Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Pricing CMS spreads in the Libor market model

Denis Belomestny[1], Anastasia Kolodko[1], John Schoenmakers[1]

submitted: 12th December 2008

[1] Weierstrass Institute for Applied Analysis and Stochastics,
Mohrenstr. 39, 10117 Berlin;
E-mail: belomest@wias-berlin.de, kolodko@wias-berlin.de, schoenma@wias-berlin.de.

No. 1386
Berlin 2008

2000 *Mathematics Subject Classification.* 60G44, 60G51, 91B28.

Key words and phrases. CMS spread option, Margrabe's formula, Libor market model.

Partial support by the Deutsche Forschungsgemeinschaft through the SFB 649 'Economic Risk' and the DFG Research Center MATHEON 'Mathematics for Key Technologies' in Berlin is gratefully acknowledged.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We present two approximation methods for pricing of CMS spread options in Libor market models. Both approaches are based on approximating the underlying swap rates with lognormal processes under suitable measures. The first method is derived straightforwardly from the Libor market model. The second one uses a convexity adjustment technique under a linear swap model assumption. A numerical study demonstrates that both methods provide satisfactory approximations of spread option prices and can be used for calibration of a Libor market model to the CMS spread option market.

1 Introduction

Swap rate dependent derivatives have become increasingly popular in the interest rate market. Typical examples are the CMS spreads and CMS spread options. A constant maturity swap (CMS) spread note is a derivative with a payoff based on the difference of two swap rates of specific maturities. For example, a CMS spread note might pay quarterly coupons based on the difference between quarterly fixings of the 10-year and 5-year semi-annual swap rates. The coupons of such a structure depend on the slope of the yield curve; the Note would therefore be traded by parties who wish to take a view on future relative changes in different parts of the yield curve. The steeper the yield curve, the greater the coupon - giving rise to the term "steepener" for certain CMS spread instruments. Since the beginning of 2006, the yield curve in both the US and Europe has flattened dramatically and has thus led to a rapidly shrinking CMS spread. This has caused the classic steepener to underperform and investors who had positions in CMS spread instruments experienced large losses due to a significantly lower coupon. For these reasons, such products have become less attractive, and so their level of liquidity is thus also decreasing. However, mark-to-market valuations still need to be performed for the many outstanding products on investors' books.

The price of CMS spread is usually computed under the forward measure (induced by next-after-maturity zero coupon bond). It is convenient to price CMS spread options in a suitable forward probability measure. If a lognormal approximation of the CMS rate and the correlation between swap rates is available under this measure, the price of the CMS spread option can be approximately computed via the well-known Margrabe's formula. In this paper we present two lognormal approximations for CMS rates in a forward measure. The first one is derived straightforwardly from the Libor market model, where the Libor drift is approximated with it's initial value ('freezing'), or in a refined way with a suitable normal approximation. The second approach is based on a convexity adjustment technique (see, e.g., [2]). It is important to note that in contrast to the convexity adjustment approach, the lognormal approximation provides a natural way to approximate the correlation between logarithms of swap rates in the CMS spread pricing formula.

The structure of the paper is as follows. First, we recall some definitions of Libor market models and CMS spread options. Then we derive approximation formulas for swap dynamics under the terminal measure. Next the approach based on the CMS convexity adjustment will be presented. Finally we compare both methods numerically and study the calibration of the Libor model to market CMS spread prices.

2 Main setup

Consider a fixed sequence of tenor dates $0 =: T_0 < T_1 < \ldots < T_n$, together with a sequence of so called day-count fractions $\delta_i := T_{i+1} - T_i,\ i = 1, \ldots, n-1$. With respect to this tenor structure we consider zero bond processes $B_i,\ i = 1, \ldots, n$, where each B_i lives on the interval $[0, T_i]$ and ends up with its face value $B_i(T_i) = 1$. With respect to this bond system we deduce a system of forward rates, called Libor rates, which are defined by

$$L_i(t) := \frac{1}{\delta_i}\left(\frac{B_i(t)}{B_{i+1}(t)} - 1\right), \quad 0 \le t \le T_i,\ 1 \le i \le n-1.$$

Note that L_i is the annualized effective forward rate to be contracted at the date t, for a loan over a forward period $[T_i, T_{i+1}]$. Based on this rate one has to pay at T_{i+1} an interest amount of $\$\delta_i L_i(T_i)$ on a \$1 notional.

For a pre-specified volatility process $\gamma_i \in \mathbb{R}^m$, adapted to the filtration generated by some standard Brownian motion $W \in \mathbb{R}^m$, the dynamics of the corresponding Libor model have the form,

$$\frac{dL_i}{L_i} = [\ldots]dt + \gamma_i^\top dW \tag{1}$$

$i = 1, ..., n-1$. The drift term, adumbrated by the dots, is known under different numeraire measures, such as the risk-neutral, spot, terminal and all measures induced by individual bonds taken as numeraire. If the processes $t \to \gamma_i(t)$ in (1) are deterministic, one speaks of a Libor *market* model (for details see [1], [4] for example).

