Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

A note on a parabolic equation with nonlinear dynamical boundary condition

Jürgen Sprekels[1], Hao Wu[1, 2]

submitted: 3 Dec 2008

[1] Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstrasse 39
10117 Berlin
Germany
E-Mail: sprekels@wias-berlin.de
wu@wias-berlin.de

[2] School of
Mathematical Sciences
Fudan University
200433 Shanghai
P.R. China
E-Mail: haowufd@yahoo.com

No. 1378
Berlin 2008

2000 *Mathematics Subject Classification.* 35B40, 35B41, 35B45..

Key words and phrases. Parabolic equation, dynamical boundary condition, global attractor, convergence to equilibrium, Łojasiewicz-Simon inequality.

Acknowledgements. H. Wu wants to thank Prof. Ch. Schwab for suggesting this problem and calling his attention to the book [28]. He is grateful to Prof. S. Zheng for his continuous support and helpful discussions, and thanks Dr. C. Gal for sharing his recent work. The research of the second author was partially supported by WIAS Postdoctoral Fellowship and China Postdoctoral Science Foundation.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider a semilinear parabolic equation subject to a nonlinear dynamical boundary condition that is related to the so-called Wentzell boundary condition. First, we prove the existence and uniqueness of global solutions as well as the existence of a global attractor. Then we derive a suitable Łojasiewicz-Simon type inequality to show the convergence of global solutions to single steady states as time tends to infinity under the assumption that the nonlinear terms f, g are real analytic. Moreover, we provide an estimate for the convergence rate.

1 Introduction

In this paper, we consider the semilinear parabolic equation

$$u_t - \Delta u + f(u) = 0, \qquad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

subject to the boundary condition

$$-\Delta_{||} u + \partial_\nu u + u + g(u) + u_t = 0, \qquad (x,t) \in \Gamma \times \mathbb{R}^+, \tag{1.2}$$

and to the initial condition

$$u|_{t=0} = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

Here, $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N},\ n \geq 2)$ is a bounded domain with smooth boundary Γ, $\Delta_{||}$ is the Laplace-Beltrami operator on Γ, and ν is the outward normal direction to the boundary. We note that in the case $n = 1$, the operator $\Delta_{||}$ simply does not appear in the boundary, and that our result still holds with some modifications in the proof (cf. also [31]).

The boundary condition (1.2) is inspired by some problems that are related to multidimensional diffusion processes (cf. [28]) in a certain sense. It is well known that the infinitesimal generator of a Feller semigroup, which is related to a certain strong Markov process, can be described analytically by a Waldenfels operator $\mathbf{W}$ and a Wentzell boundary operator $\mathbf{L}$. If no jump phenomenon in the interior of Ω is considered, the operator $\mathbf{W}$ in the simplest case can be chosen as the Laplacian Δ (which denotes the diffusion of the particle). In terms of local coordinates $x' = (x_1, ..., x_{n-1})$, the Wentzell boundary operator $\mathbf{L}$ may have the following form:

$$\mathbf{L}u = Qu(x') - \mu(x')\partial_\nu u(x') + \gamma(x')u(x') - \delta(x')\mathbf{W}u(x'). \tag{1.4}$$

Here, we also neglect the jump phenomenon on the boundary and the inward jump phenomenon from the boundary. $\gamma, \mu, \delta \in C^\infty(\Gamma)$ satisfy $\gamma \leq 0,\ \mu \geq 0,\ \delta \geq 0$. Q is a second order (possibly degenerate) elliptic operator on the boundary that corresponds to the diffusion phenomenon along the boundary (like the Brownian motion on Γ). The second to fourth terms in (1.4) represent absorption, reflection and sticking (or viscosity) in the boundary condition, respectively (cf. [28, Chapter 7] for details). A simple example for $\mathbf{L}$ might be

$$\mathbf{L}u = \Delta_{||}u - \partial_\nu u - u - \Delta u|_\Gamma, \tag{1.5}$$

where $\Delta|_\Gamma$ represents the restriction of the Laplace operator Δ to the boundary Γ, which should not be confused with the Laplace-Beltrami operator $\Delta_{||}$. For the linear heat equation $u_t - \Delta u = 0$ subject to the boundary condition $\mathbf{L}u = 0$, the Wentzell boundary condition can be reduced to a linear dynamical condition (cf. [4]). Furthermore, if one considers a semilinear parabolic equation like (1.1), then the boundary condition can be (formally) transformed into

$$-\Delta_{||}u + \partial_\nu u + u + f(u) + u_t = 0, \tag{1.6}$$

which is a special case of (1.2) with $g(u) = f(u)$. Since the nonlinearity g on the boundary Γ is arbitrary, our results below also apply for a class of nonlinear Wentzell boundary conditions. For extensive discussions of the heat equation subject to various Wentzell type boundary conditions, we refer to [3–5] and the references therein.

On the other hand, the dynamical boundary condition (1.2) is also related to some phase transition problems in materials science. Various models describing isothermal / non-isothermal phase transitions in a bounded spatial region have been introduced and studied in the literature. As far as the Cahn-Hilliard equation is concerned, some physicists have recently pointed out that for certain materials a dynamical interaction with the wall (i.e., Γ) must be taken into account (see, e.g., [21] and references therein). This fact corresponds to considering a free energy functional that also contains a boundary contribution. As a consequence, one deduces a (linear) dynamic boundary condition of the form

$$u_t = \Delta_{||}u - \partial_\nu u - u, \quad \text{on } (0, +\infty) \times \Gamma. \tag{1.7}$$

Phenomenologically speaking, the boundary condition (1.7) means that the density at the surface relaxes towards equilibrium with a rate proportional to the driving force given by the Fréchet derivative of the free energy functional. The dynamical boundary condition can also be associated with the well-known Caginalp model, which is a non-isothermal second-order phase-field system in a certain bounded domain in $\mathbb{R}^n$ $(n \leq 3)$(cf. [1, 8, 9, 15]). There, a nonlinear dynamical boundary condition like (1.2) was derived as a variational boundary condition such that the system tends to minimize its total free energy. Our problem (1.1)–(1.3) can in some sense be regarded as a subsystem of the Caginalp model in [8,9], subject to nonlinear dynamical boundary conditions for the order parameter. In [8, 9], the authors proved global existence and uniqueness of the solution and analyzed the asymptotic behavior of the solutions; they showed the existence of a global attractor, as well as of an exponential attractor, and convergence to equilibrium as time tends to infinity. We refer to [1, 6–11, 13, 15] for extensive discussions on various evolution systems subject to nonlinear dynamical boundary conditions.

In this note, we shall first prove global existence and uniqueness for problem (1.1)–(1.3), as well as the existence of a global attractor. Then, we are interested in the question whether the global solution to (1.1)–(1.3) will converge to an equilibrium as time tends to infinity.

Before stating our main results, we make some assumptions on the nonlinearities f, g.
(F1a) $f, g \in C^1(\mathbb{R})$.

To prove the result on convergence to equilibrium, we will instead of **(F1a)** need a stronger condition:
(F1b) f, g are real analytic on $\mathbb{R}$.

We further assume:
(F2)

$$|f'(s)| \le c(1+|s|^p), \quad \forall s \in \mathbb{R},\ p \in [0,\alpha),$$

where

$$\alpha := \begin{cases} +\infty, & \text{if } n=2, \\ \frac{4}{n-2}, & \text{if } n \ge 3, \end{cases}$$

and

$$|g'(s)| \le c(1+|s|^q), \quad \forall s \in \mathbb{R},\ q \in [0,\tilde{\alpha}),$$

where

$$\tilde{\alpha} := \begin{cases} +\infty, & \text{if } n=2,3, \\ \frac{4}{n-3}, & \text{if } n \ge 4. \end{cases}$$

(F3)

$$\liminf_{|s|\to+\infty} f'(s) > 0, \quad \liminf_{|s|\to+\infty} g'(s) > 0.$$

Remark 1.1. *Assumption (**F1a**) is sufficient for the result on wellposedness. Assumption (**F1b**) is designed so that we can derive an extended Łojasiewicz-Simon type inequality and use it to prove the convergence to equilibrium. Assumption (**F2**) implies that the nonlinear terms have a subcritical growth. Actually from the argument in [8], we do not have to suppose any growth assumption for $n \le 3$ due to the validity of a certain version of maximum principle, which yields an L^∞ estimate (cf. [23, Lemma A.2]). Assumption (**F3**) is a dissipative condition necessary to ensure global existence of a solution to our problem* (1.1)–(1.3)*. For simplicity of exposition, the nonlinear terms f, g are assumed to depend only on u. However, the results in this paper still remain true for nonlinear terms of the form $f(x,u), g(x,u)$, under proper additional smoothness assumptions with respect to x.*

Throughout this paper, we simply denote by $\|\cdot\|$ the norm on $L^2(\Omega)$. $C > 0$ denotes a generic constant that may depend on Ω, Γ and the initial data, and it may be different even in the same line. The specific dependence of C in the subsequent text will be pointed out explicitly. Sometimes in the proofs, we denote the solution $u(t)$ by u for the sake of simplicity.

As in [26], let V_1 be the Hilbert space, which is the completion of $C^1(\bar{\Omega})$ with respect to the following inner product and the associated norm:

$$(u,v)_{\mathrm{V}_1} = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Gamma (\nabla_\| u \cdot \nabla_\| v + uv) dS, \quad \forall\, u,v \in \mathrm{V}_1. \tag{1.8}$$

Here, $\nabla_{||}$ denotes the covariant gradient operator on Γ. Similarly, let V_0 be the Hilbert space that is the completion of $C^0(\bar{\Omega})$ with respect to the inner product

$$(u,v)_{\mathrm{V}_0} = \int_\Omega uv dx + \int_\Gamma uv dS, \quad \forall\, u,v \in \mathrm{V}_0. \tag{1.9}$$

Moreover, we introduce the space (cf. [6–8]) $\mathrm{V}_i = H^i(\Omega) \times H^i(\Gamma), (i = 2, 3...)$, which is the completion of $C^i(\bar{\Omega})$ with respect to the norm

$$\|u\|^2_{\mathrm{V}_i} = \|u\|^2_{H^i(\Omega)} + \|u|_\Gamma\|^2_{H^i(\Gamma)}.$$

In general, any vector $u \in \mathrm{V}_i$ will be of the form (u_1, u_2) with $u_1 \in H^i(\Omega)$ and $u_2 \in H^i(\Gamma)$, and there needs not be any connection between u_1 and u_2. We denote $\tilde{\mathrm{V}}_i = \{u = (u, u|_\Gamma) : u \in \mathrm{V}_i\}$ $(i \geq 1)$.

We are now in a position to state our main results.

Theorem 1.1. *Suppose that (**F1a**), (**F2**), (**F3**) are satisfied. Then for any initial datum $u_0 \in \tilde{\mathrm{V}}_2$, problem (1.1)–(1.3) admits a unique global solution such that*

$$u \in C([0, +\infty); \tilde{\mathrm{V}}_2) \cap C^1([0, +\infty); \mathrm{V}_0) \cap H^1(0, +\infty; \tilde{\mathrm{V}}_1).$$

Theorem 1.2. *Suppose that (**F1a**), (**F2**), (**F3**) are satisfied. Problem (1.1)–(1.3) admits a compact global attractor $\mathcal{A} \subset \tilde{\mathrm{V}}_2$.*

Theorem 1.3. *Suppose that (**F1b**), (**F2**), (**F3**) are satisfied. Then for any initial datum $u_0 \in \tilde{\mathrm{V}}_2$, the unique global solution to problem (1.1)–(1.3) converges to an equilibrium ψ such that*

$$\lim_{t\to+\infty} (\|u(t) - \psi\|_{\mathrm{V}_2} + \|u_t(t)\|_{\mathrm{V}_0}) = 0. \tag{1.10}$$

Moreover, we have the estimate

$$\|u(t) - \psi\|_{\mathrm{V}_2} + \|u_t(t)\|_{\mathrm{V}_0} \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \qquad \forall\, t \geq 0. \tag{1.11}$$

Here, $C \geq 0$ depends only on $\|u_0\|_{\mathrm{V}_2}$, $\|\psi\|_{\mathrm{V}_2}$, $|\Omega|$, and $|\Gamma|$. ψ is an equilibrium to problem (1.1)–(1.3), i.e., a strong solution to the following nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta\psi + f(\psi) = 0, & \text{in } \Omega, \\ -\Delta_{||}\psi + \partial_\nu \psi + \psi + g(\psi) = 0, & \text{on } \Gamma. \end{cases} \tag{1.12}$$

In (1.11), *$\theta \in (0, \frac{1}{2})$ is a constant that depends on ψ (cf. Lemma 3.1).*

