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Abstract. We study the asymptotic behaviour of a stochastic particle system that is determined by an independent motion of each particle and by an interaction mechanism between pairs of particles. The limit of the empirical measures of the system is characterized by a nonlinear equation, which is related to the Boltzmann equation. Using a uniqueness result for the limiting equation, we establish a law of large numbers. We also investigate the convergence of moments of the empirical measures.

Contents

1. Introduction

We consider a Markov particle system

$$Z(t) = (Z_1(t), \ldots, Z_n(t)), \quad t \geq 0, \tag{1.1}$$

with the generator

$$\mathcal{A}^{(n)}(\Phi)(\bar{z}) = \sum_{i=1}^{n} A_{0,z_i}(\Phi)(\bar{z}) + \tag{1.2}$$
$$\frac{1}{n} \sum_{1 \leq i \neq j \leq n} \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[\Phi(J(\bar{z}, i, j, \tilde{z}_1, \tilde{z}_2)) - \Phi(\bar{z})\Big]\, Q(z_i, z_j, d\tilde{z}_1, d\tilde{z}_2),$$

where $\bar{z} = (z_1, \ldots, z_n) \in (\mathcal{R}^d)^n$, Φ is an appropriate function on $(\mathcal{R}^d)^n$, and

$$\Big[J(\bar{z}, i, j, \tilde{z}_1, \tilde{z}_2)\Big]_k = \begin{cases} z_k, \text{ if } & k \neq i, j, \\ \tilde{z}_1, \text{ if } & k = i, \\ \tilde{z}_2, \text{ if } & k = j. \end{cases} \tag{1.3}$$

We assume that A_0 is the generator of a Markov process with the state space $\mathcal{R}^d$, and that $Q(z, \tilde{z}, ., .)$ are uniformly bounded measures on $\mathcal{R}^d \times \mathcal{R}^d$. Thus, the time evolution of the particle system is characterized by an independent motion of the particles (governed by the generator A_0) and by a pairwise interaction (governed by the kernel Q).

Let

$$\mu^{(n)}(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{Z_i(t)} \tag{1.4}$$

be the empirical measures associated with the Markov process $Z(t)$, where the symbol δ_z denotes the Dirac measure concentrated in z. We show, under suitable assumptions, that the limit (as $n \to \infty$) of the empirical measures is a deterministic function characterized as the unique solution of a nonlinear equation. This equation has the form

$$\langle \varphi, \lambda(t) \rangle = \langle \varphi, \lambda_0 \rangle + \int_0^t \langle A_0(\varphi), \lambda(s) \rangle\, ds + \tag{1.5}$$
$$\int_0^t \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2) - \varphi(z_1) - \varphi(z_2)\Big] \times$$
$$Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \Big\}\, \lambda(s, dz_1)\, \lambda(s, dz_2)\, ds, \quad \forall t \geq 0, \quad \forall \varphi \in \mathcal{D}(A_0),$$

where the measure λ_0 on $\mathcal{R}^d$ is a given initial value and $\mathcal{D}(A_0)$ is the domain of the generator A_0.

Particle systems of the form (1.1)–(1.3) have been studied for a long time. Early references are papers by M. A. Leontovich [14] and M. Kac [12]. Concerning the case $A_0 = 0$, we refer to the review papers [11] and [17] (see also the historical comments and the reference list in [19]). Various models of the form (1.1)–(1.3), for special choices of A_0 (usually generating a drift) and Q, were considered in [16], [4], [7], [9], [13], [1], [15], [19]. Such particle systems were used in [18] to analyze stochastic numerical schemes for the Boltzmann equation.

Convergence of the empirical measures to the solution of a nonlinear equation of the form (1.5) with a special kernel Q was established in [16]. The proof was based on the approximation of the independent motion by jump processes.

In [13], [15] a concrete model related to the mollified Boltzmann equation (cf. Section 3 of the present paper) was studied via the correlation function approach. The k–marginals were shown to factorize in the limit $n \to \infty$. This gives a law of large numbers for fixed time.

The purpose of [1], [19] was to generalize the approach from [16] to cover the case of the mollified Boltzmann equation. However, the proofs were not very transparent mainly due to the lack of a uniqueness result for the limiting equation (1.5).

This gap is filled in the present paper. Our proof of the convergence theorem is divided into three steps – relative compactness of the sequence of empirical measures, characterization of the limiting points of the sequence as solutions to the limiting equation (1.5), and uniqueness for the solution of the limiting equation. Our results cover the case of A_0 generating a diffusion and of fairly general continuous and bounded Q. Besides a law of large numbers we also study the behaviour of moments (unbounded functionals) of the empirical measures. This problem is of interest for numerical applications.

The paper is organized as follows. The main assumptions and results are formulated in Section 2. In Section 3 we consider some examples of systems (1.1)–(1.3), for which the assumptions of the convergence theorem are fulfilled. In particular, the relationship with the Boltzmann equation is discussed. The proofs of the results are given in Section 4. Finally, Section 5 contains some concluding remarks.

2. The convergence theorem

We denote the state space of a single particle by $\mathcal{Z} = \mathcal{R}^d$. Let $B(\mathcal{Z})$ be the Banach space of bounded Borel measurable functions on $\mathcal{Z}$ with the norm $\|\varphi\| = \sup_{z\in\mathcal{Z}} |\varphi(z)|$, and let $\hat{C}(\mathcal{Z})$ denote the subspace of continuous functions vanishing at infinity. Furthermore, let $\mathcal{M}(\mathcal{Z})$ be the space of finite, positive Borel measures on $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ and $\mathcal{P}(\mathcal{Z})$ be the space of probability measures on $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$, where $\mathcal{B}_{\mathcal{Z}}$ denotes the Borel–σ–algebra. We write

$$\langle \varphi, \nu \rangle = \int_{\mathcal{Z}} \varphi(z)\, \nu(dz)\,, \quad \text{where } \varphi \in B(\mathcal{Z})\,,\ \nu \in \mathcal{M}(\mathcal{Z})\,.$$

On $\mathcal{P}(\mathcal{Z})$, we consider the bounded Lipschitz metric

$$\varrho(\nu_1, \nu_2) = \sup_{\varphi\in B(\mathcal{Z}):\|\varphi\|_L\leq 1} |\langle \varphi, \nu_1 \rangle - \langle \varphi, \nu_2 \rangle|\,, \tag{2.1}$$

where

$$\|\varphi\|_L = \max\left(\sup_{x\in\mathcal{Z}} |\varphi(x)|\,,\ \sup_{x,y\in\mathcal{Z}\,,\,x\neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x-y\|} \right)\,. \tag{2.2}$$

The metric ϱ is equivalent to weak convergence (cf. [8, p. 150]). Let $D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$ denote the space of $\mathcal{P}(\mathcal{Z})$–valued right continuous functions with left limits (cf. [8, Sect. 3.5]).

For the convergence theorem, we need the following assumptions.

A1: The semigroup $(T_0(t) : t \geq 0)$, which corresponds to A_0, is a Feller semigroup, i.e. $T_0(t) : \hat{C}(\mathcal{Z}) \to \hat{C}(\mathcal{Z})$ are positive contractions and

$$\lim_{t\to 0} \|T_0(t)\varphi - \varphi\| = 0\,, \quad \forall \varphi \in \hat{C}(\mathcal{Z})\,. \tag{2.3}$$

Moreover the process is conservative so that the semigroup and the generator can be extended to include functions with non-zero limits at infinity with $T_0(t)1 = 1$ for all $t \geq 0$.

A2: There exists a core (cf. [8, Section 1.3]) D for A_0 such that $\varphi^2 \in D$ if $\varphi \in D$.

A3: There exists a function ψ_0 on $[0,\infty)$ satisfying

$$\psi_0(0) = 0\,, \tag{2.4}$$

$$0 < \psi_0(x) \le \psi_0(y) \le \tilde{c}_0 < \infty\,, \quad \forall\, 0 < x \le y\,, \tag{2.5}$$

$$\lim_{x \to 0} \psi_0(x) = 0\,, \tag{2.6}$$

and such that

$$\tilde{\varphi}_y \in \mathcal{D}(A_0)\,, \quad \forall y \in \mathcal{Z}\,, \quad \text{and} \quad \sup_{y \in \mathcal{Z}} \|A_0(\tilde{\varphi}_y)\| = \tilde{c}_1 < \infty\,, \tag{2.7}$$

where the functions $\tilde{\varphi}_y$ are defined as

$$\tilde{\varphi}_y(z) = \psi_0(\|z - y\|)\,. \tag{2.8}$$

A4: The kernel $Q(z_1, z_2, ., .)$ is weakly continuous in (z_1, z_2).
A5: The kernel Q satisfies

$$Q(z_1, z_2, \mathcal{Z}, \mathcal{Z}) \le C_{Q,max}\,, \quad \forall z_1, z_2 \in \mathcal{Z}\,. \tag{2.9}$$

A6: Suppose $T_0(t)$ is given by the transition function U_0 (cf. [8, Ch. 4, Sect. 1]). There exists a function $\bar{\psi}$ on $[0, \infty)$ satisfying

$$0 \le \bar{\psi}(x) \le \bar{\psi}(y)\,, \quad \forall 0 \le x \le y\,, \tag{2.10}$$

$$\lim_{x \to \infty} \bar{\psi}(x) = \infty\,, \tag{2.11}$$

and such that

$$\Big| \int_{\mathcal{Z}} \bar{\psi}(\|\tilde{z}\|)\, U_0(s, z, d\tilde{z}) - \bar{\psi}(\|z\|) \Big| \le c_0 + c_1\, \bar{\psi}(\|z\|)\,, \tag{2.12}$$
$$\forall z \in \mathcal{Z}\,,\ \forall s \in [0, t]\,, \quad \text{for some} \quad t > 0 \quad \text{and} \quad c_1 \in (0, 1)\,,$$

$$\int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[\bar{\psi}(\|\tilde{z}_1\|) + \bar{\psi}(\|\tilde{z}_2\|)\Big]\, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \le \tag{2.13}$$
$$c_2 \Big[1 + \bar{\psi}(\|z_1\|) + \bar{\psi}(\|z_2\|)\Big]\,, \quad \forall z_1, z_2 \in \mathcal{Z}\,,$$

and

$$\limsup_{n \to \infty} E^{(n)} \int_{\mathcal{Z}} \bar{\psi}(\|z\|)\, \mu^{(n)}(0, dz) < \infty\,. \tag{2.14}$$

Theorem 2.1 *Suppose that the assumptions A1–A6 are satisfied. Let $\mu^{(n)}$ be the empirical measures defined in (1.4).*

If for some $\lambda_0 \in \mathcal{P}(\mathcal{Z})$

$$\lim_{n\to\infty} E^{(n)}|\langle\varphi, \mu^{(n)}(0)\rangle - \langle\varphi, \lambda_0\rangle| = 0\,, \quad \forall\varphi \in \hat{C}(\mathcal{Z})\,, \tag{2.15}$$

then

$$\lim_{n\to\infty} P^{(n)} = \delta_\lambda\,, \tag{2.16}$$

where $P^{(n)}$ denote the measures on $D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$ associated with $\mu^{(n)}$, λ is the unique solution of Eq. (1.5), and $E^{(n)}$ denotes mathematical expectation.