An interest rate swap is a contract to exchange a series of floating interest payments in return for a series of fixed rate payments. Consider a series of payment dates between T_{p+1} and T_q, $q > p$. The fixed leg of the swap pays $\delta_j K$ at each time T_{j+1}, $j = p, \ldots, q-1$ where $\delta_j = T_{j+1} - T_j$. In return, the floating leg pays $\delta_j L_j(T_j)$ at time T_{j+1}, where $L_j(T_j)$ is the rate fixed at time T_j for payment at T_{j+1}. Thus, the time t value of the interest rate swap is

$$\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)(L_j(t) - K).$$

The swap rate $S_{p,q}(t)$ is the value of the fixed rate K, such that the present value of the contract is zero, hence after some rearranging

$$S_{p,q}(t) = \frac{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t) L_j(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)} = \frac{B_p(t) - B_q(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)}. \tag{2}$$

So $S_{p,q}$ is a martingale under the probability measure $P_{p,q}$, induced by the annuity numeraire $B_{p,q} = \sum_{j=p}^{q-1} \delta_j B_{j+1}(t)$. A *constant maturity swap (CMS)* is a contract to exchange floating swap rate in return to a fixed rate (or Libor) at a settlement dates. Pricing of *CMS spread options* essentially comes down to the valuation of a pay-off of the form

$$(S_{p,q'}(T_p) - S_{p,q}(T_p) - K)^+ \tag{3}$$

at a future date T_r, $T_r \geq T_p$, where typically $q' > q > p$.

Assume that a close lognormal approximations for $S_{p,q}$ and $S_{p,q'}$ under the probability measure P_r (induced by the zero bond B_r) are available,

$$\begin{aligned} S_{p,q}(T_p) &\approx S^*_{p,q}(0) \exp(-\frac{1}{2}(\sigma^*_{p,q})^2 T_p + \sigma^*_{p,q} W^r_{T_p}), \\ S_{p,q'}(T_p) &\approx S^*_{p,q'}(0) \exp(-\frac{1}{2}(\sigma^*_{p,q'})^2 T_p + \sigma^*_{p,q'} W'^r_{T_p}), \end{aligned} \tag{4}$$

where W^r and W'^r are standard Brownian motions under P_r with

$$Cor(W^r, W'^r) =: \rho.$$

Then, the value of CMS spread option with zero strike can be approximated via the well-known explicit Margrabe's formula,

$$E_r[S_{p,q'}(T_p) - S_{p,q}(T_p)]^+ \approx S^*_{p,q'}(0)\mathcal{N}(b_1) - S^*_{p,q}(0)\mathcal{N}(b_2) \tag{5}$$

with

$$b_1 := \frac{\ln \frac{S^*_{p,q'}(0)}{S^*_{p,q}(0)} + \frac{1}{2}\left[\left(\sigma^*_{p,q}\right)^2 + \left(\sigma^*_{p,q'}\right)^2 - 2\sigma^*_{p,q}\sigma^*_{p,q'}\rho\right] T_p}{\sqrt{\left[\left(\sigma^*_{p,q}\right)^2 + \left(\sigma^*_{p,q'}\right)^2 - 2\sigma^*_{p,q}\sigma^*_{p,q'}\rho\right] T_p}},$$

$$b_2 := \frac{\ln \frac{S^*_{p,q'}(0)}{S^*_{p,q}(0)} - \frac{1}{2}\left[\left(\sigma^*_{p,q}\right)^2 + \left(\sigma^*_{p,q'}\right)^2 - 2\sigma^*_{p,q}\sigma^*_{p,q'}\rho\right] T_p}{\sqrt{\left[\left(\sigma^*_{p,q}\right)^2 + \left(\sigma^*_{p,q'}\right)^2 - 2\sigma^*_{p,q}\sigma^*_{p,q'}\rho\right] T_p}}.$$

For non-zero strikes, the Margrabe's formula involves one-dimensional integration

$$E_r[\varepsilon(S_{p,q'}(T_p) - S_{p,q}(T_p) - K)]^+ \approx \int_{\mathbb{R}} \varepsilon[X_2 \mathcal{N}(\varepsilon d_1) - (X_1 + K)\mathcal{N}(\varepsilon d_2)]\varphi(x)\, dx \tag{6}$$

with $\varepsilon = 1$ for a caplet, $\varepsilon = -1$ for a flooret,

$$X_1 := S^*_{p,q}(0) \exp\left[\sigma^*_{p,q} x \sqrt{T_p} - \frac{1}{2}\left(\sigma^*_{p,q}\right)^2 T_p\right],$$

$$X_2 := S^*_{p,q'}(0) \exp\left[\sigma^*_{p,q'} x \rho \sqrt{T_p} - \frac{1}{2}\left(\sigma^*_{p,q'}\right)^2 \rho^2 T_p\right],$$

$$d_{1,2} := \frac{\ln \frac{X_2}{X_1+K} \pm \frac{1}{2}\left(\sigma^*_{p,q'}\right)^2 T_p(1-\rho^2)}{\sqrt{\left(\sigma^*_{p,q'}\right)^2 T_p(1-\rho^2)}},$$

and φ is the density of the standard normal distribution.

Thus, to price CMS spread via Margrabe's formula, we need (i) to find a close lognormal approximation for swap rates $S_{p,q}$ and $S_{p,q'}$ under the probability measure P_r, and (ii) to estimate the correlation between $\ln S_{p,q}$ and $\ln S_{p,q'}$.

3 Lognormal swap rate approximations in the LMM

In general the dynamics of swap rates $S_{p,q}$ under the swap measure $P_{p,q}$ is given by

$$dS_{p,q} = S_{p,q}\sigma_{p,q}^{\top} dW^{p,q},$$

for some adaptive process $\sigma_{p,q}$. Our goal is to obtain the dynamics of $S_{p,q}$ under some measure P_r, $p \le r < q$. With the Radon-Nykodym martingale defined by

$$M_t := E^{\mathcal{F}_t}_{p,q} \frac{dP_r}{dP_{p,q}}(T_r) = \frac{B_{p,q}(0)}{B_r(0)} \frac{B_r(t)}{B_{p,q}(t)}, \quad t \le T_r,$$

we have by Girsanov's theorem that

$$S_{p,q} - \langle S_{p,q}, \ln M\rangle = S_{p,q} - \langle S_{p,q}, \ln \frac{B_r}{B_{p,q}}\rangle \tag{7}$$

