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A stochastic fractal model of the Universe related to the fractional Laplacian

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Abstract

A new stochastic fractal model based on a fractional Laplace equation is developed. Exact representation for the spectral and correlation functions under random boundary excitation are obtained. Randomized spectral expansion is constructed for simulation of the solution of the fractional Laplace equation. We present calculations for 2D and 3D spaces for a series of fractional parameters showing a strong memory effect: the decay of correlations is several order of magnitudes less compared to the conventional Laplace equation model.

1 Introduction

The fractal concept in cosmography is based on the assumption that the Universe reveals a spatial self-similarity with respect to scale, known also as scaling [14]. This concept in a sense extends the homogeneity. Clearly, an infinite homogeneous medium possesses the scaling property since it is homogeneous at all scales. However there exist non-homogeneous self-similar systems, called fractals, where more and more structures appear at larger scales in a self-similar way so that all the structures are similar to the one at small scales. The next step in the generalization of the fractal idea is the definition of a random fractal [12]. It is introduced by the concept of a probability density function (pdf) characterizing the fluctuations of a spatial structure, e.g., the density of mass. The fractality in this case is treated via a stochastic similarity of structures, so for instance when analyzing random fractals of a set of points, one assumes that a normalized number of objects in a volume has the same pdf at all distances, and all points are statistically equivalent, while the mean number of objects in a ball of radius R is proportional to R^D where D is a fractal dimension. There are a few estimates of the fractal dimension of the Universe, all of them lie between 2 and 3.

The mean density of stochastic fractals is postulated to have a power asymptotic, $f(\mathbf{x}) = Ar^{D-3}$, where A is some normalizing constant, and r is the distance to some fixed point (say, a mass center“) of the fractal. Applying a Fourier transform to $f(\mathbf{x})$ one can find that the mean density satisfies a fractional Laplace equation (e.g., see [14], [12], [21]):

$$(-\Delta)^{D/2} f(\mathbf{x}) = A_D \delta(\mathbf{x}), \quad \mathbf{x} \in G = \mathbb{R}^n \tag{1}$$

with $A_D = 4\pi\,\Gamma(D-1)\,|\cos(\pi D/2)|$, where $\Gamma(\cdot)$ is the Euler Γ-function. This equation implies that $f(\mathbf{x})$ is a fundamental solution of the fractional Laplace equation. It should be noted that this function is often called a Green function even if G is not the whole space (e.g., say, [15], [25]) say, a half-space $\mathbb{R}^n_+$, which leads to misunderstanding between physicists and mathematicians, because, in mathematics, in addition to (1), the Green function should satisfy a boundary condition, say, in the case of a Dirichlet boundary conditions, $f|_{\partial G} = 0$. One might be interested to define uniquely the Green function from (1) and the boundary condition, by analogy to the classical Laplace equation, but here we face a difficulty related to the fact that the pseudo-differential equations (like (1)) are of non-local type, and the theory of existence and uniqueness of boundary value problems is not developed yet (e.g., see [15], [2], [4]).

In this paper we undertake a different approach, which defines a fractional Green function for the half-space with a given Dirichlet boundary condition via an extension in a fractional space. This enables to uniquely define the Green function, and to evaluate explicitly the correlation function of the solution in response to the stochastic white-noise fluctuations on the boundary. The derivation is based on a generalized Poisson formula for the half-space. We construct also simple Monte Carlo simulation formulae of the random solution itself which opens new possibilities in studying more complicated statistical characteristics like many point statistical moments, and the probability that the solution exceeds a fixed critical level. We mention in conclusion that the stochastic correlation analysis of PDEs with random boundary conditions is used in different fields of science and technology, for instance, flows in porous media [9], [18], turbulence [5], [10], diffusion and transport [23], [22], elasticity and elastography [13], [19], [20], and many others (e.g., see the bibliography in [7] and [19]). The present paper is the first study extending this approach to a stochastic fractional PDE, with the relevant applications to stochastic fractal models.

1.1 Fractional Laplacian

Suppose we are given a smooth bounded function $\mathbb{R}^n \to \mathbb{R}$ to be extended smoothly to the space $\mathbb{R}^n \times (0, \infty)$ so that

$$\Delta u(\mathbf{x}, y) = 0, \quad \mathbf{x} \in \mathbb{R}^n, \qquad y > 0 \;, u(\mathbf{x}, 0) = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{\mathbf{n}} \;. \tag{2}$$

The fractional Laplacian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as follows (e.g., see ([3], [25]):

$$(-\Delta)^s f(\mathbf{x}) = C_{n,s} \int\limits_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^{n+2s}} \, d\boldsymbol{\xi} \tag{3}$$

where the parameter s is a real number between 0 and 1, and $C_{n,s}$ is some normalization constant.