Corollary 2.2 *Suppose that the assumptions of Theorem 2.1 are satisfied with a continuous function $\bar{\psi}$. Let φ be a continuous function such that*

$$\lim_{\|z\|\to\infty} \frac{|\varphi(z)|}{\bar{\psi}(\|z\|)} = 0\,. \tag{2.17}$$

Then

$$\lim_{n\to\infty} E^{(n)}|\langle\varphi, \mu^{(n)}(t)\rangle - \langle\varphi, \lambda(t)\rangle| = 0\,, \quad \forall t \geq 0\,. \tag{2.18}$$

3. Examples

Consider the generator A_0 of the form

$$A_0(\varphi)(z) = \sum_{i=1}^{d} b_i(z)\,\frac{\partial}{\partial z^{(i)}}\varphi(z) + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(z)\,\frac{\partial^2}{\partial z^{(i)}\,\partial z^{(j)}}\varphi(z)\,, \tag{3.1}$$

where $z = (z^{(1)},\ldots,z^{(d)}) \in \mathcal{R}^d$. Assume $b : \mathcal{R}^d \to \mathcal{R}^d$ is Lipschitz continuous, and $a : \mathcal{R}^d \to \mathcal{R}^d \times \mathcal{R}^d$ is such that $a_{ij} \in C^2(\mathcal{R}^d)$, $i,j = 1,\ldots d$ and $\frac{\partial^2}{\partial z^{(i)}\partial z^{(j)}} a_{kl}$, $i,j,k,l = 1,\ldots d$ are bounded. According to [8, Ch. 8, Th. 2.5], A_0 generates a Feller semigroup, and $C_c^\infty(\mathcal{R}^d)$ is a core. Thus, assumptions A1 and A2 are fulfilled, and assumption A3 can be checked easily.

Assume further that

$$\sum_{i,j=1}^{d} |a_{ij}(z)| \leq \bar{c}\,(1 + \|z\|^2)\,, \quad \forall z \in \mathcal{R}^d\,, \tag{3.2}$$

and

$$|(z, b(z))| \leq \bar{c}(1 + \|z\|^2), \quad \forall z \in \mathcal{R}^d. \tag{3.3}$$

Then, according to [8, Ch. 5, Th. 3.10], the corresponding process $\hat{z}(t)$ can be represented as the solution of a stochastic integral equation

$$\hat{z}(t) = \hat{z}(0) + \int_0^t b(\hat{z}(s))\, ds + \int_0^t \sigma(\hat{z}(s))\, dW(s),$$

where $a = \sigma \sigma^T$. Applying Ito's formula to the function $f(z) = \|z\|^2$, one obtains

$$E\|\hat{z}(t)\|^2 = E\|\hat{z}(0)\|^2 + 2\, E \int_0^t (\hat{z}(s), b(\hat{z}(s)))\, ds + E \sum_{i=1}^d \int_0^t a_{ii}(\hat{z}(s))\, ds.$$

Then, using (3.2), (3.3), and Gronwall's inequality, one easily obtains that (2.12) is fulfilled for the function

$$\bar{\psi}(x) = x^2, \quad x \in [0, \infty). \tag{3.4}$$

Consider the case of pure drift

$$A_0(\varphi)(z) = \sum_{i=1}^d b_i(z) \frac{\partial}{\partial z^{(i)}} \varphi(z), \quad z = (z^{(1)}, \ldots, z^{(d)}) \in \mathcal{R}^d. \tag{3.5}$$

Then $\quad U_0(t, z, \Gamma) = \delta_{F(t,z)}(\Gamma), \quad$ where

$$F(t, z) = z + \int_0^t b(F(s, z))\, ds.$$

Assume that $b : \mathcal{R}^d \to \mathcal{R}^d$ is Lipschitz continuous and satisfies

$$|(z, b(z))| \leq \bar{c}(1 + \|z\|^2), \quad \forall z \in \mathcal{R}^d. \tag{3.6}$$

Then one easily checks that (2.12) is fulfilled for the function

$$\bar{\psi}(x) = x^\alpha, \quad x \in [0, \infty), \quad \alpha > 0. \tag{3.7}$$

Consider the special case of $z = (x, v) \in \mathcal{R}^3 \times \mathcal{R}^3$, where x and v are interpreted as the position and the velocity of a particle. Let

$$A_0(\varphi)(z) = (v, \nabla_x)\, \varphi(x, v) + (\beta(x, v), \nabla_v)\, \varphi(x, v), \tag{3.8}$$

where the function β describes an external force acting on the particles. Condition (3.6) takes the form

$$|(x,v)+(v,\beta(x,v))| \le \bar{c}\,(1+\|x\|^2+\|v\|^2)\,.$$

Suppose that the collision kernel Q has the form

$$Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2) = \qquad (3.9)$$
$$\delta_{x_1}(d\tilde{x}_1)\,\delta_{x_2}(d\tilde{x}_2)\int_{\mathcal{S}^2}\delta_{v_1^*}(d\tilde{v}_1)\,\delta_{v_2^*}(d\tilde{v}_2)\,\frac{1}{2}\,h(x_1,x_2)\,B(v_1,v_2,e)\,de\,,$$

where $z_1=(x_1,v_1)\,,\ z_2=(x_2,v_2)\,,\ d\tilde{z}_1=(d\tilde{x}_1,d\tilde{v}_1)\,,\ d\tilde{z}_2=(d\tilde{x}_2,d\tilde{v}_2)\,,$

$$v_1^*=v_1+e(e,v_2-v_1)\,,\quad v_2^*=v_2+e(e,v_1-v_2)\,, \qquad (3.10)$$

h and B are bounded continuous functions, $\mathcal{S}^2$ denotes the unit sphere in $\mathcal{R}^3$ and de is uniform surface measure. Then assumptions A4 and A5 are fulfilled. We show that (2.13) is fulfilled for the function (3.7). Note that the transformation (3.10) preserves momentum and energy, i.e.

$$v_1^*+v_2^*=v_1+v_2\,,\qquad \|v_1^*\|^2+\|v_2^*\|^2=\|v_1\|^2+\|v_2\|^2\,. \qquad (3.11)$$

Using (3.11) and the inequality

$$(x^\alpha+y^\alpha)^{1/\alpha}\le \bar{c}_1\,(x^2+y^2)^{1/2}\le \bar{c}_2\,(x^\alpha+y^\alpha)^{1/\alpha}\,,\quad x,y\ge 0\,,$$

one obtains

$$\begin{aligned}
&\int_{\mathcal{Z}}\int_{\mathcal{Z}}\left[\bar{\psi}(\|\tilde{z}_1\|)+\bar{\psi}(\|\tilde{z}_2\|)\right]Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\le\\
&\le \int_{\mathcal{Z}}\int_{\mathcal{Z}}\bar{c}_1^\alpha\left[\|\tilde{z}_1\|^2+\|\tilde{z}_2\|^2\right]^{\alpha/2}Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\\
&= \int_{\mathcal{S}^2}\bar{c}_1^\alpha\left[\|x_1\|^2+\|v_1^*\|^2+\|x_2\|^2+\|v_2^*\|^2\right]^{\alpha/2}\frac{1}{2}\,h(x_1,x_2)\,B(v_1,v_2,e)\,de\\
&\le \bar{c}_2^\alpha\left[\|z_1\|^\alpha+\|z_2\|^\alpha\right]\frac{1}{2}\,h(x_1,x_2)\int_{\mathcal{S}^2}B(v_1,v_2,e)\,de\\
&\le c_2\left[\bar{\psi}(\|z_1\|)+\bar{\psi}(\|z_2\|)\right],\qquad \forall z_1,z_2\in\mathcal{Z}\,.
\end{aligned}$$

With A_0 defined in (3.8) and Q defined in (3.9), Eq. (1.5) takes the form

$$\langle\varphi,\lambda(t)\rangle=\langle\varphi,\lambda_0\rangle+\int_0^t\langle A_0(\varphi),\lambda(s)\rangle\,ds+ \qquad (3.12)$$
$$\int_0^t\int_{\mathcal{R}^6}\int_{\mathcal{R}^6}\int_{\mathcal{S}^2}\frac{1}{2}\,h(x_1,x_2)\,B(v_1,v_2,e)\left[\varphi(x_1,v_1^*)+\varphi(x_2,v_2^*)\right.$$
$$\left.-\varphi(x_1,v_1)-\varphi(x_2,v_2)\right]de\,\lambda(s,dx_2,dv_2)\,\lambda(s,dx_1,dv_1)\,ds\,.$$