is a P_r-martingale. Note that

$$\frac{B_r}{B_{p,q}} = \frac{B_r}{B_q} \frac{1}{\sum_{j=p}^{q-1} \delta_j B_{j+1}/B_q} = \frac{\prod_{k=r}^{q-1}(1+\delta_k L_k)}{\sum_{j=p}^{q-1} \delta_j \prod_{k=j+1}^{q-1}(1+\delta_k L_k)}$$

hence modulo finite variation dt-terms,

$$d \ln \frac{B_r}{B_{p,q}} = \sum_{j=r}^{q-1} \frac{\delta_j dL_j}{1+\delta_j L_j} - \frac{\sum_{j=p}^{q-1} \delta_j d \prod_{k=j+1}^{q-1}(1+\delta_k L_k)}{\sum_{j=p}^{q-1} \delta_j \prod_{k=j+1}^{q-1}(1+\delta_k L_k)}.$$

After some transformations, we get (modulo dt-terms)

$$d \ln \frac{B_r}{B_{p,q}} = \sum_{k=p}^{n} \varsigma_k^{p,q,r} \frac{\delta_k dL_k}{1 + \delta_k L_k}$$

with

$$\varsigma_k^{p,q,r} := 1_{r=k=p} - 1_{r>p,\, p+1 \le k<r} + \frac{B_{k,q}}{B_{p,q}} 1_{p<k<q}. \tag{8}$$

As the process (7) is P_r-martingale, we have (with $\varsigma_k \equiv \varsigma_k^{p,q,r}$ for short)

$$\frac{dS_{p,q}}{S_{p,q}} = \sum_{k=p}^{q-1} \frac{\delta_k L_k}{1 + \delta_k L_k} \varsigma_k \sigma_{p,q}^{\mathsf{T}} \gamma_k \; dt + \sigma_{p,q}^{\mathsf{T}} dW^r, \tag{9}$$

or

$$\ln \frac{S_{p,q}(T_p)}{S_{p,q}(0)} = -\frac{1}{2} \int_0^{T_p} |\sigma_{p,q}|^2 \, dt + \sum_{k=p}^{q-1} \int_0^{T_p} \frac{\delta_k L_k}{1 + \delta_k L_k} \varsigma_k \sigma_{p,q}^{\mathsf{T}} \gamma_k \; dt + \int_0^{T_p} \sigma_{p,q}^{\mathsf{T}} dW^r. \tag{10}$$

Using formula (1.29) in [4] we may express $\sigma_{p,q}$ via Libor volatility structure,

$$\sigma_{p,q} = \sum_{l=p}^{q-1} \frac{\delta_l L_l}{1 + \delta_l L_l} \left(\sum_{r=l}^{q-1} w_r^{p,q} + \frac{B_q}{B_p - B_q} \right) \gamma_l =: \frac{1}{S_{p,q}} \sum_{l=p}^{q-1} v_l^{p,q} L_l \gamma_l, \tag{11}$$

with

$$v_l^{p,q} := \frac{\delta_l B_{l,q}}{B_{p,q}} \frac{S_{p,q} - S_{l,q}}{1 + \delta_l L_l} + w_l^{p,q}, \quad w_l^{p,q} := \frac{\delta_l B_l}{B_{p,q}}.$$

Note that all dynamics are 'exact' so far. Now we use a normal approximation from [3] of the form

$$\frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \doteq \delta_k L_k(0)(1 + \int_0^t \gamma_k^{\mathsf{T}} dW^r), \tag{12}$$

We then obtain from (10),

$$\begin{aligned}
\ln \frac{S_{p,q}(T_p)}{S_{p,q}(0)} &= -\frac{1}{2} \int_0^{T_p} |\sigma_{p,q}|^2 \, dt \\
&+ \sum_{k=p}^{q-1} \delta_k L_k(0) \int_0^{T_p} (1 + \int_0^t \gamma_k^{\mathsf{T}} dW^{p+1}) \varsigma_k \sigma_{p,q}^{\mathsf{T}} \gamma_k \; dt + \int_0^{T_p} \sigma_{p,q}^{\mathsf{T}} dW^r \\
&= \int_0^{T_p} \left(-\frac{1}{2} |\sigma_{p,q}|^2 + \sum_{k=p}^{q-1} \delta_k L_k(0) \varsigma_k \sigma_{p,q}^{\mathsf{T}} \gamma_k \right) dt \\
&\quad + \int_0^{T_p} \left(\sum_{k=p}^{q-1} \delta_k L_k(0) \int_s^{T_p} \varsigma_k \sigma_{p,q}^{\mathsf{T}} \gamma_k \; du \; \gamma_k^{\mathsf{T}} + \sigma_{p,q}^{\mathsf{T}} \right) dW^r.
\end{aligned} \tag{13}$$

In order to obtain a lognormal approximation for (13) we use the deterministic approximations

$$\sigma_{p,q}(s)=\sum_{l=p}^{q-1}\left(\frac{v_l^{p,q}L_l}{S_{p,q}}\gamma_l\right)(s)\approx\sum_{l=p}^{q-1}\left(\frac{v_l^{p,q}L_l}{S_{p,q}}\right)(0)\,\gamma_l(s)=:\widehat{\sigma}_{p,q}, \tag{14}$$

$$\varsigma_k(s)\approx\varsigma_k(0),$$

by freezing weight related coefficients at $t=0$. This yields the lognormal approximation

$$\begin{aligned}\ln\frac{S_{p,q}(T_p)}{S_{p,q}(0)}&\approx\int_0^{T_p}\left(-\frac{1}{2}\left|\widehat{\sigma}_{p,q}\right|^2+\sum_{k=p}^{q-1}\delta_kL_k(0)\varsigma_k(0)\widehat{\sigma}_{p,q}^{\mathsf{T}}\gamma_k\right)dt\\&+\int_0^{T_p}\left(\sum_{k=p}^{q-1}\delta_kL_k(0)\varsigma_k(0)\int_s^{T_p}\widehat{\sigma}_{p,q}^{\mathsf{T}}\gamma_k\;du\;\gamma_k^{\mathsf{T}}+\widehat{\sigma}_{p,q}^{\mathsf{T}}\right)dW^r\\&=:A_{p,q}+Z_{p,q}\end{aligned} \tag{15}$$

with $Z_{p,q}$ being a zero-mean Gaussian random variable.

