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Hölder index for density states of $(\alpha, 1, \beta)$-superprocesses at a given point

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Abstract. A Hölder regularity index at *given points* for density states of $(\alpha, 1, \beta)$-superprocesses with $\alpha > 1+\beta$ is determined. It is shown that this index is strictly greater than the optimal index of *local* Hölder continuity for those density states.

1. Introduction and statement of results

For $0 < \alpha \leq 2$ and $1+\beta \in (1,2)$, the (α, d, β)-*superprocess* $X = \{X_t : t \geq 0\}$ in R^d is a finite measure-valued process related to the log-Laplace equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u \;=\; \boldsymbol{\Delta}_\alpha u + au - bu^{1+\beta}, \tag{1}$$

where $a \in \mathsf{R}$ and $b > 0$ are any fixed constants. Its underlying motion is described by the fractional Laplacian $\boldsymbol{\Delta}_\alpha := -(-\boldsymbol{\Delta})^{\alpha/2}$ determining a symmetric α–stable motion in R^d of index $\alpha \in (0,2]$ (Brownian motion if $\alpha = 2$), whereas its continuous-state branching mechanism

$$v \;\mapsto\; -av + bv^{1+\beta}, \quad v \geq 0, \tag{2}$$

belongs to the domain of attraction of a stable law of index $1+\beta \in (1,2)$ (the branching is critical if $a = 0$).

From now on we assume that $d < \frac{\alpha}{\beta}$. Then X has a.s. *absolutely continuous states* $X_t(\mathrm{d}x)$ at fixed times $t > 0$. Moreover, as is shown in Fleischmann, Mytnik, and Wachtel [FMW08], there is a *dichotomy* for their density function (also denoted by X_t): There is a continuous version $\tilde{X}_t$ of the density function if $d = 1$ and $\alpha > 1+\beta$, but otherwise the density function X_t is locally unbounded on open sets of positive $X_t(\mathrm{d}x)$-measure. (Partial results had been derived earlier in Mytnik and Perkins [MP03].)

In the case of continuity, Hölder regularity properties of $\tilde{X}_t$ had been studied in [FMW08], too. Let us first recall the notion of an optimal Hölder index at a point.

We say a function f is Hölder continuous with index $\eta \in (0,1]$ *at the point* x if there is an open neighborhood $U(x)$ and a constant C such that

$$\bigl|f(y) - f(x)\bigr| \;\leq\; C\,|y-x|^\eta \quad \text{for all } y \in U(x). \tag{3}$$

The *optimal Hölder index* $H(x)$ of f at the point x is defined as

$$H(x) \;:=\; \sup\bigl\{\eta \in (0,1] :\; f \text{ is Hölder continuous at } x \text{ with index } \eta\bigr\}, \tag{4}$$

and set to 0 if f is not Hölder continuous at x.

Going back to the continuous (random) density function $\tilde{X}_t$, in what follows, $H(x)$ will denote the (random) Hölder index of $\tilde{X}_t$ at $x \in \mathsf{R}$. In [FMW08], the so-called *optimal index* for *local* Hölder continuity of $\tilde{X}_t$ had been determined by

$$\eta_{\mathrm{c}} \;:=\; \frac{\alpha}{1+\beta} - 1 \;\in\; (0,1). \tag{5}$$

This means that in any non-empty open set $U \subset \mathsf{R}$ with $X_t(U) > 0$ one can find (random) points x such that $H(x) = \eta_{\mathrm{c}}$. This however left unsolved the question whether there are points $x \in U$ such that $H(x) > \eta_{\mathrm{c}}$.

The *purpose* of this note is to verify the following theorem conjectured in [FMW08, Section 1.3]. To formulate it, let $\mathcal{M}_{\mathrm{f}}$ denote the set of finite measures on R^d, and $B_\epsilon(x)$ the open ball of radius $\epsilon > 0$ around $x \in \mathsf{R}^d$:

Theorem 1 (**Hölder continuity at a given point**). *Fix* $t > 0$, $z \in \mathsf{R}$, *and* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$. *Let* $d = 1$ *and* $\alpha > 1 + \beta$. *Then with probability one, for each* $\eta > 0$ *satisfying*

$$\eta < \bar{\eta}_{\mathrm{c}} := \min\Big\{\frac{1+\alpha}{1+\beta} - 1,\, 1\Big\}, \tag{6}$$

the continuous version $\tilde{X}_t$ *of the density is Hölder continuous of order* η *at the point* z :

$$\sup_{x \in B_\epsilon(z),\, x \neq z} \frac{\big|\tilde{X}_t(x) - \tilde{X}_t(z)\big|}{|x-z|^\eta} < \infty, \quad \epsilon > 0. \tag{7}$$

Consequently, since $\eta_{\mathrm{c}} < \bar{\eta}_{\mathrm{c}}$, at each given point $z \in \mathsf{R}$ the density state $\tilde{X}_t$ allows some Hölder exponents η larger than η_{c}, the optimal Hölder index for local domains. Thus, Theorem 1 nicely complements the main result of [FMW08].

