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Interface Conditions for Limits of the Navier–Stokes–Korteweg Model

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Abstract

In this contribution we will study the behaviour of the pressure across phase boundaries in liquid–vapour flows. As mathematical model we will consider the static version of the Navier–Stokes–Korteweg model which belongs to the class of diffuse interface models. From this static equation a formula for the pressure jump across the phase interface can be derived. If we perform then the sharp interface limit we see that the resulting interface condition for the pressure seems to be inconsistent with classical results of hydrodynamics. Therefore we will present two approaches to recover the results of hydrodynamics in the sharp interface limit at least for special situations.

1 The Navier-Stokes-Korteweg Model

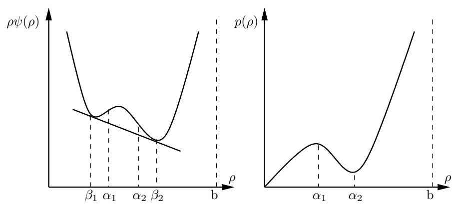
In this paper we will consider a mathematical model for liquid-vapour flows including phase transition which was proposed by Korteweg already in 1901 [17] and which is known as the Navier-Stokes-Korteweg model. It is an extension of the compressible Navier-Stokes equation and given by the following system.

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0 \\ \partial_t(\rho v) + \nabla \cdot (\rho v v^t + p(\rho) I) &= \mu \Delta v + \gamma \varepsilon^2 \rho \nabla \Delta \rho. \end{aligned} \tag{1}$$

This is a one fluid model where $\rho, v, p(\rho)$ and μ denote the density, velocity, pressure and the viscosity of the fluid/vapour respectively. Compared to the original Navier–Stokes equation the system (1) contains the term $\gamma\varepsilon^2\rho\nabla\Delta\rho$ which is supposed to model capillarity effects close to phase transitions. The pressure $p(\rho)$ as a function of the density ρ is defined as

$$p(\rho) = \rho^2 \psi'(\rho) \tag{2}$$

where ψ is a smooth function of ρ such that $\rho\psi(\rho)$ is the total free energy density and of the following form:



The values α_1 and α_2 are defined by the extrema of p and β_1 and β_2 are the Maxwell points at which the tangent line on $\rho\psi(\rho)$ is equal to the difference quotient. The conservation of energy is neglected in (1). Different phases of the fluid are defined by the size of ρ. If $\rho \leq \alpha_1$ we are in the vapour phase and if $\rho \geq \alpha_2$ we are in the liquid phase. The equation (2) is known as the van der Waals equation of state.

For a rigorous derivation of the system (1) one has to consider the equations for conservation of mass, momentum, energy and the entropy production equation (second law of thermodynamics). Special conditions for the stress tensor P, which appears in the equations for conservation of momentum and energy, ensure that the entropy production is nonnegative [7], [2]. This will lead to the Navier-Stokes-Korteweg model. Then neglecting the equation for the energy we end up with (1).

An alternative derivation of the static case can be described as follows. First let fix some notations:

$$\begin{aligned} \tilde{W}(\rho) : &= \text{free energy density (double well)} = \rho\psi(\rho) \\ \tilde{E}_0(\rho) : &= \text{total energy.} \end{aligned} \tag{3}$$

For a moment let us consider the minimizers of

$$\tilde{E}_0(\rho) = \int_\Omega \tilde{W}(\rho)\, dx \tag{4}$$

under the constraint

$$\int_\Omega \rho\, dx = M \quad \text{(conservation of mass)} \tag{5}$$

as a mathematical model for a two-phase fluid at rest. Then there exists β_1, β_2 and a linear function l such that $l(\beta_i) = \tilde{W}(\beta_i)$ and $l'(\beta_i) = \tilde{W}'(\beta_i),\ i = 1, 2$. Let $l(\rho) =: d_0\rho + d_1$, define $W(\rho) := \rho\psi(\rho) - l(\rho)$ and

$$E_0(\rho) := \int_\Omega W(\rho(x)))dx. \tag{6}$$

Then the functionals $\tilde{E}_0(\rho)$ and $E_0(\rho)$ differ only in a constant $\tilde{M} = d_0M + d_1|\Omega|$ and have the same minimizers:

$$E_0(\rho) = \tilde{E}_0(\rho) - \tilde{M}. \tag{7}$$

The solution is not unique. All functions ρ with

$$\rho(x) = \beta_1 \quad \text{for} \quad x \in \Omega_1, \tag{8}$$

$$\rho(x) = \beta_2 \quad \text{for} \quad x \in \Omega_2, \tag{9}$$

such that $\Omega_1 \cup \Omega_2 = \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$, $|\Omega_1|\beta_1 + |\Omega_2|\beta_2 = M$ are minimizers of (6) under the constraint (5). The minimum of E_0 is equal to 0 and the minimum of $\tilde{E}_0$ is given by $\tilde{M} = d_0M + d_1|\Omega|$. The length of the interfaces is not minimized and the energy due to the curvature and the surface tension is not included.

It is known that already van der Waals [27] has recognized first the non-uniqueness of this approach. There are infinitely many ways to distribute mass of densities β_1 and β_2 in the domain Ω such that (4) and (5) are satisfied. He proposed to penalize the occurrence of free boundaries between the phases by adding a term of the form

$$\int_\Omega \gamma\varepsilon^2 \frac{|\nabla\rho|^2}{2}\, dx$$

and to consider instead of (4), (5) or (6), (5) the following problem.

Minimize

$$\tilde{J}_\varepsilon(\rho) := \int_\Omega \left(\tilde{W}(\rho) + \gamma\varepsilon^2 \frac{|\nabla\rho|^2}{2}\right) dx \qquad \textit{(total energy)}$$

under the constraint $\int_\Omega \rho \; dx = M$ *(conservation of mass).*

Now it is easy to see that the Euler–Lagrange equation for this variational problem is just

$$\tilde{W}'(\rho) \;=\; \gamma\varepsilon^2\Delta\rho + \lambda_\varepsilon, \tag{10}$$

where λ_ε is the Lagrange multiplier corresponding to the mass constraint. Taking the gradient of both sides in (10) and multiplying with ρ implies

$$\rho\tilde{W}''(\rho)\nabla\rho \;=\; \gamma\varepsilon^2\rho\nabla\Delta\rho. \tag{11}$$

The definition of $\tilde{W}$ and a simple calculation using (2) shows that $p'(\rho) = \rho\tilde{W}''(\rho)$ and therefore we get from (11)

$$\nabla p(\rho) \;=\; \gamma\varepsilon^2\rho\nabla\Delta\rho. \tag{12}$$

This is just the static form of (1).