For pricing of spread options we assume (15) for q and q', with $q'>q>p$. We then have

$$E_r\,Z_{p,q}Z_{p,q'}=(I)+(II)+(III),$$

where

$$E_r\,Z_{p,q}Z_{p,q'}=\int_0^{T_p}\left(\sum_{k=p}^{q-1}\delta_kL_k(0)\varsigma_k(0)\int_s^{T_p}\widehat{\sigma}_{p,q}^{\mathsf{T}}\gamma_k\;du\;\gamma_k^{\mathsf{T}}+\widehat{\sigma}_{p,q}^{\mathsf{T}}\right)\times$$

$$\left(\sum_{k'=p}^{q'-1}\delta_{k'}L_{k'}(0)\varsigma_k'(0)\int_s^{T_p}\widehat{\sigma}_{p,q'}^{\mathsf{T}}\gamma_{k'}\;du\;\gamma_{k'}^{\mathsf{T}}+\widehat{\sigma}_{p,q'}^{\mathsf{T}}\right)dt=(I)+(II)+(III)$$

$$\begin{aligned}(I)\;&=\;\sum_{k=p}^{q-1}\sum_{k'=p}^{q'-1}\varsigma_k^{p,q}(0)\varsigma_{k'}(0)\delta_k\delta_{k'}L_k(0)L_{k'}(0)\times\\&\quad\int_0^{T_p}\gamma_k^{\mathsf{T}}\gamma_{k'}ds\int_s^{T_p}\left(\widehat{\sigma}_{p,q}^{\mathsf{T}}\gamma_k\right)dt\int_s^{T_p}\left(\widehat{\sigma}_{p,q'}\gamma_{k'}\right)dt'\\&=\;\frac{1}{S_{p,q}S_{p,q'}}\sum_{k=p}^{q-1}\sum_{k'=p}^{q'-1}\sum_{l=p}^{q-1}\sum_{l'=p}^{q'-1}\varsigma_k\varsigma_{k'}'v_l^{p,q}v_{l'}^{p,q'}\delta_k\delta_{k'}L_kL_{k'}L_lL_{l'}\times\\&\quad\int_0^{T_p}\gamma_k^{\mathsf{T}}\gamma_{k'}ds\int_s^{T_p}\gamma_l^{\mathsf{T}}\gamma_kdt\int_s^{T_p}\gamma_{l'}^{\mathsf{T}}\gamma_{k'}dt',\end{aligned}$$

$$
\begin{aligned}
(II) &= \sum_{k=p}^{q-1} \varsigma_k(0)\delta_k L_k(0) \int_0^{T_p} \sigma_{p,q'}^{\mathsf{T}}\gamma_k ds \int_s^{T_p} \left(\sigma_{p,q}^{\mathsf{T}}\gamma_k\right) dt \\
&+ \sum_{k'=p+1}^{q'-1} \varsigma'_{k'}(0)\delta_{k'} L_{k'}(0) \int_0^{T_p} \sigma_{p,q}^{\mathsf{T}}\gamma_{k'} ds \int_s^{T_p} \left(\sigma_{p,q'}\gamma_{k'}\right) dt' \\
&= \frac{1}{S_{p,q}S_{p,q'}} \sum_{k=p}^{q-1}\sum_{l=p}^{q-1}\sum_{l'=p}^{q'-1} \varsigma_k \upsilon_l^{p,q}\upsilon_{l'}^{p,q'}\delta_k L_k L_l L_{l'} \int_0^{T_p} \gamma_{l'}^{\mathsf{T}}\gamma_k ds \int_s^{T_p} \gamma_l^{\mathsf{T}}\gamma_k dt \\
&+ \frac{1}{S_{p,q}S_{p,q'}} \sum_{k'=p}^{q'-1}\sum_{l=p}^{q-1}\sum_{l'=p}^{q'-1} \varsigma'_{k'} \upsilon_l^{p,q}\upsilon_{l'}^{p,q'}\delta_{k'} L_{k'} L_l L_{l'} \int_0^{T_p} \gamma_l^{\mathsf{T}}\gamma_{k'} ds \int_s^{T_p} \gamma_{l'}^{\mathsf{T}}\gamma_{k'} dt',
\end{aligned}
$$

$$
(III) \approx \frac{1}{S_{p,q}S_{p,q'}} \sum_{k=p}^{q-1}\sum_{k'=p}^{q'-1} \upsilon_k^{p,q}\upsilon_{k'}^{p,q'} L_k L_{k'} \int_0^{T_p} \gamma_k^{\mathsf{T}}\gamma_{k'} dt. \tag{16}
$$

To apply the Margrabe's formula (5), or (6), straightforward, it is convenient to rewrite (15) in the form (4) with

$$
\sigma_{p,q}^* := \sqrt{\frac{E_r Z_{p,q}^2}{T_p}}, \quad S_{p,q}^*(0) := S_{p,q}(0)\exp\left(A_{p,q} + \frac{\sigma_{p,q}^*}{2}T_p\right),
$$

and

$$
\rho := \frac{E_r Z_{p,q} Z_{p,q'}}{\sqrt{E_r Z_{p,q}^2}\sqrt{E_r Z_{p,q'}^2}} \tag{17}
$$