Remark 1.2. *Under assumptions (**F1b**), (**F2**), (**F3**), for any* $u_0 \in \tilde{\mathrm{V}}_2$*, we can actually show that*

$$\|u(t)-\psi\|_{\mathrm{V}_{m+2}} + \|u_t(t)\|_{\mathrm{V}_m} \le C_\delta (1+t)^{-\frac{\theta}{1-2\theta}}, \qquad \forall\, t \ge \delta > 0, \tag{1.13}$$

where $m \ge 1$ *is any positive integer, and* C_δ *is a positive constant depending only on* $\|u_0\|_{\mathrm{V}_2}$*,* $\|\psi\|_{\mathrm{V}_2}$ *and* δ*.*

Before ending this section, we point out some features of our problem. First, problem (1.1)–(1.3) can be viewed as a subsystem of the Caginalp model studied in [8, 9]. In these papers, the space dimension is assumed to be less or equal to three; the authors then are able to apply a proper maximum principle that implies an L^∞ estimate and enables them to remove the growth assumption on the nonlinearities. Unfortunately, the proof of this maximum principle relies on the continuous embedding $H^2 \hookrightarrow C$ and thus is only valid in lower dimension case ($n \le 3$). Here, we do not have any restriction on the space dimension n. It would be natural to suppose proper growth assumptions on the nonlinearities (cf. (**F2**)). We could prove the results by more delicate *a priori* estimates without using the maximum principle mentioned before. Second, as far as the problem on convergence to equilibrium is concerned, it is well known that in the case of higher spatial dimension, the situation is quite complicated in that the stationary states can form a continuum (see, e.g., [17]). Moreover, a counterexample for semilinear parabolic equations with C^∞ nonlinearities was given in the literature (see [25]), to show that there is a bounded global solution whose ω-limit set is diffeomorphic to the unit circle S^1. A breakthrough was made in [27], namely that, using the so-called Łojasiewicz-Simon inequality, any bounded global solution of a semilinear parabolic equation will converge to a single stationary state provided that the nonlinearity f is real analytic. Since then, a number of contributions have been made for various types of nonlinear evolution equations. Concerning our problem (1.1)–(1.3), since (1.2) is now a nonlinear dynamical boundary condition, the classical Łojasiewicz-Simon inequality for homogeneous (Dirichlet/Neumann/Robin) boundary conditions in the literature fails to apply. Moreover, it seems that it is not straightforward to extend the proof for the Łojasiewicz-Simon inequality corresponding to the linear dynamical boundary condition (cf. [2, 34]) to the nonlinear case. As a result, a nontrivial modification is required to treat the present problem. This can be viewed as an extension of some previous results for the linear dynamical boundary condition (cf. [1, 2, 14, 31, 34, 35]) and can be used to prove the convergence result for other evolution equations with boundary condition (1.2) (cf. [6–8]). We would like to mention that in the very recent manuscripts [9, 11], a proper Ł-S type inequality was used to prove convergence result for some Cagnalp systems with nonlinear dynamical boundary condition. Our proof is different from the ones given there (see Section 3). Third, by delicate energy estimates and constructing proper differential inequalities, we are able to obtain the same estimates for convergence rate in both lower and higher order norms. This in some sense improves previous results in the literature (see, for instance, [19, 37]), and our approach can apply to other problems (cf. [9, 14, 30, 32, 33]).

The remaining part of this paper is organized as follows: Section 2 is devoted to the proof of the existence and uniqueness of a global solution, and of the existence of a global attractor. In Section 3, an extended Łojasiewicz-Simon inequality is derived. In the final Section 4, we prove the convergence to equilibrium and provide an estimate for the convergence rate.

2 Wellposedness and Global Attractor

First we look at the linear problem

$$u_t - \Delta u = h_1(x,t), \quad (x,t) \in \Omega \times (0,\infty), \tag{2.1}$$
$$-\Delta_{||} u + \partial_\nu u + u + u_t = h_2(x,t), \quad (x,t) \in \Gamma \times (0,\infty), \tag{2.2}$$
$$u|_{t=0} = u_0, \quad x \in \Omega. \tag{2.3}$$

Problem (2.1)–(2.3) has been studied in the literature (see, for instance, [8, 26]). In particular, we have (cf. [8, Theorem 2.3]):

Proposition 2.1. *Let* $u_0 \in \tilde{\mathrm{V}}_2$, $h_1 \in H^1([0,T]; L^2(\Omega))$, $h_2 \in H^1([0,T]; L^2(\Gamma))$. *Set* $u_1 = (u_{11}, u_{12})$, *where*

$$u_{11} := \Delta u_0 + h_1(0) \in L^2(\Omega), \quad u_{12} := \Delta_{||} u_0 - \partial_\nu u_0 - u_0 + h_2(0) \in L^2(\Gamma).$$

Then the linear problem (2.1)–(2.3) *admits a unique solution such that*

$$u \in C([0,T]; \tilde{\mathrm{V}}_2) \cap C^1([0,T]; \mathrm{V}_0) \cap H^1([0,T]; \tilde{\mathrm{V}}_1). \tag{2.4}$$

Moreover, for $t \in [0,T]$ *the following estimates hold:*

$$\|u(t)\|^2_{\mathrm{V}_1} + \int_0^t \|u_t(\tau)\|^2_{\mathrm{V}_0} d\tau \le C \left[\|u_0\|^2_{\mathrm{V}_1} + \int_0^t \left(\|h_1(\tau)\|^2 + \|h_2(\tau)\|^2_{L^2(\Gamma)} \right) d\tau \right], \tag{2.5}$$

$$\|u_t(t)\|^2_{\mathrm{V}_0} + \int_0^t \|u_t(\tau)\|^2_{\mathrm{V}_1} d\tau \le C \left[\|u_1\|^2_{\mathrm{V}_0} + \int_0^t \left(\|h_{1t}(\tau)\|^2 + \|h_{2t}(\tau)\|^2_{L^2(\Gamma)} \right) d\tau \right], \tag{2.6}$$

$$\|u(t)\|^2_{\mathrm{V}_2} \le C \left[\|u_1\|^2_{\mathrm{V}_0} + \|h_1\|^2 + \|h_2\|^2_{L^2(\Gamma)} + \int_0^t \left(\|h_{1t}(\tau)\|^2 + \|h_{2t}(\tau)\|^2_{L^2(\Gamma)} \right) d\tau \right], \tag{2.7}$$

where $C > 0$ *is a constant that depends only on* $|\Omega|, |\Gamma|$ *and is independent of* T, t, u_0, u_1, h_1, h_2.

Next, we state a local wellposedness result for problem (1.1)–(1.3). This can be proved by using the idea in [8], namely to apply a proper contraction mapping principle together with an approximation procedure and the uniqueness result (see Lemma 2.2 below). The details can be omitted here.

Theorem 2.1. *Assume that **(F1a)**, **(F2)**, **(F3)** hold. If* $u_0 \in \tilde{\mathrm{V}}_2$, *then problem* (1.1)–(1.3) *admits a unique local solution such that*

$$u \in C([0,T_{max}); \tilde{\mathrm{V}}_2) \cap C^1([0,T_{max}); \mathrm{V}_0) \cap H^1([0,T_{max}); \tilde{\mathrm{V}}_1). \tag{2.8}$$

In order to prove global existence, it suffices to obtain uniform *a priori* estimates. In what follows, we will derive (high-order) uniform bounds of $u(t)$ via a formal argument. However, this procedure can be made rigorous using an appropriate regularization scheme (e.g. [8, 37]).

The following lemma is useful for the derivation of *a priori* estimates.

Lemma 2.1. *[23, Appendix: Corollary A.1] Consider the linear problem*

$$\begin{cases} -\Delta\phi = h_1(x), \quad x \in \Omega, \\ -\Delta_{||}\phi + \partial_\nu\phi + \phi = h_2(x), \quad x \in \Gamma. \end{cases} \tag{2.9}$$

Let the functions h_1 and h_2 belong to $H^s(\Omega)$ and $H^s(\Gamma)$ respectively, where $s \geq 0$ and $s + \frac{1}{2}$ is not an integer. Then the solution $(\phi, \phi|_\Gamma)$ to problem (2.9) belongs to $H^{s+2}(\Omega) \times H^{s+2}(\Gamma)$ and the following estimate holds:

$$\|\phi\|_{H^{s+2}(\Omega)} + \|\phi|_\Gamma\|_{H^{s+2}(\Gamma)} \leq C(\|h_1\|_{H^s(\Omega)} + \|h_2\|_{H^s(\Gamma)}), \tag{2.10}$$

where C is a constant independent of ϕ and s.

Theorem 2.2. *[Global Existence] Assume that **(F1a)**, **(F2)**, **(F3)** hold. Then the solution u obtained in Theorem 2.1 exists globally, i.e., $T_{max} = +\infty$.*

Proof. Denote

$$E(u(t)) = \frac{1}{2}\|u(t)\|^2_{V_1} + \int_\Omega F(u(t))dx + \int_\Gamma G(u(t))dS, \tag{2.11}$$

where $F(z) = \int_0^z f(s)ds$, $G(z) = \int_0^z g(s)ds$. It is easy to see that $\forall\ t \in (0, T_{max})$,

$$\frac{d}{dt}E(u(t)) + \|u_t(t)\|^2_{V_0} = 0, \tag{2.12}$$

which implies that $E(u(t))$ is decreasing with respect to time. From the Sobolev embedding theorem and the growth assumption (**F2**), we get

$$\begin{aligned} \int_\Omega F(u(t))dx &= \int_\Omega\int_0^{u(t)} f(s)ds \leq C\int_\Omega\int_0^{|u(t)|}(1+|s|^{p+1})dsdx \\ &\leq C\int_\Omega(|u(t)| + |u(t)|^{p+2})dx \leq C\left(\|u(t)\|_{L^1(\Omega)} + \|u\|^{p+2}_{L^{p+2}(\Omega)}\right) \\ &\leq C(\|u(t)\|_{H^1(\Omega)}). \end{aligned} \tag{2.13}$$

Similarly,

$$\int_\Gamma G(u(t))dS \leq C(\|u(t)\|_{H^1(\Gamma)}). \tag{2.14}$$

(2.12)-(2.14) imply that

$$E(u(t)) + \int_0^t \|u_t(\tau)\|^2_{V_0}d\tau \leq E(u_0) \leq C(\|u_0\|_{V_1}), \quad \forall\ t \geq 0. \tag{2.15}$$

By assumption (**F3**), there exists a constant $M > 0$ such that $f(s) < 0$ for $s < -M$ and $f(s) > 0$ for $s > M$. Consequently, the function F is bounded from below by a certain constant C_F. Hence,

$$\int_\Omega F(u(t))dx \geq C_F|\Omega|. \tag{2.16}$$

Analogously,

$$\int_\Gamma G(u(t))dS \geq C_G|\Gamma|. \tag{2.17}$$

Thus we can deduce that

$$E(u(t)) \geq \frac{1}{2}\|u(t)\|^2_{\mathrm{V}_1} + C_F|\Omega| + C_G|\Gamma|. \tag{2.18}$$

Therefore, we have the uniform estimate

$$\begin{aligned} \|u(t)\|^2_{\mathrm{V}_1} &\leq 2(E(u(t)) - C_F|\Omega| - C_G|\Gamma|) \\ &\leq 2(C(\|u_0\|_{\mathrm{V}_1}) - C_F|\Omega| - C_G|\Gamma|), \quad \forall\, t \geq 0. \end{aligned} \tag{2.19}$$

Differentiating (1.1) and (1.2) with respect to t, we get

$$u_{tt} - \Delta u_t + f'(u)u_t = 0, \tag{2.20}$$

$$-\Delta_{||}u_t + \partial_\nu u_t + u_t + g'(u)u_t + u_{tt} = 0. \tag{2.21}$$

Multiplying (2.20) by u_t and integrating over Ω, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{\mathrm{V}_0} + \|u_t\|^2_{\mathrm{V}_1} + \int_\Omega f'(u)u_t^2dx + \int_\Gamma g'(u)u_t^2dS = 0. \tag{2.22}$$

It follows from (**F3**) that

$$-\int_\Omega f'(u)u_t^2dx \leq C_f\|u_t\|^2, \quad -\int_\Gamma g'(u)u_t^2dS \leq C_g\|u_t\|^2_{L^2(\Gamma)}, \tag{2.23}$$

where C_f, C_g are constants depending only on f, g, respectively. Hence, we infer from the assumption on u_0 and (2.15) that

$$\begin{aligned} &\|u_t(t)\|^2_{\mathrm{V}_0} + 2\int_0^t \|u_t(\tau)\|^2_{\mathrm{V}_1}d\tau \\ \leq\ & \|u_t|_{t=0}\|^2_{\mathrm{V}_0} + 2C_f\int_0^t \|u_t(\tau)\|^2d\tau + 2C_g\int_0^t \|u_t(\tau)\|^2_{L^2(\Gamma)}d\tau \\ =\ & \|-\Delta u_0 + f(u_0)\|^2 + \|\Delta_{||}u_0 - \partial_\nu u_0 - u_0 - g(u_0)\|^2_{L^2(\Gamma)} + 2C_f\int_0^t \|u_t(\tau)\|^2d\tau \\ & +2C_g\int_0^t \|u_t(\tau)\|^2_{L^2(\Gamma)}d\tau \\ \leq\ & C, \end{aligned} \tag{2.24}$$

where C depends on $\|u_0\|_{\mathrm{V}_2}$ and is independent of T_{max}.