The mathematical model for the dynamical case $\partial_t v \neq 0$ can be obtained as follows (see [22], [23]). The Lagrangian is given by

$$L(\rho, v) := \frac{1}{2}\rho|v|^2 - \tilde{W}(\rho) - \frac{\gamma\varepsilon^2}{2}|\nabla\rho|^2$$

and the Euler-Lagrange equations for the action functional with respect to the constraint $\partial_t\rho + \nabla\cdot(\rho v) = 0$

$$\int_0^T \int_{\mathbb{R}^3} L(\rho(x,t), v(x,t))\, dx\, dt$$

by

$$\partial_t v + v\nabla v = \nabla\left(-\tilde{W}'(\rho) + \gamma\varepsilon^2\Delta\rho\right).$$

Using $p'(\rho) = \rho\tilde{W}''(\rho)$ and conservation of mass we get

$$\partial_t(\rho v) + \nabla\cdot(\rho v v^t + p(\rho)I) = \gamma\varepsilon^2\rho\nabla\Delta\rho.$$

Add some scaled viscosity and obtain

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0 \\ \partial_t (\rho v) + \nabla \cdot (\rho v v^t + p(\rho) I) &= \mu \Delta v + \gamma \varepsilon^2 \rho \nabla \Delta \rho. \end{aligned}$$

This is the Navier-Stokes-Korteweg system (1).

In [3] the authors consider the Cauchy problem for the non–dissipative isothermal case of (1) in multiple space dimensions (Euler–Korteweg problem). They also allow that the third order term $\gamma \varepsilon^2 \rho \nabla \Delta \rho$ depends even nonlinearly on ρ. They prove the wellposedness of the Cauchy problem. The corresponding one-dimensional isothermal, inviscid initial value problem has been considered in [4]. Uniqueness and global existence of solutions, close to a stable equilibrium and furthermore local in time existence for (1) has been obtained in [6]. The existence of global weak solutions and periodic boundary conditions without any smallness assumptions on the data has been shown in [5]. Global existence results for weak solutions of (1) in 1–D with $\mu = 0$ and $\gamma = 0$ are available in [1]. Kotschote considers existence of the corresponding initial boundary value problem to (1) in [18].

Equation (1) can be also considered as a diffusive-dispersive regularization. The analytical and numerical background for the diffusive-dispersive regularizations for scalar conservation laws with non convex flux functions is the main subject in [13], [12]. In [26] the author studies a system of conservation laws as a simplest model for one dimensional isothermal elastodynamics with no body forces and constant reference density. In this system he extends the usual non convex stress by viscous and capillary stresses and obtains a model for phase transition. Then for traveling wave solutions the limit if the viscosity and the capillarity coefficient tend to zero can be controlled. The system reduces to an overdetermined boundary value problem of second order on the whole $\mathbb{R}$. Kinetic relations are then derived which gives the desired informations for the admissible boundary values.

Usually sharp interface models are derived as the limit of a diffuse ones. In [25] they do it just in the other way. They derive a diffuse interface model for the direct simulation of two-phase flows with surface tension, phase change and different viscosities in the two phases. For this they use ensemble averaging procedure on an atomic scale.

In classical hydrodynamics the zone between two phases or between two immiscible fluids is represented as a discontinuity. Due to [20] this is a good approximation if the thickness of the interface is small compared with other characteristic scales of the flow. This model breaks down if the thickness of the interface is comparable to the curvature or the distance between surfaces.

Slemrod indicates in [24] that the term $\rho \nabla \Delta \rho$ in (1) is necessary to describe phase transitions within this context.

A different approach for the modeling of two phase flows with phase transition can be found in [20]. In addition to the density, pressure and velocity as in (1) they use also an equation for the mass concentration of the fluids and end up with the so called Navier-Stokes-Cahn-Hilliard equations. In some sense this is a physically motivated regularization of the Euler equations. In a second part of the paper they consider also quasi-incompressible versions of the Navier-Stokes-Cahn-Hilliard equations.

In this paper we will study the behaviour of the pressure across the interface. Since a rigorous theory about this question is not available and difficult, we will concentrate on the static version of (1). In particular we will study the behaviour of the pressure in the limit if $\varepsilon \to 0$. In Section 2 we will quote some recent results which show that the difference of the pressures on both sides of the interface is of order ε. This seems to contradict the classical result of Landau and Lifschitz [19], which says that the differenceof the pressures on both sides of the interface is proportional tothe mean curvature of the interface. Jamet [15] tries toovercome this problem by defining a modified thermodynamic free energy density. The main idea consists in the definition of the function $\psi = \psi(r)$ in [15],(19). But from that paper it is not clear, how this function and its derivatives behave in neighborhoods next to $r = 0$ and $r = 1$ quantitatively. In this paper we will show that we get the expected jump relation for the differenceof the pressures on both sides of the interface if we use either a scaled surface tension or a modified definition of the pressure on the basis of a special scaling of the free energy density (see Section 5). In this context we will see that the scaling/capillarity quantity $\gamma\varepsilon^2$ can be related to the Mach number under certain conditions. Using this dependence we achieve an asymptotic expansion of p in the Mach number and the expected jump condition for p_2, the second order coefficient in the Mach number expansion of p, (see Section 4).
The arguments in Section 4 for the system (1) are prepared in Section 3 for the usual compressible Navier-Stokes equations.

2 Phase transition and the sharp interface condition

Similar as before we define (assume that $\gamma = 1$)

$$J_\varepsilon(\rho) := \int_\Omega \Big(W(\rho(x)) + \frac{\varepsilon^2}{2}|\nabla\rho(x)|^2\Big)dx \tag{13}$$

Then the functionals $\tilde{J}_\varepsilon$, see (10), and J_ε differ only in a constant $\tilde{M} = d_0 M + d_1|\Omega|$ under the constraint $\int_\Omega \rho(x)dx = M$ and have the same minimizers.

For $\varepsilon > 0$ the functional J_ε, in particular the term $\frac{\varepsilon^2}{2}|\nabla\rho(x)|^2$, penalizes the occurrence of a large interface. In this case the minimizers are characterized by the following theorem.