Alternatively, we can approximate (10) with a lognormal process by simple freezingLL-ibor rates at $t = 0$,

$$
\frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \doteq \delta_k L_k(0),
$$

instead of using approximation (12). Analogously to (15), we than obtain the log-normal approximation

$$
\ln \frac{S_{p,q}(T_p)}{S_{p,q}(0)} \approx A_{p,q} + Z_{p,q}^0, \tag{18}
$$

where $Z_{p,q}^0$ is a zero-mean Gaussian random variable and for q', with $q' > q > p$, we have

$$
E_r\, Z_{p,q}^0 Z_{p,q'}^0 = (III),
$$

where (III) is given by (16). Then, we price CMS spread option via Margrabe's formula (5), or (6), with

$$
\sigma_{p,q}^* := \sqrt{\frac{E_r (Z_{p,q}^0)^2}{T_p}}, \quad S_{p,q}^*(0) := S_{p,q}(0)\exp\left(A_{p,q} + \frac{\sigma_{p,q}^*}{2}T_p\right),
$$

and

$$\rho^0 := \frac{E_r Z^0_{p,q} Z^0_{p,q'}}{\sqrt{E_r (Z^0_{p,q})^2}\sqrt{E_r (Z^0_{p,q'})^2}}. \tag{19}$$

In Section 5 we compare the two approximations (15) and (18).

4 Convexity adjustment approach

Now we show an alternative way to construct a lognormal approximation of a swap rate process (2) under measure P_r, $p \le r < q$. Let us consider the valuation of a cash payment $S_{p,q}(T_p)$ (in \$ for instance) at date T_r seen at time $T_0 = 0$. We thus consider

$$E_r\, S_{p,q}(T_p) = \frac{B_{p,q}(0)}{B_r(0)} E_{p,q}\, S_{p,q}(T_p) \frac{B_r(T_p)}{B_{p,q}(T_p)}. \tag{20}$$

For evaluating (20) we here recall a convexity approach in [2] based on the assumption of a *linear swap market model:* For fixed p and q it is assumed that

$$\frac{B_r(T_p)}{B_{p,q}(T_p)} = \alpha + \beta_r S_{p,q}(T_p), \quad p \le r, \tag{21}$$

where α and β_r are constants which are determined as follows. By taking the expectation of (21) under the annuity measure we get

$$\frac{B_r(0)}{B_{p,q}(0)} = \alpha + \beta_r S_{p,q}(0),$$

hence

$$\beta_r = \frac{1}{S_{p,q}(0)} \left(\frac{B_r(0)}{B_{p,q}(0)} - \alpha \right)$$

It holds,

$$\begin{aligned} 1 &= \sum_{r=p}^{q-1} \frac{\delta_r B_{r+1}(T_p)}{B_{p,q}(T_p)} = \sum_{r=p}^{q-1} \delta_r \left(\alpha + \beta_{r+1} S_{p,q}(T_p)\right) \\ &= \alpha \left(1 - \frac{S_{p,q}(T_p)}{S_{p,q}(0)}\right) \sum_{r=p}^{q-1} \delta_r + \frac{S_{p,q}(T_p)}{S_{p,q}(0)}, \end{aligned}$$

so we obtain

$$\alpha = \frac{1}{\sum_{r=p}^{q-1} \delta_r}, \quad \beta_r = \frac{\frac{B_r(0)}{B_{p,q}(0)} - \frac{1}{\sum_{r=p}^{q-1} \delta_r}}{S_{p,q}(0)} = \frac{B_r(0) - \frac{B_{p,q}(0)}{\sum_{r=p}^{q-1} \delta_r}}{B_p(0) - B_q(0)}.$$

Under assumption (21), we now derive from (20) straightforwardly,

$$E_r\, S_{p,q}(T_p) = S_{p,q}(0) \left(1 + \frac{\beta_r Var_{p,q}\left[S_{p,q}(T_p)\right]}{S_{p,q}(0)\left(\alpha + \beta_r S_{p,q}(0)\right)}\right) \tag{22}$$

(see [2]).

In a similar way we may derive for the second moment,

$$E_r\, S^2_{p,q}(T_p) = \frac{B_{p,q}(0)}{B_r(0)} E_{p,q}\left(S^2_{p,q}\frac{B_r}{B_{p,q}}\right)(T_p) \tag{23}$$
$$= \frac{1}{\alpha + \beta_{p+1}S_{p,q}(0)}\ \left(\alpha E_{p,q}S^2_{p,q}(T_p) + \beta_{p+1}E_{p,q}S^3_{p,q}(T_p)\right).$$

Under the market model for swaptions, $S_{p,q}$ is a log-normal martingale under the annuity measure $P_{p,q}$ induced by the annuity numeraire $B_{p,q}$. We so have

$$S_{p,q}(T_p) = S_{p,q}(0)\exp(\sigma^B_{p,q}W^{p,q}_{T_p} - \frac{1}{2}\left(\sigma^B_{p,q}\right)^2 T_p) \tag{24}$$

where $\sigma^B_{p,q}$ is the so called Black volatility, and thus

$$E_{p,q}S^2_{p,q}(T_p) = S^2_{p,q}(0)\exp\left[\left(\sigma^B_{p,q}\right)^2 T_p\right].$$