Assume the measures $\lambda(t)$ are absolutely continuous with respect to Lebesgue measure, h is symmetric and the kernel B has the properties

$$B(v_1, v_2, e) = B(v_2, v_1, e) = B(v_1^*, v_2^*, e).$$

Then, after the substitution of the integration variables (v_1, v_2) by (v_1^*, v_2^*) and removing the test function φ, Eq. (3.12) reduces to

$$\begin{aligned} &\frac{\partial}{\partial t} p(t,x,v) + (v, \nabla_x)\, p(t,x,v) + \\ &\quad + (\beta(x,v), \nabla_v)\, p(t,x,v) = \int_{\mathcal{R}^3} dy \int_{\mathcal{R}^3} dw \int_{S^2} de \\ &\quad h(x,y)\, B(v,w,e) \Big[p(t,x,v^*)\, p(t,y,w^*) - p(t,x,v)\, p(t,y,w) \Big]. \end{aligned} \tag{3.13}$$

Replacing formally h by the delta–function, one obtains the **Boltzmann equation**

$$\begin{aligned} &\frac{\partial}{\partial t} p(t,x,v) + (v, \nabla_x)\, p(t,x,v) + (\beta(x,v), \nabla_v)\, p(t,x,v) = \\ &\quad \int_{\mathcal{R}^3} dw \int_{S^2} de\, B(v,w,e) \Big[p(t,x,v^*)\, p(t,x,w^*) - p(t,x,v)\, p(t,x,w) \Big]. \end{aligned}$$

This is the basic equation of the kinetic theory of dilute (monatomic) gases. It describes the time evolution of a density function $p(t,x,v)$ that depends on a time variable $t \geq 0$, on coordinates $x \in \mathcal{R}^3$ representing the possible positions of the gas particles, and on coordinates $v \in \mathcal{R}^3$ representing the possible velocities of the gas particles. The function B is called the collision kernel, and the function β describes an external force acting on the particles. The objects v^* and w^* are defined as in (3.10) and are interpreted as the post–collision velocities of two particles with the pre–collision velocities v and w. We refer to [5] concerning more information about the Boltzmann equation. Eq. (3.13) is a **mollified Boltzmann equation** (with h called the mollifier) (cf. [5, Ch. 8, Sect. 3]).

4. Proof of the convergence theorem

The proof of Theorem 2.1 is based on the following three results.

Theorem 4.1 (relative compactness) *Suppose the assumptions* A3, A5 *and* A6 *are fulfilled.*

Then the sequence $(P^{(n)})$ *is relatively compact.*

Theorem 4.2 (characterization of limiting points) *Let* P^∞ *be any limiting point of the sequence* $(P^{(n)})$. *Let* $\bar{\Omega}$ *denote the set of all* $\omega \in D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$ *satisfying Eq. (1.5). Suppose the assumptions* A1, A2, A4 *and* A5 *are fulfilled.*

Then $P^\infty(\bar{\Omega}) = 1$.

Theorem 4.3 (uniqueness of solution) *Suppose the assumptions* A1 *and* A5 *are fulfilled.*

Then there exists at most one solution of Eq. (1.5).

Proof of Theorem 2.1. From Theorem 4.2 and Theorem 4.3 we obtain that any limiting point of the sequence $(P^{(n)})$ is concentrated on the set $\bar{\Omega} = \{\lambda\}$. Thus, there is at most one limiting point, and (2.16) follows from Theorem 4.1. □

The proofs of Theorems 4.1–4.3 are prepared by several lemmas.

Lemma 4.4 *Consider the function*

$$\Phi(\bar{z}) = \frac{1}{n}\sum_{i=1}^{n} \varphi_i(z_i), \quad \bar{z} = (z_1, \ldots, z_n) \in \mathcal{Z}^n, \tag{4.1}$$

where $\varphi_i \in \mathcal{D}(A_0)$, $i = 1, \ldots, n$.

Then

$$\mathcal{A}^{(n)}(\Phi)(\bar{z}) = \frac{1}{n}\sum_{i=1}^{n} A_0(\varphi_i)(z_i) + \tag{4.2}$$
$$\frac{1}{n^2}\sum_{1\le i\ne j\le n} \int_{\mathcal{Z}}\int_{\mathcal{Z}} \Big[\varphi_i(\tilde{z}_1) + \varphi_j(\tilde{z}_2) - \varphi_i(z_i) - \varphi_j(z_j)\Big]\, Q(z_i, z_j, d\tilde{z}_1, d\tilde{z}_2) .$$

If, in addition,

$$\varphi_i^2 \in \mathcal{D}(A_0), \quad i = 1, \ldots, n, \tag{4.3}$$

then

$$|\mathcal{A}^{(n)}(\Phi^2)(\bar{z}) - 2\,\Phi(\bar{z})\,\mathcal{A}^{(n)}(\Phi)(\bar{z})| \le \tag{4.4}$$
$$\frac{2}{n}\max_i \|\varphi_i\, A_0(\varphi_i)\| + \frac{1}{n}\max_i \|A_0(\varphi_i^2)\| + \frac{1}{n}\, 16\, C_{Q,max}\, (\max_i \|\varphi_i\|)^2 .$$

Proof. We introduce the notations

$$\mathcal{A}_0^{(n)}(\Phi)(\bar{z}) = \sum_{i=1}^{n} A_{0,z_i}(\Phi)(\bar{z}) \tag{4.5}$$

and

$$\mathcal{Q}^{(n)}(\Phi)(\bar{z}) = \frac{1}{n} \sum_{1\le i\neq j\le n} \int_{\mathcal{Z}} \int_{\mathcal{Z}} \left[\Phi(J(\bar{z},i,j,\tilde{z}_1,\tilde{z}_2)) - \Phi(\bar{z})\right] Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2)\,, \tag{4.6}$$

so that the generator (1.2) takes the form

$$\mathcal{A}^{(n)} = \mathcal{A}_0^{(n)} + \mathcal{Q}^{(n)}\,. \tag{4.7}$$

Taking into account (1.3), we obtain

$$\Phi(J(\bar{z},i,j,\tilde{z}_1,\tilde{z}_2)) = \Phi(\bar{z}) + \frac{1}{n}\left[\varphi_i(\tilde{z}_1) + \varphi_j(\tilde{z}_2) - \varphi_i(z_i) - \varphi_j(z_j)\right], \tag{4.8}$$

and thus

$$\begin{aligned}\Phi^2(J(\bar{z},i,j,\tilde{z}_1,\tilde{z}_2)) &= \Phi^2(\bar{z}) + 2\,\Phi(\bar{z})\,\frac{1}{n}\left[\varphi_i(\tilde{z}_1) + \varphi_j(\tilde{z}_2) - \varphi_i(z_i) - \varphi_j(z_j)\right] \\ &\quad + \frac{1}{n^2}\left[\varphi_i(\tilde{z}_1) + \varphi_j(\tilde{z}_2) - \varphi_i(z_i) - \varphi_j(z_j)\right]^2. \end{aligned} \tag{4.9}$$

By (4.6), (4.8), and (4.9), we have

$$\mathcal{Q}^{(n)}(\Phi)(\bar{z}) = \frac{1}{n^2} \sum_{1\le i\neq j\le n} \int_{\mathcal{Z}} \int_{\mathcal{Z}} \left[\varphi_i(\tilde{z}_1) + \varphi_j(\tilde{z}_2) - \varphi_i(z_i) - \varphi_j(z_j)\right] Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2) \tag{4.10}$$

and

$$\mathcal{Q}^{(n)}(\Phi^2)(\bar{z}) = 2\,\Phi(\bar{z})\,\mathcal{Q}^{(n)}(\Phi)(\bar{z}) + \frac{1}{n^3} \sum_{1\le i\neq j\le n} \int_{\mathcal{Z}} \int_{\mathcal{Z}} \left[\varphi_i(\tilde{z}_1) + \varphi_j(\tilde{z}_2) - \varphi_i(z_i) - \varphi_j(z_j)\right]^2 Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2)\,. \tag{4.11}$$

Furthermore, it follows from (4.1) and (4.5) that

$$\mathcal{A}_0^{(n)}(\Phi)(\bar{z}) = \frac{1}{n} \sum_{i=1}^{n} A_0(\varphi_i)(z_i)\,. \tag{4.12}$$

Since

$$\Phi^2(\bar{z}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}\varphi_i(z_i)\,\varphi_j(z_j) + \frac{1}{n^2}\sum_{i=1}^{n}\varphi_i^2(z_i)\,,$$

we obtain

$$\begin{aligned} A_{0,z_k}(\Phi^2)(\bar{z}) &= \frac{1}{n^2}\sum_{j\neq k}A_0(\varphi_k)(z_k)\,\varphi_j(z_j) + \\ &\quad \frac{1}{n^2}\sum_{i\neq k}\varphi_i(z_i)\,A_0(\varphi_k)(z_k) + \frac{1}{n^2}A_0(\varphi_k^2)(z_k)\,, \end{aligned}$$

and, by (4.5),

$$\begin{aligned} \mathcal{A}_0^{(n)}(\Phi^2)(\bar{z}) &= \frac{2}{n^2}\sum_{k=1}^{n}\sum_{j\neq k}A_0(\varphi_k)(z_k)\,\varphi_j(z_j) + \frac{1}{n^2}\sum_{k=1}^{n}A_0(\varphi_k^2)(z_k) \qquad (4.13)\\ &= 2\,\Phi(\bar{z})\,\mathcal{A}_0^{(n)}(\Phi)(\bar{z}) - \frac{2}{n^2}\sum_{k=1}^{n}A_0(\varphi_k)(z_k)\,\varphi_k(z_k) + \frac{1}{n^2}\sum_{k=1}^{n}A_0(\varphi_k^2)(z_k)\,. \end{aligned}$$

Now (4.2) follows from (4.7), (4.12), (4.10), and (4.4) is a consequence of (4.7), (4.13), (4.11), and (2.9). □

Lemma 4.5 *Let ψ be a function on $[0,\infty)$ satisfying (2.5). Let $\varphi \in B(\mathcal{Z})$ be such that $\|\varphi\|_L \le 1$ (cf. (2.2)).*

Then

$$|\varphi(z) - \varphi(\tilde{z})| \le \varepsilon + \frac{2}{\psi(\varepsilon)}\psi(\|z-\tilde{z}\|)\,, \quad \forall\varepsilon > 0\,, \quad \forall z,\tilde{z}\in\mathcal{Z}\,.$$

Proof. If $\|z-\tilde{z}\| \le \varepsilon$, then $|\varphi(z)-\varphi(\tilde{z})| \le \varepsilon$.

