Control problems with state constraints for the Penrose–Fife phase–field model

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Control Problems with State Constraints for the Penrose-Fife Phase-field Model

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Abstract

This article gives an optimality system for a control problem with state constraints for a Penrose-Fife model for phase transitions.

1 Introduction

In this article, we consider optimal control problems governed by the following system of quasi-linear parabolic equations,

$$\phi_t \;=\; K_1\Delta\phi - s_0'(\phi) - \frac{\lambda(\phi)}{T}\,, \tag{1}$$

$$T_t \;=\; -M_1\Delta\left(\frac{1}{T}\right) - \lambda(\phi)\phi_t + v\,, \tag{2}$$

in $Q = \Omega\times(0,t^*)$, where $\Omega\subset\mathbb{R}^3$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. s_0 denotes a double well potential. We let $\partial Q = \partial\Omega\times(0,t^*)$, and we impose the boundary conditions

$$\frac{\partial T}{\partial n} = -\alpha\left(T - w\right)\,, \qquad \text{on} \qquad \partial Q \tag{3}$$

$$\frac{\partial \phi}{\partial n} = 0\,, \qquad \text{on} \qquad \partial Q\,, \tag{4}$$

as well as the initial conditions

$$\phi(x,0) = \phi_0(x), \quad T(x,0) = T_0(x), \quad \forall\, x\in\overline{\Omega}\,. \tag{5}$$

These equations arise in a model for phase transitions introduced by Penrose and Fife [10]. In this setting, T denotes the absolute temperature, and ϕ is a non-conserved order-parameter.

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Several papers have appeared in connection with the existence and uniqueness of solutions to this system, as well as other analytical aspects of this system and related systems. We refer the reader to [7, 13, 4, 6, 8, 9] for some specific treatments. A more general discussion of systems of this type can be found in [2].
We will make similar assumptions in this article as in [7, 13], namely, for the potential s_0 we will assume that either

- **(A)** $s_0 \in C^3(\mathbb{R})$ and there exists a constant $C > 0$ such that $s_0''(\phi) > -C$ for all $\phi \in \mathbb{R}$. or
- **(B)** $s_0 = \phi \log \phi + (1-\phi)\log(1-\phi)$.

Furthermore, we will make the following simplifying assumptions:

- $\lambda(\phi) = a\phi + b$, for a positive constant a. To simplify notations, we will, without loss of generality, use $a = 1$ and $b = 0$, i.e. we use $\lambda(\phi) = \phi$.
- In the boundary conditions, we let $\alpha = 1$.

To state an existence result, we have to make some regularity assumptions and compatibility conditions. In particular, we assume that

(H1) $\phi_0 \in H^4(\Omega); \frac{\partial \phi}{\partial n}(x) = 0, \forall x \in \partial\Omega; \frac{\partial}{\partial n}\left(-s_0'(\phi_0) + \frac{\phi_0}{T_0} + \Delta\phi_0\right)(x) = 0, \forall x \in \partial\Omega.$

(H2) $T_0 \in H^3(\Omega); \tilde{T}(x) = \frac{\partial T_0}{\partial n}(x) + T_0(x) > 0, \forall x \in \partial\Omega; T_0(x) > 0, \forall x \in \overline{\Omega}.$

Finally, we introduce some Banach spaces which will be widely used throughout this article.

$$\begin{aligned}
X_1 &= C([0,t^*]; H^4(\Omega)) \cap C^1([0,t^*]; H^2(\Omega)) \cap C^2([0,t^*]; L^2(\Omega)), \\
X_2 &= C([0,t^*]; H^3(\Omega)) \cap C^1([0,t^*]; H^1(\Omega)) \cap H^{4,2}(Q), \\
V &= H^2(0,t^*; L^2(\Omega)) \cap H^1(0,t^*; H^2(\Omega)), \\
W &= H^2(0,t^*; H^{\frac{3}{2}}(\partial\Omega)).
\end{aligned}$$

Using these conditions, one can prove the following existence result (cf. [7, 13]).

Proposition 1 *Suppose that* **(H1)** *and* **(H2)** *are satisfied. Then there exists a unique global smooth solution* $(\phi, T) \in X_1 \times X_2$ *to the initial-boundary value problem* (1)–(5). *Furthermore, there exists a constant* $c_{t^*} > 0$ *such that* $T(x,t) \geq c_{t^*}$ *for all* $(x,t) \in \overline{Q}$, *and in the case* **(B)** *there exist constants* $0 < a_{t^*} < b_{t^*} < 1$, *such that* $a_{t^*} \leq \phi(x,t) \leq b_{t^*}$ *for all* $(x,t) \in \overline{Q}$.

In Section 2 of this article, we will state the optimal control problem with state constrains and discuss it. In Section 3, we will investigate the related observation operator and prove its differentiability in the setting of Section 2. Finally, we will derive the necessary conditions for optimality in Section 4 of this paper.

2 The Optimal Control Problem

The state equations (1)–(2) give rise to several interesting optimal control problems. In this article, we want to control the state (ϕ, T) by using the source term v in (2) and the boundary term w in (4) as controls. However, we want to put local constraints on the state, as well.

In order to formulate this problem in a precise manner, we need to introduce some additional notation. We start by defining the cost functional

$$\begin{aligned} I(\phi, T; v, w) \;=\; & \frac{\alpha_1}{2} \left\| \phi(t^*) - \hat{\phi}(t^*) \right\|^2_{L^2(\Omega)} + \frac{\alpha_2}{2} \left\| T - \hat{T} \right\|^2_{L^2(Q)} \\ & + \frac{\alpha_3}{2} \|v\|^2_{L^2(Q)} + \frac{\alpha_4}{2} \int_0^{t^*} \|w(t)\|^2_{L^2(\partial\Omega)} \, dt, \end{aligned} \tag{6}$$

for given target functions $\hat{\phi} \in X_1$ and $\hat{T} \in X_2$. Next, let

$$\begin{aligned} \tilde{W} \;=\; \Big\{ w \in W : \quad & w(x,0) = \tilde{T}(x), \quad \forall x \in \partial\Omega; \\ & w(x,t) \geq \beta, \quad |w_t(x,t)| < k, \quad \forall (x,t) \in \partial Q \Big\}, \end{aligned}$$

where $\tilde{T}$ is the function introduced in **(H2)** and β and k are suitably chosen positive constants. We use this set to introduce

$$K = V \times \tilde{W} .$$

The set $\mathcal{U}_{\text{ad}}$ of admissible controls is a closed, convex and bounded subset of K.