On the other hand, Theorem 1 is also only a partial result, since it does not yet claim that $\bar{\eta}_{\mathrm{c}}$ is optimal. So let us add here the following conjecture.

Conjecture 2 (**Optimality of** $\bar{\eta}_{\mathrm{c}}$). Under the conditions of Theorem 1, for each $\eta \geq \bar{\eta}_{\mathrm{c}}$ with probability one,

$$\sup_{x \in B_\epsilon(z),\, x \neq z} \frac{\big|\tilde{X}_t(x) - \tilde{X}_t(z)\big|}{|x-z|^\eta} = \infty \quad \text{whenever } X_t(z) > 0, \quad \epsilon > 0. \qquad \diamond \tag{8}$$

Statements (7) and (8) together just say by definition that $\bar{\eta}_{\mathrm{c}}$ is the optimal index $H(z)$, for Hölder continuity of $\tilde{X}_t$ at given points $z \in \mathsf{R}$ where $\tilde{X}_t(z) > 0$.

The full program however would include proving that for any $\eta \in (\eta_{\mathrm{c}}, \bar{\eta}_{\mathrm{c}})$ there are (random) points $x \in \mathsf{R}$ such that the optimal Hölder index $H(x)$ of $\tilde{X}_t$ at x is exactly η. Moreover, we would like to establish the *Hausdorff dimension*, say $D(\eta)$, of the (random) set $\{x : H(x) = \eta\}$. The function $\eta \mapsto D(\eta)$ reveals the so-called *multifractal structure* related to the optimal Hölder index at points. As we already mentioned in [FMW08, Conjecture 1.3], we *conjecture* that

$$\lim_{\eta \downarrow \eta_{\mathrm{c}}} D(\eta) = 0 \quad \text{and} \quad \lim_{\eta \uparrow \bar{\eta}_{\mathrm{c}}} D(\eta) = 1. \tag{9}$$

The investigation of such multifractal structure is left for future work.

Note also that in the case $\alpha = 2$ for the optimal exponents η_{c} and $\bar{\eta}_{\mathrm{c}}$ we have

$$\eta_{\mathrm{c}} \downarrow 0 \quad \text{and} \quad \bar{\eta}_{\mathrm{c}} \downarrow \tfrac{1}{2} \quad \text{as } \beta \uparrow 1, \tag{10}$$

whereas for continuous super-Brownian motion one would have $\eta_{\mathrm{c}} = \frac{1}{2} = \bar{\eta}_{\mathrm{c}}$. This discontinuity reflects the essential differences between continuous and discontinuous super-Brownian motion concerning Hölder continuity properties of density states, as discussed already in [FMW08, Section 1.3].

After some preparation in the next section, the proof of Theorem 1 will be given in Section 3.

2. Preparation for the proof

Let p^α denote the continuous α–stable transition kernel related to the fractional Laplacian $\boldsymbol{\Delta}_\alpha = -(-\boldsymbol{\Delta})^{\alpha/2}$ in R^d, and S^α the related semigroup. Fix $X_0 = \mu \in \mathcal{M}_{\mathrm{f}} \setminus \{0\}$.

First we want to recall the *martingale decomposition* of the (α, d, β)-superprocess X (see, e.g., [FMW08, Lemma 1.5]): For all sufficiently smooth bounded non-negative functions φ on R^d and $t \geq 0$,

$$\langle X_t, \varphi\rangle \;=\; \langle \mu, \varphi\rangle + \int_0^t \mathrm{d}s\;\langle X_s, \boldsymbol{\Delta}_\alpha \varphi\rangle + M_t(\varphi) + a\, I_t(\varphi) \tag{11}$$

with discontinuous martingale

$$t \;\mapsto\; M_t(\varphi) \;:=\; \int_{(0,t]\times\mathsf{R}^d\times\mathsf{R}_+} \tilde{N}\big(\mathrm{d}(s,x,r)\big)\, r\,\varphi(x) \tag{12}$$

and increasing process

$$t \;\mapsto\; I_t(\varphi) \;:=\; \int_0^t \mathrm{d}s\;\langle X_s, \varphi\rangle. \tag{13}$$

Here $\tilde{N} := N - \hat{N}$, where $N\big(\mathrm{d}(s,x,r)\big)$ is a random measure on $\mathsf{R}_+ \times \mathsf{R}^d \times \mathsf{R}_+$ describing all the jumps $r\delta_x$ of X at times s at sites x of size r (which are the only discontinuities of the process X). Moreover,

$$\hat{N}\big(\mathrm{d}(s,x,r)\big) \;=\; \varrho\;\mathrm{d}s\, X_s(\mathrm{d}x)\, r^{-2-\beta}\mathrm{d}r \tag{14}$$

is the compensator of N, where $\varrho := b\,(1+\beta)\beta/\Gamma(1-\beta)$ with Γ denoting the Gamma function.