If $\|z-\tilde{z}\| > \varepsilon$, then $\psi(\|z-\tilde{z}\|) \ge \psi(\varepsilon)$ and consequently

$$|\varphi(z)-\varphi(\tilde{z})| \le 2 = \frac{2}{\psi(\varepsilon)}\psi(\varepsilon) \le \frac{2}{\psi(\varepsilon)}\psi(\|z-\tilde{z}\|)\,. \quad \square$$

Lemma 4.6 *Suppose there exists a function ψ_0 on $[0,\infty)$ satisfying (2.4), (2.5), and such that (2.7) holds.*

Then

$$E^{(n)}\left[\varrho(\mu^{(n)}(t+u),\mu^{(n)}(t))|\mathcal{F}_t^{(n)}\right] \le \varepsilon + \frac{2\,u}{\psi_0(\varepsilon)}\left[\tilde{c}_1 + 4\,C_{Q,max}\,\tilde{c}_0\right]\,, \quad (4.14)$$
$$\forall t\ge 0\,,\ \forall u \ge 0\,,\ \forall\varepsilon > 0\,,\ \forall n\,.$$

Proof. Using Lemma 4.5, one obtains

$$\begin{aligned} |\langle\varphi,\mu^{(n)}(t+u)\rangle - \langle\varphi,\mu^{(n)}(t)\rangle| &\le \frac{1}{n}\sum_{i=1}^{n}|\varphi(Z_i(t+u)) - \varphi(Z_i(t))| \\ &\le \varepsilon + \frac{2}{\psi_0(\varepsilon)}\frac{1}{n}\sum_{i=1}^{n}\psi_0(\|Z_i(t+u) - Z_i(t)\|), \end{aligned}$$

and, according to (2.1),

$$\varrho(\mu^{(n)}(t+u),\mu^{(n)}(t)) \le \varepsilon + \frac{2}{\psi_0(\varepsilon)}\frac{1}{n}\sum_{i=1}^{n}\psi_0(\|Z_i(t+u) - Z_i(t)\|). \quad (4.15)$$

Let $E^{(n)}_{\bar{z}^0}$ denote the conditional expectation under the condition

$$Z(t) = \bar{z}^0, \quad \text{where} \quad \bar{z}^0 = \left(z^0_1,\ldots,z^0_n\right) \in \mathcal{Z}^n.$$

Consider the functions

$$\varphi_i(z) = \psi_0(\|z - z^0_i\|), \quad i = 1,\ldots,n,$$

and the function Φ defined in (4.1). The assumption of Lemma 4.4 is fulfilled because of (2.7), (2.8). Using the fact

$$E^{(n)}_{\bar{z}^0}\Phi(Z(t+u)) = \Phi(\bar{z}^0) + E^{(n)}_{\bar{z}^0}\int_t^{t+u}\mathcal{A}^{(n)}(\Phi)(Z(t+s))\,ds$$

and the inequality

$$|\mathcal{A}^{(n)}(\Phi)(\bar{z})| \le \tilde{c}_1 + 4\,C_{Q,max}\,\tilde{c}_0,$$

which follows from (4.2), (2.7), and (2.5), we obtain

$$E^{(n)}_{\bar{z}^0}\Phi(Z(t+u)) \le \Phi(\bar{z}^0) + u\,[\tilde{c}_1 + 4\,C_{Q,max}\,\tilde{c}_0]\,. \quad (4.16)$$

Now assertion (4.14) follows from (4.15), (4.16), (2.4), and the Markov property. □

Lemma 4.7 *Let φ be a nonnegative function on $\mathcal{Z}$ such that*

$$|\int_{\mathcal{Z}}\varphi(\tilde{z})\,U_0(s,z,d\tilde{z}) - \varphi(z)| \le c_0 + c_1\,\varphi(z), \quad \forall z\in\mathcal{Z},\ \forall s\in[0,t], \quad (4.17)$$

for some $t > 0$ *and* $c_1 \in (0,1)$, *and*

$$\int_{\mathcal{Z}} \int_{\mathcal{Z}} [\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2)] \, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \le \\ c_2 \, [1 + \varphi(z_1) + \varphi(z_2)] \,, \quad \forall z_1, z_2 \in \mathcal{Z} \,. \tag{4.18}$$

Then

$$E^{(n)} \langle \varphi, \mu^{(n)}(u+t) \rangle \le c_3 \left[E^{(n)} \langle \varphi, \mu^{(n)}(u) \rangle + 1 \right] , \quad \forall u \ge 0 \,. \tag{4.19}$$

Proof. Consider the function

$$\Phi(\bar{z}) = \frac{1}{n} \sum_{i=1}^{n} [\varphi(z_i) + 1] \,.$$

We will show that

$$\mathcal{T}^{(n)}(t)(\Phi)(\bar{z}) \le c_3 \, \Phi(\bar{z}) \,, \quad \forall \bar{z} \in \mathcal{Z}^n \,, \tag{4.20}$$

where $\mathcal{T}^{(n)}(t)$ is the semigroup corresponding to the process $Z(t)$. Assertion (4.19) follows from (4.20), since

$$\Phi(Z(s)) = \langle \varphi, \mu^{(n)}(s) \rangle + 1 \,, \quad \forall s \ge 0 \,.$$

Define $\tilde{\mathcal{Q}}^{(n)} = \mathcal{Q}^{(n)} + (n-1)\, C_{Q,max} \, \mathrm{Id}$. Then $\tilde{\mathcal{Q}}^{(n)}$ is a positive operator. We use the series representation

$$\mathcal{T}^{(n)}(t) = \exp\left(-C_{Q,max}\,(n-1)\,t\right) \Big[\mathcal{T}_0^{(n)}(t) + \\ \sum_{l=1}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{l-1}} dt_l \, \mathcal{T}_0^{(n)}(t-t_1) \, \tilde{\mathcal{Q}}^{(n)} \ldots \mathcal{T}_0^{(n)}(t_{l-1}-t_l) \, \tilde{\mathcal{Q}}^{(n)} \mathcal{T}_0^{(n)}(t_l) \Big] \,, \tag{4.21}$$

where $\mathcal{T}_0^{(n)}(t)$ denotes the semigroup corresponding to the generator $\mathcal{A}_0^{(n)}$ defined in (4.5). We proceed in two steps showing

$$\mathcal{T}_0^{(n)}(t)(\Phi)(\bar{z}) \le c_4 \, \Phi(\bar{z}) \,, \quad \forall \bar{z} \in \mathcal{Z}^n \,, \tag{4.22}$$

and

$$|\mathcal{Q}^{(n)}(\mathcal{T}_0^{(n)}(s)(\Phi))(\bar{z})| \le c_5 \, \mathcal{T}_0^{(n)}(s)(\Phi)(\bar{z}) \,, \quad \forall \bar{z} \in \mathcal{Z}^n \,, \ \forall s \,. \tag{4.23}$$

Using (4.23), one obtains

$$\tilde{\mathcal{Q}}^{(n)}(\mathcal{T}_0^{(n)}(s)(\Phi))(\bar{z}) = C_{Q,max}\,(n-1)\,\mathcal{T}_0^{(n)}(s)(\Phi)(\bar{z}) + \mathcal{Q}^{(n)}(\mathcal{T}_0^{(n)}(s)(\Phi))(\bar{z})$$
$$\leq [C_{Q,max}\,(n-1) + c_5]\;\mathcal{T}_0^{(n)}(s)(\Phi)(\bar{z})\,.$$

Consequently, it follows from (4.21) and (4.22) that

$$\begin{aligned}
&\mathcal{T}^{(n)}(t)(\Phi)(\bar{z}) \leq \exp\left(-C_{Q,max}\,(n-1)\,t\right)\,\Big[\mathcal{T}_0^{(n)}(t)(\Phi)(\bar{z}) + \\
&\quad\sum_{l=1}^{\infty}\int_0^t dt_1 \ldots \int_0^{t_{l-1}} dt_l\,\mathcal{T}_0^{(n)}(t)(\Phi)(\bar{z})\,[C_{Q,max}\,(n-1)+c_5]^l\Big] \\
&= \mathcal{T}_0^{(n)}(t)(\Phi)(\bar{z})\,\exp\left([-C_{Q,max}\,(n-1) + C_{Q,max}\,(n-1) + c_5]\,t\right) \\
&\leq c_4\,\exp(c_5\,t)\,\Phi(\bar{z})\,.
\end{aligned}$$

First we show (4.22). We have $\mathcal{T}_0^{(n)}(t) = \prod_{i=1}^n T_{0,z_i}(t)$, where $T_{0,z_i}(t)$ denotes the semigroup corresponding to the generator A_0, because of the independence of the components. Consequently,

$$\mathcal{T}_0^{(n)}(t)\Phi(\bar{z}) = \frac{1}{n}\sum_{i=1}^n [1 + T_0(t)\varphi(z_i)]\,. \tag{4.24}$$

It follows from (4.17) that

$$T_0(s)(\varphi)(z) \leq \varphi(z) + c_0 + c_1\,\varphi(z)\,, \quad \forall s \in [0,t]\,. \tag{4.25}$$

From (4.24) and (4.25), one obtains

$$\mathcal{T}_0^{(n)}(t)\Phi(\bar{z}) \leq [1 + c_0 + c_1]\,\Phi(\bar{z})\,.$$

Next we show (4.23). We have from (4.24) and (4.10)

$$\mathcal{Q}^{(n)}(\mathcal{T}_0^{(n)}(s)(\Phi))(\bar{z}) = \frac{1}{n^2}\sum_{1\leq i\neq j\leq n}\int_{\mathcal{Z}}\int_{\mathcal{Z}}$$
$$\Big[T_0(s)\varphi(\tilde{z}_1) + T_0(s)\varphi(\tilde{z}_2) - T_0(s)\varphi(z_i) - T_0(s)\varphi(z_j)\Big]\,Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2)$$

and

$$\begin{aligned}
&|\mathcal{Q}^{(n)}(\mathcal{T}_0^{(n)}(s)(\Phi))(\bar{z})| \leq 2\,C_{Q,max}\,\mathcal{T}_0^{(n)}(s)(\Phi)(\bar{z}) + \\
&\quad\frac{1}{n^2}\sum_{1\leq i\neq j\leq n}\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[T_0(s)\varphi(\tilde{z}_1) + T_0(s)\varphi(\tilde{z}_2)\Big]\,Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2)\,.
\end{aligned} \tag{4.26}$$

Suppose we have

$$\int_{\mathcal{Z}}\int_{\mathcal{Z}} [T_0(s)\varphi(\tilde{z}_1) + T_0(s)\varphi(\tilde{z}_2)]\, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \\ \leq c_6\, [1 + T_0(s)\varphi(z_1) + T_0(s)\varphi(z_2)]\,. \tag{4.27}$$

Then one obtains from (4.26)

$$|\mathcal{Q}^{(n)}(\mathcal{T}_0^{(n)}(s)(\Phi))(\bar{z})| \leq 2\, C_{Q,max}\, \mathcal{T}_0^{(n)}(s)(\Phi)(\bar{z}) + 2\, c_6\, \mathcal{T}_0^{(n)}(s)(\Phi)(\bar{z})\,,$$

and (4.23) follows.