To state the local state constraints, we use constants $0 < K_1 < K_2$ and $K_3 < K_4$ to define

$$\mathcal{Y}_{\text{ad}} = \{(\phi, T) \in X_1 \times X_2 : K_1 \leq T(x,t) \leq K_2 \wedge K_3 \leq \phi(x,t) \leq K_4, \forall (x,t) \in Q\}, \tag{7}$$

the set of admissible states. Note that this set has a nonempty interior.

We can now state the optimal control problem under consideration.

Optimal Control Problem (CP)

Minimize $I(\phi, T; v, w)$ under the following conditions:

1. (ϕ, T) satisfies the state equations (1)–(2) and the initial and boundary conditions (3)–(5).
2. $(v, w) \in \mathcal{U}_{\text{ad}}$.
3. $(\phi, T) \in \mathcal{Y}_{\text{ad}}$.

Remarks:

- Clearly the initial values (ϕ_0, T_0) must also satisfy the constraints $K_1 \leq T(x) \leq K_2$ and $K_3 \leq \phi_0(x) \leq K_4$ for all $x \in \Omega$.
- The authors of [14] considered a similar but weaker control problem. In particular, they did not impose local constraints on the state.

  Moreover, their treatment focused on the function $s_0'(\phi) = \phi - \phi^3$. However, this latter restriction can easily be removed, and their arguments extend to the cases **(A)** and **(B)** investigated here (see [5], for a sketch of this argument). We can therefore use the results of [14] whenever they are applicable.
- Note that state constraints have not been considered in [14], so that a larger space of observations with a coarser topology could be used.

In the study of the control problem **(CP)** it is useful to introduce the observation operator S. To this end, we define the space of observations B by

$$B = (C([0,t^*]; H^2(\Omega))) \times (C([0,t^*]; H^2(\Omega))). \tag{8}$$

Next, define

$$\begin{aligned} S &: K \to B \qquad (9)\\ S &: (v,w) \mapsto (\phi, T), \qquad (10) \end{aligned}$$

that is, S assigns to every pair $(v,w) \in K$ the pair (ϕ,T) which solves (1)–(5) for the given v and w. Since $X_1 \times X_2 \subset B$, and by virtue of Proposition 1, this operator S is well defined. Using this operator, one sees that the cost functional $I(\phi, T; v, w)$ depends only on the controls v and w, i.e. we can rewrite it as

$$J(v,w) = I(\phi, T; v, w)|_{(\phi,T)=S(v,w)}\,.$$

In the following section, we will study the properties of this operator S. In Section 4 these properties will be used to derive the necessary conditions of optimality.

3 Differentiability of the Observation Operator

We now turn our attention to the observation operator S defined in (9)–(10). This operator is well-defined, and – also due to Proposition 1 – there exist positive constants α and γ satisfying

$$\begin{aligned} \|\phi\|_{X_1} + \|T\|_{X_2} &\leq \alpha, \quad \forall (v,w) \in \mathcal{U}_{\mathrm{ad}}, \qquad (11)\\ T(x,t) \geq \gamma &> 0, \quad \forall (x,t) \in \overline{Q}. \qquad (12) \end{aligned}$$

Moreover, if $s_0(\phi)$ is of the form given in case **B**, there exist constants $0 < \hat{a}_{t^*} < \hat{b}_{t^*} < 1$ such that

$$\hat{a}_{t^*} \leq \phi(x,t) \leq \hat{b}_{t^*}, \quad \forall (x,t) \in \overline{Q}. \tag{13}$$

In order to prove differentiability of the observation operator S, one first has to improve the stability result of [14]. To this end, we let $(\phi_i, T_i) = S(v_i, w_i)$, $\quad i = 1, 2$, and $(v_i, w_i) \in \mathcal{U}_{\mathrm{ad}}$. We define $\overline{\phi} = \phi_1 - \phi_2$, $\overline{T} = T_1 - T_2$, $\overline{v} = v_1 - v_2$, and $\overline{w} = w_1 - w_2$. With these notations, we have the following result.

Proposition 2 *There exists a constant $C > 0$ such that*

$$\begin{aligned} \max_{0<t<t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{\phi}(t)\right\|_{H^3}^2 + \left\|\overline{T}\right\|_{H^2}^2 + \left\|\overline{T}_t(t)\right\|_{L^2}^2 \right) \;&+\; \int_0^{t^*} \left\|\overline{\phi}_{tt}(t)\right\|^2 dt \\ + \int_0^{t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{T}_t(t)\right\|_{H^1}^2 \right) dt \;&\leq\; C\,\overline{G}(\overline{v}, \overline{w}), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \overline{G}(\overline{v}, \overline{w}) = \int_0^{t^*} &\left(\|\overline{w}_t(t)\|_{L^2(\partial\Omega)}^2 + \|\overline{v}_t(t)\|^2 + \|\overline{v}(t)\|^2 \right) dt \\ &+ \|\overline{v}(0)\|^2 + \|\overline{w}\|_{H^1(0,t^*;L^2(\partial\Omega))}^2 + \max_{0\leq t\leq t^*} \|\overline{w}(t)\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 . \end{aligned} \tag{15}$$

Proof: From Theorem 2.1 of [14] we know that there exists a constant $\hat{C} > 0$ satisfying

$$\begin{aligned} \max_{0<t<t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{\phi}(t)\right\|_{H^3}^2 + \left\|\overline{T}\right\|_{H^1}^2 \right) \;&+\; \int_0^{t^*} \left(\left\|\overline{\phi}_{tt}(t)\right\|^2 + \left\|\overline{T}_t(t)\right\|^2 \right) dt \\ + \int_0^{t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{T}(t)\right\|_{H^2}^2 \right) dt \;&\leq\; \hat{C}\, G(\overline{v}, \overline{w}), \end{aligned} \tag{16}$$

where

$$G(\overline{v}, \overline{w}) = \int_0^{t^*} \|\overline{v}(t)\|^2 \, dt + \|\overline{w}\|_{H^1(0,t^*;L^2(\partial\Omega))}^2 . \tag{17}$$

As in that paper, $\overline{T}$ satisfies the following linear parabolic boundary value problem.