This definition can be replaced with the definition commonly used in treating the pseudo-differential equations using the Fourier transform [24]

$$\widehat{(-\Delta)^s} f(\mathbf{k}) = |\xi|^{2s} \hat{f}(\mathbf{k}) \;.$$

Here the Fourier transforms $f(\mathbf{x}) = F[\hat{f}](\mathbf{x})$ and $\hat{f}(\mathbf{k}) = F^{-1}[f](\mathbf{k})$ are related by

$$\hat{f}(\mathbf{k}) = \int\limits_{\mathbb{R}^n} e^{-\mathrm{i}\, 2\pi (\mathbf{x} \cdot \mathbf{k})} f(\mathbf{x}) \, d\mathbf{x}, \quad f(\mathbf{x}) = \int\limits_{\mathbb{R}^n} e^{\mathrm{i}\, 2\pi (\mathbf{x} \cdot \mathbf{k})} \hat{f}(\mathbf{k}) \, d\mathbf{k} \;.$$

The fractional Laplacian is also defined in a distributional sense for functions that are not differentiable as long as $\hat{f}$ is not too singular at the origin, i.e., provided

$$\int\limits_{\mathbb{R}^n} \frac{|f(\mathbf{x})|}{(1 + |\mathbf{x}|)^{n+2s}} \, d\mathbf{x} < \infty \;.$$

The relation of the fractional Laplacian to the above extension problem (2) is suggested in [3] on the basis of the following arguments. The derivative $-u_y(\mathbf{x},0)$ can be formally identified with $(-\Delta)^{1/2}f(\mathbf{x})$, hence the operator $T: f \to -u_y(\mathbf{x},0)$ can be considered as a realization of $(-\Delta)^{1/2}f(\mathbf{x})$. This is easy to show by applying the operator T twice. First, replacing the Dirichlet condition f with $-u_y(\mathbf{x},0)$ we obtain $-u_y(\mathbf{x},y)$ instead of u as the solution of the problem (2). Then,

$$T(T(f))(\mathbf{x}))(\mathbf{x}) = T(-u_y(\mathbf{x},0))(\mathbf{x}) = u_{yy}(\mathbf{x},0) = -\Delta_{\mathbf{x}} f(\mathbf{x}) \ .$$

Thus to show that $T = (-\Delta)^{1/2}$ it remains only to mention that T is a positive operator which is easy to check by a simple integration by parts.

An analogous extension problem for fractional Laplacians is formulated as follows.

For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the extension $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ that satisfies the equation

$$\Delta_{\mathbf{x}} u + \frac{a}{y} u_y + u_{yy} = 0, \qquad u(\mathbf{x},0) = f(\mathbf{x}) \ . \tag{4}$$

The equation (4) can also be written as

$$\mathrm{div}\ (y^a \nabla\, u) = 0 \ .$$

This in turn is the Euler-Lagrange equation for the functional

$$J(u) = \int\limits_{y>0} |\nabla\, u|^2\, y^a\, d\mathbf{x}\, dy \ .$$

To obtain a fundamental solution to (4), we introduce the notation $X = (\mathbf{x}, y)$ and consider the fundamental solution in $n+1+a$ dimensions. For $n-1+a > 1$ we know that (e.g., see [3]):

$$\mathcal{E}_a(X) = C_{n+1+a} \frac{1}{|X|^{n-1+a}}$$

where the constant C_{n+1+a} is defined by $C_k = \pi^{k/2}\Gamma(k/2-1)/4$. For $n = 1$, the logarithmic function should be taken. By direct evaluations we can see that $\mathcal{E}$ is a solution of (4) when $y \neq 0$, and $\lim_{y\to 0^+} y^a u_y = -C\delta(0)$ for some constant C, hence $\mathcal{E}_a(\mathbf{x},0) = \frac{C}{|x|^{n-1+a}}$ is the fundamental solution of the fractional Laplace $(-\Delta)^{\frac{1-a}{2}}$ for some appropriate constant C depending on n and a.