It remains to show (4.27). Using (4.25) and assumption (4.18), we obtain

$$\begin{aligned}
&\int_{\mathcal{Z}}\int_{\mathcal{Z}} [T_0(s)\varphi(\tilde{z}_1) + T_0(s)\varphi(\tilde{z}_2)]\, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \leq \\
&\leq\ 2\, c_0\, C_{Q,max} + [1 + c_1] \int_{\mathcal{Z}}\int_{\mathcal{Z}} [\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2)]\, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \\
&\leq\ 2\, c_0\, C_{Q,max} + [1 + c_1]\, c_2\, [1 + \varphi(z_1) + \varphi(z_2)]\,.
\end{aligned} \tag{4.28}$$

From (4.17) we have

$$\varphi(z) \leq T_0(s)\, \varphi(z) + c_0 + c_1\, \varphi(z)\,, \quad \forall s \in [0, t]\,,$$

or

$$\varphi(z) \leq [1 - c_1]^{-1}\, [c_0 + T_0(s)\, \varphi(z)]\,, \quad \forall s \in [0, t]\,. \tag{4.29}$$

Consequently, by (4.28) and (4.29),

$$\begin{aligned}
&\int_{\mathcal{Z}}\int_{\mathcal{Z}} [T_0(s)\varphi(\tilde{z}_1) + T_0(s)\varphi(\tilde{z}_2)]\, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \leq \\
&\quad 2\, c_1\, C_{Q,max} + [1 + c_1]\, c_2\, [1 - c_1]^{-1}\, [1 + 2\, c_0 + T_0(s)\, \varphi(z_1) + T_0(s)\, \varphi(z_2)]\,,
\end{aligned}$$

and (4.27) follows. □

Proof of Theorem 4.1. According to [8, Ch. 3, Theorem 8.6]), it is sufficient to check the conditions

(a) $\forall \eta > 0\,,\ \forall t \geq 0\ \ \exists$ compact $\Gamma_{\eta,t} \subset \mathcal{P}(\mathcal{Z})$ such that

$$\liminf_{n\to\infty} \mathrm{Prob}\{\mu^{(n)}(t) \in \Gamma_{\eta,t}\} \geq 1 - \eta \tag{4.30}$$

and

(b) $\forall T > 0 \ \ \exists\, \beta > 0$ and $\exists$ random variables $\gamma_n(\delta) \geq 0$, $\delta \in (0,1)$, $n \geq 1$ such that

$$E^{(n)}[\varrho^\beta(\mu^{(n)}(t+u), \mu^{(n)}(t)) | \mathcal{F}_t^{(n)}] \ \leq \ E^{(n)}[\gamma_n(\delta) | \mathcal{F}_t^{(n)}], \quad \forall t \in [0,T], \ \forall u \in [0,\delta], \tag{4.31}$$

and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} E^{(n)} \gamma_n(\delta) = 0 . \tag{4.32}$$

To check condition (a), we note that the sets

$$K_c = \left\{ \nu \in \mathcal{P}(\mathcal{Z}) \ : \ \int_{\mathcal{Z}} \bar{\psi}(\|z\|)\, \nu(dz) \leq c \right\}, \quad c \geq 0,$$

are compact in $\mathcal{P}(\mathcal{Z})$, where the function $\bar{\psi}$ is supposed to satisfy (2.10) and (2.11). Thus, condition (a) will be fulfilled if
$\forall \eta > 0$, $\forall t \geq 0 \ \ \exists\, c(\eta,t) \geq 0$ such that

$$\liminf_{n \to \infty} \text{Prob} \left\{ \mu^{(n)}(t) \in K_{c(\eta,t)} \right\} \geq 1 - \eta ,$$

or,

$$\limsup_{n \to \infty} \text{Prob} \left\{ \int_{\mathcal{Z}} \bar{\psi}(\|z\|)\, \mu^{(n)}(t,dz) > c(\eta,t) \right\} \leq \eta . \tag{4.33}$$

From (4.33) and Chebyshev's inequality, one obtains that the condition

$$\limsup_{n \to \infty} E^{(n)} \int_{\mathcal{Z}} \bar{\psi}(\|z\|)\, \mu^{(n)}(t,dz) < \infty$$

is sufficient. This is assured by Lemma 4.7 and the assumption A6.

It remains to check condition (b). According to Lemma 4.6 and assumption A3, we obtain

$$E^{(n)}[\varrho(\mu^{(n)}(t+u), \mu^{(n)}(t)) | \mathcal{F}_t^{(n)}] \leq \varepsilon + \frac{2\,\delta}{\psi_0(\varepsilon)} \left[\tilde{c}_0 + 4\, C_{Q,max}\, c_0\right], \quad \forall t \geq 0, \ \forall u \in [0,\delta], \ \forall \varepsilon > 0, \ \forall n .$$

We choose $\varepsilon(\delta) = \psi_0^{-1}(\sqrt{\delta})$ (the inverse function exists because of (2.4)–(2.6)) and denote

$$\gamma_n(\delta) = \varepsilon(\delta) + 2\sqrt{\delta}\, \left[\tilde{c}_0 + 4\, C_{Q,max}\, c_0\right] .$$

Then (4.31) is fulfilled with $\beta = 1$, and (4.32) holds because of (2.4) and (2.6). $\square$

Lemma 4.8 *Suppose Q satisfies (2.9) and assumption* A4, *and let $\varphi \in \mathcal{D}(A_0)$ be such that φ and $A_0(\varphi)$ are bounded and continuous.*

Then the mapping $\Psi_\varphi : D_{\mathcal{P}(\mathcal{Z})}[0,\infty) \to D_{\mathcal{R}}[0,\infty)$, defined as

$$\begin{aligned}\Psi_\varphi(\omega)(t) &= \langle\varphi,\omega(t)\rangle - \langle\varphi,\omega(0)\rangle - \int_0^t \langle A_0(\varphi),\omega(s)\rangle\, ds - \\ &\int_0^t \int_{\mathcal{Z}}\int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}}\int_{\mathcal{Z}} \big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_1)-\varphi(z_2)\big] \times \\ &Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\Big\}\, \omega(s,dz_1)\,\omega(s,dz_2)\, ds\,,\end{aligned} \tag{4.34}$$

is continuous.

Proof. First we notice that $\lim_{n\to\infty}\omega_n(0)=\omega(0)$ in $\mathcal{P}(\mathcal{Z})$, if $\lim_{n\to\infty}\omega_n=\omega$ in $D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$, according to [8, Ch. 3, Prop. 5.2], since 0 is a continuity point for any ω. Thus, the mapping

$$\Psi^{(1)}_\varphi(\omega)(t) = \langle\varphi,\omega(0)\rangle \tag{4.35}$$

is continuous.

Next, we use the fact (cf. [8, p. 153]) that the mapping

$$f(x)(t) = \int_0^t x(s)\, ds$$

from $D_{\mathcal{R}}[0,\infty)$ into $D_{\mathcal{R}}[0,\infty)$ is continuous. Therefore, it remains to show that

$$\Psi^{(2)}_\varphi(\omega)(t) = \langle\varphi,\omega(t)\rangle\,, \quad \Psi^{(3)}_\varphi(\omega)(t) = \langle A_0(\varphi),\omega(t)\rangle\,,$$

and

$$\begin{aligned}\Psi^{(4)}_\varphi(\omega)(t) &= \int_{\mathcal{Z}}\int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}}\int_{\mathcal{Z}} \big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_1)-\varphi(z_2)\big] \times \\ &Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\Big\}\, \omega(t,dz_1)\,\omega(t,dz_2)\end{aligned}$$

are continuous mappings from $D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$ into $D_{\mathcal{R}}[0,\infty)$. This is fulfilled (cf. [8, p. 151]), if

$$\hat{\Psi}^{(2)}_\varphi(\nu) = \langle\varphi,\nu\rangle\,, \qquad \hat{\Psi}^{(3)}_\varphi(\nu) = \langle A_0(\varphi),\nu\rangle\,,$$

and

$$\hat{\Psi}^{(4)}_{\varphi}(\nu) = \int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big\{\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_1)-\varphi(z_2)\Big] \times$$
$$Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\Big\}\,\nu(dz_1)\,\nu(dz_2)$$

are continuous mappings from $\mathcal{P}(\mathcal{Z})$ into $\mathcal{R}$. These properties are fulfilled because the functions φ and $A_0(\varphi)$ are bounded and continuous, and the kernel Q satisfies (2.9) and assumption A4. □