We consider the elliptic boundary value problem

$$\begin{cases} -\Delta u = -u_t - f(u), & x \in \Omega, \\ -\Delta_{\|} u + \partial_\nu u + u = -u_t - g(u), & x \in \Gamma. \end{cases} \tag{2.25}$$

From Lemma 2.1, it turns out that

$$\|u\|_{\mathrm{V}_2} \le C(\|u_t+f(u)\|+\|u_t+g(u)\|_{L^2(\Gamma)}) \le C(\|u_t\|_{\mathrm{V}_0}+\|f(u)\|+\|g(u)\|_{L^2(\Gamma)}), \tag{2.26}$$

where C is certain positive constant depending only on $|\Omega|$, $|\Gamma|$.
According to (**F2**), we have

$$|f(s)| \le C(1+|s|^{p_1}), \quad s \in \mathbb{R},\ p_1 \in [0, \alpha_1), \tag{2.27}$$

with

$$\alpha_1 := \begin{cases} +\infty, & \text{if } n = 1, 2, \\ \frac{n+2}{n-2}, & \text{if } n \ge 3. \end{cases}$$

Thus,

$$\|f(u)\| \le C + C\|u\|^{p_1}_{L^{2p_1}(\Omega)}. \tag{2.28}$$

In what follows, we consider the case $n \ge 4$. For the easier cases $n = 2, 3$, we refer to Remark 2.1 below.

(1) If $p_1 \in [0, \frac{n}{n-2}]$, by the continuous embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, $1 \le q \le \frac{2n}{n-2}$, and (2.19),

$$\|f(u)\| \le C + C\|u\|^{p_1}_{L^{2p_1}(\Omega)} \le C + C\|u\|^{p_1}_{H^1(\Omega)} \le C. \tag{2.29}$$

(2) If $p_1 \in (\frac{n}{n-2}, \frac{n+2}{n-2})$, by the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^{2p_1}(\Omega)} \le C\|u\|^a_{H^2(\Omega)}\|u\|^{1-a}_{L^{\frac{2n}{n-2}}(\Omega)}, \quad a = \frac{n-2}{2} - \frac{n}{2p_1} \in (0,1). \tag{2.30}$$

Using the continuous embedding $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and (2.19), we get

$$\|f(u)\| \le C + C\|u\|^{(1-a)p_1}_{H^1(\Omega)}\|u\|^{ap_1}_{H^2(\Omega)} \le C + C\|u\|^{ap_1}_{H^2(\Omega)}, \tag{2.31}$$

where

$$q_1 := ap_1 = p_1\frac{n-2}{2} - \frac{n}{2} = 1 - \frac{n-2}{2}\left(\frac{n+2}{n-2} - p_1\right) \in (0,1). \tag{2.32}$$

A similar argument shows that for properly chosen $q_2 \in (0,1)$, we have

$$\|g(u)\|_{L^2(\Gamma)} \le C + C\|u\|^{q_2}_{H^2(\Gamma)}. \tag{2.33}$$

By Young's inequality and (2.24),

$$\begin{aligned} \|u(t)\|_{\mathrm{V}_2} &\le C(\|u_t(t)\|_{\mathrm{V}_0} + \|f(u(t))\| + \|g(u(t))\|_{L^2(\Gamma)}) \\ &\le C + C\|u(t)\|^{q_1}_{H^2(\Omega)} + C\|u(t)\|^{q_2}_{H^2(\Gamma)} \le C + \frac{1}{2}\|u(t)\|_{\mathrm{V}_2}. \end{aligned} \tag{2.34}$$

As a result, we obtain the uniform estimate

$$\|u(t)\|_{\mathrm{V}_2} \le C, \tag{2.35}$$

for all $t \in [0, T_{max})$, where C is a constant that depends on $\|u_0\|_{\mathrm{V}_2}, f, g, \Omega, \Gamma$, but not on T_{max}. Now, we can conclude that the local solution u is indeed a global one.

The proof is complete. □

Remark 2.1. *If $n = 2, 3$, the situation is easier. Actually, due to **(F3)**, we can apply a maximum principle (cf. [23, Appendix: Lemma A.2]) to problem* (2.25) *to obtain that*

$$\|u\|_{L^\infty(\Omega)} + \|u|_\Gamma\|_{L^\infty(\Gamma)} \le C(1 + \|u_t\|_{\mathrm{V}_0}) \le C.$$

This is sufficient to prove (2.35)*. Moreover, in this case, we actually do not have to assume any growth assumption on f, g (cf. [8]). However, for the case of higher dimension, we do not have this property.*

The following result yields the continuous dependence on the initial data, which also implies the uniqueness of the solution to (1.1)–(1.3).

Lemma 2.2. *Let u_1, u_2 be two solutions to* (1.1)–(1.3) *corresponding to the initial data u_{01}, $u_{02} \in \tilde{\mathrm{V}}_2$, respectively. Then it holds*

$$\|u_1(t) - u_2(t)\|_{\mathrm{V}_1}^2 + \int_0^t (\|u_1(\tau) - u_2(\tau)\|_{\mathrm{V}_2}^2 + \|(u_1 - u_2)_t(\tau)\|_{\mathrm{V}_0}^2) d\tau \le Ce^{Lt}\|u_{01} - u_{02}\|_{\mathrm{V}_1}^2, \tag{2.36}$$

where C and L are positive constants depending on the norms of the initial data, on Ω and Γ, but not on time.

Proof. Subtracting the equations for u_1 from the equations for u_2, we get

$$(u_1 - u_2)_t - \Delta(u_1 - u_2)u + f(u_1) - f(u_2) = 0, \tag{2.37}$$

$$-\Delta_{||}(u_1 - u_2) + \partial_\nu(u_1 - u_2) + (u_1 - u_2) + g(u_1) - g(u_2) + (u_1 - u_2)_t = 0. \tag{2.38}$$

Multiplying (2.37) by $(u_1 - u_2)_t$, integrating over Ω, and using the boundary condition (2.38), we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|(u_1 - u_2)\|_{\mathrm{V}_1}^2 + \|(u_1 - u_2)_t\|_{\mathrm{V}_0}^2 \\ = & -\int_\Omega (f(u_1) - f(u_2))(u_1 - u_2)_t dx - \int_\Gamma (g(u_1) - g(u_2))(u_1 - u_2)_t dS \\ \le & \frac{1}{2}\|(u_1 - u_2)_t\|_{\mathrm{V}_0}^2 + \|f(u_1) - f(u_2)\|^2 + \|g(u_1) - g(u_2)\|_{L^2(\Gamma)}^2. \end{aligned} \tag{2.39}$$

The mean value formula gives

$$f(u_1) - f(u_2) = \int_0^1 f'(su_1 + (1 - s)u_2)(u_1 - u_2) ds. \tag{2.40}$$

Hence,

$$\|f(u_1)-f(u_2)\| \le \max_{0\le s\le 1} \|f'(su_1+(1-s)u_2)(u_1-u_2)\|. \tag{2.41}$$

By Hölder's inequality and the Sobolev embedding $H^{2-\eta}(\Omega) \hookrightarrow L^{\frac{2n}{n-4+2\eta}}(\Omega)$, $\eta\in[0,1]$,

$$\begin{aligned}\|f(u_1)-f(u_2)\| &\le \max_{0\le s\le 1} \|f'(\xi_s)\|_{L^{\frac{n}{2-\eta}}(\Omega)} \|u_1-u_2\|_{L^{\frac{2n}{n-4+2\eta}}(\Omega)} \\ &\le \max_{0\le s\le 1} \|f'(\xi_s)\|_{L^{\frac{n}{2-\eta}}(\Omega)} \|u_1-u_2\|_{H^{2-\eta}(\Omega)},\end{aligned} \tag{2.42}$$

where $\xi_s = su_1+(1-s)u_2$, $s\in[0,1]$. From (**F2**), we have

$$\|f'(\xi_s)\|_{L^{\frac{n}{2-\eta}}(\Omega)} \le C\left(1+\|\xi_s\|^{\frac{4}{n-2}}_{L^{\frac{4n}{(2-\eta)(n-2)}}(\Omega)}\right). \tag{2.43}$$

When $n\le 4$, we can just take $\eta=1$ while for $n>5$, we take $\eta\in\left(0,\min\{1,\frac{4}{n-2}\}\right]$. Moreover,

$$\frac{4n}{(2-\eta)(n-2)} < \frac{2n}{n-4}, \quad n>4.$$

Then we have

$$\begin{aligned}\|f(u_1)-f(u_2)\| &\le \max_{0\le s\le 1} \|f'(\xi_s)\|_{L^{\frac{n}{2-\eta}}(\Omega)} \|u_1-u_2\|_{H^{2-\eta}(\Omega)} \\ &\le C(\|u\|_{\mathrm{V}_2}, \|\psi\|_{\mathrm{V}_2}) \|u_1-u_2\|_{H^{2-\eta}(\Omega)} \\ &\le C(\eta)\|u_1-u_2\|^{1-\eta}_{H^2(\Omega)} \|u_1-u_2\|^{\eta}_{H^1(\Omega)} \\ &\le \varepsilon\|u_1-u_2\|_{H^2(\Omega)} + C_\varepsilon \|u_1-u_2\|_{H^1(\Omega)}.\end{aligned} \tag{2.44}$$

A similar argument shows that

$$\|g(u_1)-g(u_2)\|_{L^2(\Gamma)} \le \varepsilon\|u_1-u_2\|_{H^2(\Gamma)} + C_\varepsilon\|u_1-u_2\|_{H^1(\Gamma)}. \tag{2.45}$$

Applying Lemma 2.1 to problem (2.37)–(2.38), we obtain that

$$\begin{aligned}&\|u_1-u_2\|_{\mathrm{V}_2} \\ \le\ & C(\|(u_1-u_2)_t + f(u_1)-f(u_2)\| + \|(u_1-u_2)_t + g(u_1)-g(u_2)\|_{L^2(\Gamma)}) \\ \le\ & C(\|(u_1-u_2)_t\|_{\mathrm{V}_0} + \|f(u_1)-f(u_2)\| + \|g(u_1)-g(u_2)\|_{L^2(\Gamma)}) \\ \le\ & \frac{1}{2}\|u_1-u_2\|_{\mathrm{V}_2} + C(\|(u_1-u_2)_t\|_{\mathrm{V}_0} + \|u_1-u_2\|_{\mathrm{V}_1}).\end{aligned} \tag{2.46}$$

As a result,

$$\|u_1-u_2\|_{\mathrm{V}_2} \le C_*(\|(u_1-u_2)_t\|_{\mathrm{V}_0} + \|u_1-u_2\|_{\mathrm{V}_1}), \tag{2.47}$$

where C_* is a constant depending on $\|u_1\|_{\mathrm{V}_2}, \|u_2\|_{\mathrm{V}_2}$. The above estimate also implies that

$$\frac{1}{2C_*^2}\|u_1-u_2\|^2_{\mathrm{V}_2} \le \|(u_1-u_2)_t\|^2_{\mathrm{V}_0} + \|u_1-u_2\|^2_{\mathrm{V}_1}. \tag{2.48}$$

Thus, we can infer from (2.44), (2.45), (2.39), and (2.48), that

$$\begin{aligned}
&\frac{d}{dt}\|(u_1-u_2)\|^2_{\mathrm{V}_1}+\frac{1}{2}\|(u_1-u_2)_t\|^2_{\mathrm{V}_0}+\frac{1}{4C_*}\|u_1-u_2\|^2_{\mathrm{V}_2}\\
\leq\ &\frac{1}{2}\|u_1-u_2\|^2_{\mathrm{V}_1}+\|f(u_1)-f(u_2)\|^2+\|g(u_1)-g(u_2)\|^2_{L^2(\Gamma)}\\
\leq\ &\frac{1}{8C_*}\|u_1-u_2\|^2_{\mathrm{V}_2}+L\|u_1-u_2\|^2_{\mathrm{V}_1}. \qquad (2.49)
\end{aligned}$$

The conclusion then follows from Gronwall's inequality. □

We can see that the global solution u to problem (1.1)–(1.3) defines a semigroup $S(t):\tilde{\mathrm{V}}_2\to\tilde{\mathrm{V}}_2$ such that $S(t)u_0=u(t)$.