Theorem 1 *(see [21]) Let $\beta_1|\Omega| \le M \le \beta_2|\Omega|$, where β_1, β_2 are defined as above and let ρ_ε be a global minimizer of (13) with $\int_\Omega \rho dx = M$. Then the following statements hold:*

a) There exists a sequence $(\varepsilon_k)_k$, $\varepsilon_k > 0$, with $lim_{k\to\infty}\varepsilon_k = 0$ such that the corresponding sequence $(\rho_{\varepsilon_k})_k$ of global minimizers ρ_{ε_k} converges in $L^1(\Omega)$ as $k \to \infty$.

b) If $\rho_{\varepsilon_j} \to \rho_0$ in $L^1(\Omega)$ as $j \to \infty$ then $\rho_0(x) = \beta_1$ or $\rho_0(x) = \beta_2$ for a.e. $x \in \Omega$ where $\beta_1|A| + \beta_2|\Omega\backslash A| = M$ and $A := \{x \in \Omega | \rho_0(x) = \beta_1\}$.

c) The set A is a solution of the following geometric variational problem:

$$P_\Omega(A) = min\left\{P_\Omega(F) : F \subset \Omega, |F| = \frac{\beta_2|\Omega| - M}{\beta_2 - \beta_1}\right\},$$

where $P_\Omega(A)$ is the perimeter of A in Ω, see [11] for the definition.
(Roughly speaking this result expresses the fact that the boundary ∂A of A has minimal area since it can be shown by the theory of minimal surfaces that the reduced boundary $\partial^* A$ is smooth and $\mathcal{H}^{n-1}((\partial A \backslash \partial^* A) \cap \Omega) = 0$, where $\mathcal{H}^{n-1}$ is the $(n-1)$–dimensional Hausdorff measure in $\mathbb{R}^n$, (cf. [11]). Note, for sets G with Lipschitz–boundary we have $P_\Omega(G) = H_{n-1}(\partial G \cap \Omega)$.)

d) If $\rho_{\varepsilon_j} \to \rho_0$ in $L^1(\Omega)$ as $j \to \infty$ then the energy J_{ε_j} satisfies

$$J_{\varepsilon_j}(\rho_{\varepsilon_j}) = \int_\Omega \frac{\varepsilon_j^2}{2} |\nabla \rho_{\varepsilon_j}|^2 + W(\rho_{\varepsilon_j}) dx = \sqrt{2} c_0 P_\Omega(A) \varepsilon_j + o(\varepsilon_j),$$

where $c_0 := \int_{\beta_1}^{\beta_2} \sqrt{W(t)} dt$.

Remark 2 *Item b) of this theorem, in particular the properties of ρ_0 and of the sets A, B, indicate that equation (12) can be considered as a model for two phase flows with phase transition.*

Remark 3 *Notice that in the case of Theorem 1 the energy J_{ε_j} in the limit $\varepsilon_j \to 0$ is the same as for E_0 . This implies that in this model there is no contribution of some interfacial energy in the sharp limit.*

Furthermore we obtain that in the limit $\varepsilon_j \to 0$ the pressure $p(\rho)$ across the interphase is continuous. More precisely we get:

Theorem 4 *(see [8], Theorem 3.5) Let us assume that we are in the situation of Theorem 1, a) and b). Let $U \subset\subset A, V \subset\subset \Omega \backslash \bar{A}$ be open sets and ψ be sufficiently smooth. Then we have*

$$p_+(\rho_{\varepsilon_k}(x_2)) - p_-(\rho_{\varepsilon_k}(x_1)) = -\sqrt{2} c_0 (n-1) k_m \varepsilon_k + o(\varepsilon_k) \tag{14}$$

for $x_1 \in U$ and $x_2 \in V$ as $k \to \infty$, where the indexes $-$ and $+$ stand for the enclosed phase and the surrounding phase, see Fig. 1. The symbol k_m denotes the constant mean curvature of the (reduced) boundary of A which is given by the sum of the principle curvatures divided by $(n-1)$, i.e. $k_m = \mathrm{div_I}\nu/(\mathrm{n}-1)$, where the unit normal ν of the interface points into the direction of the matrix.

Remark 5 *Equation (14) implies in particular that for a two–phase system the pressure of the enclosed phase is always higher than the pressure of the surrounding phase.*

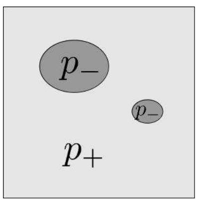


Figure 1: Pressure condition

At first glance the results of Theorem 4 seem not to be consistent with classical results of hydrodynamics. Landau and Lifschitz ([19], page 301) pointed out that in reality there is a layer of finite but small thickness between the two media which are in contact. But the layer is so small that it can be approximated as a curvilinear surface. For curvilinear interfaces between two media the pressure in both media is different. They derive in [19] that

$$p_+ - p_- = -\text{const } k_m, \tag{15}$$

holds on the interface, where k_m is the mean curvature of the curvilinear surface and $p_+ - p_-$ is the pressure jump across the interface. This seems to contradict the result of Theorem 4 which indicates that in the limit $\varepsilon \to 0$ we have on the interface

$$p_+ - p_- = 0. \tag{16}$$

The corresponding interface condition for the dynamical case with phase transition is

$$p_+ - p_- = \sigma k_m - [\rho(v_\nu - v_I)^2]_-^+ + \left[\mu \frac{\partial v_\nu}{\partial \nu}\right]_-^+ \tag{17}$$

and can be found in [9], formular (13).

In the next two sections we will look for a special scaling of (1) in order to recover (15) at least for special situations. First we will briefly repeat the limit of the compressible Navier-Stokes equations if the Mach number tends to zero. This will then be generalized to the Navier-Stokes-Korteweg equations in Section 4.

3 Zero Mach number limit for the compressible Navier-Stokes equations

In this section we will consider the zero Mach number limit for the compressible Navier-Stokes equations. It turns out that we get the incompressible Navier-Stokes equations in the limit (for low Mach number). Let us briefly discuss the main ideas. For the non-dimensionalization of

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0 \\ \partial_t(\rho v) + \nabla \cdot (\rho v v^t) + \nabla p(\rho) &= \mu \Delta v \end{aligned}$$

we will introduce the following characteristic quantities: $x_{ref}, t_{ref}, v_{ref}, p_{ref}, \rho_{ref}, c_{ref} := \sqrt{\frac{\gamma p_{ref}}{\rho_{ref}}}$ and the Mach number $M := \frac{v_{ref}}{c_{ref}}$. After non-dimensionalization the form of the compressible Navier-Stokes equations is given by

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0 \\ \partial_t(\rho v) + \nabla \cdot (\rho v v^t) + \frac{1}{M^2} \nabla p(\rho) &= \frac{1}{Re} \Delta v. \end{aligned} \tag{18}$$

Now we will formally consider the limit $M \to 0$ and assume that the following asymptotic expansions hold.