Proof of Theorem 4.2. If $\Phi \in \mathcal{D}(\mathcal{A}^{(n)})$, then

$$\Phi(Z(t)) = \Phi(Z(0)) + \int_0^t \mathcal{A}^{(n)}(\Phi)(Z(s))\,ds + M(t)\,, \tag{4.36}$$

where $M(t)$ is a martingale (cf., e.g., [8, Ch. 4, Prop. 1.7]). If $\Phi^2 \in \mathcal{D}(\mathcal{A}^{(n)})$, then one can show that

$$E^{(n)}M(t)^2 = E^{(n)}\int_0^t[\mathcal{A}^{(n)}(\Phi^2)-2\,\Phi\,\mathcal{A}^{(n)}(\Phi)](Z(s))\,ds\,. \tag{4.37}$$

We consider the function (4.1) with $\varphi_i = \varphi$, where $\varphi \in \mathcal{D}(A_0)$ is such that $\varphi^2 \in \mathcal{D}(A_0)$. Note that assumption (4.3) is fulfilled, $\Phi \in \mathcal{D}(\mathcal{A}^{(n)})$, $\Phi^2 \in \mathcal{D}(\mathcal{A}^{(n)})$, and

$$\Phi(Z(t)) = \langle\varphi,\mu^{(n)}(t)\rangle\,. \tag{4.38}$$

Using Lemma 4.4, (4.36), (4.37), and (4.38), we obtain

$$\langle\varphi,\mu^{(n)}(t)\rangle = \langle\varphi,\mu^{(n)}(0)\rangle + \int_0^t\langle A_0(\varphi),\mu^{(n)}(s)\rangle\,ds + \tag{4.39}$$
$$\int_0^t\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big\{\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_1)-\varphi(z_2)\Big] \times$$
$$Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\Big\}\,\mu^{(n)}(s,dz_1)\,\mu^{(n)}(s,dz_2)\,ds + R^{(n)}(\varphi,t) + M^{(n)}(\varphi,t)\,,$$

where

$$|R^{(n)}(\varphi,t)| \le \frac{1}{n}\,4\,\|\varphi\|\,C_{Q,max}\,t \quad \text{a.s.}\,, \tag{4.40}$$

and $M^{(n)}$ is a martingale such that

$$E^{(n)}M^{(n)}(\varphi,t)^2 \le \Big\{\frac{2}{n}\,\|\varphi\,A_0(\varphi)\| + \frac{1}{n}\,\|A_0(\varphi^2)\| + \frac{1}{n}\,16\,\|\varphi\|^2\,C_{Q,max}\Big\}\,t\,. \tag{4.41}$$

Consider the mapping Ψ_φ defined in (4.34). It follows from (4.39) that

$$\Psi_\varphi(\mu^{(n)})(t) = R^{(n)}(\varphi, t) + M^{(n)}(\varphi, t)$$

and from (4.40) and (4.41) that

$$\lim_{n\to\infty} E^{(n)} \sup_{0\le s\le t} |\Psi_\varphi(\mu^{(n)})(s)| = 0\,, \quad \forall t > 0\,. \tag{4.42}$$

It is easy to see from the definition of the Skorokhod topology (cf. [8, p. 117]) that (4.42) implies

$$\lim_{n\to\infty} E^{(n)} d_{\mathcal{R}}(\Psi_\varphi(\mu^{(n)}), 0) = 0\,,$$

or

$$\lim_{n\to\infty} \langle \psi, P^{(n)} \rangle = 0\,, \tag{4.43}$$

where the function ψ is defined as

$$\psi(\omega) = d_{\mathcal{R}}(\Psi_\varphi(\omega), 0)\,, \quad \omega \in D_{\mathcal{P}(\mathcal{Z})}[0, \infty)\,, \tag{4.44}$$

and $d_{\mathcal{R}}$ is a bounded metric in $D_{\mathcal{R}}[0,\infty)$ giving the Skorokhod topology.

The function (4.44) is bounded and continuous because of Lemma 4.8 and the obvious inequality $|d_{\mathcal{R}}(x,0) - d_{\mathcal{R}}(y,0)| \le d_{\mathcal{R}}(x,y)$. Therefore, we obtain

$$\lim_{n\to\infty} \langle \psi, P^{(n)} \rangle = \langle \psi, P^{\infty} \rangle\,. \tag{4.45}$$

From (4.43) and (4.45), one obtains

$$P^{\infty}(\{\omega : \Psi_\varphi(\omega) = 0\}) = 1\,. \tag{4.46}$$

Next, we use the hypothesis (2.15) on the initial conditions. The function

$$\psi_1(\omega) = |\langle \varphi, \omega(0) \rangle - \langle \varphi, \lambda_0 \rangle|\,, \quad \omega \in D_{\mathcal{P}(\mathcal{Z})}[0, \infty)\,,$$

is continuous and bounded (cf. (4.35)). Thus, we obtain

$$P^{\infty}(\{\omega : \langle \varphi, \omega(0) \rangle = \langle \varphi, \lambda_0 \rangle\}) = 1\,. \tag{4.47}$$

Remembering the definition (4.34) of Ψ_φ and denoting

$$\Omega_\varphi = \{\omega : \Psi_\varphi(\omega) = 0 \text{ and } \langle \varphi, \omega(0) \rangle = \langle \varphi, \lambda_0 \rangle\}\,, \tag{4.48}$$

we obtain from (4.46) and (4.47) that

$$\langle \varphi, \omega(t)\rangle = \langle \varphi, \lambda_0\rangle + \int_0^t \langle A_0(\varphi), \omega(s)\rangle \, ds + \int_0^t \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2) - \varphi(z_1) - \varphi(z_2)\Big] \times Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2)\Big\} \omega(s, dz_1)\, \omega(s, dz_2)\, ds\,, \quad \forall t \geq 0\,, \quad \forall \omega \in \Omega_\varphi\,, \tag{4.49}$$

and $P^\infty(\Omega_\varphi) = 1$, for any $\varphi \in \mathcal{D}(A_0)$ such that $\varphi^2 \in \mathcal{D}(A_0)$.

Let D be the core of the generator A_0 given by assumption A2. Consider the sets

$$\mathcal{G}_D(A_0) = \{(\varphi, A_0(\varphi)) \,:\, \varphi \in D\} \quad \text{and} \quad \mathcal{G}(A_0) = \{(\varphi, A_0(\varphi)) \,:\, \varphi \in \mathcal{D}(A_0)\}\,.$$

Then

$$\mathcal{G}_D(A_0) \subset \mathcal{G}(A_0) \subset \hat{C}(\mathcal{Z}) \times \hat{C}(\mathcal{Z})\,.$$

Since $\hat{C}(\mathcal{Z}) \times \hat{C}(\mathcal{Z})$ is separable, the subspace $\mathcal{G}_D(A_0)$ is also separable. Let (ψ_n) be a dense subset of $\mathcal{G}_D(A_0)$, where

$$\psi_n = (\varphi_n, A_0\varphi_n)\,, \quad n = 1, 2, \ldots\,.$$

Then (ψ_n) is also dense in $\mathcal{G}(A_0)$, because D is a core. Thus,

$$\forall \psi \in \mathcal{G}(A_0) \quad \exists\, (\psi_{n_k}) \quad \text{such that} \quad \lim_{k\to\infty} \psi_{n_k} = \psi\,,$$

and

$$\forall \varphi \in \mathcal{D}(A_0) \quad \exists\, (\varphi_{n_k}) \in D \quad \text{such that} \quad \lim_{k\to\infty} \varphi_{n_k} = \varphi\,, \quad \text{and} \quad \lim_{k\to\infty} A_0(\varphi_{n_k}) = A_0(\varphi)\,. \tag{4.50}$$

Consider the set $\Omega_0 = \cap_{n=1}^\infty \Omega_{\varphi_n}$ (cf. (4.48)). Then $P^\infty(\Omega_0) = 1$, and Eq. (4.49) holds for all $\omega \in \Omega_0$ and φ_n, $n = 1, 2, \ldots$. By (4.50), it holds for all $\varphi \in \mathcal{D}(A_0)$ so that $\Omega_0 \subset \bar{\Omega}$, and the assertion of the theorem follows. □

Notation: If $f \in \hat{C}(\mathcal{Z} \times [0, \infty))$ and $t \in [0, \infty)$ then write $f_t : \mathcal{Z} \to \mathcal{R}$ for the function defined by $f_t(z) = f(z, t)$.

Let $\bar{T}_0(t) : \hat{C}(\mathcal{Z} \times [0,\infty)) \to \hat{C}(\mathcal{Z} \times [0,\infty))$ be defined by

$$\bar{T}_0(t)(f)(z,r) = T_0(t)(f_{t+r})(z).$$

It is straightforward to check that the collection $(\bar{T}_0(t) : t \geq 0)$ forms a Feller semigroup. It is the semigroup of the process with time added as an extra coordinate. Now let $\bar{A}_0$ be the generator of $(\bar{T}_0(t) : t \geq 0)$. The following lemma calculates the value of the generator on sufficiently nice test functions.

Lemma 4.9 *If for* $\psi : \mathcal{Z} \times [0,\infty) \to \mathcal{R}$ *the maps*

$$s \to \psi_s\,, \quad s \to A_0(\psi_s)\,, \quad s \to (\partial\psi/\partial t)_s$$

exist, are continuous from $[0,\infty) \to \hat{C}(\mathcal{Z})$ *and converge in norm to zero as* $s \to \infty$, *then*

$$\psi \in \mathcal{D}(\bar{A}_0) \quad \text{and} \quad \bar{A}_0(\psi)(z,s) = (\partial\psi/\partial t)(z,s) + A_0(\psi_s)(z).$$

The proof is omitted.

Lemma 4.10 *If* $\mathcal{C} \subseteq \mathcal{D}(\bar{A}_0)$ *is dense and* $\bar{T}_0(t) : \mathcal{C} \to \mathcal{C}$ *for all* $t \geq 0$ *then* $\mathcal{C}$ *is a core i.e. for all* $\psi \in \mathcal{D}(\bar{A}_0)$ *there exist* $\psi_n \in \mathcal{C}$ *with* $(\psi_n, \bar{A}_0\psi_n) \to (\psi, \bar{A}_0\psi)$ *in norm in* $\hat{C}(\mathcal{Z} \times [0,\infty)) \times \hat{C}(\mathcal{Z} \times [0,\infty))$.