In the remaining part of this section, we want to prove Theorem 1.2, namely that $S(t)$ has a global attractor $\mathcal{A}$ in the phase space $\tilde{\mathrm{V}}_2$. For this propose, we first show that there exists a bounded absorbing set in $\tilde{\mathrm{V}}_2$.

Lemma 2.3. *Assume that **(F1a)**, **(F2)**, **(F3)** hold. There exists a positive constant R_0 such that the ball*

$$\mathcal{B}_0:=\{u\in\tilde{\mathrm{V}}_2\mid \|u\|_{\mathrm{V}_2}\leq R_0\}$$

is an absorbing set. In fact, for any bounded set $\mathcal{B}\in\tilde{\mathrm{V}}_2$, there is some $t_0=t_0(\mathcal{B})\geq 0$ such that $S(t)\mathcal{B}\subset\mathcal{B}_0$ for every $t\geq t_0$.

Proof. In the proof, we shall denote by c_i $(i=1,2,...)$ constants that are independent of u. Multiplying (1.1) by u, and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\mathrm{V}_0}+\|u\|^2_{\mathrm{V}_1}=-\int_\Omega f(u)u dx-\int_\Gamma g(u)u dS. \qquad (2.50)$$

Set

$$Y_1(t)=E(u(t))+\|u(t)\|^2_{\mathrm{V}_0}+C_F|\Omega|+C_G|\Gamma|. \qquad (2.51)$$

We infer from (2.18) that

$$Y_1(t)\geq\frac{1}{2}\|u(t)\|^2_{\mathrm{V}_1}+\|u(t)\|^2_{\mathrm{V}_0}. \qquad (2.52)$$

It follows from (2.12) and (2.50) that

$$\frac{d}{dt}Y_1(t)+\|u_t\|^2_{\mathrm{V}_0}+2\|u\|^2_{\mathrm{V}_1}\leq 2\left(-\int_\Omega f(u)u dx-\int_\Gamma g(u)u dS\right). \qquad (2.53)$$

Owing to assumption (**F3**), there exists some $M_1>0$ such that $f'(s)>0$, $g'(s)>0$, for $|s|>M_1$. Furthermore, there exists some $M_2\geq M_1$ such that $f(s)s>0$, $g(s)s>0$, for $|s|>M_2$. Consequently,

$$\begin{aligned}
-\int_\Omega f(u)u dx &= -\int_{|u|>M_2} f(u)u dx-\int_{|u|\leq M_2} f(u)u dx\leq-\int_{|u|\leq M_2} f(u)u dx\\
&\leq |\Omega|\max_{|s|\leq M_2}|f(s)s|:=C_{M_2,f}. \qquad (2.54)
\end{aligned}$$

Similarly,

$$-\int_\Gamma g(u)udS \le |\Gamma| \max_{|s|\le M_2} |g(s)s| := C_{M_2,g}. \tag{2.55}$$

Again from (**F3**), we have (cf. [8])

$$F(s) - f(s)s \le C_{f_1}s^2 + C_{f_2}, \quad G(s) - g(s)s \le C_{g_1}s^2 + C_{g_2}, \tag{2.56}$$

where $C_{f_1}, C_{f_2}, C_{g_1}, C_{g_2}$ are independent of u. Hence,

$$\int_\Omega (F(u) - f(u)u)dx \le C_{f_1}\|u\|^2 + C_{f_2}|\Omega|, \tag{2.57}$$

$$\int_\Gamma (G(u) - g(u)u)dS \le C_{g_1}\|u\|^2_{L^2(\Gamma)} + C_{g_2}|\Gamma|. \tag{2.58}$$

We can conclude that there exist positive constants $c_1 \ll 1$, c_2, independent of u, such that

$$\frac{d}{dt}Y_1(t) + c_1Y_1(t) \le c_2. \tag{2.59}$$

Therefore,

$$Y_1(t) \le e^{-c_1t}Y_1(0) + \frac{c_2}{c_1}, \tag{2.60}$$

which, together with (2.52), implies that

$$\|u(t)\|^2_{\mathrm{V}_1} \le C(\|u_0\|_{\mathrm{V}_1})e^{-c_3t} + c_4. \tag{2.61}$$

Set

$$Y_2(t) = Y_1(t) + \mu\|u_t(t)\|^2_{\mathrm{V}_0}, \tag{2.62}$$

where $\mu > 0$ is a constant to be determined later. We infer from (2.18) that

$$Y_2(t) \ge \frac{1}{2}\|u(t)\|^2_{\mathrm{V}_1} + \|u(t)\|^2_{\mathrm{V}_0} + \mu\|u_t(t)\|^2_{\mathrm{V}_0}. \tag{2.63}$$

It follows from (2.12), (2.22), (2.50), that

$$\begin{aligned} &\frac{d}{dt}Y_2(t) + (1 - 2\mu\max\{C_f, C_g\})\|u_t\|^2_{\mathrm{V}_0} + 2\mu\|u_t\|^2_{\mathrm{V}_1} + 2\|u\|^2_{\mathrm{V}_1} \\ \le\ & 2\left(-\int_\Omega f(u)udx - \int_\Gamma g(u)udS\right). \end{aligned} \tag{2.64}$$

Taking

$$\mu = \frac{1}{4\max\{C_f, C_g\}}, \tag{2.65}$$

by a similar argument as before, there exist positive constants c_5, c_6 such that

$$Y_2(t) \le e^{-c_5t}Y_2(0) + \frac{c_6}{c_5}. \tag{2.66}$$

This together with (2.61) yields

$$\|u_t(t)\|_{\mathrm{V}_0} \leq C(\|u_0\|_{\mathrm{V}_2})e^{-c_7 t} + c_8. \tag{2.67}$$

According to (2.26), in order to estimate $\|u(t)\|_{\mathrm{V}_2}$, we have to estimate $\|f(u(t))\|$ and $\|g(u(t))\|_{L^2(\Gamma)}$. From (2.31) and (2.61), we find that by the Gagliardo-Nirenberg inequality

$$\begin{aligned}\|f(u(t))\| &\leq c_9 + c_{10}\|u(t)\|_{H^1(\Omega)}^{(1-a)p_1}\|u(t)\|_{H^2(\Omega)}^{ap_1} \\ &\leq c_9 + \frac{1}{2}\|u(t)\|_{H^2(\Omega)} + c_{11}\|u(t)\|_{H^1(\Omega)}^{(1-a)p_1(1-ap_1)} \\ &\leq c_{12} + C(\|u_0\|_{H^1(\Omega)})e^{-c_{13}t} + \frac{1}{2}\|u(t)\|_{H^2(\Omega)}. \end{aligned} \tag{2.68}$$

Here, a is the same as in (2.30). Similarly,

$$\|g(u(t))\|_{L^2(\Gamma)} \leq c_{14} + C(\|u_0\|_{H^1(\Gamma)})e^{-c_{15}t} + \frac{1}{2}\|u(t)\|_{H^2(\Gamma)}. \tag{2.69}$$

Then it follows from (2.26) and (2.67)-(2.69) that

$$\|u(t)\|_{\mathrm{V}_2} \leq c_{16} + C(\|u_0\|_{\mathrm{V}_2})e^{-c_{17}t}, \tag{2.70}$$

which proves the assertion. □

The following lemma implies the compactness of the solution u in V_2 for $t > 0$.

Lemma 2.4. *It holds*

$$\|u(t)\|_{\mathrm{V}_3} \leq C_\delta, \quad \forall\, t \geq \delta > 0, \tag{2.71}$$

where δ is arbitrary and C_δ depends on δ, $\|u_0\|_{\mathrm{V}_2}$.

Proof. Multiplying (2.20) by $-\Delta u_t$, and integrating over Ω, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_{\mathrm{V}_1}^2 + \|\Delta u_t\|^2 + \|u_{tt}\|_{L^2(\Gamma)}^2 = \int_\Omega f'(u)u_t\Delta u_t dx - \int_\Gamma g'(u)u_t u_{tt} dS. \tag{2.72}$$

First, we look at the first term on the right-hand side.
(1) If $n = 2, 3$, it follows from Remark 2.1 that

$$\left|\int_\Omega f'(u)u_t\Delta u_t dx\right| \leq \|f'(u)\|_{L^\infty(\Omega)}\|u_t\|\|\Delta u_t\| \leq \frac{1}{2}\|\Delta u_t\|^2 + C\|u_t\|^2. \tag{2.73}$$

(2) If $n = 4, 5, 6$, we have

$$\left|\int_\Omega f'(u)u_t\Delta u_t dx\right| \leq \|f'(u)\|_{L^{q_1}(\Omega)}\|u_t\|_{L^{q_2}(\Omega)}\|\Delta u_t\|, \tag{2.74}$$

whenever $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$. Take

$$q_1 = n, \quad q_2 = \frac{2n}{n-2}. \tag{2.75}$$

Then we can check that

$$q_1 \frac{4}{n-2} = 8 < \infty, \quad \text{if } n = 4, \tag{2.76}$$

$$q_1 \frac{4}{n-2} \le \frac{2n}{n-4}, \quad \text{if } n = 5, 6. \tag{2.77}$$

By the Sobolev embedding theorem,

$$\left| \int_\Omega f'(u) u_t \Delta u_t dx \right| \le C(\|u\|_{H^2(\Omega)}) \|u_t\|_{H^1(\Omega)} \|\Delta u_t\| \le \frac{1}{2} \|\Delta u_t\|^2 + C \|u_t\|^2_{H^1(\Omega)}. \tag{2.78}$$

(3) If $n \ge 7$, we take in (2.74)

$$q_1 = \frac{n(n-2)}{2(n-4)}, \quad q_2 = \frac{2n(n-2)}{n^2 - 6n + 16}. \tag{2.79}$$

Since

$$q_1 \frac{4}{n-2} = \frac{2n}{n-4}, \tag{2.80}$$

we can see that

$$\|f'(u)\|_{L^{q_1}(\Omega)} \le C(\|u\|_{H^2(\Omega)}). \tag{2.81}$$

By the Gagliardo-Nirenberg inequality,

$$\|u_t\|_{L^{q_2}(\Omega)} \le C \|\Delta u_t\|^a \|u_t\|^{1-a}_{L^{\frac{2n}{n-2}}(\Omega)} + C \|u_t\|_{L^{\frac{2n}{n-2}}(\Omega)}, \tag{2.82}$$

where $a = \frac{n-6}{n-2} \in (0, 1)$. As a result,

$$\begin{aligned} \left| \int_\Omega f'(u) u_t \Delta u_t dx \right| &\le C(\|u\|_{H^2(\Omega)}) \left(\|\Delta u_t\|^a \|u_t\|^{1-a}_{L^{\frac{2n}{n-2}}(\Omega)} + \|u_t\|_{L^{\frac{2n}{n-2}}(\Omega)} \right) \|\Delta u_t\| \\ &\le \frac{1}{2} \|\Delta u_t\|^2 + C \|u_t\|^2_{H^1(\Omega)}. \end{aligned} \tag{2.83}$$

Similarly, we can prove that

$$\left| \int_\Gamma g'(u) u_t u_{tt} dS \right| \le \frac{1}{2} \|u_{tt}\|^2_{L^2(\Gamma)} + C \|u_t\|^2_{H^1(\Gamma)}. \tag{2.84}$$

Summing up, we have

$$\frac{d}{dt} \|u_t\|^2_{\mathrm{V}_1} + \|\Delta u_t\|^2 + \|u_{tt}\|^2_{L^2(\Gamma)} \le C \|u_t\|^2_{\mathrm{V}_1}. \tag{2.85}$$

Multiplying (2.85) by t, and integrating with respect to the time variable, we get

$$t \|u_t(t)\|^2_{\mathrm{V}_1} + \int_0^t \tau \left(\|\Delta u_t(\tau)\|^2 + \|u_{tt}(\tau)\|^2_{L^2(\Gamma)} \right) d\tau \le \int_0^t (1 + C\tau) \|u_t(\tau)\|^2_{\mathrm{V}_1} d\tau. \tag{2.86}$$

It follows from (2.24) that

$$\|u_t(t)\|_{\mathrm{V}_1}^2 \le C\left(1+\frac{1}{t}\right), \quad \forall\, t>0. \tag{2.87}$$