Thus the above extension problem (4) is related to the fractional Laplacian as follows (up to a constant factor):

$$\lim_{y\to 0} y^a u_y(\mathbf{x},y) = -(-\Delta)^s\, f(\mathbf{x}) = \int\limits_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^{n+2s}}\, d\boldsymbol{\xi} \ .$$

This can be shown using the Poisson integral formula which gives the exact representation of the solution of the extension problem (4):

$$u(\mathbf{x},y) = \int\limits_{\mathbb{R}^n} P_a(\mathbf{x} - \boldsymbol{\xi}, y) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \tag{5}$$

where the Poisson kernel P_a must be a solution to (4) where $y > 0$ and $\lim_{y\to 0} P_a(\mathbf{x}, y) = \delta(0)$, so the correct choice is $P_a(\mathbf{x}, y) = -y^{-a}\partial_y \mathcal{E}_a(\mathbf{x}, y)$, hence,

$$P_a(\mathbf{x}, y) = C_{n,a}\,\frac{y^{1-a}}{\left(|\mathbf{x}|^2 + |y|^2\right)^{\frac{n+1-a}{2}}}\ .$$

The normalizing constant $C_{n,a}$ is chosen to be consistent with the dimension of the problem so that $\int\limits_{\mathbb{R}^n} P_a(\mathbf{x}, y)d\mathbf{x} = 1$. We will show in the Theorem below that

$$C_{n,a} = \frac{\Gamma\left(\frac{n+1-a}{2}\right)}{\pi^{\frac{n+1-a}{2}}} \times \frac{\pi^{\frac{1-a}{2}}}{\Gamma\left(\frac{1-a}{2}\right)} = \frac{\Gamma\left(\frac{n+1-a}{2}\right)}{\pi^{\frac{n}{2}}\,\Gamma\left(\frac{1-a}{2}\right)}\ .$$

2 Boundary white noise

Suppose the given boundary function $f(\mathbf{x})$ is a homogeneous random field with its correlation function $B_f(\mathbf{x}_2 - \mathbf{x}_1)$. We are interested in the structure of the correlation function of the solution $B_u(X_1, X_2) = \langle u(X_1)u(X_2)\rangle$. Recall that we use the notation $X = (\mathbf{x}, y)$.

Let us start with the case when f is a zero mean Gaussian white noise, with $B_f = \delta(\mathbf{x}_2 - \mathbf{x}_1)$.

Theorem. *The random field $u(X)$, $X \in \left(\mathbb{R}^n \times [0, \infty)\right)$ solving the equation (4) with the Gaussian white noise f is partially isotropic, that is, its correlation function depends on $x = |\mathbf{x}_1 - \mathbf{x}_2|$, and on y_1, y_2. It is uniquely defined by its correlation function which has the form*

$$B_u(X_1, X_2) = C_{n,a}^2\, A(y_1, y_2) \int\limits_0^1 \frac{t^{\frac{n-a-1}{2}}(1-t)^{\frac{n-a-1}{2}}\,dt}{\left[y_1^2 t + y_2^2(1-t) + t(1-t)|\mathbf{x}_1 - \mathbf{x}_2|^2\right]^{\frac{n}{2}+1-a}} \tag{6}$$

where

$$A(y_1, y_2) = \frac{\pi^{n/2}\,\Gamma\left(\frac{n}{2} + 1 - a\right)}{\left[\Gamma\left(\frac{n+1-a}{2}\right)\right]^2}\, y_1^{1-a} y_2^{1-a}\ .$$

The partial spectral function is explicitly given by

$$S(\mathbf{k}; y_1, y_2) = C^2\,(|\mathbf{k}|y_1)^{\frac{1-a}{2}}\,(|\mathbf{k}|y_2)^{\frac{1-a}{2}}\,K_{\frac{1-a}{2}}(2\pi|\mathbf{k}|y_1)\,K_{\frac{1-a}{2}}(2\pi|\mathbf{k}|y_2) \tag{7}$$

where $C = 2\pi^{\frac{1-a}{2}}/\Gamma(\frac{1-a}{2})$, and $K_\nu(z)$ is the modified Bessel function of the second kind, also known as the Macdonalds function (e.g., see [8]): $K_\nu(z) = \frac{\pi}{2}\,\frac{I_\nu - I_\nu(z)}{\sin(\nu\pi}$), where $I_\nu(z) = \mathrm{i}^\nu J_\nu(\mathrm{i}z)$ is the modified Bessel function in turn defined via J_ν, the Bessel function of a pure imaginary argument iz.