$$
\begin{aligned}
\rho(x,t) &= \rho_0(x,t) + M\rho_1(x,t) + M^2\rho_2(x,t) + O(M^3) \\
v(x,t) &= v_0(x,t) + Mv_1(x,t) + M^2 v_2(x,t) + O(M^3) \\
p(x,t) &= p(\rho(x,t)) = p_0(x,t) + Mp_1(x,t) + M^2 p_2(x,t) + O(M^3).
\end{aligned}
\tag{19}
$$

In case of the corresponding inviscid systems the asymptotic expansions have been proved rigorously in [16]. For the viscous system the arguments are only formal. Using the asymptotic expansion (19) in (18) and comparing terms with the coefficient M^{-2} and M^{-1} we obtain: $\nabla p_0(x,t) = 0$ and $\nabla p_1(x,t) = 0$ respectively. This implies $p_0 = p_0(t)$ and $p_1 = p_0(t)$. Conservation of mass in the whole set Ω implies $\int_{\partial\Omega} \nu v_0 = \int_\Omega \nabla \cdot v_0 = 0$ and since $\int_\Omega \partial_t \rho_0 + \rho_0(t) \int_\Omega \nabla \cdot v_0 = 0$ we have $\partial_t \rho_0(t) = 0$ and therefore $\rho_0(t) = \text{const}$ and $p_0(t) = \text{const}$. Again we use $\partial_t \rho_0 + \rho_0(t) \nabla \cdot v_0 = 0$ in order to obtain

$$
\nabla \cdot v_0 = 0. \tag{20}
$$

The momentum equation

$$
\partial_t(\rho v) + \nabla \cdot (\rho v v^t) + \frac{1}{M^2}\nabla p = \frac{1}{Re}\Delta v
$$

implies for terms of order M^0:

$$
\rho_0 \partial_t v_0 + \rho_0 \nabla \cdot (v_0 v_0^t) + \nabla p_2 = \frac{1}{Re}\Delta v_0. \tag{21}
$$

The equations (20) and (21) are just the system of the incompressible Navier-Stokes equations. Notice: The pressure which appears in the incompressible Navier-Stokes equations is p_2 while we have p for the compressible one. The relation between p_2 and p is given by (19).

4 Zero Mach number limit for the Navier-Stokes Korteweg equations

In this section we will repeat the arguments from Section 3 for the system (1). In order to shorten the notation we replace $\gamma\varepsilon^2$ by λ and $\mu = 1$. We consider

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho v) &= 0 \\
\partial_t(\rho v) + \nabla \cdot (\rho v v^t + p(\rho) I) &= \Delta v + \lambda \rho \nabla \Delta \rho.
\end{aligned}
\tag{22}
$$

In this case the non-dimensionalization form of the Navier-Stokes- Korteweg system is given by

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho v) &= 0 \\
\partial_t(\rho v) + \nabla \cdot (\rho v v^t + \frac{1}{M^2} p(\rho) I) &= \frac{1}{Re}\Delta v + \frac{\lambda b^2}{M^2}\rho \nabla \Delta \rho
\end{aligned}
$$

where $b^2 := \frac{\rho_{ref}^2}{x_{ref}^2 p_{ref}}$. The corresponding dimensionless total energy has the form

$$\mathcal{E}(v,\rho) = \int_\Omega \Big(M^2 \frac{\rho}{2} v^2 + \rho\psi(\rho) + \frac{\lambda}{2} |\nabla\rho|^2 \Big).$$

Next we want to concentrate on solutions of (1) for v and ρ such that the scaled energy

$$\frac{1}{\sqrt{\lambda}} (\mathcal{E}(v,\rho) - \tilde{M}) = \int_\Omega \Big(\frac{M^2}{\sqrt{\lambda}} \frac{\rho}{2} v^2 + \frac{1}{\sqrt{\lambda}} \rho\psi(\rho) + \frac{\sqrt{\lambda}}{2} |\nabla\rho|^2 \Big)$$

is uniformly bounded as $\lambda \to 0$. This leads to the condition $M^4 \le c_1 \lambda$, $c_1 > 0$ some constant, and implies for λ the following ansatz:

$$\lambda = c_2 M^\delta$$

with $0 < \delta \le 4$ and $c_2 > 0$ some constant if $M \ll 1$. In the following we will choose for δ the upper bound such that

$$\lambda = \frac{M^4}{b^2} \tag{23}$$

and obtain

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0 \\ \partial_t(\rho v) + \nabla \cdot (\rho v v^t + \frac{1}{M^2} p(\rho) I) &= \frac{1}{Re} \Delta v + M^2 \rho \nabla \Delta \rho. \end{aligned} \tag{24}$$

Using asymptotic expansions as in (19) we obtain in each phase the same equations as in (20), (21), i.e.

$$\nabla \cdot v_0 = 0 \tag{25}$$

and

$$\rho_0 \partial_t v_0 + \rho_0 \nabla \cdot (v_0 v_0^t) + \nabla p_2 = \frac{1}{Re} \Delta v_0. \tag{26}$$

In order to derive the pressure condition on the interface between the two phases we will only consider the static case which we obtain from (24) and which is given by

$$\nabla p = M^4 \rho \nabla \Delta \rho.$$

This implies as before (see (10), (11), (12)):

$$\begin{aligned} \rho \nabla W'(\rho) &= M^4 \rho \nabla \Delta \rho \\ \nabla W'(\rho) &= M^4 \nabla \Delta \rho \end{aligned}$$

and in particular

$$W'(\rho) \quad = \quad M^4\Delta\rho + c_0(M). \tag{27}$$

The last equation (27) corresponds to (10) with M^2 instead of ε. Therefore we can apply the whole theory to (27) as in [8] to study the behaviour of ρ for the limit $M \to 0$. We obtain

$$c_0(M) = c_1 k_m M^2$$

and therefore the condition

$$p(\rho(x_l)) - p(\rho(x_v)) \quad = \quad c_1 k_m M^2 + o(M^2).$$

Using the asymptotic expansion $p(x,t) = p_0(x,t) + Mp_1(x,t) + M^2p_2(x,t) + O(M^3)$ we derive

$$\begin{aligned} p(\rho(x_l)) - p(\rho(x_v)) &= p_2(\rho(x_l)) - p_2(\rho(x_v)) + o(M^2) \\ &= c_1 k_m M^2 + o(M^2). \end{aligned} \tag{28}$$

This shows that we have recovered the relation (15) for the pressure p_2 at least for the special scaling of λ as in (23). Notice that we obtain this relation for the pressure p_2 (sometimes called hydrodynamic pressure) which enters the incompressible equation

$$\rho_0\partial_t v_0 + \rho_0\nabla\cdot(v_0 v_0^t) + \nabla p_2(\rho) \quad = \quad \frac{1}{Re}\Delta v_0. \tag{29}$$

The equation has been obtained from the compressible Navier-Stokes equations in the limit $M \to 0$.