Applying Lemma 2.1 to (2.25) with $s=1$, we obtain that

$$\begin{aligned} \|u\|_{\mathrm{V}_3} &\le C(\|u_t+f(u)\|_{H^1(\Omega)}+\|u_t+g(u)\|_{H^1(\Gamma)}) \\ &\le C(\|u_t\|_{\mathrm{V}_1}+\|f(u)\|_{H^1(\Omega)}+\|g(u)\|_{H^1(\Gamma)}). \end{aligned} \tag{2.88}$$

Similar as for $\|f'(u)u_t\|$ above, we can prove that

$$\|f(u)\|_{H^1(\Omega)} \le C + C\|u\|_{H^3(\Omega)}^{a_1}, \quad \|g(u)\|_{H^1(\Gamma)} \le C + C\|u\|_{H^3(\Gamma)}^{a_2}, \tag{2.89}$$

with some $a_1, a_2 \in (0,1)$. The conclusion (2.71) follows from (2.31), (2.87)–(2.89) and Young's inequality. The proof is complete. □

Proof of Theorem 1.2. On account of Lemma 2.3, Lemma 2.4 and a classical result in dynamical systems (cf. [29, Theorem I.1.1]), we conclude the validity of Theorem 1.2, i.e., problem (1.1)–(1.3) possesses a compact global attractor $\mathcal{A}$ in $\tilde{\mathrm{V}}_2$. □

Remark 2.2. *Following the argument in [8], it is possible to prove existence of an exponential attractor, which also implies that $\mathcal{A}$ has finite fractal dimension. We will not report the details here.*

3 An Extended Łojasiewicz-Simon Inequality

In this section, we always assume that (**F1b**) holds. First, we collect some results on the stationary problem. To this end, we consider the functional

$$E(u) = \frac{1}{2}\|u\|_{\mathrm{V}_1}^2 + \int_\Omega F(u)dx + \int_\Gamma G(u)dS. \tag{3.1}$$

Proposition 3.1. *Assume that (**F1b**), (**F2**) hold. If $\psi \in \tilde{\mathrm{V}}_2$ is a strong solution to problem (1.12), then ψ is a critical point of the functional $E(u)$ in $\tilde{\mathrm{V}}_1$. Conversely, if ψ is a critical point of the functional $E(u)$ in $\tilde{\mathrm{V}}_1$, then $\psi \in \tilde{\mathrm{V}}_2$ is a strong solution to problem (1.12). Moreover, $\psi \in C^\infty(\bar{\Omega})$.*

Proof. If $\psi \in \tilde{\mathrm{V}}_2$ satisfies (1.12), then it follows from (1.12) that for any $v \in \tilde{\mathrm{V}}_1$

$$\int_\Omega (-\Delta\psi + f(\psi))v dx + \int_\Gamma (-\Delta_{\|}\psi + \partial_\nu\psi + \psi + g(\psi))v dS = 0. \tag{3.2}$$

By integration by parts, and using the boundary condition in (1.12), we get

$$\int_\Omega (\nabla\psi\cdot\nabla v + f(\psi)v)dx + \int_\Gamma (\nabla_{\|}\psi\nabla_{\|}v + \psi v + g(\psi)v)dS = 0, \tag{3.3}$$

which, by a straightforward calculation, just means that

$$\left.\frac{dE(\psi+\varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0}=0. \tag{3.4}$$

Thus, ψ is a critical point of $E(\psi)$ on $\tilde{\mathrm{V}}_1$.

Conversely, if ψ is a critical point of E with respect to $\tilde{\mathrm{V}}_1$, then (3.3) holds for any $v\in\tilde{\mathrm{V}}_1$. It turns out that ψ is a weak solution to problem (1.12). By the regularity result (for instance, Lemma 2.1), and using the bootstrap argument, we can conclude that $\psi\in C^\infty$, and that ψ is a classical solution to problem (1.12). The proof is complete. □

In the following, we are going to derive an extended Łojasiewicz-Simon type inequality. Since we are now dealing with a nonlinear boundary condition (in the presence of g), it seems that we are not able to adapt the argument for a linear boundary condition (cf. [2, 30, 34]) to conclude our result directly. Hence, nontrivial modifications have to be made. Our proof is based on ideas in [18] and can also apply to other problems (cf. [14]).

Let us denote

$$A:=\begin{pmatrix}-\Delta & 0\\ \partial_\nu & -\Delta_{||}+I\end{pmatrix}. \tag{3.5}$$

It has been shown in [26] that A is a strictly positive self-adjoint unbounded operator from

$$D(A)=\{u\in\tilde{\mathrm{V}}_1:\ Au\in\mathrm{V}_0\} \tag{3.6}$$

into V_0 (cf. also [12]).

Standard spectral theory allows us to define the power A^s ($s\in\mathbb{R}$), and we infer that there exist a complete orthonormal family $\{\phi_j\}, j\in\mathbb{N}$, in V_0, with $\phi_j\in D(A^s)$ ($s\in\mathbb{R}$), as well as a sequence of eigenvalues $0<\lambda_1\le\lambda_2\le\ldots$, $\lambda_j\to\infty$ as j tends to infinity, such that

$$A\phi_j=\lambda_j\phi_j,\quad j\in\mathbb{N}. \tag{3.7}$$

In particular, $D(A^{\frac{1}{2}})=\tilde{\mathrm{V}}_1$, $D(A)\subset\tilde{\mathrm{V}}_2$. By a bootstrap argument, we have $\phi_j\in C^\infty(\bar{\Omega})$, for all $j\in\mathbb{N}$ (cf. [26]).

Following the idea in [18], we now introduce the orthogonal projection P_m in V_0 onto $K_m:=span\{\phi_1,\ldots,\phi_m\}$. For any $u\in\tilde{\mathrm{V}}_1$, we have

$$(Au,u)_{\mathrm{V}_1',\mathrm{V}_1}=\|u\|^2_{\mathrm{V}_1}, \tag{3.8}$$

where $(\cdot,\cdot)_{\mathrm{V}_1',\mathrm{V}_1}$ denotes the dual product between V_1 and its dual space V_1'. On the other hand, it is easy to see that

$$\begin{aligned}(Au,u)_{\mathrm{V}_1',\mathrm{V}_1} &= (AP_mu,P_mu)_{\mathrm{V}_1',\mathrm{V}_1}+(A(u-P_mu),(u-P_mu))_{\mathrm{V}_1',\mathrm{V}_1}\\ &\ge \lambda_1\|P_mu\|^2_{\mathrm{V}_0}+\lambda_m\|u-P_mu\|^2_{\mathrm{V}_0}.\end{aligned} \tag{3.9}$$

Therefore, it follows from (3.8) and (3.9) that

$$(Au,u)_{\mathrm{V}_1',\mathrm{V}_1} \geq \frac{1}{2}\|u\|_{\mathrm{V}_1}^2 + \frac{1}{2}\lambda_m\|u-P_m u\|_{\mathrm{V}_0}^2. \tag{3.10}$$

Then we have

$$\begin{aligned}(Au+\lambda_m P_m u,u)_{\mathrm{V}_1',\mathrm{V}_1} &\geq \frac{1}{2}\|u\|_{\mathrm{V}_1}^2 + \frac{1}{2}\lambda_m\|u-P_m u\|_{\mathrm{V}_0}^2 + \lambda_m\|P_m u\|_{\mathrm{V}_0}^2 \\ &\geq \frac{1}{2}\|u\|_{\mathrm{V}_1}^2 + \frac{1}{4}\lambda_m\|u\|_{\mathrm{V}_0}^2. \end{aligned}\tag{3.11}$$

Let ψ be a critical point of $E(u)$. For any $v \in \tilde{\mathrm{V}}_2$, we consider the following linearized operator:

$$L(v) := \begin{pmatrix} -\Delta & 0 \\ \partial_\nu & -\Delta_{\|}+I \end{pmatrix} + \begin{pmatrix} f'(v+\psi) & 0 \\ 0 & g'(v+\psi) \end{pmatrix}, \tag{3.12}$$

with domain $\mathcal{D} = D(A)$. In analogy to [34, Lemma 2.3], we can easily show that $L(v)$ is self-adjoint on $\tilde{\mathrm{V}}_1$.

Associated with $L(0)$, we define the following bilinear form $b(w_1,w_2)$ on $\tilde{\mathrm{V}}_1$: for any $w_1, w_2 \in \tilde{\mathrm{V}}_1$,

$$b(w_1,w_2) = \int_\Omega (\nabla w_1\cdot\nabla w_2 + f'(\psi)w_1w_2)dx + \int_\Gamma \big(\nabla_{\|}w_1\cdot\nabla_{\|}w_2 + w_1w_2 + g'(\psi)w_1w_2\big)\,dS. \tag{3.13}$$

It follows from (3.11) that

$$L(0)+\lambda_m P_m = A + \begin{pmatrix} f'(\psi) & 0 \\ 0 & g'(\psi) \end{pmatrix} + \lambda_m P_m \tag{3.14}$$

is coercive in V_1, provided that

$$\frac{1}{4}\lambda_m > \max\{\|f'(\psi)\|_{L^\infty(\Omega)}, \|g'(\psi)\|_{L^\infty(\Gamma)}\}. \tag{3.15}$$

Since $\psi \in C^\infty$, we can choose m so large that (3.15) is satisfied. Denote $\lambda_m P_m$ by Π, that is,

$$\Pi := \lambda_m P_m : \ \mathrm{V}_0 \to \mathrm{V}_0. \tag{3.16}$$

We define $\mathcal{L}(v) : \mathcal{D} \to \mathrm{V}_0$ by setting

$$\mathcal{L}(v)h = \Pi h + L(v)h. \tag{3.17}$$

It follows from (3.15) that $\mathcal{L}(0) : \mathcal{D} \to \mathrm{V}_0$ is bijective. Moreover, it holds:

Lemma 3.1. *For any $w \in \mathrm{V}_k$ $(k = 0,1,2...)$, the equation*

$$\mathcal{L}(0)h = w \tag{3.18}$$

admits a unique solution $h \in \tilde{\mathrm{V}}_{k+2}$ such that

$$\|h\|_{\mathrm{V}_{k+2}} \leq C\|w\|_{\mathrm{V}_k}, \tag{3.19}$$

where $C > 0$ is independent of k.

Proof. Since $\mathcal{L}(0) : \mathcal{D} \to \mathrm{V}_0$ is bijective, for any $w \in \mathrm{V}_k \subset \mathrm{V}_0$, (3.18) admits a unique solution h. Moreover, we have

$$\frac{1}{2}\|h\|_{\mathrm{V}_1}^2 \le (\mathcal{L}(0)h, h)_{\mathrm{V}_0} = (w, h)_{\mathrm{V}_0} \le C\|h\|_{\mathrm{V}_0}\|w\|_{\mathrm{V}_0}, \tag{3.20}$$

which implies that

$$\|h\|_{\mathrm{V}_1} \le C\|w\|_{\mathrm{V}_0}. \tag{3.21}$$

We rewrite the equation $\mathcal{L}(0)h = w$ in the form

$$\begin{cases} -\Delta h = -f'(\psi)h - (\Pi h)_1 + w_1, \quad x \in \Omega, \\ -\Delta_{||}h + \partial_\nu h + h = -g'(\psi)h - (\Pi h)_2 + w_2, \quad x \in \Gamma, \end{cases} \tag{3.22}$$

where $\Pi h = ((\Pi h)_1, (\Pi u)_2)$ with $(\Pi h)_2 = \Pi h|_\Gamma$, $w = (w_1, w_2)$. It follows from Lemma 2.1 and (3.21) that for $k = 0$,

$$\begin{aligned} \|h\|_{\mathrm{V}_2} &\le C(\| - f'(\psi)h - (\Pi h)_1 + w_1\|_{L^2(\Omega)} + \| - g'(\psi)h - (\Pi h)_2 + w_2\|_{L^2(\Gamma)}) \\ &\le C(\|h\|_{\mathrm{V}_0} + \|w\|_{\mathrm{V}_0}) \le C\|w\|_{\mathrm{V}_0}. \end{aligned} \tag{3.23}$$

Since $\psi \in C^\infty$, we can apply Lemma 2.1 with $k = 1, 2...$, and a bootstrap argument shows that (3.19) holds. □

Now set

$$u = v + \psi. \tag{3.24}$$

We denote

$$\mathcal{E}(v) = E(u) = E(v + \psi), \tag{3.25}$$

and

$$\mathcal{M}(u) = \begin{pmatrix} -\Delta u + f(u) \\ -\Delta_{||}u + \partial_\nu u + u + g(u) \end{pmatrix}. \tag{3.26}$$

In the following we will use the equivalent form of $\mathcal{M}(u)$:

$$M(v) = \mathcal{M}(u) = \mathcal{M}(v + \psi). \tag{3.27}$$

It is obvious that for any $u \in \tilde{\mathrm{V}}_2$, it holds $\mathcal{M}(u) \in \mathrm{V}_0$ as well as

$$\left.\frac{dE(u + \varepsilon h)}{d\varepsilon}\right|_{\varepsilon=0} = (\mathcal{M}(u), h)_{\mathrm{V}_0}, \quad \forall\, h \in \tilde{\mathrm{V}}_1. \tag{3.28}$$

Moreover, $M(0) = \mathcal{M}(\psi) = 0$ (cf. Proposition 3.1).