The random field $u(\mathbf{x}, y)$ has the following Randomized Spectral approximation:

$$u(\mathbf{x}, y) \approx V(\mathbf{x}, y) = C\,\frac{(|\mathbf{k}|y)^{\frac{1-a}{2}}\,K_{\frac{1-a}{2}}(2\pi\,|\mathbf{k}|y)}{\sqrt{p(\mathbf{k})}}\left[\xi\cos(2\pi\mathbf{k}\cdot\mathbf{x}) + \eta\sin(2\pi\mathbf{k}\cdot\mathbf{x})\right] \tag{8}$$

where $\mathbf{k}$ is a random vector in $\mathbb{R}^n$ distributed with a density $p(\mathbf{k})$ which can be chosen quite arbitrarily, satisfying the condition $p(\mathbf{k}) \neq 0$ if $K_{\frac{1-a}{2}}(|k|y) \neq 0$, and ξ, η are independent *standard Gaussian random variables. Since the random field is isotropic, it is convenient to choose* $\mathbf{k} = \kappa\boldsymbol{\omega}$ *where* $\boldsymbol{\omega}$ *is a random unit isotropic vector in* $I\!R^n$*, and* $\kappa = |\mathbf{k}|$ *is a random variable distributed in* $(0,\infty)$ *with a density* $\frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}\, p(\kappa)\kappa^{n-1}$*. The approximation* $u(\mathbf{x},y) \approx V(\mathbf{x},y)$ *means that the correlation functions of* $u(\mathbf{x},y)$ *and* $V(\mathbf{x},y)$ *coincide, and moreover, a sum of* N *independent realizations of* $V(\mathbf{x},y)$ *converges to the solution* $u(\mathbf{x},y)$*, as* $N \to \infty$*.*

Proof. Let us first show that (6) is true. By (5) we get

$$
\begin{aligned}
B_u(X_1,X_2) &= \langle u(\mathbf{x}_1,y_1)u(\mathbf{x}_2,y_2)\rangle \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} P_a(\mathbf{x}_1-\boldsymbol{\xi},y_1)P_a(\mathbf{x}_2-\boldsymbol{\xi}',y_2)\langle f(\boldsymbol{\xi})f(\boldsymbol{\xi}')\rangle\, d\boldsymbol{\xi}\, d\boldsymbol{\xi}' \\
&= C_{n,a}^2 \int\limits_{\mathbb{R}^n} \frac{y_1^{1-a}\, y_2^{1-a}\, d\boldsymbol{\xi}}{\left(|\mathbf{x}_1-\boldsymbol{\xi}|^2+|y_1|^2\right)^{\frac{n+1-a}{2}}\left(|\mathbf{x}_2-\boldsymbol{\xi}|^2+|y_2|^2\right)^{\frac{n+1-a}{2}}} \qquad (9)\\
&= C_{n,a}^2\, A(y_1,y_2)\int\limits_0^1 \frac{t^{\frac{n-a-1}{2}}(1-t)^{\frac{n-a-1}{2}}\, dt}{\left[y_1^2 t + y_2^2(1-t) + t(1-t)|\mathbf{x}_1-\mathbf{x}_2|^2\right]^{\frac{n}{2}+1-a}} \qquad (10)
\end{aligned}
$$

where

$$A(y_1,y_2) = \frac{\pi^{n/2}\,\Gamma\left(\frac{n}{2}+1-a\right)}{\left[\Gamma\left(\frac{n+1-a}{2}\right)\right]^2}\, y_1^{1-a} y_2^{1-a}\ .$$

Here we first changed the product of the velocities by a double integral, substituted the delta-correlation function, while the step from (9) to (10) is done by using n-dimensional polar coordinates, see [16], p. 594, 7.

Now let us prove (7). Note that the correlation function is written in (9) in the convolution form

$$B_u(\mathbf{x}_1,y_1;\mathbf{x}_2,y_2) = P_a(\cdot,y_1) * P_a(\cdot - (\mathbf{x}_1-\mathbf{x}_2),y_2)\ . \qquad (11)$$

To derive the partial spectral density we take the inverse Fourier transform $F^{-1}[\cdot]$ of both sides of 11) and use the Fourier transform property for convolutions. This yields:

$$S(\mathbf{k};y_1,y_2) = F^{-1}[P_a(\cdot,y_1)](\mathbf{k},y_1)\ F^{-1}[P_a(\cdot,y_2)](\mathbf{k},y_2)\ . \qquad (12)$$