Hence, under the lognormal swap rate assumption (24), (22) yields

$$E_r\, S_{p,q}(T_p) = S_{p,q}(0)\left(1 + \frac{\beta_r S_{p,q}(0)\left(\exp\left[\left(\sigma^B_{p,q}\right)^2 T_p\right] - 1\right)}{(\alpha + \beta_r S_{p,q}(0))}\right). \tag{25}$$

and further, since $E_{p,q}S^3_{p,q}(T_p) = S^3_{p,q}(0)\exp\left[3\left(\sigma^B_{p,q}\right)^2 T_p\right]$, (23) yields

$$E_r S^2_{p,q}(T_p) = \frac{1}{\alpha + \beta_r S_{p,q}(0)}\ S^2_{p,q}(0)\left(\alpha\exp\left[\left(\sigma^B_{p,q}\right)^2 T_p\right]\right. \tag{26}$$
$$\left. + \beta_r S_{p,q}(0)\exp\left[3\left(\sigma^B_{p,q}\right)^2 T_p\right]\right).$$

If we now also assume (though inconsistently) that $S_{p,q}(T_p)$ is (approximately) log-normal distributed under P_r, i.e.,

$$S_{p,q}(T_p) = S^*_{p,q}(0)\exp\left[-\frac{1}{2}\left(\sigma^*_{p,q}\right)^2 T_p + \sigma^*_{p,q}W^r_{T_p}\right], \tag{27}$$

for some $S^*_{p,q}(0) > 0$ and $\sigma^*_{p,q} > 0$, we obtain by matching the first and second moment of (27) to (25) and (26) respectively,

$$S^*_{p,q}(0) = S_{p,q}(0)\left(1 + \frac{\beta_r S_{p,q}(0)\left(\exp\left[\left(\sigma^B_{p,q}\right)^2 T_p\right] - 1\right)}{\alpha + \beta_r S_{p,q}(0)}\right), \tag{28}$$

$$\left(\sigma^*_{p,q}\right)^2 = \left(\sigma^B_{p,q}\right)^2 + \frac{1}{T_p}\ln\left[\frac{\alpha + \beta_r S_{p,q}(0)\left(\alpha + \beta_r S_{p,q}(0)\exp\left[2\left(\sigma^B_{p,q}\right)^2 T_p\right]\right)}{\left(\alpha + \beta_r S_{p,q}(0)\exp\left[\left(\sigma^B_{p,q}\right)^2 T_p\right]\right)^2}\right].$$

Here, the implied Black's volatility can be expressed via Libor volatility structure (see, e.g., [4]),

$$\sigma^B_{p,q} = \frac{1}{T_p}\int_0^{T_p} \sigma^2_{p,q}\, dt \approx \frac{1}{T_p}\int_0^{T_p} \widehat{\sigma}^2_{p,q}\, dt = \frac{1}{T_p}(III),$$

with integral (III) given by (16).

Finally we note that the convexity adjustment technique does not give the correlation between $\ln S_{p,q}$ and $\ln S_{p,q'}$. To price CMS spread via Margrabe's formula, we assume that the correlation is given by (17), or (19).

5 Numerical comparison of methods

Now we illustrate the algorithms by pricing the spread of 10-year and 2-year CMS for different maturities T_p under probability measure P_{p+1}. Further we show a calibration of the Libor model to market CMS spread prices.

In the market Libor model the dynamics of the forward Libors $L_i(t)$ defined in the interval $[0, T_p]$ for $p \leq i < n$, in the measure P_{p+1} are governed by the following system of SDE's,

$$dL_i = \sum_{j=p+1}^{i} \frac{\delta_j L_i L_j\, \gamma_i^\top \gamma_j}{1+\delta_j L_j}\, dt + L_i\, \gamma_i^\top dW^{(p+1)}, \tag{29}$$

where $t \to \gamma_i(t) = (\gamma_{i,1}(t), \ldots, \gamma_{i,n-1}(t))$, $0 \leq t \leq T_i$, are deterministic volatility vector functions. For experiments we take $W^{(p+1)}$ to be a standard $n-1$-dimensional Wiener process under P_{p+1} in (29), and use the volatility structure

$$\gamma_i(t) = cg(T_i - t)e_i, \quad \text{where} \quad g(s) = g_\infty + (1 - g_\infty + as)e^{-bs},$$

where e_i are n-dimensional unit vectors, decomposing an input correlation matrix

$$\begin{aligned}\rho_{ij} \quad &= \quad \exp\Big[-\frac{|j-i|}{m-1}\Big(-\ln\rho_\infty \\ &\quad +\eta\frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)}\Big)\Big], \\ &\quad 1 \leq i,j \leq m, \quad \eta > 0,\ 0 < \eta < -\ln\rho_\infty.\end{aligned}$$

Further we take $n = 30$, $\delta_i \equiv 1$yr, and the parameter values in Table 1. The parameters are chosen to be more or less typical with Libor volatilities ranging between 15% and 25%.

Table 1.

a	b	g_∞	η	ρ_∞	c
1.190	1.550	0.587	0.086	0.449	0.264

In Tables 2-4, we compare the values of the CMS spreads

$$E_{p+1}\left[\varepsilon(S_{p,p+10}-S_{p,p+2}-\text{strike})\right]^{+}, \quad \varepsilon=1 \text{ for caplet}, \quad \varepsilon=-1 \text{ for flooret},$$

by the convexity adjustment approach (CA-method) and the lognormal swap rate approximation (LN-method) with ("true") Monte Carlo simulated prices (MC), using small enough time steps. We denote by LN^0 and CA^0 the prices via the approximations (18) and (27)-(28), respectively, with the correlation given by (19). The prices via refinedäpproximations (15) and (27)-(28) with the correlation (17) are denoted by LN and CA, respectively. From the tables we see that overall both CA- and LN-methods give a reasonable approximations of the spread option prices. The pricing error is at most 3 base points for 20yr maturity. Remarkably, for the floor examples the (LN) method performs slightly better, whereas for the caps the (CA) approach gives somewhat better results. The reason might be that the biases of both methods are in opposite directions. Note that the refinedäpproximations CA and LN can give a significant improvement only for large maturity ($T_p > 10$), and require higher computational time than approximations CA^0 and LN^0. Therefore, the more crudeäpproximations CA^0 and LN^0 are preferable for calibration procedure.