5 Phase field like scaling

In Section 1, (4), (6) we have considered

$$\tilde{E}_0(\rho) = \int_\Omega \rho\psi(\rho)dx \to \text{Minimum}. \tag{30}$$

We have seen that (30) does not take into account the length of the interfaces. Furthermore the surface tension of the interface is not included. The minimum of (30) is denoted by $\tilde{M}$.

In order to minimize the length of the interface we considered the functional (see also (10))

$$\tilde{J}_\varepsilon(\rho) := \int_\Omega \rho\psi(\rho) + \frac{\varepsilon^2}{2}|\nabla\rho|^2 dx \to \text{Minimum}. \tag{31}$$

This is equivalent to

$$\tilde{J}_\varepsilon(\rho) := \int_\Omega W(\rho) + \frac{\varepsilon^2}{2}|\nabla\rho|^2 dx + \tilde{M} \to \text{Minimum}. \tag{32}$$

Denote the minimum by $\tilde{M}_\varepsilon$. We have for $\varepsilon \to 0$:

$$\tilde{M}_\varepsilon \to \tilde{M}.$$

This means, in the limit $\varepsilon \to 0$ we have for (30) the same energy as for (32), i.e. there is no contribution which is due to surface energy. However, we obtain from Theorem 1 the minimal area property of the interface.

In the following we introduce two different ways of scalings to get from the phase field model (31) a corresponding sharp interface model which includes surface energy. This, in turn, leads to a non–vanishing jump condition for the pressures at the interface.

(i) The scaled surface tension $\hat{\sigma}$
From Theorem 1 we conclude that $\tilde{J}_\varepsilon(\rho_\varepsilon)$ has the following asymptotic behaviour

$$\tilde{J}_\varepsilon(\rho_\varepsilon) = \sqrt{2}c_0\varepsilon \int_{\partial^* A} d\mathcal{H}^{n-1} + \int_A \beta_1\psi(\beta_1) + \int_{\Omega\backslash A} \beta_2\psi(\beta_2) + o(\varepsilon)$$

as $\varepsilon \to 0$. This means that for this model the surface tension is related to the width ε of the interface, i.e. $\sigma = \sqrt{2}c_0\varepsilon$.
In order to get more insight into equilibrium conditions of that kind of functionals, let us study necessary conditions for minimizers of the energy functional

$$\hat{J}_\varepsilon(\rho) = \hat{\sigma}\varepsilon \int_{\partial^* A} d\mathcal{H}^{n-1} + \int_\Omega \rho\psi(\rho)dx$$

under the condition $\int_\Omega \rho\, dx = M$.

Theorem 6 *Let $\Omega \subset \mathbb{R}^N$ be a domain with C^1–boundary and let $\rho \in BV(\Omega)$, where $BV(\Omega)$ denotes the space of functions of bounded variation. Furthermore, let $A \subset \Omega$ be a non–empty open set. Then any minimizer $\hat{\rho}_\varepsilon$ of the energy functional*

$$\hat{J}_\varepsilon(\rho) = \hat{\sigma}\varepsilon \int_{\partial^* A} d\mathcal{H}^{n-1} + \int_\Omega \rho\psi(\rho)dx, \qquad \int_\Omega \rho dx = M,$$

fulfills the condition

(i) $p(\hat{\rho}_\varepsilon^+) - p(\hat{\rho}_\varepsilon^-) = -2\hat{\sigma}\varepsilon(n-1)k_m$ *on $\partial^* A$, where $\hat{\rho}_\varepsilon^+$ and $\hat{\rho}_\varepsilon^-$ denote the traces of $\hat{\rho}_\varepsilon$ in Ω_- and Ω_+ respectively.*

If, in addition, $\hat{\rho}_\varepsilon$ is a global minimizer with $\hat{\rho}_\varepsilon(x) \in (-\infty, \alpha_1]$ in A and $\hat{\rho}_\varepsilon(x) \in [\alpha_2, \infty)$ in $\Omega\backslash\overline{A}$. Then

(ii) $\hat{\rho}_\varepsilon(x) = \beta_1$ *for a.e.* $x \in \Omega$ *and* $\hat{\rho}_\varepsilon(x) = \beta_2$ *for a.e.* $x \in \Omega\backslash\overline{A}$.

Proof: To obtain the pressure condition we choose variations by means of a one parametric family of diffeomorphisms of Ω given by the initial value problem

$$\Phi(0,x)=x \qquad \text{and} \qquad \Phi_{,\tau}(\tau,x)=\xi(\Phi(\tau,x))$$

for $x\in\Omega$, where $\xi\in C_c^\infty(\Omega,\mathbb{R}^n)$ is arbitrary. Then Φ fulfills the following properties:

(i) $\Phi(\tau,\cdot)$ is the inverse of $\Phi(-\tau,\cdot)$, i. e. $\Phi(\tau,\Phi(-\tau,x))=x$. In consequence,

$$Id=\Phi_{,x}\big(\tau,\Phi(-\tau,x)\big)\,\Phi_{,x}(-\tau,x).$$

(ii)

$$\frac{d}{d\tau}\big(\det\Phi_{,x}(\tau,x)\big)|_{\tau=0}=\big(\nabla\cdot\xi\big)(x).$$

(iii)

$$\frac{d}{d\tau}\Big(\big(\Phi_{,x}(\tau,x)\big)^{-1}\Big)\Big|_{\tau=0}=-\nabla\xi(x).$$

We set $x=\Phi(-\tau,y)$ and consider for any fixed $h\in BV(\Omega)$ with $\int_\Omega h dx\neq 0$:

$$\begin{aligned} j(\tau,\omega) &:= \int_\Omega\Big(\hat\rho_\varepsilon(\Phi(-\tau,y))+\omega h(\Phi(-\tau,y))\Big)dy-M\\ &= \int_\Omega\Big(\hat\rho_\varepsilon(x)+\omega h(x)\Big)\,\mathrm{det}\Phi_{,x}(\tau,x)dx-M. \end{aligned}$$