Suppose again $d < \frac{\alpha}{\beta}$ and fix $t > 0$. Then the random measure $X_t(\mathrm{d}x)$ is a.s. absolutely continuous. From the Green's function representation related to (11) (see, e.g., [FMW08, (1.9)]) we obtain the following representation of a version of the density function of $X_t(\mathrm{d}x)$ (see, e.g., [FMW08, (1.12)]):

$$\begin{aligned} X_t(x) \;=\; \mu{*}p_t^\alpha\,(x) \;+\; &\int_{(0,t]\times\mathsf{R}^d} M\big(\mathrm{d}(s,y)\big)\, p_{t-s}^\alpha(x-y) \\ +\, a\int_{(0,t]\times\mathsf{R}^d} I\big(\mathrm{d}(s,y)\big)\, p_{t-s}^\alpha(x-y) &=:\; Z_t^1(x) + Z_t^2(x) + Z_t^3(x), \quad x \in \mathsf{R}^d, \end{aligned} \tag{15}$$

(with notation in the obvious correspondence). Here $M\big(\mathrm{d}(s,y)\big)$ is the martingale measure related to (12) and $I\big(\mathrm{d}(s,y)\big)$ the random measure related to (13).

Let $\Delta X_s := X_s - X_{s-}\,,\; s > 0,$ denote the jumps of the measure-valued process X. Recall that they are of the form $r\delta_x$. By an abuse of notation, we write $r =: \Delta X_s(x)$. As a further preparation we prove the following analogous of [FMW08, Lemma 2.14]:

Lemma 3 (**Total jump mass around a given point** z)**.** *Fix* $t > 0,\; z \in \mathsf{R}$, *and* $X_0 = \mu \in \mathcal{M}_\mathrm{f}\backslash\{0\}$. *Suppose* $d = 1$ *and* $\alpha > 1 + \beta$. *Let* $\varepsilon > 0$ *and* $\gamma \in \big(0, (1+\beta)^{-1}\big)$. *There exists a constant* $c_{(16)} = c_{(16)}(\varepsilon, \gamma)$ *such that*

$$\mathbf{P}\Big(\Delta X_s(x) > c_{(16)}\,\big((t-s)|z-x|\big)^\lambda \text{ for some } s < t \text{ and } x \in B_2(z)\Big) \leq \varepsilon, \tag{16}$$

where

$$\lambda \;:=\; \frac{1}{1+\beta} - \gamma. \tag{17}$$

Proof. For any $c > 0$ (later to be chosen as $c_{(16)}$) set

$$Y \;:=\; N\Big((s,x,r):\; (s,x) \in [0,t) \times (z-2, z+2),\; r \geq c\,\big((t-s)|z-x|\big)^\lambda\Big),$$

Clearly,

$$\mathbf{P}\Big(\Delta X_s(x) > c\,\big((t-s)|z-x|\big)^{\lambda} \text{ for some } s<t \text{ and } x \in B_2(z)\Big) \\ = \mathbf{P}(Y \geq 1) \leq \mathbf{E}Y, \tag{18}$$

where in the last step we have used the classical Markov inequality. From (14),

$$\begin{aligned}\mathbf{E}Y &= \varrho\,\mathbf{E}\int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}x)\,1_{B_2(z)}(x) \int_{c\left(|z-x|(t-s)\right)^{\lambda}}^{\infty} \mathrm{d}r\; r^{-2-\beta} \\ &= \varrho\,\frac{c^{-1-\beta}}{1+\beta}\int_0^t \mathrm{d}s\;(t-s)^{-1+\gamma(1+\beta)} \int_{\mathsf{R}} \mathbf{E}X_s(\mathrm{d}x)\,1_{B_2(z)}(x)\,|z-x|^{-1+\gamma(1+\beta)}.\end{aligned}$$

Now, writing C for a generic constant (which may change from place to place),

$$\begin{aligned}&\int_{\mathsf{R}} \mathbf{E}X_s(\mathrm{d}x)\,1_{B_2(z)}(x)\,|z-x|^{-1+\gamma(1+\beta)} \\ &\leq\; \mathrm{e}^{|a|t}\int_{\mathsf{R}}\mu(\mathrm{d}y)\int_{\mathsf{R}}\mathrm{d}x\;p_s^{\alpha}(x-y)\,1_{B_2(z)}(x)\,|z-x|^{-1+\gamma(1+\beta)} \\ &\leq\; C\,\mu(\mathsf{R})\,s^{-1/\alpha}\int_{\mathsf{R}}\mathrm{d}x\;1_{B_2(z)}(x)\,|z-x|^{-1+\gamma(1+\beta)} \;=:\; c_{(19)}s^{-1/\alpha}, \qquad (19)\end{aligned}$$

where $c_{(19)} = c_{(19)}(\gamma)$. Consequently,

$$\mathbf{E}Y \;\leq\; \varrho\,c_{(19)}\,c^{-1-\beta}\int_0^t \mathrm{d}s\,s^{-1/\alpha}\,(t-s)^{-1+\gamma(1+\beta)} \;=:\; c_{(20)}\,c^{-1-\beta} \tag{20}$$

with $c_{(20)} = c_{(20)}(\gamma)$. Choose now c such that the latter expression equals ε and write $c_{(16)}$ instead of c. Recalling (18), the proof is complete. $\square$