Now we can state the main result of this section:

Theorem 3.1. *Let ψ be a critical point of $E(u)$. Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on ψ, such that for any $u \in \mathcal{D}$ satisfying $\|u - \psi\|_{\mathrm{V}_2} < \beta$ we have*

$$\| - \Delta u + f(u)\| + \| - \Delta_{||}u + \partial_\nu u + u + g(u)\|_{L^2(\Gamma)} \ \ge\ |E(u) - E(\psi)|^{1-\theta}. \tag{3.29}$$

Proof. Let $\mathcal{N} : \mathcal{D} \mapsto \mathrm{V}_0$ be the nonlinear operator defined by

$$\mathcal{N}(w) = \Pi_K w + M(w). \tag{3.30}$$

Then $\mathcal{N}$ is Fréchet differentiable with derivative $D\mathcal{N}(w) = \mathcal{L}(w)$.

We shall use the notion of analyticity of functions introduced in [36]. We also recall the following result in [24]:

Lemma 3.2. *The mappings* $L^\infty(\Omega) \ni u \mapsto f(u) \in L^\infty(\Omega)$ *and* $L^\infty(\Gamma) \ni u \mapsto g(u) \in L^\infty(\Gamma)$ *are analytic.*

We now restrict $\mathcal{N}$ to higher order spaces: $\mathcal{N}_k := \mathcal{N}|_{\mathrm{V}_{k+2}} : \mathrm{V}_{k+2} \cap \mathcal{D} \mapsto \mathrm{V}_k$, with integer $k \geq \max\{0, \left[\frac{n}{2} - 2\right] + 1\}$. Since $\mathrm{V}_{k+2} \hookrightarrow L^\infty(\Omega) \times L^\infty(\Gamma)$, it follows from Lemma 3.2 that $\mathcal{N}_k(w)$ is analytic in $w \in \mathrm{V}_{k+2}$. Because $\mathcal{L}(0)$ is invertible, by the implicit function theorem (for the analytic version see, e.g., [36, Corallary 4.37, p.172]), there exist small neighborhoods of the origins, $W_1(0) \subset \mathrm{V}_{k+2} \cap \mathcal{D}$, $W_2(0) \subset \mathrm{V}_k$, and an analytic inverse $\Psi = \mathcal{N}_k^{-1}$ such that $\Psi : W_2(0) \to W_1(0)$ is bijective. Besides, we have

$$\mathcal{N}_k(\Psi(g)) = g \quad \forall g \in W_2(0), \tag{3.31}$$
$$\Psi(\mathcal{N}_k(v)) = v \quad \forall v \in W_1(0), \tag{3.32}$$

and

$$\|\Psi(g_1) - \Psi(g_2)\|_{\mathrm{V}_{k+2}} \leq C\|g_1 - g_2\|_{\mathrm{V}_k}, \qquad \forall g_1, g_2 \in W_2(0), \tag{3.33}$$
$$\|\mathcal{N}_k(v_1) - \mathcal{N}_k(v_2)\|_{\mathrm{V}_k} \leq C\|v_1 - v_2\|_{\mathrm{V}_{k+2}}, \qquad \forall v_1, v_2 \in W_1(0). \tag{3.34}$$

Since Ψ is analytic, it turns out that

$$\Gamma(\xi) := \mathcal{E}\left(\Psi\left(\sum_{i=1}^m \xi_i\phi_i\right)\right) : \ \mathbb{R}^m \to \mathbb{R} \tag{3.35}$$

is analytic with respect to $\xi = (\xi_1, ..., \xi_m)$, if $|\xi|_{\mathbb{R}^m}$ is sufficiently small such that $\Pi v := \sum_{i=1}^m \xi_i\phi_i \in W_2(0)$.

On the other hand, for $\mathcal{N} : \mathcal{D} \mapsto \mathrm{V}_0$, by the classical local inversion theorem, we can see that $\mathcal{N}$ is still invertible. A similar argument in [18] shows that there exist a neighborhood $U_1(0)$ of 0 in $\mathrm{V}_2 \cap \mathcal{D}$, and a neighborhood $U_2(0)$ of 0 in V_0, such that

$$\|\mathcal{N}^{-1}(g_1) - \mathcal{N}^{-1}(g_2)\|_{\mathrm{V}_2} \leq C\|g_1 - g_2\|_{\mathrm{V}_0}, \quad \forall g_1, g_2 \in U_2(0), \tag{3.36}$$
$$\|\mathcal{N}(v_1) - \mathcal{N}(v_2)\|_{\mathrm{V}_0} \leq C\|v_1 - v_2\|_{\mathrm{V}_2}, \qquad \forall v_1, v_2 \in U_1(0). \tag{3.37}$$

In particular,

$$\mathcal{N}^{-1} = \Psi = \mathcal{N}_k^{-1}, \quad \text{on } W_2(0) \cap U_2(0). \tag{3.38}$$

In what follows, we will show that for any v satisfying $v \in U_1(0)$ and $\Pi v = \sum_{i=1}^{m} \xi_i \phi_i \in W_2(0) \cap U_2(0)$, there holds

$$|\nabla \mathfrak{J}(\xi)|_{\mathbb{R}^m} \leq C\|M(v)\|_{\mathrm{V}_0}. \tag{3.39}$$

Indeed, a straightforward calculation and integration by parts yield that

$$\begin{aligned}\frac{\partial \mathfrak{J}(\xi)}{\partial \xi_i} &= (M(\Psi(\Pi v)), D\Psi(\Pi v)\phi_i)_{\mathrm{V}_0} \\ &\leq \|M(\Psi(\Pi v))\|_{\mathrm{V}_0}\|D\Psi(\Pi v)\phi_i\|_{\mathrm{V}_0} \\ &\leq C\|M(\Psi(\Pi v))\|_{\mathrm{V}_0} \\ &\leq C(\|M(\Psi(\Pi v)) - M(v)\|_{\mathrm{V}_0} + \|M(v)\|_{\mathrm{V}_0}).\end{aligned} \tag{3.40}$$

Recalling that $v = \mathcal{N}^{-1}(M(v) + \Pi v)$, and using (3.36), (3.37), (3.38), we find that

$$\begin{aligned}&\|M(\Psi(\Pi v)) - M(v)\|_{\mathrm{V}_0} \\ \leq\ &\|\mathcal{N}(\Psi(\Pi v)) - \mathcal{N}(v)\|_{\mathrm{V}_0} + \|\Pi\Psi(\Pi v) - \Pi v\|_{\mathrm{V}_0} \\ \leq\ &C\|\mathcal{N}^{-1}(\Pi v + M(v)) - \Psi(\Pi v)\|_{\mathrm{V}_2} \\ \leq\ &C\|M(v)\|_{\mathrm{V}_0}.\end{aligned} \tag{3.41}$$

Combining (3.40) with (3.41) yields that

$$|\nabla \mathfrak{J}(\xi)|_{\mathbb{R}^m} = \left(\sum_{i=1}^{m} \left|\frac{\partial \mathfrak{J}(\xi)}{\partial \xi_i}\right|^2\right)^{\frac{1}{2}} \leq C\|M(v)\|_{\mathrm{V}_0}. \tag{3.42}$$

We now proceed to estimate $|\mathcal{E}(\Psi(\Pi v)) - \mathcal{E}(v)|$. By the Newton-Leibniz formula, we have

$$|\mathcal{E}(\Psi(\Pi v)) - \mathcal{E}(v)| \leq \left|\int_0^1 (M(v + t(\Psi(\Pi v) - v)), \Psi(\Pi v) - v)_{\mathrm{V}_0} dt\right|. \tag{3.43}$$

Using $v = \mathcal{N}^{-1}(M(v) + \Pi v)$, and referring to (3.41), (3.36), we obtain that

$$\begin{aligned}&|\ \mathcal{E}(\Psi(\Pi v)) - \mathcal{E}(v)\ | \\ \leq\ &\max_{0\leq t\leq 1} \|M(v + t(\Psi(\Pi v) - v))\|_{\mathrm{V}_0}\|\Psi(\Pi v) - v\|_{\mathrm{V}_0} \\ \leq\ &C\|M(v)\|_{\mathrm{V}_0}\left(\max_{0\leq t\leq 1}\|M(v + t(\Psi(\Pi v) - v)) - M(v)\|_{\mathrm{V}_0} + \|M(v)\|_{\mathrm{V}_0}\right) \\ \leq\ &C\|M(v)\|_{\mathrm{V}_0}\left(C\max_{0\leq t\leq 1}\|v + t(\Psi(\Pi v) - v) - v\|_{\mathrm{V}_2} + \|M(v)\|_{\mathrm{V}_0}\right) \\ \leq\ &C\|M(v)\|^2_{\mathrm{V}_0}.\end{aligned} \tag{3.44}$$

Since $\mathfrak{J}(\xi) = \mathcal{E}(\Psi(\Pi v)) :\ \mathbb{R}^m \to \mathbb{R}$ is real analytic for small $|\xi|_{\mathbb{R}^m}$, and since $\nabla\mathfrak{J}(0) = 0$, we have the following Łojasiewicz inequality for analytic functions defined on $\mathbb{R}^m$ (cf. [22]): for $|\xi| < \beta_1$,

$$|\nabla \mathfrak{J}(\xi))|_{\mathbb{R}^m} \geq |\mathfrak{J}(\xi) - \mathfrak{J}(0)|^{1-\theta}, \tag{3.45}$$

where $\theta \in (0, \frac{1}{2})$, $\beta_1 > 0$.
Thus, from (3.42), (3.45) and (3.44) we can infer that

$$\begin{aligned} C\|M(v)\|_{\mathrm{V}_0} &\geq |\nabla \mathfrak{J}(\xi)|_{\mathbb{R}^m} \geq |\mathfrak{J}(\xi) - \mathfrak{J}(0)|^{1-\theta} \\ &= |\mathfrak{J}(\xi) - \mathcal{E}(v) + \mathcal{E}(v) - \mathfrak{J}(0)|^{1-\theta} \\ &\geq \frac{1}{2}|\mathcal{E}(v) - \mathfrak{J}(0)|^{1-\theta} - C|\mathfrak{J}(\xi) - \mathcal{E}(v)|^{1-\theta} \\ &\geq \frac{1}{2}|\mathcal{E}(v) - \mathfrak{J}(0)|^{1-\theta} - C\|M(v)\|_{\mathrm{V}_0}^{2(1-\theta)}. \end{aligned} \tag{3.46}$$

Hence,

$$|\mathcal{E}(v) - \mathfrak{J}(0)|^{1-\theta} \leq \|M(v)\|_{\mathrm{V}_0}\left(2C + 2C\|M(v)\|_{\mathrm{V}_0}^{2(1-\theta)-1}\right). \tag{3.47}$$

By properly choosing smaller θ and $\beta \in (0, \beta_1)$, we can show that for any $v \in \mathcal{D}$ satisfying $\|v\|_{\mathrm{V}_2} < \beta$, it holds

$$\|M(v)\|_{\mathrm{V}_0} \geq |\mathcal{E}(v) - \mathfrak{J}(0)|^{1-\theta} = |E(u) - E(\psi)|^{1-\theta}. \tag{3.48}$$

By the definition of $M(v)$, (3.48) yields

$$\|-\Delta u + f(u)\| + \|-\Delta_{\|}u + \partial_\nu u + u + g(u)\|_{L^2(\Gamma)} \geq |E(u) - E(\psi)|^{1-\theta}. \tag{3.49}$$

The proof is complete. □

4 Convergence to Equilibrium

After the previous preparations, we are ready to finish the proof of Theorem 1.3.

4.1 Convergence to Equilibrium

From (2.12) it is clear that the energy $E(u(t))$ is decreasing with respect to time. On the other hand, (2.18) implies that $E(u(t))$ is bounded from below. Consequently, $E(u(t))$ serves as a Lyapunov functional for system (1.1)–(1.3).