$$\overline{T}_t - \Delta\left(\overline{T}\zeta\right) \;=\; \phi_{1,t}\overline{\phi} - \phi_2\overline{\phi}_t + \overline{v}, \tag{18}$$

$$\left.\frac{\partial\overline{T}}{\partial n} + \overline{T}\right|_{\partial\Omega} = \overline{w}|_{\partial\Omega}\,, \qquad \overline{T}(x,0) = 0, \quad \forall x \in \overline{\Omega}\,, \tag{19}$$

where $\zeta = (T_1T_2)^{-1}$. Observe that we have $\zeta \in L^\infty(Q)$ and $\nabla\zeta_t \in L^2(Q)$, because of the regularity properties of T_i from the existence and uniqueness results (cf. [7, 13]). We can now take the time derivative of (18) and (19) to obtain

$$\begin{aligned} \overline{T}_{tt} - \Delta\left(\overline{T}\zeta\right)_t \;&=\; \phi_{1,tt}\overline{\phi} - \phi_2\overline{\phi}_{tt} + \overline{\phi}_t^2 + \overline{v}_t && (20) \\ &=\; f, && (21) \\ \left.\frac{\partial\overline{T}_t}{\partial n} + \overline{T}_t\right|_{\partial\Omega} \;&=\; \overline{w}_t|_{\partial\Omega} \,. && (22) \end{aligned}$$

For the initial values of $\overline{T}_t$ observe that

$$\begin{aligned} \overline{T}_t(x,0) \;&=\; \left(\Delta\left(\overline{T}\zeta\right) + \phi_{1,t}\overline{\phi} + \phi_2\overline{\phi}_t\right)(x,0) + \overline{v}(x,0) \\ &=\; \overline{v}(x,0). \end{aligned}$$

Furthermore, we observe that

$$\int_0^{t^*} \|f(t)\|^2 \, dt \le c_1 \, G(\overline{v}, \overline{w}) + \int_0^{t^*} \|\overline{v}_t(t)\|^2 \, dt, \tag{23}$$

by the previous results. To continue our proof, we multiply (20) by $\overline{T}_t$ and integrate the resulting equation over Ω to arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\overline{T}_t(t)\right\|^2 &+ \int_\Omega \nabla\overline{T}_t(t)\cdot\nabla\left(\zeta(t)\overline{T}(t)\right)_t dx - \int_{\partial\Omega} \overline{T}_t(t)\frac{\partial\left(\zeta(t)\overline{T}(t)\right)_t}{\partial n}\, dx \\ &\le \frac{\delta_1}{2}\|f(t)\|^2 + \frac{1}{2\delta_1}\left\|\overline{T}_t(t)\right\|^2, \end{aligned} \tag{24}$$

after applying (23) and Young's inequality. The value of δ_1 will be determined later. Next, we observe that

$$\begin{aligned} \int_\Omega \nabla\overline{T}_t(t)\cdot\nabla\left(\zeta(t)\overline{T}(t)\right)_t dx &= \int_\Omega \nabla\overline{T}_t(t)\cdot\nabla\left(\zeta_t(t)\overline{T}(t) + \zeta(t)\overline{T}_t(t)\right) dx \\ &= \int_\Omega \zeta(t)\left|\nabla\overline{T}_t(t)\right|^2 dx + \int_\Omega \overline{T}(t)\nabla\overline{T}_t(t)\cdot\nabla\zeta_t(t)\, dx \\ &\quad + \int_\Omega \zeta_t(t)\nabla\overline{T}_t(t)\cdot\nabla\overline{T}(t)\, dx \\ &\quad + \int_\Omega \overline{T}_t(t)\nabla\overline{T}(t)\cdot\nabla\zeta(t)\, dx \\ &= \int_\Omega \zeta(t)\left|\nabla\overline{T}_t(t)\right|^2 dx + I_1(t) + I_2(t) + I_3(t)\,. \end{aligned}$$

We can estimate the terms on the right of this last inequality individually as follows.

$$\begin{aligned} |I_1(t)| &\le \left\|\nabla\overline{T}_t(t)\right\| \left\|\overline{T}(t)\right\|_{L^4(\Omega)} \|\nabla\zeta_t(t)\|_{L^4(\Omega)} \\ &\le \frac{\delta_2}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_2}\|\nabla\zeta_t(t)\|^2_{H^1(\Omega)}\left\|\overline{T}(t)\right\|^2_{H^1(\Omega)}, \\ |I_2(t)| &\le \left\|\nabla\overline{T}_t(t)\right\| \left\|\nabla\overline{T}(t)\right\|_{L^4(\Omega)} \|\zeta_t(t)\|_{L^4(\Omega)} \\ &\le \frac{\delta_3}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_3}\|\zeta_t(t)\|^2_{H^1(\Omega)}\left\|\overline{T}(t)\right\|^2_{H^2(\Omega)}, \\ |I_3(t)| &\le \left\|\nabla\overline{T}_t(t)\right\| \left\|\overline{T}_t(t)\right\| \|\nabla\zeta(t)\|_{L^\infty(\Omega)} \\ &\le \frac{\delta_4}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_4}\|\nabla\zeta(t)\|^2_{L^\infty(\Omega)}\left\|\overline{T}_t(t)\right\|^2. \end{aligned}$$

In each of these inequalities, one can estimate the integral over t of the second term on the right via $G(\overline{v}, \overline{w})$. The values for δ_i will be determined later. For the boundary term we observe that

$$\begin{aligned} -\int_{\partial\Omega} \overline{T}_t \frac{\partial}{\partial t}\left(\frac{\partial\left(\zeta\overline{T}\right)}{\partial n}\right) dx &= -\int_{\partial\Omega} \overline{T}_t \frac{\partial}{\partial t}\left(\zeta\left(\overline{w} - \overline{T}\right)\right) dx \\ &\quad - \int_{\partial\Omega} \overline{T}_t \frac{\partial}{\partial t}\left(\overline{T}\zeta^2\left(T_1\left(w_2 - T_2\right) + T_2\left(w_1 - T_1\right)\right)\right) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega} \zeta \overline{T}_t^2 \, dx - \int_{\partial\Omega} \overline{T}_t \left(\overline{w} - \overline{T}\right) \zeta_t \, dx - \int_{\partial\Omega} \zeta \overline{T}_t \overline{w}_t \, dx \\
&\quad + \int_{\partial\Omega} \overline{T}_t^2 \left(\zeta^2 \left(T_1 \left(w_2 - T_2\right) + T_2 \left(w_1 - T_1\right)\right)\right) \, dx \\
&\quad\quad + \int_{\partial\Omega} \overline{T}_t \overline{T} \left(\zeta^2 \left(T_1 \left(w_2 - T_2\right) + T_2 \left(w_1 - T_1\right)\right)\right)_t \, dx \\
&= \int_{\partial\Omega} \zeta \overline{T}_t^2 \, dx + J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned}
$$

Again, we can estimate the terms individually as follows.