3. Proof of Theorem 1

We will use some ideas from the proofs in Section 3 of [FMW08]. However, to be adopted to our case, those proofs require significant changes. Let $d=1$ and fix $t, z, \mu, \alpha, \beta, \eta$ as in the theorem. Consider an $x \in B_1(z)$. For simplicity we will assume $t \leq 1$ and $x > z$. By definition (15) of Z_t^2,

$$Z_t^2(z) - Z_t^2(x) \;=\; \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\,\varphi_+(s,y) - \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\,\varphi_-(s,y), \tag{21}$$

where $\varphi_+(s,y)$ and $\varphi_-(s,y)$ are the positive and negative parts of $p_{t-s}^{\alpha}(z-y) - p_{t-s}^{\alpha}(x-y)$. It is easy to check that φ_+ and φ_- satisfy the assumptions in [FMW08, Lemma 2.15]. Thus, there exist spectrally positive stable processes L^1 and L^2 such that

$$Z_t^2(z) - Z_t^2(x) \;=\; L^1_{T_+} - L^2_{T_-}\,, \tag{22}$$

where $T_{\pm} := \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\,\big(\varphi_{\pm}(s,y)\big)^{1+\beta}$. Fix any $\varepsilon \in (0,1/3)$ and $\gamma \in \big(0,(1+\beta)^{-1}\big)$. Also fix some $J = J(\gamma)$ and

$$0 =: \rho_0 < \rho_1 < \cdots < \rho_J := 1/\alpha \tag{23}$$

such that

$$\rho_\ell\,(\alpha+1) - \frac{\rho_{\ell+1}}{1+\beta} \;\geq\; -\frac{\gamma}{2}, \quad 0 \leq \ell \leq J-1. \tag{24}$$

According to [FMW08, Lemma 2.11], there exists a constant c_ε such that $\mathbf{P}(V \leq c_\varepsilon) \geq 1-\varepsilon$, where

$$V := \sup_{0 \leq s \leq t,\, y \in B_2(z)} S_{2^\alpha(t-s)} X_s(y) \tag{25}$$

(note that there is no difference in using $B_2(z)$ or its closure for taking the supremum). By Lemma 3 we can fix $c_{(16)}$ sufficiently large such that the probability of the event

$$A^{\varepsilon,1} := \Big\{ \Delta X_s(y) \leq c_{(16)} \big((t-s)|z-y|\big)^\lambda \text{ for all } s<t \text{ and } y \in B_2(z) \Big\} \tag{26}$$

is larger than $1-\varepsilon$. Moreover, according to [FMW08, Lemma 2.14], there exists a constant $c^* = c^*(\varepsilon,\gamma)$ such that the probability of the event

$$A^{\varepsilon,2} := \Big\{ \Delta X_s(y) \leq c^*(t-s)^\lambda \text{ for all } s<t \text{ and } y \in \mathsf{R} \Big\} \tag{27}$$

is larger than $1-\varepsilon$. Set

$$A^\varepsilon := A^{\varepsilon,1} \cap A^{\varepsilon,2} \cap \{V \leq c_\varepsilon\}. \tag{28}$$

Evidently,

$$\mathbf{P}(A^\varepsilon) \geq 1-3\varepsilon. \tag{29}$$

Define $Z_t^{2,\varepsilon} := Z_t^2 \, 1(A^\varepsilon)$. We first show that $Z_t^{2,\varepsilon}$ has a version which is locally Hölder continuous of all orders η less than $\bar{\eta}_c$. It follows from (22) that

$$\begin{aligned} &\mathbf{P}\Big(\big|Z_t^{2,\varepsilon}(z) - Z_t^{2,\varepsilon}(x)\big| \geq 2\,|z-x|^\eta \Big) \\ &\leq \mathbf{P}\big(L^1_{T_+} \geq |z-x|^\eta,\, A^\varepsilon\big) + \mathbf{P}\big(L^2_{T_-} \geq |z-x|^\eta,\, A^\varepsilon\big). \end{aligned} \tag{30}$$