The ω-limit set of $u_0 \in \tilde{\mathrm{V}}_2$ is defined as follows:

$$\omega(u_0) = \{\psi(x) \in \tilde{\mathrm{V}}_2 : \exists \{t_n\}_{n=1}^{\infty},\ t_n \to +\infty, \text{ such that } u(t_n) \to \psi \text{ in } \mathrm{V}_2\}.$$

Then from well-known results on dynamic systems (cf. [29, Lemma I.1.1]), we can conclude the following result.

Lemma 4.1. *The ω-limit set of u_0 is a non-empty compact connected subset in $\tilde{\mathrm{V}}_2$. Furthermore, (i) it is invariant under the nonlinear semigroup $S(t)$ defined by the solution $u(t)$, i.e., $S(t)\omega(u_0) = \omega(u_0)$ for all $t \geq 0$. (ii) $E(u)$ is constant on $\omega(u_0)$. Moreover, $\omega(u_0)$ consists of equilibria.*

We now first prove the convergence of u_t.

Lemma 4.2. *It holds*

$$\lim_{t\to+\infty}\|u_t(t)\|_{\mathrm{V}_0}=0. \tag{4.1}$$

Proof. Using Young's inequality, we infer from (2.22), (2.23), that

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_{\mathrm{V}_0}^2\le C\|u_t\|_{\mathrm{V}_0}^2\le\|u_t\|_{\mathrm{V}_0}^4+C. \tag{4.2}$$

Besides, (2.15) implies that $\int_0^\infty\|u_t(t)\|_{\mathrm{V}_0}^2dt<\infty$. Then we can conclude (4.1) by applying [37, Lemma 6.2.1]. □

We proceed to prove the convergence of u, following a simplified argument introduced in [20], in which the key observation is that after a certain time t_0 the solution u will fall into a small neighborhood of a certain equilibrium ψ, and stay there forever. Since this procedure has by now become a standard argument, we just sketch the proof. There is a sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n\to+\infty$, such that

$$u(t_n)\to\psi,\qquad\text{in }\mathrm{V}_2, \tag{4.3}$$

where $\psi\in\omega(u_0)$ is a certain equilibrium. Therefore,

$$\lim_{t_n\to+\infty}E(u(t_n))=E(\psi). \tag{4.4}$$

On the other hand, since $E(u(t))$ is decreasing in time (cf. (2.12)) and bounded from below, there exists some $E_\infty\in\mathbb{R}$ such that

$$\lim_{t\to+\infty}E(u(t))=E_\infty. \tag{4.5}$$

As a result, $E(\psi)=E_\infty$, and

$$\lim_{t\to+\infty}E(u(t))=E(\psi). \tag{4.6}$$

We aim to show that

$$\lim_{t\to+\infty}\|u(t)-\psi\|_{\mathrm{V}_2}=0. \tag{4.7}$$

To this end, we consider all possibilities.

(1). If there is a $t_0>0$ such that $E(u(t_0))=E(\psi)$, then we deduce from (2.12) that u is independent of time for all $t>t_0$. Since (4.3) holds, we conclude (4.7).

(2). If there is some $t_0>0$ such that u satisfies the condition of Lemma 3.1 for all $t\ge t_0$, i.e., $\|u(t)-\psi\|_{\mathrm{V}_2}<\beta$, then with the constant $\theta\in(0,\frac{1}{2})$ introduced in Lemma 3.1, we have

$$\frac{d}{dt}(E(u(t))-E(\psi))^\theta=\theta(E(u(t))-E(\psi))^{\theta-1}\frac{dE(u(t))}{dt}. \tag{4.8}$$

Combining this with (3.29), (2.12) yields that

$$\frac{d}{dt}(E(u(t)) - E(\psi))^\theta + \frac{\theta}{2}\|u_t\|_{\mathrm{V}_0} \ \le \ 0. \tag{4.9}$$

Integrating from t_0 to t, we find that

$$(E(u(t)) - E(\psi))^\theta + \frac{\theta}{2}\int_{t_0}^{t} \|u_t(\tau)\|_{\mathrm{V}_0} d\tau \le (E(u(t_0)) - E(\psi))^\theta. \tag{4.10}$$

Since $E(u(t)) - E(\psi) \ge 0$, we have

$$\int_{t_0}^{\infty} \|u_t(\tau)\|_{\mathrm{V}_0} d\tau < +\infty, \tag{4.11}$$

which yields that $u(t)$ converges in V_0 as $t \to +\infty$. Because the orbit is compact in V_2 (see Lemma 2.4), we can deduce from uniqueness of limit that (4.7) holds.

(3). It follows from (4.3) that for any $\varepsilon > 0$ with $\varepsilon < \beta$, there exists an integer N such that for $n \ge N$, it holds

$$\|u(t_n) - \psi\|_{\mathrm{V}_0} \le \|u(t_n) - \psi\|_{\mathrm{V}_2} < \frac{\varepsilon}{2}, \tag{4.12}$$

$$\frac{1}{\theta}(E(u(t_n)) - E(\psi))^\theta < \frac{\varepsilon}{4}. \tag{4.13}$$

Define

$$\bar{t}_n := \sup\{\ t > t_n \mid \ \|u(s) - \psi\|_{\mathrm{V}_2} < \beta, \ \forall s \in [t_n, t]\}. \tag{4.14}$$

It follows from (4.12), and from the continuity of the orbit in V_2 for $t > 0$, that $\bar{t}_n > t_n$ for all $n \ge N$. Then there are two possibilities:

(i). If there exists some $n_0 \ge N$ such that $\bar{t}_{n_0} = +\infty$, then, from the previous arguments in (1) and (2), the theorem is proved.

(ii) Otherwise, for all $n \ge N$, we have $t_n < \bar{t}_n < +\infty$. Moreover, for all $t \in [t_n, \bar{t}_n]$, $E(\psi) < E(u(t))$. Then from (4.10), with t_0 being replaced by t_n, and t being replaced by $\bar{t}_n$, we deduce that

$$\int_{t_n}^{\bar{t}_n} \|u_t(\tau)\|_{\mathrm{V}_0} d\tau \le \frac{2}{\theta}(E(u(t_n)) - E(\psi))^\theta < \frac{\varepsilon}{2}. \tag{4.15}$$

Thus, it follows that

$$\|u(\bar{t}_n) - \psi\|_{\mathrm{V}_0} \le \ \ \|u(t_n) - \psi\|_{\mathrm{V}_0} + \int_{t_n}^{\bar{t}_n} \|u_t(\tau)\|_{\mathrm{V}_0} d\tau < \varepsilon, \tag{4.16}$$

which implies that for $n \to +\infty$, it holds

$$u(\bar{t}_n) \to \psi \quad \text{in} \ \ \mathrm{V}_0. \tag{4.17}$$

Since $\bigcup_{t\ge\delta} u(t)$ is relatively compact in V_2, there exists a subsequence of $\{u(\bar{t}_n)\}$, still denoted by $\{u(\bar{t}_n)\}$, that converges to ψ in V_2. Namely, if n is sufficiently large,

$$\|u(\bar{t}_n) - \psi\|_{\mathrm{V}_2} \ < \beta, \tag{4.18}$$

which contradicts the fact that by the definition of $\bar{t}_n$ it holds $\|u(\bar{t}_n) - \psi\|_{\mathrm{V}_2} = \beta$.

Summing up, we have proved the strong convergence of u in V_2 (cf. (4.7)). □

4.2 Estimates for the Convergence Rate

In this subsection, we shall show the estimate (1.11) for the convergence rate. This can be achieved in several steps.

Step 1. V_0-estimate. As has been shown in the literature (cf., for instance, [19,37]), an estimate for the convergence rate in a certain lower-order norm can be obtained directly from the Łojasiewicz-Simon approach. From Lemma 3.1 and (4.9), we have

$$\frac{d}{dt}(E(u(t)) - E(\psi)) + C(E(u(t)) - E(\psi))^{2(1-\theta)} \le 0, \quad \forall\, t \ge t_0, \tag{4.19}$$

which implies that

$$E(u(t)) - E(\psi) \le C(1+t)^{-\frac{1}{1-2\theta}}, \quad \forall\, t \ge t_0. \tag{4.20}$$

Integrating (4.9) over (t, ∞), where $t \ge t_0$, it follows that

$$\int_t^\infty \|u_t(\tau)\|_{\mathrm{V}_0} d\tau \le C(1+t)^{-\frac{\theta}{1-2\theta}}. \tag{4.21}$$

By adjusting the constant C properly, we obtain that

$$\|u(t) - \psi\|_{\mathrm{V}_0} \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad t \ge 0. \tag{4.22}$$

Step 2. V_1-estimate. Subtracting the stationary problem (1.12) from the evolution equations (1.1)–(1.2), we get

$$(u-\psi)_t - \Delta(u-\psi) + f(u) - f(\psi) = 0, \tag{4.23}$$

$$-\Delta_{||}(u-\psi) + \partial_\nu(u-\psi) + (u-\psi) + g(u) - g(\psi) + (u-\psi)_t = 0. \tag{4.24}$$

Multiplying (4.23) by u_t, and integrating over Ω, we obtain that

$$\begin{aligned}\frac{d}{dt}\Bigg[\frac{1}{2}\|u-\psi\|_{\mathrm{V}_1}^2 &+ \int_\Omega (F(u) - F(\psi) - f(\psi)u + f(\psi)\psi)\, dx \\ &+ \int_\Gamma (G(u) - G(\psi) - g(\psi)u + g(\psi)\psi)\, dS\Bigg] + \|u_t\|_{\mathrm{V}_0}^2 = 0.\end{aligned} \tag{4.25}$$

Multiplying (4.23) by $u-\psi$, and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-\psi\|_{\mathrm{V}_0}^2 + \|u-\psi\|_{\mathrm{V}_1}^2 &+ \int_\Omega (f(u) - f(\psi))(u-\psi)dx \\ &+ \int_\Gamma (g(u) - g(\psi))(u-\psi)dS = 0.\end{aligned} \tag{4.26}$$

The Newton-Leibniz formula

$$F(u) = F(\psi) + f(\psi)(u-\psi) + \int_0^1\int_0^1 f'(szu + (1-sz)\psi)(u-\psi)^2 ds dz \tag{4.27}$$

yields that

$$\begin{aligned}
&\left|\int_\Omega (F(u)-F(\psi)-f(\psi)u+f(\psi)\psi)\,dx\right| \\
= &\left|\int_\Omega\int_0^1\int_0^1 f'(szu+(1-sz)\psi)(u-\psi)^2 dsdzdx\right|. \qquad (4.28)
\end{aligned}$$

Let $\xi=szu+(1-sz)\psi$, $(s,z\in[0,1])$.
(1) If $n=2,3$, then it follows from Remark 2.1 that

$$\|f'(\xi)\|_{L^\infty(\Omega)}\le C. \qquad (4.29)$$

Therefore,

$$\begin{aligned}
&\left|\int_\Omega (F(u)-F(\psi)-f(\psi)u+f(\psi)\psi)\,dx\right| \\
\le &\max_{s,z\in[0,1]}\|f'(szu+(1-sz)\psi)\|_{L^\infty(\Omega)}\|u-\psi\|^2\le C\|u-\psi\|^2. \qquad (4.30)
\end{aligned}$$

(2) If $n=4$, we have

$$\begin{aligned}
&\left|\int_\Omega (F(u)-F(\psi)-f(\psi)u+f(\psi)\psi)\,dx\right| \\
\le\ &\|f'(\xi)\|_{L^4(\Omega)}\|u-\psi\|_{L^4(\Omega)}\|u-\psi\| \\
\le\ &C(1+\|\xi\|^2_{L^8(\Omega)})\|u-\psi\|_{H^1(\Omega)}\|u-\psi\| \\
\le\ &C(\|\xi\|_{H^2(\Omega)})\|u-\psi\|_{H^1(\Omega)}\|u-\psi\| \\
\le\ &\frac{1}{8}\|u-\psi\|^2_{\mathrm{V}_1}+C\|u-\psi\|^2. \qquad (4.31)
\end{aligned}$$

(3) If $n>4$, then it follows that

$$\left|\int_\Omega (F(u)-F(\psi)-f(\psi)u+f(\psi)\psi)\,dx\right|\le\|f'(\xi)\|_{L^{\frac{n(n-2)}{2(n-4)}}(\Omega)}\|u-\psi\|^2_{L^{\frac{2n(n-2)}{n^2-4n+8}}(\Omega)} \qquad (4.32)$$

From (**F2**), we have

$$\|f'(\xi)\|_{L^{\frac{n(n-2)}{2(n-4)}}(\Omega)}\le C\left(1+\|\xi\|^{\frac{4}{2-n}}_{L^{\frac{2n}{n-4}}(\Omega)}\right)\le C(\|\xi\|_{H^2(\Omega)}). \qquad (4.33)$$