$$
\begin{aligned}
|J_1(t)| &\le \frac{\delta_5}{2} \left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} + \frac{c_5}{2\delta_5} \left(\|\overline{w}(t)\|^2_{L^4(\partial\Omega)} + \left\|\overline{T}(t)\right\|^2_{L^4(\partial\Omega)} \right) \|\zeta_t\|^2_{L^4(\partial\Omega)}, \\
|J_2(t)| &\le \frac{\delta_6}{2} \left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} + \frac{c_6}{2\delta_6} \|\overline{w}_t(t)\|^2_{L^2(\partial\Omega)}, \\
|J_3(t)| &\le c_7 \left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} \le \frac{\delta_7}{2} \left\|\nabla \overline{T}_t(t)\right\|^2 + \hat{c}_7 \left\|\overline{T}_t(t)\right\|^2, \\
|J_4(t)| &\le \frac{\delta_8}{2} \left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} \\
&\quad + \frac{1}{2\delta_8} \left\|\overline{T}\right\|^2_{L^4(\partial\Omega)} \left\| \left(\zeta^2 \left(T_1 \left(w_2 - T_2\right) + T_2 \left(w_1 - T_1\right)\right)\right)_t \right\|^2_{L^4(\partial\Omega)}.
\end{aligned}
$$

From the trace theorem and the Sobolev imbedding theorem (see, for example, [1] for the Sobolev theorem for fractional exponents), we have the continuous imbeddings

$$
\left\{ v : v = u|_{\partial\Omega} \, ; u \in H^1(\Omega) \right\} \hookrightarrow H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^4(\partial\Omega). \tag{25}
$$

Using this, we can bound the time integrals of the second terms on the right by $\overline{G}(\overline{v}, \overline{w})$. After choosing the δ_i's sufficiently small, we combine all the estimates to get after integration over t

$$
\begin{aligned}
\frac{1}{2} \left\|\overline{T}_t(t)\right\|^2 + \hat{c} \int_0^t \left\|\overline{T}_t(s)\right\|^2_{H^1(\Omega)} ds &\le C_1 \overline{G}(\overline{v}, \overline{w}) + \frac{1}{2} \left\|\overline{T}_t(0)\right\|^2 \\
&\le C_2 \overline{G}(\overline{v}, \overline{w}).
\end{aligned}
$$

The result now immediately follows from elliptic regularity estimates. □

In order to formulate the next result, we introduce the sets

$$
K^{\pm}(v, w) = \left\{ (h, k) \in V \times W : \exists \lambda > 0 \text{ such that } (v \pm \lambda h, w \pm \lambda k) \in \mathcal{U}_{\mathrm{ad}} \right\}, \tag{26}
$$

for $(v, w) \in \mathcal{U}_{\mathrm{ad}}$.

Proposition 3 *Suppose that* **(H1)** *and* **(H2)** *are satisfied and that* $(v, w) \in \mathcal{U}_{\mathrm{ad}}$. *Then the observation operator*

$$
S : K \to B,
$$

has a directional derivative $(\psi, \theta) = D_{(h,k)}S(v, w)$ *in the direction* (h, k)*. Furthermore, at* $S(v, w) = (\phi, T)$*, this directional derivative* $(\psi, \theta) \in X_1 \times X_2$ *is the unique solution to the linear parabolic initial-boundary value problem*

$$\begin{aligned} \psi_t - \Delta\psi &= \psi\left(\frac{1}{T} - s_0''(\phi)\right) - \frac{\phi}{T^2}\theta, \\ \theta_t - \Delta\left(\frac{\theta}{T^2}\right) &= (\phi\psi)_t + h, \\ \frac{\partial\psi}{\partial n} = 0, &\qquad \frac{\partial\theta}{\partial n} + \theta = k, \quad \text{on } \partial\Omega, \\ \psi(x, 0) &= \theta(x, 0) = 0, \quad \text{on } \overline{\Omega}. \end{aligned}$$

A corresponding result holds for the directional derivative $D_{(-h,-k)}S(v, w)$ *at* (v, w) *in the direction* $(h, k) \in K^-(v, w)$*.*

Proof: As in [14], we let

$$(\phi^\lambda, T^\lambda) = S(v + \lambda h, w + \lambda k).$$

Furthermore, we use the notation of the previous proposition and let

$$\overline{\phi} = \phi^\lambda - \phi\,, \quad \overline{T} = T^\lambda - T\,, \quad \zeta = \frac{1}{TT^\lambda}\,.$$

Finally, define

$$p = \overline{\phi} - \lambda\psi\,, \quad q = \overline{T} - \lambda\theta\,.$$

It is clear that the linear parabolic system in the statement admits a unique solution $(\psi, \theta) \in X_1 \times X_2$. To continue, suppose that $(h, k) \in K^+(v, w)$, and suppose that there is a $\overline{\lambda} > 0$ such that $(v + \lambda h, w + \lambda k) \in \mathcal{U}_{\mathrm{ad}}, \quad \forall \lambda \in (0, \overline{\lambda})$. We have to show that

$$\|(p, q)\|_B = o(\lambda), \quad \text{as } \lambda \to 0^+. \tag{27}$$

Using our notation, p and q obey the following system of linear parabolic boundary value problems.

$$\begin{aligned} p_t - \Delta p &= s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi + \frac{p}{T} - \frac{\phi}{T^2}q + \frac{\phi}{T}\overline{T}^2\zeta - \overline{\phi}\overline{T}\zeta\,, & (28) \\ q_t - \Delta\left(\frac{q}{T^2}\right) &= \phi_t p + \phi p_t + \overline{\phi\phi_t} - \Delta\left(\frac{\overline{T}^2\zeta}{T}\right), & (29) \\ \frac{\partial p}{\partial n} = 0, &\qquad \frac{\partial q}{\partial n} + q = 0\,, & (30) \\ 0 &= p(x, 0) = q(x, 0)\,. & (31) \end{aligned}$$

We prove (27) in several steps.