The inverse Fourier transform $F^{-1}[P_a]$ can be calculated explicitly using the integral cited in [5], p. 155 (see also [1]:

$$F^{-1}\Big[\frac{y^{n-\alpha}}{(y^2+|\mathbf{x}|^2)^{\frac{n-\alpha}{2}}}\Big] = y^{n-\alpha/2}\pi^{(n-\alpha)/2}\frac{2}{\Gamma\left(\frac{n-\alpha}{2}\right)}\,|\mathbf{k}|^{-\alpha/2}\, K_{\alpha/2}(2\pi y|\mathbf{k}|)\ . \qquad (13)$$

From this we find by $\alpha = a-1$ that

$$F^{-1}\left[P_a(\cdot,y)\right] = 2\,\frac{\pi^{\frac{1-a}{2}}}{\Gamma\left(\frac{1-a}{2}\right)}\,(y|\mathbf{k}|)^{\frac{1-a}{2}}\, K_{\frac{1-a}{2}}(2\pi\, y|\mathbf{k}|) \qquad (14)$$

which proves (7) in view of (12).

Note that the normalizing constant in (14) is obtained from the condition $F^{-1}[P_a(\cdot, y)]|_{k=0} = 1$, taking into account that $K_\nu(z) \to \frac{\sqrt{\pi}}{\sqrt{2\,z}}\, e^{-z}$. This yields the normalizing constant in (14).

To prove that $V(\mathbf{x}, y)$ has the same correlation function $B_u(X_1, X_2)$ it is enough to verify by direct evaluation that the partial spectral function of $V(\mathbf{x}, y)$ coincides with (7). Finally, based on this fact, the convergence of a sum of independent realizations of $V(\mathbf{x}, y)$ to the solution $u(\mathbf{x}, y)$ follows by the standard arguments of the weak convergence theorem (e.g., see [17] and the recent paper [11]). This completes the proof. $\square$

3 General homogeneous excitations on the boundary

The correlation function derived in the Theorem for the white noise boundary excitations $f = W$ can be used to obtain the correlation function for the general case of homogeneous boundary excitations defined by a correlation function $B_f(\mathbf{x})$. Indeed, let us denote by $B_u^{(W)}(\mathbf{x}, y_1, y_2)$ and $B_u^{(f)}(\mathbf{x}, y_1, y_2)$ the correlation functions of the solution $u(\mathbf{x}, y)$ under the white noise W and general homogeneous boundary excitations $f(\mathbf{x})$, respectively. Then starting again with (9) we obtain

$$B_u^{(f)}(\mathbf{x}, y_1, y_2) = \int\limits_{\mathbb{R}^n} B_u^{(W)}(\mathbf{x} - \mathbf{x}', y_1, y_2)\, B_f(\mathbf{x}')\, dx' \ .$$

From this we conclude by the convolution property of the Fourier transform that the following convenient formula for the partial spectral function holds

$$S_u^{(f)}(\mathbf{k}, y_1, y_2) = S_u^{(W)}(\mathbf{k}, y_1, y_2)\, S_f(\mathbf{k}) \tag{15}$$

where the partial spectral function $S_u^{(W)}(\mathbf{k}, y_1, y_2)$ is given by (7), and $S_f(\mathbf{k})$ is the spectral function of the general boundary excitations f.

Note that the formula (15) is especially convenient in the Monte Carlo simulation of the random field because the simulation formula of the type (8) involves only the spectral function which in this case is explicitly given by (15).

4 Correlation function calculations

We show in Figure 1 the longitudinal correlation function in the upper half-space R^3_+ ($n = 2$, left panel), and half-plane ($n = 1$, right panel), for different values of a, compared with the case of Laplace equation. It is clearly seen that the decay of the correlation function