By the Gagliardo-Nirenberg inequality,

$$\|u-\psi\|_{L^{\frac{2n(n-2)}{n^2-4n+8}}(\Omega)}\le C\|u-\psi\|^{\frac{n-4}{n-2}}_{H^1(\Omega)}\|u-\psi\|^{\frac{2}{n-2}}. \qquad (4.34)$$

Consequently,

$$\begin{aligned}
&\left|\int_\Omega (F(u)-F(\psi)-f(\psi)u+f(\psi)\psi)\,dx\right| \\
\le\ &C\|u-\psi\|^{\frac{2(n-4)}{n-2}}_{H^1(\Omega)}\|u-\psi\|^{\frac{4}{n-2}}\le\frac{1}{8}\|u-\psi\|^2_{\mathrm{V}_1}+C\|u-\psi\|^2. \qquad (4.35)
\end{aligned}$$

Summing up, in all the cases (1), (2), (3), we have

$$\left|\int_\Omega (F(u)-F(\psi)-f(\psi)u+f(\psi)\psi)\,dx\right| \le \frac{1}{8}\|u-\psi\|^2_{\mathrm{V}_1}+C\|u-\psi\|^2. \tag{4.36}$$

Similarly, we can show that

$$\begin{aligned}\left|\int_\Omega (f(u)-f(\psi))(u-\psi)dx\right| &= \left|\int_\Omega\int_0^1 f'(su+(1-s)\psi)(u-\psi)^2dsdx\right|\\ &\le \frac{1}{8}\|u-\psi\|^2_{\mathrm{V}_1}+C\|u-\psi\|^2,\end{aligned} \tag{4.37}$$

and

$$\left|\int_\Gamma (G(u)-G(\psi)-g(\psi)u+g(\psi)\psi)\,dS\right| \le \frac{1}{8}\|u-\psi\|^2_{\mathrm{V}_1}+C\|u-\psi\|^2_{L^2(\Gamma)}, \tag{4.38}$$

$$\left|\int_\Gamma (g(u)-g(\psi))(u-\psi)dS\right| \le \frac{1}{8}\|u-\psi\|^2_{\mathrm{V}_1}+C\|u-\psi\|^2_{L^2(\Gamma)}. \tag{4.39}$$

Let

$$\begin{aligned} y_1(t) &= \frac{1}{2}\|u(t)-\psi\|^2_{\mathrm{V}_0}+\frac{1}{2}\|u(t)-\psi\|^2_{\mathrm{V}_1}\\ &\quad+\int_\Omega (F(u(t))dx-F(\psi)+f(\psi)\psi dx-f(\psi)u(t))\,dx\\ &\quad+\int_\Gamma (G(u(t))-G(\psi)-g(\psi)u(t)+g(\psi)\psi)\,dS.\end{aligned} \tag{4.40}$$

(4.36) indicates that there exists a constant $C_1>0$ such that

$$y_1(t)\ge \frac{1}{4}\|u(t)-\psi\|^2_{\mathrm{V}_1}-C_1\|u(t)-\psi\|^2_{\mathrm{V}_0}. \tag{4.41}$$

Adding (4.25) and (4.26), we can conclude that

$$\frac{d}{dt}y_1(t)+C_2y_1(t)+\|u_t(t)\|^2_{\mathrm{V}_0}\le C_3\|u(t)-\psi\|^2_{\mathrm{V}_0}. \tag{4.42}$$

By Gronwall's inequality, we have (cf. [30, 33])

$$y_1(t)\le C(1+t)^{-\frac{2\theta}{1-2\theta}},\quad \forall\, t\ge 0, \tag{4.43}$$

which, together with (4.21) and (4.41), yields that

$$\|u(t)-\psi\|_{\mathrm{V}_1}\le C(1+t)^{-\frac{\theta}{1-2\theta}},\quad \forall\, t\ge 0. \tag{4.44}$$

Step 3. V_2-estimate. Multiplying (2.22) by $\eta>0$, and adding the result to (4.42), we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\eta}{2}\|u_t\|^2_{\mathrm{V}_0}+y_1(t)\right)+C_2y_1(t)+\eta\|u_t\|^2_{\mathrm{V}_1}+(1-\eta\max\{C_f,C_g\})\|u_t\|^2_{\mathrm{V}_0}\\ \le\;& C_3\|u-\psi\|^2_{\mathrm{V}_0}.\end{aligned} \tag{4.45}$$

After taking $\eta = \frac{1}{2\max\{C_f, C_g\}}$, we can use a similar argument as for the estimate of $y_1(t)$ to conclude that

$$\frac{\eta}{2}\|u_t(t)\|^2_{\mathrm{V}_0} + y_1(t) \le C(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \ge 0. \tag{4.46}$$

This and (4.41) imply that

$$\|u_t(t)\|_{\mathrm{V}_0} \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \ge 0. \tag{4.47}$$

By the elliptic estimate for (4.23)-(4.24) (cf. Lemma 2.1), we have

$$\begin{aligned} \|u(t)-\psi\|_{\mathrm{V}_2} &\le C(\|u_t(t) + f(u(t)) - f(\psi)\| + \|u_t(t) + g(u(t)) - g(\psi)\|_{L^2(\Gamma)}) \\ &\le C(\|u_t(t)\|_{\mathrm{V}_0} + \|f(u(t)) - f(\psi)\| + \|g(u(t)) - g(\psi)\|_{L^2(\Gamma)}). \end{aligned} \tag{4.48}$$

Applying the same argument as in Lemma 2.2, we find that (cf. (2.47))

$$\|u(t)-\psi\|_{\mathrm{V}_2} \le C(\|u_t(t)\|_{\mathrm{V}_0} + \|u(t)-\psi\|_{\mathrm{V}_1}) \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \ge 0. \tag{4.49}$$

The proof of Theorem 1.3 is complete. □

References

[1] L. Cherfils and A. Miranville, On the Caginalp system with dynamic boundary conditions and singular potentials, Appl. Math., to appear, 2008.

[2] R. Chill, E. Fasangova and J. Pruess, Convergence to steady states of solutions of the Cahn-Hilliard and Caginalp equations with dynamic boundary conditions, Math. Nachr., **13** (2006), 1448–1462.

[3] A. Favini, G.R. Goldstein, J.A. Goldstein and S. Romanelli, Nonlinear boundary conditions for nonlinear second order differential operators on $C[0,1]$, Arch. Math. (Basel), **76** (2001), 391–400.

[4] A. Favini, G.R. Goldstein, J.A. Goldstein and S. Romanelli, The heat equation with generalized Wentzell boundary condition, J. Evol. Equ., **2** (2002), 1–19.

[5] A. Favini, G.R. Goldstein, J.A. Goldstein, S. Romanelli, The heat equation with nonlinear general Wentzell boundary condition, Adv. Differential Equations, **11** (2006), 481–510.

[6] C. Gal, Global Well-posedness for the non-isothermal Cahn-Hilliard with dynamical boundary conditions, Adv. Differential Equations, **12**(11) (2007), 1241–1274.

[7] C. Gal, Well-posedness and long time behavior of the non-isothermal viscous Cahn-Hilliard equation with dynamic boundary conditions, Dynamics of PDE, **5**(1) (2008), 39–67.

[8] C. Gal and M. Grasselli, The non-isothermal Allen-Cahn equation with dynamical boundary condition, Disc. Conti. Dyna. Sys., **22**(4) (2008), 1009–1040.

[9] C. Gal and M. Grasselli, On the asymptotic behavior of the Cagnalp system with dynamical boundary conditions, to appear in Comm. Pure Anal. Appl., 2008.

[10] C. Gal, M. Grasselli and A. Miranville, Robust exponential attractors for singularly perturbed phase-field equations with dynamic boundary conditions, to appear in Nonlinear Diff. Eqns. Appl., 2008.

[11] C. Gal, M. Grasselli and A. Miranville, Non-isothermal Allen-Cahn equations with coupled dynamic boundary conditions, preprint, 2008.

[12] C. Gal, G.R. Goldstein, J.A. Goldstein and S. Romanelli, Fredholm alternative, semilinear elliptic problems and Wentzell boundary conditions, preprint, 2008.

[13] C. Gal and A. Miranville, Uniform global attractors for non-isothermal Cahn-Hilliard equations with dymanical boundary conditions, to appear in Nonlinear Analysis: Real World Applications, 2008.

[14] C. Gal and H. Wu, Asymptotic behavior of a Cahn-Hilliard equation with Wentzell boundary conditions and mass conversation, Disc. Conti. Dyna. Sys., **22**(4) (2008), 1041–1063.

[15] S. Gatti and A. Miranville, Asymptotic behavior of a phase-field system with dynamic boundary conditions, in 'Differential Equations: Inverse and Direct Problems'(A. Favini, A. Lorenzi, Eds.), Ser. Lect. Notes Pure Appl. Math. **251**, 149–170, Chapman & Hall/CRC, Boca Raton, 2006.

[16] M. Grasselli, H. Wu and S. Zheng, Asymptotic behavior of a non-isothermal Ginzburg-Landau model, Quart. Appl. Math., 2008, PII: S0033-569X-08-01115-9.

[17] A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Masson, Paris, 1991.

[18] A. Haraux and M.A. Jendoubi, Convergence of bounded weak solutions of the wave equation with dissipation and analytic nonlinearity, Calc. Var., **9** (1999), 95–124.

[19] A. Haraux and M.A. Jendoubi, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity, Asymptot. Anal., **26** (2001), 21–36.

[20] M.A. Jendoubi, A simple unified approach to some convergence theorem of L. Simon, J. Func. Anal., **153** (1998), 187–202.

[21] R. Kenzler, F. Eurich, P. Maass, B. Rinn, J. Schropp, E. Bohl and W. Dieterich, Phase separation in confined geometries: Solving the Cahn-Hilliard equation with generic boundary conditions, Computer Phys. Comm., **133** (2001), 139–157.

[22] S. Łojasiewicz, Une propriété topologique des sous-ensembles analytiques reés. Colloque Internationaux du C.N.R.S. **117**, Les equations aux derivées parielles (1963), 87–89.

[23] A. Miranville and S. Zelik, Exponential attractors for the Cahn-Hilliard equation with dynamical boundary conditions, Math. Meth. Appl. Sci., **28** (2005), 709–735.

[24] P. Mironescu and V. Radulescu, Nonlinear Sturm-Liouville type problems with finite number of solutions, Matarom, Laboratoire d'Analyse Numerique, Universite Pierre et Marie Curie (Paris VI), **3** (1993), 54–67.

[25] P. Polačik and F. Simondon, Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains, J. Diff. Eqns., **186** (2002), 586–610.

[26] R. Racke and S. Zheng, The Cahn-Hilliard equation with dynamical boundary conditions, Adv. Diff. Eqs., **8**(1) (2003), 83–110.

[27] L. Simon, Asymptotics for a class of nonlinear evolution equation with applications to geometric problems, Ann. of Math., **118** (1983), 525–571.

[28] K. Taira, *Semigroups, boundary value problems, and Markov process*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, 2004.

[29] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics,* Appl. Math. Sci., **68**, Springer-Verlag, New York, 1988.

[30] H. Wu, Convergence to equilibrium for a Cahn-Hilliard model with the Wentzell boundary condition, Asymptot. Anal., **54** (2007), 71–92.

[31] H. Wu, Convergence to equilibrium for the semilinear parabolic equation with dynamic boundary condition, Adv. Math. Sci. Appl., **17**(1) (2007), 67–88.

[32] H. Wu, Long-time behavior for a nonlinear plate equation with thermal memory, J. Math. Anal. Appl., **348** (2008), 650–670.

[33] H. Wu, M. Grasselli and S. Zheng, Convergence to equilibrium for a parabolic-hyperbolic phase-field system with Neumann boundary conditions, Math. Models Methods Appl. Sci., **17**(1) (2007), 1–29.

[34] H. Wu and S. Zheng, Convergence to equibrium for the Cahn-Hilliard equation with dynamic boundary conditions, J. Diff. Eqns., **204** (2004), 511–531.

[35] H. Wu and S. Zheng, Convergence to equilibrium for the damped semilinear wave equation with critical exponent and dissipative boundary condition, Quart. Appl. Math., **64**(1) (2006), 167–188.

[36] E. Zeidler, *Nonlinear functional analysis and its applications, I.*, Springer, New York, Berlin, Heidelberg, 1986.

[37] S. Zheng, *Nonlinear evolution equations*, Pitman Monographs and Surveys in Pure and Applied Mathematics, **133**, CHAPMAN & HALL/CRC, Boca Raton, Florida, 2004.