\right) \leq L \exp\left(-\frac{\log(1 + \lambda/(4L))}{\lambda L} \frac{t^2}{M\sigma^2}\right).$$

Thus, in particular, for any $C_1 \geq C$ we may take

$$t = C_1\sqrt{M}\sigma\sqrt{\log\frac{U}{\sigma}}, \quad \lambda = C_1,$$

which yields

$$\mathrm{P}\left(\sup_{f\in\mathcal{F}}\left|\sum_{m=1}^{M}(f(X_m)-\mathrm{E}\,f(X_1))\right|>C_1\sqrt{M}\sigma\sqrt{\log\frac{U}{\sigma}}\right) \tag{7.12}$$
$$\leq L\exp\left(-\frac{C_1\log(1+C_1/(4L))}{L}\log\frac{U}{\sigma}\right).$$

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