Now let us represent the set $[0,t) \times B_2(z)$ as a union of the following spaces. Define:

$$D_0 := \Big\{ (s,y) \in [0,t) \times B_2(z) :\ y \in \big(z - 2(t-s)^{1/\alpha-\rho_1},\, x + 2(t-s)^{1/\alpha-\rho_1}\big) \Big\},$$

and for $1 \leq \ell \leq J-1$,

$$\begin{aligned} \tilde{D}_\ell &:= \Big\{ (s,y) \in [0,t) \times B_2(z) :\ y \in \big(z - 2(t-s)^{1/\alpha-\rho_{\ell+1}},\, x + 2(t-s)^{1/\alpha-\rho_{\ell+1}}\big) \Big\}, \\ D_\ell &:= \tilde{D}_\ell \setminus \tilde{D}_{\ell-1}. \end{aligned}$$

If the jumps of $M\big(\mathrm{d}(s,y)\big)$ do not exceed $c_{(16)} \big((t-s)|z-y|\big)^\lambda$ on D_ℓ, then the jumps of the process $u \mapsto \int_{(0,u] \times D_\ell} M\big(\mathrm{d}(s,y)\big)\, \varphi_\pm(s,y)$ are bounded by

$$c_{(16)} \sup_{(s,y) \in D_\ell} \big((t-s)|z-y|\big)^\lambda \varphi_\pm(s,y). \tag{31}$$

Put

$$\begin{aligned} D_{\ell,1} &:= \big\{ (s,y) \in D_\ell :\ (t-s)^{1/\alpha-\rho_{\ell+1}} \leq |z-x| \big\}, \\ D_{\ell,2} &:= \big\{ (s,y) \in D_\ell :\ (t-s)^{1/\alpha-\rho_{\ell+1}} > |z-x| \big\}, \\ D_{\ell,1}(s) &:= \big\{ y \in B_2(z) : (s,y) \in D_{\ell,1} \big\}, \quad s \in [0,t), \\ D_{\ell,2}(s) &:= \big\{ y \in B_2(z) : (s,y) \in D_{\ell,2} \big\}, \quad s \in [0,t). \end{aligned} \tag{32}$$

Since obviously $D_\ell = D_{\ell,1} \cup D_{\ell,2}$ we get that (31) is bounded by

$$\begin{aligned} & c_{(16)} \sup_{s<t}(t-s)^\lambda \sup_{y\in D_{\ell,1}(s)} |z-y|^\lambda\, \varphi_\pm(s,y) \\ & \quad + c_{(16)} \sup_{s<t}(t-s)^\lambda \sup_{y\in D_{\ell,2}(s)} |z-y|^\lambda\, \varphi_\pm(s,y) \;=:\; c_{(16)}(I_1+I_2). \end{aligned} \tag{33}$$

Clearly,

$$\varphi_\pm(s,y) \;\le\; \big|p^\alpha_{t-s}(z-y) - p^\alpha_{t-s}(x-y)\big|, \quad \text{for all } s,y. \tag{34}$$

First let us bound I_1. Note that for any $(s,y)\in D_{\ell,1}$,

$$|z-y| \;\le\; |z-x| + 2\,(t-s)^{1/\alpha-\rho_{\ell+1}} \;\le\; 3\,|z-x|. \tag{35}$$

Therefore we have

$$I_1 \;\le\; 3^\lambda\, |z-x|^\lambda \sup_{s<t}(t-s)^\lambda \sup_{y\in D_{\ell,1}(s)} \big|p^\alpha_{t-s}(z-y) - p^\alpha_{t-s}(x-y)\big|. \tag{36}$$

Using [FMW08, Lemma 2.1] with $\delta = \eta_{\mathrm{c}} - 2\alpha\gamma$ gives

$$\begin{aligned} & \sup_{y\in D_{\ell,1}(s)} \big|p^\alpha_{t-s}(z-y) - p^\alpha_{t-s}(x-y)\big| \\ & \le\; C\,|z-x|^{\eta_{\mathrm{c}}-2\alpha\gamma}\,(t-s)^{-\eta_{\mathrm{c}}/\alpha+2\gamma} \sup_{y\in D_{\ell,1}(s)} \Big(p^\alpha_{t-s}\big((z-y)/2\big) + p^\alpha_{t-s}\big((x-y)/2\big)\Big) \\ & =\; C\,|z-x|^{\eta_{\mathrm{c}}-2\alpha\gamma}\,(t-s)^{-\eta_{\mathrm{c}}/\alpha+2\gamma-1/\alpha} \\ & \qquad \times \sup_{y\in D_{\ell,1}(s)} \Big(p^\alpha_1\big((t-s)^{-1/\alpha}(z-y)/2\big) + p^\alpha_1\big((t-s)^{-1/\alpha}(x-y)/2\big)\Big). \end{aligned}$$

By the tail behavior of p^α_1 this can be continued with

$$=\; C\,|z-x|^{\eta_{\mathrm{c}}-2\alpha\gamma}\,(t-s)^{-\eta_{\mathrm{c}}/\alpha+2\gamma-1/\alpha+\rho_\ell(\alpha+1)}. \tag{37}$$