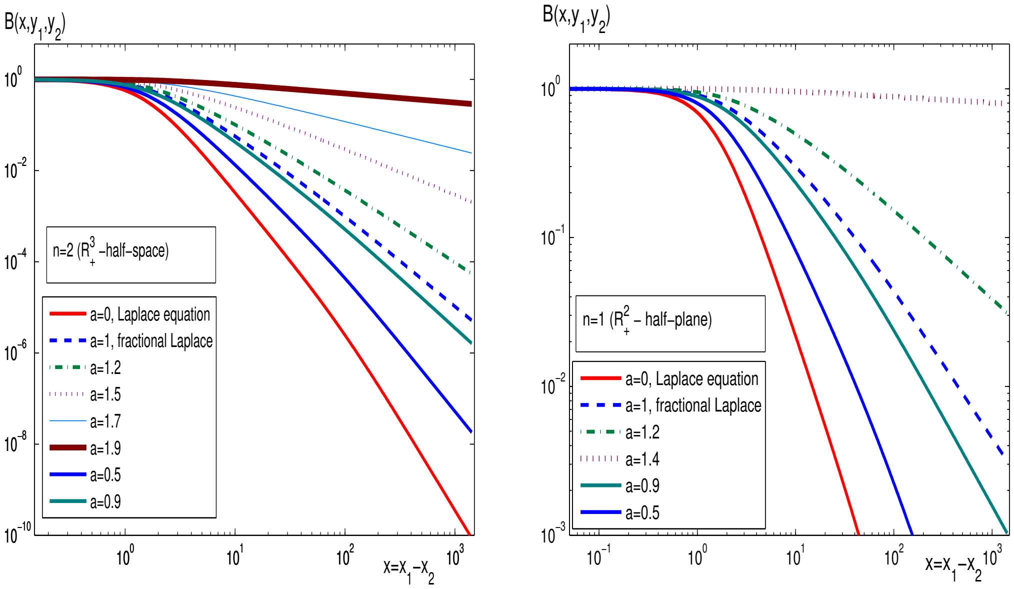

Figure 1: The correlation function $B_u(x, y_1, y_2)$, versus the longitudinal coordinate x, for fixed $y_1 = 0.5$ and $y_2 = 1$, for different values of a compared against the case of Laplace equation $a = 0$. Left panel: the half space ($n = 2$); right panel: the half-plane ($n = 1$). The curves are presented in the log-log coordinates.

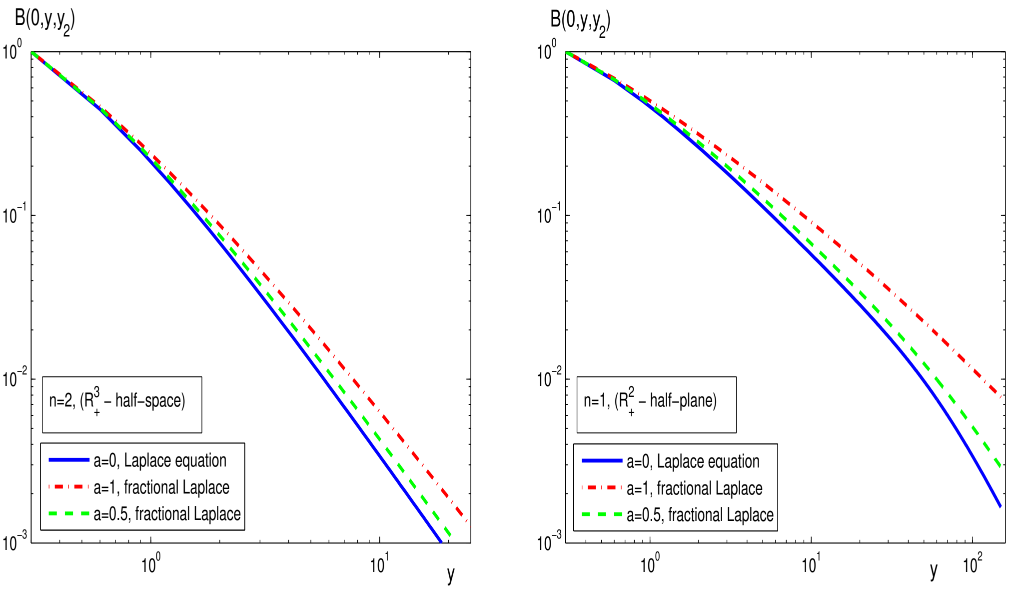

Figure 2: The correlation function $B_u(0, y, y_2)$ versus the transverse coordinate y, for fixed $y_2 = 0.2$, for different values of a compared against the case of Laplace equation $a = 0$. Left panel: the half space ($n = 2$); right panel: the half-plane ($n = 1$). The curves are presented in the log-log coordinates.

B_u, for $a = 1, a = 1.2, a = 1.5$ (as the longitudinal coordinate increases) is considerably lower compared to the case of the Laplace equation when $a = 0$. This difference becomes rapidly more dramatic as a increases. Note also that in the 3D case the correlations are obviously decreasing much faster than in the case of a half-space. In Figure 2 we show the transversal behavior of the same correlation functions. A comparison with the curves presented in Figure 1 shows that the transverse decay of the correlations is considerably slower than the longitudinal decorrelations. It is also to mention that the difference between the fractional and Laplace equations is more pronounced when comparing the longitudinal correlation functions.

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