Clearly, $j(0,0)=0$. Moreover, $j\in C^1$ and

$$\begin{aligned} \frac{\partial j}{\partial\tau}(\tau,\omega) &= \int_\Omega\Big(\hat\rho_\varepsilon(x)+\omega h(x)\Big)\nabla\cdot\xi\,dx,\\ \frac{\partial j}{\partial\omega}(\tau,\omega) &= \int_\Omega h(x)\mathrm{det}\Phi_{,x}(\tau,x)dx \quad \text{with} \quad \frac{\partial j}{\partial\omega}(0,0)\neq 0. \end{aligned}$$

Due to the Implicit Function Theorem, there exists a C^1–function $\eta:\mathbb{R}\to\mathbb{R}$ such that

$$\eta(0)=0 \qquad \text{and} \qquad j(\tau,\eta(\tau))=0 \tag{33}$$

for τ sufficiently small. Without loss of generality we may assume that (33) holds for $\tau\in[-\tau_0,\tau_0]$. Differentiating, we get

$$\frac{\partial j}{\partial\tau}(\tau,\eta(\tau))+\frac{\partial j}{\partial\omega}(\tau,\eta(\tau))\,\eta'(\tau)=0$$

Consequently,

$$\eta'(0)=-\frac{\frac{\partial j}{\partial\tau}(0,0)}{\frac{\partial j}{\partial\omega}(0,0)}.$$

Now we set

$$\hat\rho_\varepsilon^\tau(x)=\hat\rho_\varepsilon(\Phi(-\tau,x))+\eta(\tau)h(\Phi(-\tau,x)) \quad \text{for} \quad |\tau|\le\tau_0.$$

According to (33) $\hat\rho_\varepsilon^\tau$, $\tau\in[-\tau_0,\tau_0]$, are admissible comparison functions which implies

$$0=\frac{d}{d\tau}\hat J_\varepsilon(\hat\rho_\varepsilon^\tau)\big|_{\tau=0}$$

since $\hat\rho_\varepsilon = \hat\rho^0_\varepsilon$.
Next we determine the above time derivative. The first variation of the area integral, i.e. $\hat\sigma\varepsilon\int_{\partial^* A} d\mathcal{H}^{n-1}$, is computed in the setting of sets of bounded perimeter (see for instance [11] and [10]). For completeness we sketch the arguments. In the following χ denotes the characteristic function of A and $|\nabla\chi|$ the variation. As before we set $x = \Phi(-\tau, y)$ and define

$$\chi^\tau(x) = \chi(\Phi(-\tau, x)). \tag{34}$$

Then

$$\begin{aligned}
\hat\sigma\varepsilon\int_{\partial^* A} d\mathcal{H}^{n-1} &= \hat\sigma\varepsilon\int_\Omega |\nabla\chi^\tau| = \hat\sigma\varepsilon\int_\Omega |\nabla\chi(\Phi(-\tau, y))|dy \\
&= \hat\sigma\varepsilon\int_\Omega \det\Phi_{,x}(\tau, x)\big|\big(\Phi_{,x}(-\tau, x)\big)^T\nabla\chi(x)\big|dx \\
&= \hat\sigma\varepsilon\int_\Omega |(\Phi_{,x}(\tau, x))^{-T}\nu|\det\Phi_{,x}(\tau, x)|\nabla\chi|dx,
\end{aligned}$$

where $\nu = -\frac{\nabla\chi}{|\nabla\chi|}$ is the generalized unit normal which is a $|\nabla\chi|$–measurable function. ¿From the properties (i) – (iii) we conclude

$$\frac{d}{d\tau}\Big(\int_\Omega |\chi^\tau|\Big)\Big|_{\tau=0} = \int_\Omega (\nabla\cdot\xi - \nu\cdot\nabla\xi\nu)|\nabla\chi|.$$

Applying Gauß' theorem on manifolds gives

$$\begin{aligned}
\int_\Omega (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)|\nabla\chi| &= \int_{\partial^* A} (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)d\mathcal{H}^{n-1} \\
&= \int_{\partial^* A} \big(\mathrm{div}_{\partial^* A}\xi\big)\, d\mathcal{H}^{n-1} = \int_{\partial^* A} \big(\mathrm{div}_{\partial^* A}\nu\big)(\xi\cdot\nu)d\mathcal{H}^{n-1},
\end{aligned}$$

where the symbol $\mathrm{div}_{\partial^* A}$ denotes the tangential divergence with respect to the interface $\partial^* A$ and ν is the unit outer normal vector pointing into the direction of the +–phase. Now we compute $\frac{d}{d\tau}\int_\Omega \hat\rho^\tau_\varepsilon\psi(\hat\rho^\tau_\varepsilon)dy$:

$$\begin{aligned}
&\frac{d}{d\tau}\int_\Omega \hat\rho^\tau_\varepsilon\psi(\hat\rho^\tau_\varepsilon)dy\Big|_{\tau=0} \\
&= \frac{d}{d\tau}\int_\Omega \Big(\hat\rho_\varepsilon(x) + \eta(\tau)h(x)\Big)\psi\Big(\hat\rho_\varepsilon(x) + \eta(\tau)h(x)\Big)\det\Phi_{,x}(\tau, x)\,dx\Big|_{\tau=0} \\
&= \int_\Omega \hat\rho_\varepsilon\psi(\hat\rho_\varepsilon(x))\,\nabla\cdot\xi(x)\,dx + \int_\Omega \frac{\partial\hat\rho_\varepsilon\psi(\hat\rho_\varepsilon(x))}{\partial\hat\rho_\varepsilon}\eta'(0)h(x)dx \\
&= \int_\Omega \hat\rho_\varepsilon\psi(\hat\rho_\varepsilon)\,\nabla\cdot\xi\,dx + \int_\Omega \frac{\partial\hat\rho_\varepsilon\psi(\hat\rho_\varepsilon)}{\partial\hat\rho_\varepsilon}\,\frac{\int_\Omega\hat\rho_\varepsilon\,\nabla\cdot\xi\,dx}{\int_\Omega h dx}\,h\,dx \\
&= \int_\Omega \hat\rho_\varepsilon\psi(\hat\rho_\varepsilon)\,\nabla\cdot\xi\,dx + \lambda\int_\Omega \hat\rho_\varepsilon\,\nabla\cdot\xi dx,
\end{aligned}$$

with $\lambda = \int_\Omega \frac{\partial \hat\rho_\varepsilon \psi(\hat\rho_\varepsilon)}{\partial \hat\rho_\varepsilon} h dx / \int_\Omega h dx$. By trace–arguments of BV–functions we derive