Lemma 4.10 is a direct consequence of [8, Chapter 1, Prop. 3.3].

Lemma 4.11 *Suppose* $\lambda : [0,\infty) \to \mathcal{P}(\mathcal{Z})$ *is measurable and satisfies for all* $\varphi \in \mathcal{D}(A_0)\,, \quad t \geq 0$

$$\langle \lambda(t), \varphi \rangle = \langle \lambda_0, \varphi \rangle + \int_0^t \langle \lambda(s), A_0\varphi - c\varphi \rangle\, ds + \int_0^t \langle K(\lambda(s)), \varphi \rangle\, ds \qquad (4.51)$$

where $c \in [0,\infty)$ *and* $K : \mathcal{M}(\mathcal{Z}) \to \mathcal{M}(\mathcal{Z})$ *is measurable and*

$$K(\mu)(\mathcal{Z}) \leq \text{const}\,, \quad \forall \mu \in \mathcal{P}(\mathcal{Z}). \qquad (4.52)$$

If $\psi : \mathcal{Z} \times [0,t] \to \mathcal{R}$ *satisfies*
H1*: the limit* $\lim_{h\to 0}(\psi_{s+h} - \psi_s)/h$ *exists in norm for each* $s \in [0,t]$ *(right and left hand derivatives at* $s = 0, t$*)*
H2*:* $\psi_s \in \mathcal{D}(A_0)$ *for all* $s \in [0,t]$

H3: *the maps* $s \to \psi_s$, $s \to A_0(\psi_s)$, $s \to (\partial\psi/\partial t)_s$ *are continuous from* $[0,t]$ *to* $\hat{C}(\mathcal{Z})$

then for $s \in [0,t]$

$$\begin{aligned} \langle \lambda(s), \psi_s \rangle = \langle \lambda_0, \psi_0 \rangle + \\ + \int_0^s \left[\langle \lambda(r), A_0\psi_r + (\partial\psi/\partial t)_r - c\psi_r \rangle + \langle K(\lambda(r)), \psi_r \rangle \right] dr. \end{aligned} \tag{4.53}$$

Proof. Define

$$\mathcal{C} = \left\{ \psi \in \hat{C}(\mathcal{Z} \times [0,\infty)) : \begin{array}{l} \psi(s,x) = \sum_{i=1}^n \psi_i(s)\varphi_i(x) \text{ where} \\ \psi_i \text{ are differentiable with } \psi_i, \psi_i' \in \hat{C}([0,\infty)) \\ \text{and } \varphi_i \in \mathcal{D}(A_0) \end{array} \right\}.$$

For $\psi \in \mathcal{C}$ we can use integration by parts to directly calculate that for all s

$$\begin{aligned} \langle \lambda(s), \psi_s \rangle &= \langle \lambda_0, \psi_0 \rangle \\ &\quad + \int_0^s \left[\langle \lambda(r), A_0\psi_r + (\partial\psi/\partial t)_r - c\psi_r \rangle + \langle K(\lambda(r)), \psi_r \rangle \right] dr \\ &= \langle \lambda_0, \psi_0 \rangle + \int_0^s \left[\langle \lambda(r), (\bar{A}_0\psi)_r - c\psi_r \rangle + \langle K(\lambda(r)), \psi_r \rangle \right] dr. \end{aligned} \tag{4.54}$$

The aim is to extend this to all $\psi \in \mathcal{D}(\bar{A}_0)$ by checking that $\mathcal{C}$ is a core for $\bar{A}_0$. Let $\bar{\mathcal{C}}$ be the norm closure of $\mathcal{C}$. Then $\bar{\mathcal{C}}$ contains all functions ψ of the same form as in $\mathcal{C}$ but with $\varphi_i \in \hat{C}(\mathcal{Z})$ (since $\mathcal{D}(A_0)$ is dense in $\hat{C}(\mathcal{Z})$) and so separates points of $\mathcal{Z} \times [0,\infty)$. So $\bar{\mathcal{C}}$ contains an algebra which satisfies the assumptions of the Stone-Weierstrass theorem. So $\mathcal{C}$ is dense in $\hat{C}(\mathcal{Z} \times [0,\infty))$. Since $T_0(t) : \mathcal{D}(A_0) \to \mathcal{D}(A_0)$ it follows easily that $\bar{T}_0(t) : \mathcal{C} \to \mathcal{C}$ for each $t \geq 0$. Lemma 4.10 then shows that $\mathcal{C}$ is a core i.e. if $\psi \in \mathcal{D}(\bar{A}_0)$ then there exist $\psi_n \in \mathcal{C}$ so that $(\psi_n, \bar{A}_0\psi_n) \to (\psi, \bar{A}_0\psi)$ in norm. We may then pass to the limit in (4.54) to see that it holds for all $\psi \in \mathcal{D}(\bar{A}_0)$.

Finally take ψ satisfying the hypotheses H1–H3. Take also $\varrho_n : [0,\infty) \to [0,1]$ smooth, decreasing with $\varrho(x) = 0$ for $x \geq t$ and $\varrho(x) = 1$ for $x \in [0, t-n^{-1}]$. Then $\psi_n = \psi\varrho_n$ satisfies the hypotheses of Lemma 4.9 so that $\psi_n \in \mathcal{D}(\bar{A}_0)$ and

$$(\bar{A}_0\psi_n)_s = (\partial\psi_n/\partial t)_s + A_0(\psi_n)_s.$$

Applying (4.54) for ψ_n and noting that $\psi_n(s) = \psi(s)$ for $s \in [0, t-n^{-1}]$ we have that equation (4.53) holds on $[0, t-n^{-1})$. Therefore by continuity it holds on $[0,t]$ completing the proof. □

Remark: The technique used in Lemma 4.11 has been used in branching measure valued processes and was first shown to one of the authors by Ed Perkins.

Lemma 4.12 *If* $\varphi \in \mathcal{D}(A_0)$ *then*

$$\psi(s,x) = e^{-c(t-s)} T_0(t-s)\varphi(x) \quad \text{for} \quad s \in [0,t]$$

satisfies H1,H2,H3 *of Lemma 4.11 and*

$$A_0\psi_s + (\partial\psi/\partial t)_s - c\,\psi_s = 0 \;\; \text{for} \;\; s \in [0,t]\,. \tag{4.55}$$

Proof. Recall that for $\varphi \in \mathcal{D}(A_0)$ (see [8, Chapter 1, Prop. 1.5])

$$T_0(t)\varphi = \varphi + \int_0^t T_0(s)A_0\varphi\, ds \tag{4.56}$$

and that if $\hat{\psi}(s,z) = T_0(s)\varphi(z)$ then

$$(\partial\hat{\psi}/\partial t)_s = T_0(s)A_0\varphi = A_0T_0(s)\varphi = A_0\hat{\psi}_s\,.$$

The strong continuity of the semigroup implies that

$$s \to \hat{\psi}_s\,, \quad s \to A_0(\hat{\psi}_s)\,, \quad s \to (\partial\hat{\psi}/\partial t)_s$$

are all continuous on $[0,t]$. So the hypotheses H1,H2,H3 are satisfied for $\hat{\psi}$ and this then implies that they hold for ψ in the lemma. Equation (4.55) follows from (4.56) and a little calculus. □

Proof of Theorem 4.3. We introduce a kernel

$$Q_{max}(z_1,z_2,\Gamma_1,\Gamma_2) = Q(z_1,z_2,\Gamma_1,\Gamma_2) + \\ [C_{Q,max} - Q(z_1,z_2,\mathcal{Z},\mathcal{Z})]\,\delta_{z_1}(\Gamma_1)\,\delta_{z_2}(\Gamma_2)\,, \quad z_1,z_2 \in \mathcal{Z}\,, \quad \Gamma_1,\Gamma_2 \in \mathcal{B}_{\mathcal{Z}}\,.$$

Furthermore, we define the dual semigroup

$$T_0(t)^*(\mu)(\Gamma) = \int_{\mathcal{Z}} \mu(dz)\,U_0(t,z,\Gamma)\,, \quad \mu \in \mathcal{M}(\mathcal{Z})\,, \quad \Gamma \in \mathcal{B}_{\mathcal{Z}}\,, \tag{4.57}$$

and a function

$$K(\mu)(\Gamma) = \int_{\mathcal{Z}}\int_{\mathcal{Z}} \Big[Q_{max}(z_1,z_2,\Gamma,\mathcal{Z}) + \\ Q_{max}(z_1,z_2,\mathcal{Z},\Gamma)\Big]\,\mu(dz_1)\,\mu(dz_2)\,, \quad \mu \in \mathcal{M}(\mathcal{Z})\,, \quad \Gamma \in \mathcal{B}_{\mathcal{Z}}\,. \tag{4.58}$$

It is straightforward to see that (4.57) and (4.58) define operators $\mathcal{M}(\mathcal{Z}) \to \mathcal{M}(\mathcal{Z})$, justifying the notations.

On $\mathcal{M}(\mathcal{Z})$, we consider the distance induced by the total variation norm on the space of finite signed Borel measures (cf. [8, p. 495]),

$$\|\nu_1 - \nu_2\| = \sup_{\varphi \in B(\mathcal{Z}):\|\varphi\| \le 1} |\langle \varphi, \nu_1 \rangle - \langle \varphi, \nu_2 \rangle| .$$

It follows immediately from the definitions that

$$\|T_0(t)^*(\nu_1) - T_0(t)^*(\nu_2)\| \le \|\nu_1 - \nu_2\| , \quad \forall \nu_1 , \nu_2 \in \mathcal{M}(\mathcal{Z}), \tag{4.59}$$

and

$$\begin{aligned} &\|K(\nu_1) - K(\nu_2)\| \le \\ &\quad \|\nu_1 - \nu_2\| \, 4\, C_{Q,max} \max(\nu_1(\mathcal{Z}), \nu_2(\mathcal{Z})) , \quad \forall \nu_1, \nu_2 \in \mathcal{M}(\mathcal{Z}) . \end{aligned} \tag{4.60}$$

Notice that

$$\begin{aligned} &\int_{\mathcal{Z}} \varphi(z) \, K(\mu)(dz) = 2\, C_{Q,max} \, \mu(\mathcal{Z}) \int_{\mathcal{Z}} \varphi(z) \, \mu(dz) + \\ &\quad \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}} \int_{\mathcal{Z}} \big[\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2) - \varphi(z_1) - \varphi(z_2)\big] \times \\ &\quad Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2)\Big\} \, \mu(dz_1) \, \mu(dz_2) , \quad \forall \varphi \in B(\mathcal{Z}) , \quad \forall \mu \in \mathcal{M}(\mathcal{Z}) , \end{aligned} \tag{4.61}$$

so that Eq. (1.5) takes the form (4.51) with $c = 2\, C_{Q,max}$.