Step1: In [14] the authors show that

$$\max_{0\le t\le t^*}\left(\|p(t)\|_{H^1}^2+\|q(t)\|^2\right)+\int_0^{t^*}\left(\|p_t(s)\|^2+\|q(s)\|_{H^1}^2+\|p(s)\|_{H^2}^2\right)\,ds\le C\lambda^4, \tag{32}$$

for a suitable constant $C>0$. We continue from there by multiplying (29) by $\left(\frac{q}{T^2}\right)_t$. After integrating the resulting equation over $\Omega\times[0,t]$, we obtain

$$\begin{aligned}\int_0^t\left\|\frac{q_t}{T}\right\|^2\,ds\;+\;\frac{1}{2}\left\|\nabla\left(\frac{q}{T^2}\right)(t)\right\|^2-\int_0^t\int_{\partial\Omega}\left(\frac{q}{T^2}\right)_t\frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)\,dxds \\ =\;\int_0^t\int_\Omega f\left(\frac{q_t}{T^2}-2\frac{qT_t}{T^3}\right)\,dxds-2\int_0^t\int_\Omega\frac{q_tqT_t}{t^3}\,dxds\,,\end{aligned} \tag{33}$$

where f is given by

$$\phi_tp+\phi p_t+\overline{\phi\phi}_t-\Delta\left(\frac{\overline{T}^2\zeta}{T}\right). \tag{34}$$

From Proposition 1 and the earlier estimates we see that

$$\int_0^{t^*}\|f(s)\|^2\,ds\le C_1\lambda^4,$$

for a suitable constant $C_1>0$. Furthermore, we have

$$\int_0^{t^*}\left\|\frac{qT_t}{T^3}(s)\right\|^2\,ds\le C_2\lambda^4,$$

due to earlier estimates. For the boundary term we observe

$$\frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)=\frac{q}{T^2}\left(\frac{w}{T}-1\right).$$

Therefore, we have

$$\begin{aligned}\left|\int_0^t\int_{\partial\Omega}\left(\frac{q}{T^2}\right)_t\frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)\,dxds\right| &= \left|\int_0^t\int_{\partial\Omega}\left(\frac{q}{T^2}\right)_t\frac{q}{T^2}\left(1-\frac{w}{T}\right)\,dxds\right|\\ \le\; c_1\left\|\frac{q}{T^2}(t)\right\|^2\;+\;c_2\int_0^t\int_{\partial\Omega}q^2\left|\left(\frac{w}{T}\right)_t\right|\,dxds &\\ \le\; c_1\delta\left\|\nabla\left(\frac{q}{T^2}(t)\right)\right\|^2+c_3\|q(t)\|^2\;+\;c_2\int_0^t\|q(s)\|_{L^4(\partial\Omega)}^2\left\|\left(\frac{w}{T}\right)_t\right\|\,ds &\\ \le\; c_1\delta\left\|\nabla\left(\frac{q}{T^2}(t)\right)\right\|^2+c_4\lambda^4+c_5\int_0^t\|q(s)\|_{H^1}^2\,ds\,. &\end{aligned}$$

In the last line of this estimate we have used (25). Combining these estimates, using Young's inequality, and choosing $\delta>0$ sufficiently small, we obtain

$$\max_{0\le t\le t^*}\left\|\nabla\left(\frac{q}{T^2}\right)(t)\right\|^2+\int_0^{t^*}\left\|\frac{q_t}{T}\right\|^2\,ds\le C_3\lambda^4.$$

It immediately follows

$$\max_{0\le t\le t^*} \|q(t)\|_{H^1}^2 + \int_0^{t^*} \|q_t\|^2 \, ds \le C_4\lambda^4. \tag{35}$$

<u>*Step 2:*</u> In the next step, we take the derivative of (28) with respect to t to get

$$p_{tt} - \Delta p_t = \Big(s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi\Big)_t + \left(\frac{p}{T} - \frac{\phi}{T^2}q + \frac{\phi}{T}\overline{T}^2\zeta - \overline{\phi}\overline{T}\zeta\right)_t . \tag{36}$$

We observe that

$$\begin{aligned} |F_{1,t}| &= \left|\Big(s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi\Big)_t\right| \\ &\le \left|\phi_t\left(s_0''(\phi) - s_0''(\phi^\lambda) - s_0'''(\phi)\overline{\phi}\right)\right| + |s_0'''(\phi)\phi_t p| \\ &\quad + |s_0''(\phi)p_t| + \left|\Big(s_0''(\phi^\lambda) - s_0''(\phi)\Big)\,\overline{\phi}_t\right| . \end{aligned}$$

Using the mean-value theorem, one easily sees that

$$\int_0^{t^*} \|F_{1,t}(s)\|^2 \, ds \le C_5\lambda^4, \tag{37}$$

for a suitable constant $C_5 > 0$. Next, we observe that

$$\begin{aligned} F_{2,t} &= \frac{p_t}{T} - \frac{pT_t}{T^2} - \frac{\phi_t}{T^2}q + 2\frac{\phi T_t}{T^3}q - \frac{\phi}{T^2}q_t + \frac{\phi_t}{T}\overline{T}^2\zeta - \frac{\phi}{T^2}T_t\overline{T}_2\zeta \\ &\quad + 2\frac{\phi}{T}\overline{T}\overline{T}_t\zeta + \frac{\phi}{T}\overline{T}^2\zeta_t - \overline{\phi}_t\overline{T}\zeta - \overline{\phi}\overline{T}_t\zeta - \overline{\phi}\overline{T}\zeta_t . \end{aligned}$$

Since both ϕ_t and T_t are elements of $C([0,t^*];H^1(\Omega))$, we see that

$$\int_0^{t^*} \|F_{2,t}(s)\|^2 \, ds \le C_6\lambda^4, \tag{38}$$

for a suitable constant $C_6 > 0$. Hence, if one multiplies (36) by p_t and integrates the result over $\Omega \times [0,t]$, one immediately gets

$$\max_{0\le t\le t^*} \|p_t(t)\|^2 + \int_0^{t^*} \|p_t(s)\|_{H^1}^2 \, ds \le C_7\lambda^4 . \tag{39}$$

We can now apply the standard elliptic regularity estimates to obtain

$$\max_{0\le t\le t^*} \|p(t)\|_{H^2}^2 \le C_8\lambda^4. \tag{40}$$

Furthermore, we can multiply (36) by p_{tt}, integrate the result over $\Omega \times [0,t]$ and use (37) and (38) again, to get

$$\max_{0\le t\le t^*} \|p_t\|_{H^1}^2 + \int_0^{t^*} \|p_{tt}(s)\|^2 \, ds \le C_9\lambda^4, \tag{41}$$

for a suitable constant $C_9 > 0$.