Table 2. CMS spread caplet, strike 0.5%.

Maturity	CA	LN	MC(SD)
1	25.52	25.62	25.46(0.02)
2	28.00	28.20	27.90(0.03)
3	28.43	28.78	28.43(0.05)
4	27.27	27.75	27.35(0.05)
5	25.50	26.11	25.70(0.08)
6	23.46	24.20	23.73(0.08)
7	23.16	24.01	23.50(0.08)
8	23.47	24.49	23.69(0.14)
9	23.86	25.03	24.06(0.15)
10	24.58	25.91	24.79(0.16)
15	27.21	29.29	27.43(0.20)
20	32.45	35.23	32.50(0.25)

Table 3. CMS spread flooret, strike 0.5%.

Maturity	CA^0	CA	LN^0	LN	MC(SD)
1	6.49	6.49	6.45	6.42	6.22(0.04)
2	10.82	10.84	10.74	10.69	10.36(0.05)
3	15.01	15.05	14.85	14.78	14.51(0.07)
4	20.61	20.68	20.36	20.25	20.01(0.08)
5	27.16	27.28	26.81	26.65	26.69(0.10)
6	34.73	34.88	34.27	34.02	34.44(0.12)
7	40.15	40.34	39.62	39.25	39.56(0.13)
8	44.26	44.48	43.67	43.18	43.71(0.14)
9	48.00	48.25	47.38	46.73	47.14(0.11)
10	50.88	51.16	50.25	49.42	50.19(0.16)
15	62.24	62.62	61.89	59.97	60.84(0.19)
20	63.81	64.24	64.03	61.02	61.91(0.20)

Table 4. CMS spread flooret, strike -0.5%.

Maturity	CA^0	CA	LN^0	LN	MC(SD)
1	0.11	0.11	0.11	0.11	0.08(0.00)
2	0.64	0.64	0.64	0.63	0.49(0.01)
3	1.40	1.41	1.4	1.37	1.01(0.01)
4	2.65	2.68	2.65	2.58	2.17(0.02)
5	4.45	4.49	4.44	4.32	3.83(0.04)
6	6.93	7.01	6.93	6.71	6.03(0.05)
7	9.14	9.25	9.16	8.8	8.12(0.06)
8	11.03	11.17	11.08	10.66	9.80(0.12)
9	12.89	13.06	12.97	12.32	11.66(0.13)
10	14.45	14.65	14.59	13.78	13.00(0.14)
15	20.85	21.17	21.48	19.66	19.13(0.19)
20	22.24	22.61	23.53	20.76	20.00(0.20)

In Figures 1-2 we show the calibration of the Libor market model to the market CMS spread prices for a fixed strikes. To find the optimal parameters for Libors market model, we minimize the root-mean-square (RMS) error

$$\frac{1}{N}\sqrt{\sum_{i=1}^{N}(V^{mar}(\mathcal{T}_i)-V^{app}(\mathcal{T}_i;\ a,b,g_\infty,\eta,\rho_\infty))^2}\rightarrow_{a,b,g_\infty,\eta,\rho_\infty}\min.$$

Here, V^{mar} is a vector of market CMS spread prices (for fixed strike) for N maturities. In our case,

$$N=12;\quad \mathcal{T}_1=1\text{y},\, \mathcal{T}_2=2\text{y},\, \ldots,\, \mathcal{T}_{10}=10\text{y},\, \mathcal{T}_{11}=15\text{y},\, \mathcal{T}_{12}=20\text{y};$$

V^{app} denotes an approximated CMS spread options, $V^{app}=\mathrm{CA}^0$, or $V^{app}=\mathrm{LN}^0$. Although both approximations provide a good fit for a fixed strike, the simultaneous calibration for the whole CMS spread matrix is impossible. The problem can be solved by generalizing the lognormal approximation to the model with stochastic volatility.

References

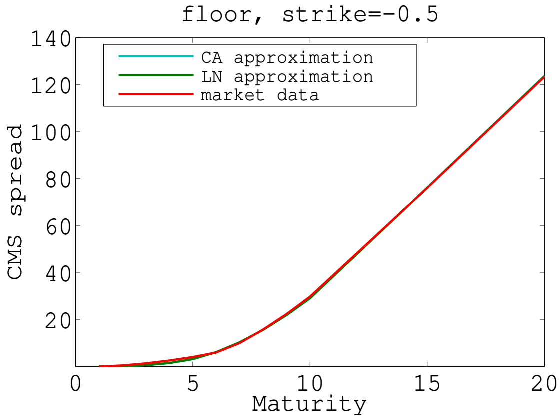
[1] Brigo, D., and F. Mercurio (2001). *Interest Rate Models: Theory and Practice*. Springer-Verlag Berlin Heidelberg New York 2001.

[2] Boenkost, W., and W. Schmidt (2003). Notes on convexity and quanto adjustments for interest rates and related options. Working Paper, Business School of Finance & Management, Frankfurt.