$$\begin{aligned}\int_\Omega \Big(\hat\rho_\varepsilon(x)\psi(\hat\rho_\varepsilon(x)) + \lambda\,\hat\rho_\varepsilon\Big)\,\nabla\cdot\xi\,dx = \int_A \Big(\frac{\partial \hat\rho_\varepsilon \psi(\hat\rho_\varepsilon)}{\partial \hat\rho_\varepsilon} + \lambda\Big)\,\xi\cdot d[D\hat\rho_\varepsilon] \\ + \int_{\Omega\setminus\overline{A}} \Big(\frac{\partial \hat\rho_\varepsilon \psi(\hat\rho_\varepsilon)}{\partial \hat\rho_\varepsilon} + \lambda\Big)\,\xi\cdot d[D\hat\rho_\varepsilon] \\ + \int_{\partial^* A} \Big(\hat\rho_\varepsilon^-\psi(\hat\rho_\varepsilon^-) - \hat\rho_\varepsilon^+\psi(\hat\rho_\varepsilon^+) + \lambda(\hat\rho_\varepsilon^- - \hat\rho_\varepsilon^+)\Big)\,\xi\cdot\nu\,d\mathcal{H}^{n-1}.\end{aligned}$$

Here, $[Df]$ denotes the derivative of f in the sense of Radon measure. Thus we obtain

$$\begin{aligned}0 = \frac{d}{d\tau}\hat J(\hat\rho_\varepsilon^\tau)\Big|_{\tau=0} = \hat\sigma\varepsilon \int_{\partial^* A} \big(\mathrm{div}_{\partial^* A}\nu\big)(\xi\cdot\nu)d\mathcal{H}^{n-1} \\ + \int_A \Big(\frac{\partial \hat\rho_\varepsilon \psi(\hat\rho_\varepsilon)}{\partial \hat\rho_\varepsilon} + \lambda\Big)\,\xi\cdot d[D\hat\rho_\varepsilon] + \int_{\Omega\setminus\overline{A}} \Big(\frac{\partial \hat\rho_\varepsilon \psi(\hat\rho_\varepsilon)}{\partial \hat\rho_\varepsilon} + \lambda\Big)\,\xi\cdot d[D\hat\rho_\varepsilon] \\ + \int_{\partial^* A} \Big(\hat\rho_\varepsilon^-\psi(\hat\rho_\varepsilon^-) - (\hat\rho_\varepsilon^+\psi(\hat\rho_\varepsilon^+) + \lambda(\hat\rho_\varepsilon^- - \hat\rho_\varepsilon^+)\Big)\,\xi\cdot\nu\,d\mathcal{H}^{n-1}.\end{aligned}$$

Since ξ may be arbitrarily chosen, we get

$$\frac{\partial \hat\rho_\varepsilon \psi(\hat\rho_\varepsilon)}{\partial \hat\rho_\varepsilon} = -\lambda \quad \text{for a.e. } x\in\Omega.$$

Now we take variations in the neighbourhood of a point of $\partial^* A$ which are of the form $\xi = g\nu$, where $g \in C_0^\infty(\Omega)$ is arbitrary. This yields

$$\hat\sigma\varepsilon\,\mathrm{div}_{\partial^* A}\nu = \hat\rho_\varepsilon^+\psi(\hat\rho_\varepsilon^+) - \hat\rho_\varepsilon^-\psi(\hat\rho_\varepsilon^-) + \lambda(\hat\rho_\varepsilon^+ - \hat\rho_\varepsilon^-) \qquad \text{on } \partial^* A.$$

In consequence,

$$p(\hat\rho_\varepsilon^+) - p(\hat\rho_\varepsilon^-) = -\hat\sigma\varepsilon\,(n-1)k_m \qquad \text{on } \partial^* A$$

since $k_m = \frac{1}{n-1}\mathrm{div}_{\partial^* A}\nu$ and $p(\hat\rho_\varepsilon) = \hat\rho_\varepsilon \frac{\partial \hat\rho_\varepsilon \psi(\hat\rho_\varepsilon)}{\partial \hat\rho_\varepsilon} - \hat\rho_\varepsilon\psi(\hat\rho_\varepsilon)$.

To item (ii): We subtract from the energy functional $\hat J_\varepsilon(\hat\rho_\varepsilon)$ the Maxwell line, i.e.

$$\acute J_\varepsilon(\hat\rho_\varepsilon) = \hat J_\varepsilon(\hat\rho_\varepsilon) - \int_\Omega d_0 + d_1\hat\rho_\varepsilon\,dx.$$

Then the minimizers of $\tilde J_\varepsilon$ and $\acute J_\varepsilon$ are the same due to the constraint $\int_\Omega \rho dx = M$. The global minimizers of $\tilde J_\varepsilon$ are obviously of the structure:

$$\acute\rho_\varepsilon(x) = \beta_1 \text{ for a.e. } x\in A \quad \text{ and } \quad \acute\rho_\varepsilon(x) = \beta_2 \text{ for a.e. } x\in\Omega\setminus\overline{A}.$$

This completes the proof. ■

Remark 7 *From Theorem 1 and Theorem 6 we can extract the following conclusions.*

If we replace the small layer of thickness ε between the two media by a curvilinear surface as Landau and Lifschitz suggested, we also have to rescale the surface tension. The dimensionless surface tension is then given by the free energy per unit length, i.e. $\hat\sigma = \sigma/\varepsilon$. Correspondingly we have to replace the difference of the pressures by their dimensionless ones, i.e.

$$\hat p_+ - \hat p_- = \frac{p_+}{\varepsilon} - \frac{p_-}{\varepsilon} = 2\hat\sigma k_m.$$

(ii) The scaled free energy $\rho\psi_\varepsilon(\rho)$
Another possibility is to scale already the free energy density in the phase field model, cf. (32), such that we obtain in the sharp limit $\varepsilon \to 0$ a non–vanishing contribution of the surface energy. This type of scaling is in particular for many numerical applications of crucial importance.

We will modify the energy functional of (32) in such a way that we keep the structure of the minimizers but the corresponding limit for $\varepsilon \to 0$ is different from $\tilde{M}$. This can be obtained by a suitable scaling of

$$W(\rho) + \frac{\varepsilon^2}{2}|\nabla\rho|^2$$

by some power of ε. In order to get some contribution which is different from 0 and ∞ we have to scale with $\frac{1}{\varepsilon}$.