Taking equation (4.53) with the choice of ψ in Lemma 4.12 gives

$$\langle \lambda(t), \varphi \rangle = e^{-ct} \langle \lambda_0, T_0(t)\varphi \rangle + \int_0^t e^{-c(t-s)} \langle K(\lambda(s)), T_0(t-s)\varphi \rangle \, ds . \tag{4.62}$$

So

$$\langle \lambda(t) - e^{-ct} \, T_0(t)^* \lambda_0 - \int_0^t e^{-c(t-s)} \, T_0(t-s)^* K(\lambda(s)) \, ds, \varphi \rangle = 0 .$$

Since this is true for all $\varphi \in \mathcal{D}(A_0)$ which is dense in $\hat{C}(\mathcal{Z})$ (true for Feller process generators) we obtain that for all $t \ge 0$

$$\lambda(t) = e^{-ct} \, T_0(t)^* \lambda_0 + \int_0^t e^{-c(t-s)} \, T_0(t-s)^* K(\lambda(s)) \, ds. \tag{4.63}$$

Uniqueness of the solution of Eq. (4.63) follows from the Lipschitz properties (4.59) and (4.60) of the operators $T_0(t)^*$ and K, respectively, and from Gronwall's inequality. □

Lemma 4.13 *Let $\bar{\psi}$ be a continuous function satisfying (2.10) and (2.11). Assume that*

$$\int_{\mathcal{Z}} \bar{\psi}(\|\tilde{z}\|)\, U_0(s,z,d\tilde{z}) \le \bar{c}_1(t)\, [1+\bar{\psi}(\|z\|)]\,, \qquad (4.64)$$
$$\forall z \in \mathcal{Z}\,,\ \forall s \in [0,t]\,, \quad \textit{for some} \quad t>0\,,$$

and that Q satisfies (2.13).

Then

$$\int_{\mathcal{Z}} \bar{\psi}(z)\, \lambda(t,dz) \le \text{const} \int_{\mathcal{Z}} \bar{\psi}(z)\, \lambda_0(dz)\,,$$

where λ is the solution of Eq. (1.5).

Proof. We consider the equivalent form (4.62) of Eq. (1.5). According to (4.61), this equation takes the form

$$\langle \lambda(t),\varphi\rangle = e^{-ct}\,\langle \lambda_0, T_0(t)\varphi\rangle + \int_0^t e^{-c(t-s)}\, c\,\langle T_0(t-s)\,\varphi, \lambda(s)\rangle ds \qquad (4.65)$$
$$+ \int_0^t \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[T_0(t-s)\,\varphi(\tilde{z}_1) + T_0(t-s)\,\varphi(\tilde{z}_2) - T_0(t-s)\,\varphi(z_1)$$
$$-T_0(t-s)\,\varphi(z_2)\Big]\, Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\Big\}\, \lambda(s,dz_1)\, \lambda(s,dz_2)\, ds\,.$$

Using bp–limits of sequences of functions from $\hat{C}(\mathcal{Z})$, one finds that Eq. (4.65) holds for arbitrary bounded continuous functions φ. Consider the function

$$\bar{\psi}_R(x) = \begin{cases} \bar{\psi}(x)\,, & \text{if} \quad x \le R\,, \\ \bar{\psi}(R)\,, & \text{if} \quad x \ge R\,. \end{cases}$$

First we show that

$$\int_{\mathcal{Z}} \bar{\psi}_R(\|\tilde{z}\|)\, U_0(s,z,d\tilde{z}) \qquad (4.66)$$
$$\le \max(1,\bar{c}_1(t))\, [1+\bar{\psi}_R(\|z\|)]\,, \quad \forall z \in \mathcal{Z}\,,\ \forall s \in [0,t]\,,\ \forall R>0\,,$$

and

$$\int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[\bar{\psi}_R(\|\tilde{z}_1\|) + \bar{\psi}_R(\|\tilde{z}_2\|)\Big]\, Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2) \le \qquad (4.67)$$
$$\max(2\, C_{Q,max}, c_2)\, \Big[1+\bar{\psi}_R(\|z_1\|)+\bar{\psi}_R(\|z_2\|)\Big]\,, \quad \forall z_1,z_2 \in \mathcal{Z}\,.$$

To show (4.66), one considers the cases $\|z\| \leq R$ and $\|z\| > R$, and uses assumption (4.64). To show (4.67), one considers the cases $\max(\|z_1\|, \|z_2\|) \leq R$ and $\max(\|z_1\|, \|z_2\|) > R$, and uses assumption (2.13). Using (4.66) and (4.67), one obtains from Eq. (4.65)

$$\int_{\mathcal{Z}} \bar{\psi}_R(\|z\|)\, \lambda(t, dz) \leq \text{const} \left[\int_{\mathcal{Z}} \bar{\psi}_R(\|z\|)\, \lambda_0(dz) + \int_0^t \int_{\mathcal{Z}} \bar{\psi}_R(\|z\|)\, \lambda(s, dz)\, ds\right],$$

and, via Gronwall's inequality,

$$\int_{\mathcal{Z}} \bar{\psi}_R(\|z\|)\, \lambda(t, dz) \leq \text{const} \int_{\mathcal{Z}} \bar{\psi}_R(\|z\|)\, \lambda_0(dz)\,. \tag{4.68}$$

The assertion follows from (4.68), since the constant does not depend on R.
□

Proof of Corollary 2.2. Using assumption (2.14) and Lemma 4.7, we obtain

$$\limsup_{n\to\infty} E^{(n)} \int_{\mathcal{Z}} \bar{\psi}(\|z\|)\, \mu^{(n)}(t, dz) < \infty\,, \quad \forall t \geq 0\,. \tag{4.69}$$

Consider the function

$$\chi_R(z) = \begin{cases} 1 & , \text{ if } \|z\| \leq R\,, \\ 1 + R - \|z\| & , \text{ if } \|z\| \in (R, R+1)\,, \\ 0 & , \text{ if } \|z\| \geq R+1\,. \end{cases}$$

Then we obtain

$$\begin{aligned} &\limsup_{n\to\infty} E^{(n)} |\langle \varphi, \mu^{(n)}(t)\rangle - \langle \varphi, \lambda(t)\rangle| \leq \\ &\quad \limsup_{n\to\infty} E^{(n)} |\langle \varphi\, \chi_R, \mu^{(n)}(t)\rangle - \langle \varphi\, \chi_R, \lambda(t)\rangle| \\ &\quad + \limsup_{n\to\infty} E^{(n)} |\langle \varphi\, (1-\chi_R), \mu^{(n)}(t)\rangle| + |\langle \varphi\, (1-\chi_R), \lambda(t)\rangle|\,. \end{aligned} \tag{4.70}$$

The first term on the right side of (4.70) equals zero, for any $R > 0$, according to Theorem 2.1, since the function $\varphi\, \chi_R$ is continuous and bounded. The second term is estimated as

$$\begin{aligned} &\limsup_{n\to\infty} E^{(n)} |\langle \varphi\, (1-\chi_R), \mu^{(n)}(t)\rangle| \leq \\ &\quad \leq \sup_{\|z\|\geq R} \left(\frac{|\varphi(z)|}{\bar{\psi}(\|z\|)}\right) \limsup_{n\to\infty} E^{(n)} \int_{\mathcal{Z}} \bar{\psi}(\|z\|)\, \mu^{(n)}(t, dz)\,, \end{aligned}$$

and tends to zero as $R \to \infty$, because of (4.69) and assumption (2.17). Finally, the third term tends to zero as $R \to \infty$ according to Lemma 4.13.
□

5. Concluding remarks

The uniqueness result for the limiting equation (1.5) allowed us to avoid the approximation of the independent motion by jump processes, which was used in [16], [1], [19]. The convergence theorem became more transparent, since the deterministic limit of the empirical measures is characterized as the unique solution of the limiting equation. The case of the independent motion being a diffusion is covered now. Also convergence of moments of the empirical measures has been obtained. Such unbounded functionals are important in the applications, since they are the basis for the calculation of physical quantities like temperature or heat flow.

The discussion of the relationship with the Boltzmann equation in Section 3 shows that an interesting and practically important problem is to consider kernels Q depending on n, i.e.

$$Q^{(n)}(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) = h^{(n)}(x_1, x_2)\, \delta_{x_1}(d\tilde{x}_1)\, \delta_{x_2}(d\tilde{x}_2)\, \hat{Q}(v_1, v_2, d\tilde{v}_1, d\tilde{v}_2)\,,$$

where $z_1 = (x_1, v_1)$, $z_2 = (x_2, v_2)$, $d\tilde{z}_1 = (d\tilde{x}_1, d\tilde{v}_1)$, $d\tilde{z}_2 = (d\tilde{x}_2, d\tilde{v}_2)$, with $h^{(n)}$ tending to the delta–function as $n \to \infty$. Under appropriate assumptions on $h^{(n)}$, one would expect convergence to the DiPerna–Lions solution [6] of the Boltzmann equation.

Results concerning convergence to the solution of an unmollified one–dimensional Boltzmann equation have been obtained in [2], [3]. They are based on a different approach, studying the interaction trees of the particle system. An interaction graph technique has been used in [10] to study general (mollified) Boltzmann type equations.

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