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Stokes flows under random boundary velocity excitations

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Abstract

A viscous Stokes flow over a disc under random fluctuations of the velocity on the boundary is studied. We give exact Karhunen-Loève (K-L) expansions for the velocity components, pressure, stress, and vorticity, and the series representations for the corresponding correlation tensors. Both the white noise fluctuations, and general homogeneous random excitations of the velocities prescribed on the boundary are studied. We analyze the decay of correlation functions in angular and radial directions, both for exterior and interior Stokes problems. Numerical experiments show the fast convergence of the K-L expansions. The results indicate that ignoring the boundary condition uncertainty dramatically underestimates the variance of the velocity and pressure in the interior/exterior of the domain.

1 Introduction

It is well known that boundary value problems for PDEs with irregular coefficients, fluctuating source terms, or randomly excited boundary conditions are used in many fields of science and technology to describe uncertainty, probabilistic distribution of irregularities, or large ensembles of measurements under similar but randomly fluctuating conditions. (e.g., see [2]), [18], [5]). We mention also the analysis of synoptic meteorological data [14], and statistical theory of fully developed turbulence [15], where the Karhunen-Loève expansions are used. Stochastic approach is intensively used in flows in porous media and soils governed by the Darcy equation with a random hydraulic conductivity coefficient [4], [24], [10], as well as biological tissues [36], and in geodesy [22], [30]. In electrical impedance tomography [9] important problem is to evaluate a global response to random boundary excitations, and to estimate local fluctuations of the solution fields. Similar analysis is made in the inverse problems of elastography [19], [26], recognition technology [7], acoustic scattering from rough surfaces [35], fluid dynamics [1], and reaction-diffusion equations with white noise boundary perturbations [32].

Flows over rough surfaces, e.g., over and in tubes with rough surfaces play an important role in a variety of applications [33]. Often the topology of such surfaces cannot be accurately described in all details due to either insufficient data or measurement errors or both. In such cases, this topological uncertainty can be efficiently handled by treating rough boundaries as random fields, so that an underlying physical phenomenon is described by deterministic or stochastic differential equations with random velocities near the boundary caused by the roughness. The results of calculations we present in this paper indicate that ignoring the boundary condition uncertainty dramatically underestimates the variance of the velocity and pressure in the interior/exterior of the domain.

Many convenient and efficient methods based on the spectral decomposition can be applied in the case when the simulated random fields are homogeneous. Among those, we mention both deterministic and randomized spectral methods (e.g., see [31], [6], [23], [12], [11]). Simulation of inhomogeneous random fields is less developed (see an overview in [25]). This complicates much the development of numerical solution of PDEs involving inhomogeneous random fields.

We deal in this paper with 2D zero mean random fields $v(r, \theta)$ in a disc $S(0, R)$ which are homogeneous with respect to one coordinate, the angular coordinate θ, and inhomogeneous in the radial direction r. Random fields with this property are called partially homogeneous [23]. The correlation function $B_v = \langle v(r_1, \theta_1) v(r_2, \theta_2) \rangle$ is then depending on the difference $\theta = \theta_1 - \theta_2$. The partial spectral function is defined as the inverse Fourier transform with respect to θ:

$$S_v(\xi, r_1, r_2) = F^{-1}[B_v](\xi, r_1, r_2) = \frac{1}{2\pi} \int\limits_0^{2\pi} B_v(\theta, r_1, r_2)\, e^{-\mathrm{i}\theta\xi}\, d\theta \ .$$

Assume we have no dependence on the variable r. The spectral representation of the random field is written in the form

$$v(\theta, \cdot) = \int_0^{2\pi} e^{\mathrm{i}\xi\theta}\, G(\xi, \cdot, \cdot)\, Z(d\xi) \tag{1}$$

where G is defined by $G(\xi, \cdot, \cdot) G^*(\xi, \cdot, \cdot) = S(\xi, \cdot, \cdot)$, the star sign stands for the complex conjugate, the sign $\cdot$ stands to recall we omit the dependence on the variable r, and $Z(d\xi)$ is a white noise on $[0, 2\pi]$.

To simulate homogeneous random field one commonly uses the Randomized Spectral methods (e.g., see [11], [23], [31]) which is based on the randomized calculation of the stochastic integral (1). Another method is based on the Riemann sums calculation with fixed cells. The integral is approximated by a finite sum

$$v(\theta, \cdot) \approx \sum_{i=1}^{N} \left[\zeta_i \sin(\xi_i\, \theta) + \eta_i \cos(\xi_i\, \theta) \right]$$

where ξ_i are deterministic nodes in the Fourier space, ζ_i and η_i are Gaussian variables with zero mean and relevant covariance.