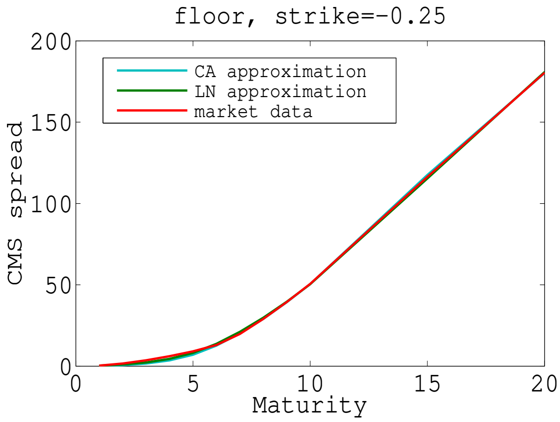
[3] Kubarmuradov, O, Sabelfeld, K., and J. Schoenmakers. Lognormal approximations to Libor market models, *Journal of Computational Finance,* 6 (Fall, 2002), pp. 69-100.

[4] Schoenmakers J. *Robust Libor Modelling and Pricing of Derivative Products.* Boca Raton London New York Singapore: Chapman & Hall – CRC Press 2005

optimal parameters	CA^0	LN^0
a	0.22	0.22
b	3.84	3.82
g_∞	0.09	0.09
η	1.36	1.36
ρ_∞	0.26	0.26
c	0.62	0.62
RMS error	0.57	0.55



optimal parameters	CA^0	LN^0
a	0.36	0.31
b	3.67	3.13
g_∞	0.22	0.24
η	0.39	0.18
ρ_∞	0.61	0.73
c	0.54	0.61
RMS error	1.21	0.93



optimal parameters	CA^0	LN^0
a	0.30	17.61
b	2.13	7.69
g_∞	0.31	0.42
η	0.02	0.20
ρ_∞	0.79	0.75
c	0.61	0.40
RMS error	1.48	1.44

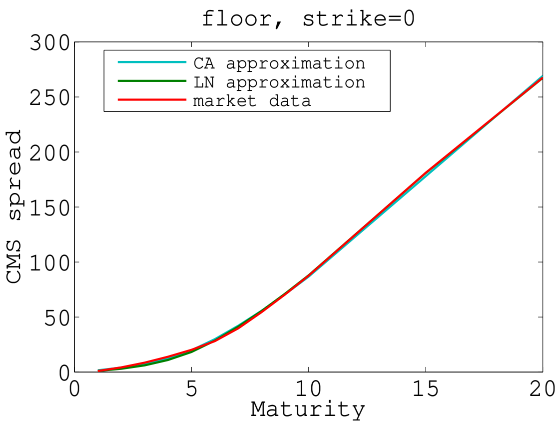
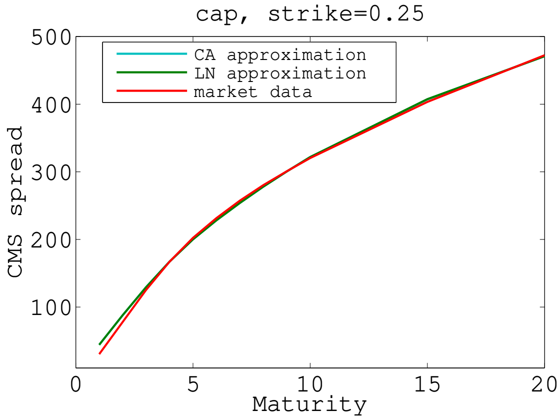
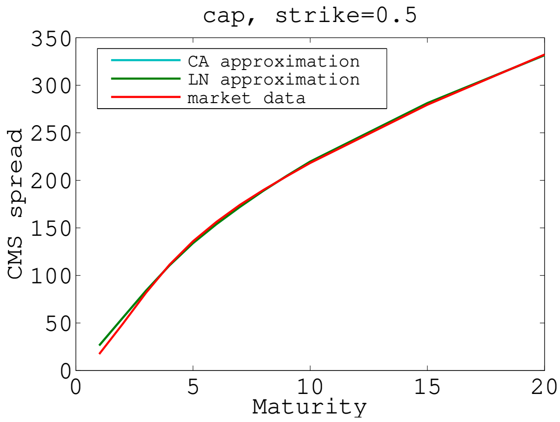


Figure 1: Calibration to CMS spread (floor) for fixed strike. The optimal values of LMM parameters are given in the tables (left).

optimal parameters	CA^0	LN^0
a	1.32	1.23
b	1.45	1.45
g_∞	0.76	0.76
η	0.88	0.86
ρ_∞	0.27	0.27
c	0.14	0.14
RMS error	6.48	5.47



optimal parameters	CA^0	LN^0
a	0.26	0.26
b	0.64	0.65
g_∞	0.29	0.29
η	1.39	1.41
ρ_∞	0.01	0.01
c	0.20	0.20
RMS error	3.52	3.52



optimal parameters	CA^0	LN^0
a	0.56	0.56
b	1.03	1.03
g_∞	2.53	2.53
η	0.72	0.74
ρ_∞	0.08	0.08
c	0.31	0.31
RMS error	2.81	2.81

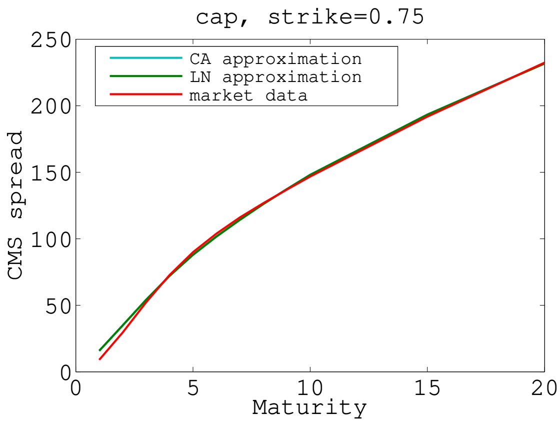


Figure 2: Calibration to CMS spread (cap) for fixed strike. The optimal values of LMM parameters are given in the tables (left).