Now let us check that

$$\sup_{s<t}\,(t-s)^\lambda\,(t-s)^{-\eta_{\mathrm{c}}/\alpha+2\gamma-1/\alpha+\rho_\ell(\alpha+1)} \;\le\; 1. \tag{38}$$

Recall that $\eta_{\mathrm{c}} = \frac{\alpha}{1+\beta} - 1$. Then one can easily get that

$$\lambda - \eta_{\mathrm{c}}/\alpha + 2\gamma - 1/\alpha + \rho_\ell(\alpha+1) \;=\; \gamma + \rho_\ell(\alpha+1) \;\ge\; \gamma, \tag{39}$$

where the last inequality follows by (24). Therefore (38) follows immediately. Combining (36) – (38) we see that

$$I_1 \;\le\; C\,|z-x|^{\lambda+\eta_{\mathrm{c}}-2\alpha\gamma} \;\le\; C\,|z-x|^{\bar\eta_{\mathrm{c}}-(2\alpha+1)\gamma}, \tag{40}$$

where we used definitions (5) and (6) of η_{c} and $\bar\eta_{\mathrm{c}}$, respectively.

Now let us bound I_2. Note that for any $(s,y)\in D_{\ell,2}$,

$$|z-y| \;\le\; |z-x| + 2\,(t-s)^{1/\alpha-\rho_{\ell+1}} \;\le\; 3\,(t-s)^{1/\alpha-\rho_{\ell+1}}. \tag{41}$$

Therefore we have

$$I_2 \;\le\; 3^\lambda \sup_{s<t}\,(t-s)^{\lambda+(1/\alpha-\rho_{\ell+1})\lambda} \sup_{y\in D_{\ell,2}(s)} \big|p^\alpha_{t-s}(z-y) - p^\alpha_{t-s}(x-y)\big|. \tag{42}$$

Using again [FMW08, Lemma 2.1] but this time with $\delta = \bar{\eta}_{\mathrm{c}} - (2\alpha+1)\gamma$ gives

$$\begin{aligned}
&\sup_{y\in D_{\ell,2}(s)} \left|p^{\alpha}_{t-s}(z-y) - p^{\alpha}_{t-s}(x-y)\right| \\
&\le C\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma}\,(t-s)^{-\bar{\eta}_{\mathrm{c}}/\alpha+2\gamma+\gamma/\alpha} \\
&\qquad \times \sup_{y\in D_{\ell,2}(s)} \Big(p^{\alpha}_{t-s}\big((z-y)/2\big) + p^{\alpha}_{t-s}\big((x-y)/2\big)\Big) \\
&= C\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma}\,(t-s)^{-\bar{\eta}_{\mathrm{c}}/\alpha+2\gamma+\gamma/\alpha-1/\alpha+\rho_\ell(\alpha+1)}. \qquad (43)
\end{aligned}$$

By definition (17) of λ,

$$\begin{aligned}
&\lambda + \Big(\frac{1}{\alpha} - \rho_{\ell+1}\Big)\lambda - \frac{\bar{\eta}_{\mathrm{c}}}{\alpha} + 2\gamma + \frac{\gamma}{\alpha} - \frac{1}{\alpha} + \rho_\ell(\alpha+1) \\
&= \frac{1}{\alpha}\Big(\frac{1+\alpha}{1+\beta} - 1 - \bar{\eta}_{\mathrm{c}}\Big) + \gamma + \gamma\rho_{\ell+1} - \frac{\rho_{\ell+1}}{1+\beta} + \rho_\ell(\alpha+1) \\
&\ge \gamma/2 \qquad (44)
\end{aligned}$$

where in the last step we used definition (6) of $\bar{\eta}_{\mathrm{c}}$ and (24). Thus

$$\sup_{s<t}\,(t-s)^{\lambda+(1/\alpha-\rho_{\ell+1})\lambda-\bar{\eta}_{\mathrm{c}}/\alpha+2\gamma+\gamma/\alpha-1/\alpha+\rho_\ell(\alpha+1)} \le 1. \qquad (45)$$

Combining estimates (42), (43), and (45), we obtain

$$I_2 \le C\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma}. \qquad (46)$$

If the jumps of $M\big(\mathrm{d}(s,y)\big)$ are smaller than $c^*(t-s)^\lambda$ on $\mathsf{R}\setminus B_2(z)$ (where c^* is from (27)), then the jumps of the process $u \mapsto \int_{(0,u]\times(\mathsf{R}\setminus B_2(z))} M\big(\mathrm{d}(s,y)\big)\,\varphi_\pm(s,y)$ are bounded by

$$c^*(t-s)^\lambda \sup_{y\in\mathsf{R}\setminus B_2(z)} \varphi_\pm(s,y). \qquad (47)$$