In our case we have however the dependence on the variable r, and the random fields are partially inhomogeneous. In [23] we have extended the Spectral Randomization method to general partially homogeneous random fields, but the inhomogeneity in y was still a problem assumed to be solved by other methods. In a sense it was a method which reduced the dimension of the problem (see for details [23]).

In the case we deal with, the partial spectral function has a special structure, namely, it depends on r_1 and r_2 as follows: $S(\xi, r_1, r_2) = G(\xi, r_1) G^*(\xi, r_2)$. This enables to construct a simple extension of the Randomization method, without solving the inhomogeneity problem in y, see [25] and [28]. This was first done in [25] by using the Karhunen-Lov̀e expansion.

Assume now, without loss of generality, that a generally inhomogeneous random field $u(x)$ has a zero mean and a variance $E\, u^2(x)$ that is bounded. The Karhunen-Loève expansion has the form [37]: $u(x) = \sum\limits_{k=1}^{\infty} \sqrt{\lambda_k}\, \zeta_k\, h_k(x)$, where λ_k and $h_k(x)$ are the eigen-values and eigen-functions of the correlation function $B(x_1, x_2) = \langle u(x_1)\, u(x_2) \rangle$, and ζ_k is a family of random variables.

By definition, $B(x_1, x_2)$ is bounded, symmetric and positive definite. For such kernels, the Hilbert-Schmidt theory says that the following spectral representation is valid

$$B(x_1, x_2) = \sum_{k=1}^{\infty} \lambda_k\, h_k(x_1)\, h_k(x_2)$$

where the eigen-values and eigen-functions are the solutions of the following eigen-value problem for the correlation operator:

$$\int B(x_1, x_2)\, h_k(x_1)\, dx_1 = \lambda_k\, h_k(x_2) \ .$$

The eigen-functions form a complete orthogonal set $\int h_i(x)\, h_j(x)\, dx = \delta_{ij}$ where δ_{ij} is the Kronecker delta-function. The family $\{\zeta_k\}$ is a set of uncorrelated random variables which are obviously related to h_k by

$$\zeta_k = \frac{1}{\sqrt{\lambda_k}} \int u(x)\, h_k(x)\, dx \ , \qquad E\,\zeta_k = 0, \qquad E\zeta_i\,\zeta_j = \delta_{ij} \ .$$

It is well known that the Karhunen-Loève expansion presents an optimal (in the mean square sense) convergence for any distribution of $u(x)$. If $u(x)$ is a zero mean Gaussian random field, then $\{\zeta_k\}$ is a family of standard Gaussian random variables. We assume in this study that the random fields are Gaussian. Some generalizations to non-gaussian random fields are reported in [20].

In [25], I studied random boundary excitations for the Laplace equation under random Dirichlet and Neumann boundary conditions, biharmonic equation, and the Lamé equation governing a 2D elastostatics problem for a disc. This study was then extended to the case of an elastic half-plane [28], [29].

In this paper I deal with the Stokes incompressible flows under random excitations of the velocity prescribed on the boundary. Note that in the Stokes flow analysis, there is a deep relation to the case of the Lamé equation studied in [25], but there are also important differences, namely, (1) the problem is exterior, (2) the velocity field is divergence-free, (3) the pressure in the Stokes flow is strongly correlated to the velocity fluctuations on the boundary which complicates the study of the cross-correlations. I consider also the interior problem where an additional boundary condition should be taken into account. Along the Karhunen-Loève expansions of the velocity and pressure, exact series representations are given for the vorticity and the stress tensor. Numerical experiments illustrate the rate of convergence of the K-L expansions, and present the details of the correlation structure of different flow characteristics.

2 Exterior Stokes problem: formulation and Poisson formula.

Let us consider a linear Stokes problem in D_R^+ which is an exterior of D_R, a disc of radius R centered at the origin. On the boundary, which is the circle $S(0,R) = \partial D_R$, the velocity components g_1 , g_2 are prescribed, and the velocity vector $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$ and the pressure $p(\mathbf{x})$ are to be found from the Stokes problem:

$$\begin{aligned} &\eta\Delta\mathbf{u}(\mathbf{x}) = \nabla p(\mathbf{x}), \quad \mathbf{x} \in D_R^+ \ , \\ &\mathrm{div}\ \mathbf{u}(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in D_R^+ \ , \\ &\mathbf{u}(\mathbf{x}') = \mathbf{g}(\mathbf{x}') \ , \quad \mathbf{x}' \in S(0,R) \end{aligned} \tag{2}$$

where η is the dynamic viscosity.