Therefore we consider the case

$$I_\varepsilon(\rho) := \frac{1}{\varepsilon}\int_\Omega W(\rho) + \frac{\varepsilon^2}{2}|\nabla\rho|^2 dx + \tilde{M} \to \text{Minimum.} \tag{35}$$

The limit $(\varepsilon \to 0)$ of the energy I_ε is now $\sqrt{2}c_0 P_\Omega(A) + \tilde{M}$, which is different from that one in (32). The minimizers are the same as in (32).

It turns out that the functional in (35) satisfies the following identities.

$$\begin{aligned}
I_\varepsilon(\rho) &= \int_\Omega \frac{1}{\varepsilon}W(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2 dx + \tilde{M} \\
&= \int_\Omega \frac{1}{\varepsilon}\left(\rho\psi(\rho) - l(\rho)\right) + \frac{\varepsilon}{2}|\nabla\rho|^2 dx + d_0 M + d_1|\Omega| \\
&= \int_\Omega \frac{1}{\varepsilon}\left(\rho\psi(\rho) - l(\rho)\right) + \frac{\varepsilon}{2}|\nabla\rho|^2 dx + \int_\Omega l(\rho) \\
&= \int_\Omega \frac{1}{\varepsilon}\left(\rho\psi(\rho) - l(\rho)\right) + l(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2 dx \\
&= \int_\Omega \rho\psi_\varepsilon(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2 dx
\end{aligned} \tag{36}$$

where $\psi_\varepsilon(\rho)$ is defined by

$$\rho\psi_\varepsilon(\rho) = \frac{1}{\varepsilon}\left(\rho\psi(\rho) - l(\rho)\right) + l(\rho). \tag{37}$$

Therefore instead of (32) we consider the functional

$$I_\varepsilon(\rho) \quad = \quad \int_\Omega \rho\psi_\varepsilon(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2 dx. \tag{38}$$

For the following arguments it is important to notice, that the minimizers of I_ε and $\tilde{J}_\varepsilon$ are the same since $I_\varepsilon - \tilde{M} = \frac{1}{\varepsilon}(\tilde{J}_\varepsilon - \tilde{M})$ but the values of the minima of I_ε are different. Let us consider the Euler-Lagrange-equation of (38):

$$\frac{d}{d\rho}(\rho\psi_\varepsilon(\rho)) - \varepsilon\Delta\rho = \tilde{\lambda}_\varepsilon \tag{39}$$

where $\tilde{\lambda}_\varepsilon$ is the Lagrange multiplier with respect to the constraint $\int_\Omega \rho = M$.
Since $p_\varepsilon(\rho) := \rho^2\psi'_\varepsilon(\rho)$ we have

$$p_\varepsilon(\rho) = \frac{1}{\varepsilon}p(\rho) + \frac{1-\varepsilon}{\varepsilon}d_1 \tag{40}$$

where $p(\rho) = \rho^2\psi'(\rho)$ is defined as in (2) (see also [8]). If $x \in U$ or $x \in V$ we can prove that $p_\varepsilon(\rho_\varepsilon(x))$ converges for a subsequence $\varepsilon \to 0$. This can be seen as follows. On U we have (notice that also ρ depends on ε)

$$\begin{aligned} p_\varepsilon(\rho) &= -\rho\psi_\varepsilon(\rho) + \frac{\rho}{\varepsilon}W'(\rho) + \rho(x)d_0 \\ &= -\frac{1}{\varepsilon}(\rho\psi(\rho) - l(\rho)) - l(\rho) + \frac{\rho}{\varepsilon}W'(\rho) + \rho(x)d_0 \\ &= -\frac{1}{\varepsilon}W(\rho(x)) + \frac{\rho}{\varepsilon}W'(\rho) - d_1. \end{aligned}$$

Due to [8] (see proof of Theorem 3.4) the convergence of $\frac{\rho}{\varepsilon}W'(\rho)$ is valid. Since W is continuous and $|\rho(x) - \beta_1| = O(\varepsilon)$ (see [8], Theorem 4.10) we have the pointwise convergence also for the term $\frac{1}{\varepsilon}W(\rho(x))$. On V we can argue in a similar way.

In order to see the relation to the static version of the Navier-Stokes- Korteweg equation we use

$$\nabla p_\varepsilon(\rho) = 2\rho\nabla\rho\psi'_\varepsilon(\rho) + \rho^2\psi''_\varepsilon(\rho)\nabla\rho. \tag{41}$$

and (39) to obtain

$$\begin{aligned} \rho\nabla\frac{d}{d\rho}(\rho\psi_\varepsilon(\rho)) &= \varepsilon\rho\nabla\Delta\rho \\ \rho\nabla\left(\psi_\varepsilon(\rho) + \rho\psi'_\varepsilon(\rho)\right) &= \varepsilon\rho\nabla\Delta\rho \\ \rho\left(\psi'_\varepsilon(\rho)\nabla\rho + \nabla\rho\psi'_\varepsilon(\rho) + \rho\psi''_\varepsilon(\rho)\nabla\rho\right) &= \varepsilon\rho\nabla\Delta\rho \\ \rho\left(2\psi'_\varepsilon(\rho)\nabla\rho + \rho\psi''_\varepsilon(\rho)\nabla\rho\right) &= \varepsilon\rho\nabla\Delta\rho \\ \nabla p_\varepsilon(\rho) &= \varepsilon\rho\nabla\Delta\rho. \end{aligned}$$

In fact this is the static form of the Navier-Stokes-Korteweg equation for the pressure p_ε as defined in (40) and the functional I_ε as in (35). From Theorem 4 we obtain for p

$$p(\rho_{\varepsilon_k}(x_2)) - p(\rho_{\varepsilon_k}(x_1)) = -\sqrt{2}c_0(n-1)k_m\varepsilon_k + o(\varepsilon_k)$$

for $x_1 \in U$ and $x_2 \in V$ as $k \to \infty$, where k_m is the mean curvature of the boundary of A. Then (40) implies for p_ε the jump condition

$$\begin{aligned} p_\varepsilon(\rho(x_2)) - p_\varepsilon(\rho(x_1)) &= \frac{1}{\varepsilon}p(\rho(x_2)) - \frac{1}{\varepsilon}p(\rho(x_1)) \\ &= -\sqrt{2}c_0(n-1)k_m + o(1). \end{aligned}$$

This means that for the pressure $p_\varepsilon(\rho)$ as defined in (40) we obtain the jump condition as in Landau and Lifschitz ([19], page 301), i.e. (15). While p is the thermodynamic pressure as defined in (2) and which appears in (1) the pressure $p_\varepsilon(\rho)$ as in (40) behaves more "incompressible", since small perturbations for ρ imply large perturbations for the pressure $p_\varepsilon(\rho)$ if ε is small.

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