Step 3: To continue, we take the time derivative of (29) to obtain

$$q_{tt} - \Delta\left(\frac{q}{T^2}\right)_t = F_{3,t}(x,t), \tag{42}$$

where

$$F_{3,t} = \left(\phi_t p + \phi p_t + \overline{\phi}\overline{\phi}_t - \Delta\left(\frac{\overline{T}^2\zeta}{T}\right)\right)_t .$$

To simplify notations, we introduce $\hat{\zeta} = \frac{\zeta}{T}$, which has the same properties as ζ. We observe that

$$\begin{aligned}\Delta\left(\overline{T}^2\hat{\zeta}\right)_t &= 2\overline{T}_t\hat{\zeta}\Delta\overline{T} + 4\hat{\zeta}\nabla\overline{T}\cdot\nabla\overline{T}_t + 4\overline{T}_t\nabla\overline{T}\cdot\nabla\hat{\zeta} + 2\overline{T}\hat{\zeta}\Delta\overline{T}_t + 4\overline{T}\nabla\overline{T}_t\cdot\nabla\hat{\zeta}\\ &\quad +2\overline{T}\overline{T}_t\Delta\hat{\zeta} + 2\left|\nabla\overline{T}\right|^2\hat{\zeta}_t + 2\hat{\zeta}_t\overline{T}\Delta\overline{T} + 4\overline{T}\nabla\overline{T}\cdot\nabla\hat{\zeta}_t + \overline{T}^2\Delta\hat{\zeta}_t.\end{aligned}$$

Using the results of Proposition 1, we can bound $\left\|\overline{T}(t)\right\|_{H^2}$ by $C_{10}\lambda$ for a sufficiently large constant $C_{10} > 0$. Furthermore, we know that $\overline{T}$ has the same regularity as $\hat{\zeta}$, which enables us to bound terms of the form

$$\int_0^{t^*}\left\|\overline{T}\right\|_{H^2}^2 ds, \quad \max_{0\le t\le t^*}\left\|\overline{T}_t(t)\right\|_{H^1},$$

by constants. Combining these properties, we see that

$$\int_0^{t^*}\left\|\Delta\left(\overline{T}^2\hat{\zeta}\right)_t(s)\right\|^2 ds \le C_{11}\lambda^2$$

for a suitable constant $C_{11} > 0$. It follows that

$$\int_0^{t^*}\|F_{3,t}(s)\|^2\, ds \le C_{12}\lambda^2. \tag{43}$$

We multiply (42) by q_t and integrate the result over $\Omega\times[0,t]$ to get

$$\begin{aligned}\frac{1}{2}\|q_t(t)\|^2 \;&+\; \int_0^{t^*}\int_\Omega \nabla q_t\cdot\nabla\left(\frac{q}{T^2}\right)_t dxds - \int_0^{t^*}\int_{\partial\Omega} q_t\frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)_t dxds\\ &\le \left(\int_0^{t^*}\|F_{3,t}(s)\|^2 ds\right)^{\frac{1}{2}}\left(\int_0^{t^*}\|q_t(s)\|^2 ds\right)^{\frac{1}{2}} \le C_{13}\lambda^3,\end{aligned}$$

for a suitable constant $C_{13} > 0$. We next observe that

$$\begin{aligned}\int_0^{t^*}\int_\Omega\nabla q_t\cdot\nabla\left(\frac{q}{T^2}\right)_t dxds &= \int_0^{t^*}\int_\Omega\nabla q_t\cdot\nabla\left(\frac{q_t}{T^2} - 2\frac{qT_t}{T^3}\right) dxds\\ &= \int_0^{t^*}\left\|\frac{\nabla q_t}{T}(s)\right\|^2 ds\\ &\quad -2\int_0^{t^*}\int_\Omega\nabla q_t\cdot\left(\frac{q}{T^3}\nabla T + \frac{T_t}{T^3}\nabla q\right) dxds\\ &\quad +2\int_0^{t^*}\int_\Omega\nabla q_t\cdot\left(3\frac{qT_t}{T^4}\nabla T - \frac{q}{T^3}\nabla T_t\right) dxds\,.\end{aligned}$$

One sees that the mixed terms on the right can be treated via Young's inequality, and that we can use the fact that

$$\int_0^{t^*} \|q\|_{H^2}^2 \, ds \le C_{14}\lambda^4,$$

and the other earlier estimates on q. Finally, we note that

$$\begin{aligned}
\int_0^{t^*}\int_{\partial\Omega} q_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)_t dxds &= \int_0^{t^*}\int_{\partial\Omega} q_t\left(\frac{1}{T^2}\frac{\partial q_t}{\partial n} - 2\frac{q_t}{T^3}\frac{\partial T}{\partial n}\right) dxds \\
&\quad +2\int_0^{t^*}\int_{\partial\Omega} q_t\left(3\frac{qT_t}{T^4}\frac{\partial T}{\partial n} - \frac{T_t}{T^3}\frac{\partial q}{\partial n} - \frac{q}{T^3}\frac{\partial T_t}{\partial n}\right) dxds \\
&= -\int_0^{t^*}\int_{\partial\Omega} \frac{q_t^2}{T^2}\left(1 + 2\frac{1}{T}\frac{\partial T}{\partial n}\right) dxds \\
&\quad -2\int_0^{t^*}\int_{\partial\Omega} \frac{qq_t}{T^3}\left(T_t + \frac{\partial T_t}{\partial n} - 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) dxds\,.
\end{aligned}$$

In the first term, we observe that

$$1 + 2\frac{1}{T}\frac{\partial T}{\partial n} \in L^\infty(\partial Q).$$

In the second term, one has

$$\frac{1}{T}\left(T_t + \frac{\partial T_t}{\partial n} - 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) \in L^2(0,t^*;L^\infty(\partial\Omega)).$$

Using this, we get

$$\left|2\int_0^{t^*}\int_{\partial\Omega} \frac{qq_t}{T^3}\left(T_t + \frac{\partial T_t}{\partial n} - 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) dxds\right| \le C_{15}\left(\int_0^{t^*} \left\|\frac{q_t}{T}\right\|_{L^2(\partial\Omega)}^2 \|q\|_{L^2(\partial\Omega)}^2 \, ds\right)^{\frac{1}{2}}.$$

Observe that

$$\|q(t)\|_{L^2(\partial\Omega)}^2 \le C_{16}\lambda^4.$$

This implies that we are left to treat a term of the form

$$\int_0^{t^*} \left\|\frac{q_t}{T}\right\|_{L^2(\partial\Omega)}^2 ds.$$

We do this by using

$$\int_0^{t^*} \|g(s)\|_{L^2(\partial\Omega)}^2 \, ds \le \delta\int_0^{t^*} \|\nabla g(s)\|^2 \, ds + \hat{C}\int_0^{t^*} \|g(s)\|^2 \, ds,$$

for a suitable constant $\hat{C} > 0$. We can now combine all the above estimates and use the properties of T to conclude that

$$\max_{0\le t\le t^*} \|q_t(t)\| + \int_0^{t^*} \|\nabla q_t\|^2 \, ds \le C_{17}\lambda^3, \tag{44}$$

for a suitable constant $C_{17} > 0$. From elliptic regularity estimates it follows that the same estimate holds for

$$\max_{0\le t\le t^*} \|q(t)\|_{H^2}^2\,.$$

This finishes the proof of the proposition. □

4 Optimality Conditions

We return to the optimal control problem **(CP)** stated in Section 2. We introduced the non-linear observation operator S in (9)–(10). We can write S in components (S_1, S_2) as follows.