Using [FMW08, Lemma 2.1] once again but this time with $\delta = \bar{\eta}_{\mathrm{c}} - 2\alpha\gamma$, we have

$$\begin{aligned}
\left|p^{\alpha}_{t-s}(z-y) - p^{\alpha}_{t-s}(x-y)\right| &\le C\,|z-x|^{\bar{\eta}_{\mathrm{c}}-2\alpha\gamma}\,(t-s)^{-\bar{\eta}_{\mathrm{c}}/\alpha+2\gamma} \\
&\qquad \times \Big(p^{\alpha}_{t-s}\big((z-y)/2\big) + p^{\alpha}_{t-s}\big((x-y)/2\big)\Big). \qquad (48)
\end{aligned}$$

Since $x \in B_1(z)$,

$$\begin{aligned}
&\sup_{y\in\mathsf{R}\setminus B_2(z)} \Big(p^{\alpha}_{t-s}\big((z-y)/2\big) + p^{\alpha}_{t-s}\big((x-y)/2\big)\Big) \\
&\le C\,(t-s)^{-1/\alpha}\,p^{\alpha}_1\big((t-s)^{-1/\alpha}/2\big) \le C\,(t-s). \qquad (49)
\end{aligned}$$

Therefore, (47), (34), (48), and (49) imply

$$\begin{aligned}
c^*(t-s)^\lambda \sup_{y\in\mathsf{R}\setminus B_2(z)} \varphi_\pm(s,y) &\le C\,|z-x|^{\bar{\eta}_{\mathrm{c}}-2\alpha\gamma}\,(t-s)^{\lambda-\bar{\eta}_{\mathrm{c}}/\alpha+2\gamma+1} \\
&\le c_{(50)}\,|z-x|^{\bar{\eta}_{\mathrm{c}}-2\alpha\gamma} \qquad (50)
\end{aligned}$$

for some constant $c_{(50)} = c_{(50)}(\varepsilon)$. Here we have used that $\bar{\eta}_{\mathrm{c}} \le (1+\alpha)/(1+\beta) - 1$ induces $\lambda - \bar{\eta}_{\mathrm{c}}/\alpha + 2\gamma + 1 \ge 1$.

Combining (31), (33), (40), (46), and (50), we see that all jumps of the process $u \mapsto \int_{(0,u]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\,\varphi_\pm(s,y)$ on the set A^ε are bounded by

$$c_{(51)}\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma} \qquad (51)$$

for some constant $c_{(51)} = c_{(51)}(\varepsilon)$. Therefore, by an abuse of notation writing $L_{T_\pm}$ for $L^1_{T_+}$ and $L^2_{T_-}$,

$$\begin{aligned}&\mathbf{P}\big(L_{T_\pm} \geq |z-x|^\eta,\, A^\varepsilon\big)\\ &= \mathbf{P}\Big(L_{T_\pm} \geq |z-x|^\eta,\, \sup_{u<T_\pm} \Delta L_u \leq c_{(51)}\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma},\, A^\varepsilon\Big)\\ &\leq \mathbf{P}\bigg(\sup_{v\leq T_\pm} L_v\, 1\Big\{\sup_{u<v}\Delta L_u \leq c_{(51)}\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma}\Big\} \geq |z-x|^\eta,\, A^\varepsilon\bigg). \end{aligned} \tag{52}$$

Since

$$T_\pm \;\leq\; \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\,\big|p^\alpha_{t-s}(z-y) - p^\alpha_{t-s}(x-y)\big|^{1+\beta}, \tag{53}$$

applying [FMW08, Lemma 2.12] with $\theta = 1+\beta$ and $\delta = 1$, we may fix $\varepsilon_1 \in (0, \alpha\gamma\beta)$ to get the bound

$$T_\pm \;\leq\; c_{(54)}\,\Big(|z-x|^{1+\beta}\,1_{\beta<(\alpha-1)/2} + |z-x|^{\alpha-\beta-\varepsilon_1}\,1_{\beta\geq(\alpha-1)/2}\Big) \quad \text{on } \{V \leq c_\varepsilon\} \tag{54}$$

for some constant $c_{(54)} = c_{(54)}(\varepsilon)$. Consequently,

$$\begin{aligned}&\mathbf{P}\big(L_{T_\pm} \geq |z-x|^\eta,\, A^\varepsilon\big)\\ &\leq \mathbf{P}\bigg(\sup_{v\leq c_{(54)}\left(|z-x|^{1+\beta}1_{\beta<(\alpha-1)/2}+|z-x|^{\alpha-\beta-\varepsilon_1}1_{\beta\geq(\alpha-1)/2}\right)} L_v\\ &\qquad\qquad \times\, 1\Big\{\sup_{u<v}\Delta L_u \leq c_{(51)}\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma}\Big\} \geq |z-x|^\eta\bigg).\end{aligned} \tag{55}$$