$$S(v,w) = \begin{pmatrix} S_1(v,w) \\ S_2(v,w) \end{pmatrix} = \begin{pmatrix} \phi \\ T \end{pmatrix}. \tag{45}$$

Proposition 3 states that this operator is Gateaux differentiable with Gateaux derivative

$$DS(v,w)(h,k) = \begin{pmatrix} DS_1(v,w)(h,k) \\ DS_2(v,w)(h,k) \end{pmatrix} = \begin{pmatrix} \psi \\ \theta \end{pmatrix}, \tag{46}$$

given by the following system of linearized equations

$$\psi_t - \Delta\psi = \psi\left(\frac{1}{T} - s_0''(\phi)\right) - \frac{\phi}{T^2}\theta, \tag{47}$$

$$\theta_t - \Delta\left(\frac{\theta}{T^2}\right) = (\phi\psi)_t + h, \tag{48}$$

$$\frac{\partial\psi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} + \theta = k, \quad \text{on } \partial\Omega, \tag{49}$$

$$\psi(x,0) = \theta(x,0) = 0, \quad \text{on } \overline{\Omega}. \tag{50}$$

An application of the Lagrange multiplier rule implies that there exist some $\lambda \geq 0$ and Borel measures $\mu_1, \mu_2, \mu_3, \mu_4$, satisfying

$$\mu_i(\{(x,t) \in \overline{Q} \mid T(x,t) \neq K_i\}) = 0, \ i = 1,2, \tag{51}$$

$$\mu_i(\{(x,t) \in \overline{Q} \mid \phi(x,t) \neq K_i\}) = 0, \ i = 3,4, \tag{52}$$

such that

$$\lambda + |\mu_1| + |\mu_2| + |\mu_3| + |\mu_4| > 0.$$

The constants K_i are the ones given in the state constraints (7). To continue, we denote $\mu = \mu_1 - \mu_2$, $\nu = \mu_3 - \mu_4$.

The abstract optimality system for the control problem under consideration is given below by $(*)$ and $(**)$. The first condition takes the form

$$(*) \qquad \forall(\zeta,\eta) \in \mathcal{Y}_{ad}: \quad \int(\eta - T)d\mu + \int(\zeta - \phi)d\nu \leq 0,$$

where $(\phi, T) = S(v,w)$ is a solution to the state equations for optimal controls $(v,w) \in \mathcal{U}_{\text{ad}}$.

For the second condition, we need to introduce some notation. We denote by $I(\phi, T; v, w)$ the cost functional, i.e. $J(v,w) = I(S_1(v,w), S_2(v,w); v,w)$. Then the gradient of the cost functional with respect to the controls takes the form

$$\begin{aligned} \langle DJ(v,w),(h,k)\rangle &= \langle D_1 I(\phi,T;v,w), D_1 S(\phi,T)(h,k)\rangle \\ &+ \langle D_2 I(\phi,T;v,w), D_2 S(\phi,T)(h,k)\rangle \\ &+ \langle D_3 I(\phi,T;v,w), h\rangle + \langle D_4 I(\phi,T;v,w), k\rangle. \end{aligned}$$

The second optimality condition is of the form

$$(**)\qquad \lambda\langle DJ(v,w),(h-v,k-w)\rangle + \langle [DS_2(v,w)]^*(h-v,k-w)],\mu\rangle \geq 0,$$

for all $(h,k)\in\mathcal{U}_{\rm ad}$, where $[DS_2(v,w)]^*$ denotes the adjoint to $[DS_2(v,w)]$.

Assuming that the Slater condition is satisfied, we can take $\lambda=1$. Note that in the present case the Slater condition (S) means that there exists some $(h_0,k_0)\in\mathcal{U}_{\rm ad}$ such that for all $(x,t)\in\overline{Q}$,

$$K_1 < T(x,t) + [DS_2(v,w)(h_0-v,k_0-w)](x,t) < K_2\,,$$
$$K_3 < \phi(x,t) + [DS_1(v,w)(h_0-v,k_0-w)](x,t) < K_4\,.$$

Furthermore, an adjoint state is introduced in order to simplify the latter optimality condition. To this end, we rewrite the linearized equations in the form

$$\mathcal{L}_{11}(\psi)+\mathcal{L}_{12}(\theta)=0, \tag{53}$$
$$\mathcal{L}_{21}(\psi)+\mathcal{L}_{22}(\theta)=h, \tag{54}$$

with the non-homogeneous boundary condition

$$\frac{\partial\theta}{\partial n}+\theta=k,\quad \text{on } \partial\Omega, \tag{55}$$

where we denote

$$\mathcal{L}_{11}(\psi)=\psi_t-\Delta\psi-\psi\left(\frac{1}{T}-s_0''(\phi)\right), \tag{56}$$
$$\mathcal{L}_{12}(\theta)=\frac{\phi}{T^2}\theta, \tag{57}$$
$$\mathcal{L}_{21}(\psi)=-\left(\phi\psi\right)_t, \tag{58}$$
$$\mathcal{L}_{22}(\theta)=\theta_t-\Delta\left(\frac{\theta}{T^2}\right)\ . \tag{59}$$

Then, for any pair of functions $(q,p)\in V\times V$ it follows that

$$(\mathcal{L}_{11}(\psi)+\mathcal{L}_{12}(\theta),q)_V=0,$$
$$(\mathcal{L}_{21}(\psi)+\mathcal{L}_{22}(\theta),p)_V=(h,p)_V,$$

and the latter term, by an application of the associated Green formula, can be written in the form

$$(\mathcal{L}_{22}(\theta),p)_V=\mathcal{A}(\theta,p)-\ell\left(\frac{\partial\theta}{\partial n}+\theta,p\right)\,, \tag{60}$$

with an appropriate bilinear form $\mathcal{A}(\cdot,\cdot)$, and a boundary form $\ell(\cdot,\cdot)$ which will be specified below. In particular, for $\frac{\partial\theta}{\partial n}+\theta=0$ it follows that

$$(\mathcal{L}_{22}(\theta),p)_V=\mathcal{A}(\theta,p)\ .$$

Hence, the system becomes

$$\begin{aligned}
&(\mathcal{L}_{11}(\psi) + \mathcal{L}_{12}(\theta), q)_V = 0, \\
&(\mathcal{L}_{21}(\psi), p)_V + \mathcal{A}(\theta, p) = (h, p)_V + \ell(k, p).
\end{aligned}$$

In order to identify the boundary form $\ell(k,p)$, we need Green formulae for the subsequent terms in the scalar product of the space $L^2(Q)$ which are given below. We have

$$-\left(\Delta\left(\frac{\theta}{T^2}\right), \phi\right)_{L^2(Q)} = \left(\nabla\left(\frac{\theta}{T^2}\right), \nabla\phi\right)_{L^2(Q)} - \int_0^{t^*} \left(\frac{\partial}{\partial n}\left(\frac{\theta}{T^2}\right), \phi\right)_{L^2(\partial\Omega)} dt,$$

and, in view of the boundary conditions, it follows that

$$\begin{aligned}
& -\left(\frac{\partial}{\partial n}\left(\frac{\theta}{T^2}\right), \phi\right)_{L^2(\partial\Omega)} \\
= & \left((\theta - k)\frac{1}{T^2}, \phi\right)_{L^2(\partial\Omega)} - \left(\theta\frac{\partial}{\partial n}\frac{1}{T^2}, \phi\right)_{L^2(\partial\Omega)} \\
= & \left(\theta\left(\frac{1}{T^2} - \frac{\partial}{\partial n}\frac{1}{T^2}\right), \phi\right)_{L^2(\partial\Omega)} - \left(k\frac{1}{T^2}, \phi\right)_{L^2(\partial\Omega)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\left(\nabla\left(\Delta\left(\frac{\theta}{T^2}\right)\right), \nabla(\phi)\right)_{L^2(\Omega)} = & \left(\Delta\left(\frac{\theta}{T^2}\right), \Delta\phi\right)_{L^2(\Omega)} \\
& - \left(\Delta\left(\frac{\theta}{T^2}\right), \frac{\partial\phi}{\partial n}\right)_{L^2(\partial\Omega)},
\end{aligned}$$

as well as

$$\begin{aligned}
-\left(\Delta\left(\Delta\left(\frac{\theta}{T^2}\right)\right), \Delta\phi\right)_{L^2(\Omega)} = & \left(\nabla\left(\Delta\left(\frac{\theta}{T^2}\right)\right), \nabla(\Delta\phi)\right)_{L^2(\Omega)} \\
& - \left(\frac{\partial}{\partial n}\Delta\left(\frac{\theta}{T^2}\right), \Delta\phi\right)_{L^2(\partial\Omega)}.
\end{aligned}$$

We also have the following relation on the boundary $\partial\Omega$, (cf. [12]),

$$\Delta\left(\frac{\theta}{T^2}\right) = \Delta_\Gamma\left(\frac{\theta}{T^2}\right) + \kappa\frac{\partial}{\partial n}\left(\frac{\theta}{T^2}\right) + \frac{\partial^2}{\partial n^2}\left(\frac{\theta}{T^2}\right),$$

where Δ_Γ is the Laplace–Beltrami operator on $\Gamma = \partial\Omega$, and where κ denotes the tangential divergence of the normal vector field on Γ, i.e. $\kappa = \mathrm{div}_\Gamma n$, in the notation of [12].
The adjoint state equations are introduced in the following way. Assume that the functions $(q,p) \in V \times V$ satisfy the variational equation

$$\begin{aligned}
& (\mathcal{L}_{11}(\zeta), q)_V + (\mathcal{L}_{12}(\eta), q)_V + (\mathcal{L}_{21}(\zeta), p)_V + (\mathcal{L}_{22}(\eta), p)_V \\
= & \langle D_1 I(\phi, T; v, w), \zeta\rangle + \int \zeta d\nu + \langle D_2 I(\phi, T; v, w), \eta\rangle + \int \eta d\mu, \qquad (61)
\end{aligned}$$

for all sufficiently smooth functions ζ, η satisfying homogeneous initial conditions and the homogeneous boundary conditions

$$\frac{\partial \zeta}{\partial n} = 0\,, \quad \frac{\partial \eta}{\partial n} + \eta = 0\,. \tag{62}$$

Using the Lions projection theorem (see e.g. [15] for a variant of this theorem), one can show that these functions are uniquely determined.

The system (61) can be rewritten in the form

$$\begin{aligned} & (\mathcal{L}_{11}(\zeta), q)_V + (\mathcal{L}_{12}(\eta), q)_V + (\mathcal{L}_{21}(\zeta), p)_V + \mathcal{A}(\eta, p) \\ = \; & \langle D_1 I(\phi, T; v, w), \zeta\rangle + \int \zeta d\nu + \langle D_2 I(\phi, T; v, w), \eta\rangle + \int \eta d\mu, \end{aligned}$$

where the boundary condition $\frac{\partial \eta}{\partial n} + \eta = 0$ is imposed directly in the equation.

If we replace ζ, η by ψ, η, it follows that

$$\begin{aligned} \langle D_1 I(\phi, T; v, w), \psi\rangle \; & + \; \langle D_2 I(\phi, T; v, w), \theta\rangle + \int \theta d\mu + \int \psi d\nu \\ & = \; (\mathcal{L}_{11}(\psi), q)_V + (\mathcal{L}_{12}(\theta), q)_V + (\mathcal{L}_{21}(\psi), p)_V + \mathcal{A}(\theta, p) \\ & = \; (\mathcal{L}_{11}(\psi), q)_V + (\mathcal{L}_{12}(\theta), q)_V \\ & \qquad + (\mathcal{L}_{21}(\psi) + \mathcal{L}_{22}(\theta), p)_V + \ell(k, p) \\ & = \; (h, p)_V + \ell(k, p)\;. \end{aligned}$$

Using the above construction, it follows that for $\lambda = 1$ the necessary optimality conditions can be given the following form.

Theorem 1 *Assume that condition* (S) *is satisfied. Then there exist* μ, ν *and the adjoint state* (q, p) *such that the optimality system for the control problem includes the state equation, the adjoint state equation, and the condition* $(*)$*, as well as the variational inequality*

$$\begin{aligned} & \langle D_3 I(\phi, T; v, w), h - v\rangle \;+\; (h - v, p)_V \;+\; \langle D_4 I(\phi, T; v, w), k - w\rangle \\ & \quad + \ell(k - w, p) \geq 0\,, \quad \forall\, (h, k) \in \mathcal{U}_{\mathrm{ad}}\,. \end{aligned}$$

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