Now use [FMW08, Lemma 2.3] with $\kappa = 1+\beta,\; t = c_{(54)}\Big(|z-x|^{1+\beta}1_{\beta<(\alpha-1)/2} + |z-x|^{\alpha-\beta-\varepsilon_1}1_{\beta\geq(\alpha-1)/2}\Big),\; |z-x|^\eta$ instead of x, and $y = c_{(51)}\,|z-x|^{\bar{\eta}_{\mathrm{c}}-(2\alpha+1)\gamma}$, and noting that

$$1+\beta-\eta-\beta(\bar{\eta}_{\mathrm{c}}-2\alpha\gamma) \;\geq\; (2\alpha+1)\gamma\beta \quad \text{on } \beta < \frac{\alpha-1}{2}, \tag{56}$$

and

$$\alpha-\beta-\varepsilon_1-\eta-\beta\big(\bar{\eta}_{\mathrm{c}}-\big(2\alpha+1\big)\gamma\big) \geq (2\alpha+1)\gamma\beta-\varepsilon_1 \geq \alpha\gamma\beta \quad \text{on } \beta \geq \frac{\alpha-1}{2}, \tag{57}$$

we obtain

$$\mathbf{P}\big(L_{T_\pm} \geq |z-x|^\eta,\, A^\varepsilon\big) \;\leq\; \big(c_{(58)}\,|z-x|^{\alpha\gamma\beta}\big)^{\left(c^{-1}_{(51)}\,|z-x|^{\eta-\bar{\eta}_{\mathrm{c}}+(2\alpha+1)\gamma}\right)} \tag{58}$$

for some constant $c_{(58)} = c_{(58)}(\varepsilon)$. Applying this bound with $\gamma = \frac{\bar{\eta}_{\mathrm{c}}-\eta}{2(2\alpha+1)}$ to the summands at the right hand side in inequality (30), and noting that $\alpha\gamma\beta$ is also a positive constant here, we have

$$\mathbf{P}\Big(\big|Z^{2,\varepsilon}_t(z) - Z^{2,\varepsilon}_t(x)\big| \geq 2\,|z-x|^\eta\Big) \;\leq\; 2\big(c_{(58)}\,|z-x|\big)^{\;c_{(58)}\,|z-x|^{(\eta-\bar{\eta}_{\mathrm{c}})/2}}. \tag{59}$$

This inequality yields

$$\mathbf{P}\Big(\big|Z^{2,\varepsilon}_t(z) - Z^{2,\varepsilon}_t(x)\big| \geq 2\,|z-x|^\eta\Big) \;\leq\; C\,|z-x|^2. \tag{60}$$

Using standard arguments, we conclude that almost surely $Z_t^{2,\varepsilon}$ has a version which is locally Hölder continuous of all orders $\eta < \bar{\eta}_c$. By an abuse of notation, from now on the symbol $Z_t^{2,\varepsilon}$ always refers to this continuous version. Consequently,

$$\lim_{k\uparrow\infty} \mathbf{P}\biggl(\sup_{x\in B_1(z),\, x\neq z} \frac{\bigl|Z_t^{2,\varepsilon}(z) - Z_t^{2,\varepsilon}(x)\bigr|}{|z-x|^\eta} > k \biggr) \;=\; 0. \tag{61}$$

Combining this with the bound

$$\begin{aligned} &\mathbf{P}\biggl(\sup_{x\in B_1(z),\, x\neq z} \frac{\bigl|Z_t^{2}(z) - Z_t^{2}(x)\bigr|}{|z-x|^\eta} > k \biggr) \\ &\quad \le \mathbf{P}\biggl(\sup_{x\in B_1(z),\, x\neq z} \frac{\bigl|Z_t^{2,\varepsilon}(z) - Z_t^{2,\varepsilon}(x)\bigr|}{|z-x|^\eta} > k,\;\; A^\varepsilon \biggr) + \mathbf{P}(A^{\varepsilon,\mathrm{c}}) \end{aligned} \tag{62}$$

(with $A^{\varepsilon,\mathrm{c}}$ denoting the complement of A^ε) gives

$$\limsup_{k\uparrow\infty} \mathbf{P}\biggl(\sup_{x\in B_1(z),\, x\neq z} \frac{\bigl|Z_t^{2}(z) - Z_t^{2}(x)\bigr|}{|z-x|^\eta} > k \biggr) \;\le\; 2\varepsilon. \tag{63}$$

Since ε may be arbitrarily small, this immediately implies

$$\sup_{x\in B_1(z),\, x\neq z} \frac{\bigl|Z_t^{2}(z) - Z_t^{2}(x)\bigr|}{|z-x|^\eta} < \infty \quad \text{almost surely.} \tag{64}$$

This is the desired Hölder continuity of Z_t^2 at z, for all $\eta < \bar{\eta}_c$. Since Z_t^1 and Z_t^3 are Lipschitz continuous (cf. [FMW08, Remark 2.13]), recalling (15), the proof of Theorem 1 is complete. □

References

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