Assume the prescribed boundary velocities $g_i, i = 1,2$ are homogeneous random processes defined on $S(0,R)$ which implies that the correlation tensor
$(B_g)_{ij} = \langle g_i(\theta_1) g_j(\theta_2)\rangle$ depends on the angular difference $\theta = \theta_1 - \theta_2$, and $\langle \mathbf{g}(\theta)\rangle = const.$

The solution of the problem (2), the velocity $\mathbf{u}$ and the pressure p are random fields, and our goal is to construct simulation formulae for these random fields, and to find their main statistical characteristics, e.g., the stress and velocity correlation tensors, and the pressure correlation function.

Here we note that from the Poisson formula (3) we present in the next section, it can be easily found that $\langle \mathbf{u} \rangle = \langle \mathbf{g} \rangle$, so without loss of generality we assume that $\langle \mathbf{g} \rangle = 0$. For simplicity, we deal here with Gaussian random fields, so we suppose that g_i are Gaussian random processes, which implies due to (3), that the solution, $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$ are also Gaussian random fields. Then, these zero mean random fields are uniquely defined by their correlation functions. In the next section we obtain exact series representations for the correlation functions.

2.1 Poisson formula in polar coordinates

It is natural to use polar coordinates, $\mathbf{x} = re^{\mathrm{i}\theta}$, where $r \geq R$ is the radial coordinate, and θ is the angular coordinate, so that $u_i = u_i(r, \theta)$, $i = 1, 2$. Moreover, we turn also to polar coordinates for the velocity components, i.e., we introduce new velocities, u_r and u_θ by the transform $\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta$, where $\mathbf{e}_r$, $\mathbf{e}_\theta$ are unit vectors in directions r and θ, respectively. Then the vectors $(u_1, u_2)^T$ and $(u_r, u_\theta)^T$ are related through a rotation:

$$\begin{pmatrix} u_1(r,\theta) \\ u_2(r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix} ,$$

and conversely,

$$\begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix} = \mathcal{R}_\theta^T \begin{pmatrix} u_1(r,\theta) \\ u_2(r,\theta) \end{pmatrix}$$

where we use the notation for the rotation matrix

$$\mathcal{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} ,$$

and $\mathcal{R}_\theta^T$ means the transpose to $\mathcal{R}_\theta$.

We will use a spherical mean value relation for the velocity vector which is a generalization of the well known Poisson integral formula

$$\begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix} = \int_0^{2\pi} \begin{pmatrix} G_{11}(r;\theta-\varphi) & G_{12}(r;\theta-\varphi) \\ G_{21}(r;\theta-\varphi) & G_{22}(r;\theta-\varphi) \end{pmatrix} \begin{pmatrix} g_r(R\,e^{\mathrm{i}\varphi}) \\ g_\theta(R\,e^{\mathrm{i}\varphi}) \end{pmatrix} d\varphi \,. \tag{3}$$

The entries G_{ij} $(i, j = 1, 2)$ can be derived explicitly [38]:

$$\begin{aligned}
G_{11}(r;\theta) &= \cos\theta\, K(r,\theta) + \frac{r^2 - R^2}{2r^2}\Big\{ \cos\theta \big(-r\frac{\partial}{\partial r}\, K(r,\theta)\big) + \sin\theta\, \frac{\partial}{\partial \theta} K(r,\theta)\Big\} \,, \\
G_{12}(r;\theta) &= \sin\theta\, K(r,\theta) + \frac{r^2 - R^2}{2r^2}\Big\{ \sin\theta \big(-r\frac{\partial}{\partial r}\, K(r,\theta)\big) - \cos\theta\, \frac{\partial}{\partial \theta} K(r,\theta)\Big\} \,, \\
G_{21}(r;\theta) &= -\sin\theta\, K(r,\theta) + \frac{r^2 - R^2}{2r^2}\Big\{ \sin\theta \big(-r\frac{\partial}{\partial r}\, K(r,\theta)\big) - \cos\theta\, \frac{\partial}{\partial \theta} K(r,\theta)\Big\} \,, \\
G_{22}(r;\theta) &= \cos\theta\, K(r,\theta) - \frac{r^2 - R^2}{2r^2}\Big\{ \cos\theta \big(-r\frac{\partial}{\partial r}\, K(r,\theta)\big) + \sin\theta\, \frac{\partial}{\partial \theta} K(r,\theta)\Big\}
\end{aligned} \tag{4}$$

while the pressure $p(r, \theta)$ is related to the velocities $u_r(R, \theta)$ and $u_\theta(R, \theta)$ on the boundary by analogous Poisson type integral formula:

$$p(r,\theta) = \frac{2\eta}{r}\Big\{ \int_0^{2\pi} P_r(r;\theta-\varphi) u_r(Re^{\mathrm{i}\varphi})\, d\varphi + \int_0^{2\pi} P_\theta(r;\theta-\varphi) u_\theta(Re^{\mathrm{i}\varphi}) d\varphi \Big\} \tag{5}$$

where

$$P_r(r;\theta) = \cos\theta\,\big(-r\frac{\partial}{\partial r}K(r,\theta)\big) + \sin\theta\,\frac{\partial}{\partial\theta}K(r,\theta)$$

$$P_\theta(r;\theta) = \sin\theta\,\big(-r\frac{\partial}{\partial r}K(r,\theta)\big) - \cos\theta\,\frac{\partial}{\partial\theta}K(r,\theta)\ .$$

In (4)-(5), $K(r,\theta)$ is the Poisson kernel for the harmonic equation in an exterior circular domain of radius R:

$$K(r,\theta) = \frac{r^2 - R^2}{2\pi(R^2 + r^2 - 2Rr\cos\theta}\ , \quad r > R\ .$$

In the Karhunen-Loève expansion we will use series expansions of the kernels which can be readily derived starting from the well known series [3]:

$$K(r,\theta) \equiv \frac{1}{2\pi}\cdot\frac{1-\rho^2}{1-2\rho\cos(\theta)+\rho^2} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho^k\,\cos(k\theta)\ , \quad \rho < 1\ , \tag{6}$$

where we introduced the notation $\rho = R/r$.

To get a series expansions for the kernels, we use (6) and the following easily verified series representations:

$$-r\frac{\partial}{\partial r}K(r,\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}k\rho^k\,\cos(k\theta)\ , \quad \frac{\partial K(r,\theta)}{\partial\theta} = -\frac{1}{\pi}\sum_{k=1}^{\infty}k\rho^k\,\sin(k\theta)$$

Substitution of these series expansions in (3) and (5) leads to the desired expansions:

$$\begin{aligned}
G_{11}(r;\theta) &= \frac{\rho}{2\pi} + \frac{1}{2\pi}\sum_{k=1}^{\infty}\Big[2\rho + \frac{k(1-\rho^2)}{\rho}\Big]\rho^k\cos(k\theta)\ ,\\
G_{12}(r;\theta) &= \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{(1-\rho^2)}{\rho}\,k\rho^k\sin(k\theta)\ ,\\
G_{21}(r;\theta) &= \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{(1-\rho^2)}{\rho}\,(k-2)\rho^k\sin(k\theta)\ ,\\
G_{22}(r;\theta) &= \frac{\rho}{2\pi} + \frac{1}{2\pi}\sum_{k=1}^{\infty}\Big[\frac{2}{\rho} - \frac{k(1-\rho^2)}{\rho}\Big]\rho^k\cos(k\theta)\ ,
\end{aligned} \tag{7}$$

and for the kernels P_r and P_θ:

$$P_r(r;\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}k\rho^k\cos\left[(k+1)\theta\right], \quad P_\theta(r;\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}k\rho^k\sin[(k+1)\theta]\ .$$

Thus we can express the pressure through the boundary velocities g_1 and g_2 by

$$p(r,\theta) = \int_0^{2\pi} P_1(r;\theta-\varphi)g_r(Re^{\mathrm{i}\varphi})\,d\varphi + \int_0^{2\pi} P_2(r;\theta-\varphi)g_\theta(Re^{\mathrm{i}\varphi})d\varphi \tag{8}$$

where

$$P_1(r;\theta) = \frac{2\eta}{\pi R}\sum_{k=1}^{\infty}k\rho^{k+1}\,\cos[(k+1)\theta]\,, \quad P_2(r;\theta) = \frac{2\eta}{\pi R}\sum_{k=1}^{\infty}k\rho^{k+1}\,\sin[(k+1)\theta]\,.$$

Let us introduce the notation

$$\lambda_{11}(\rho) = \rho + \frac{k(1-\rho^2)}{2\rho}\ , \qquad \lambda_{12}(\rho) = \frac{k(1-\rho^2)}{2\rho}\ ,$$
$$\lambda_{21}(\rho) = \frac{(k-2)(1-\rho^2)}{2\rho}\ , \quad \lambda_{22}(\rho) = \frac{1}{\rho} - \frac{k(1-\rho^2)}{2\rho}\ , \quad k = 1, 2 \ldots, \tag{9}$$

so that the series expansions (7) take the form

$$G_{11}(r;\theta) = \frac{\rho}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{11}(\rho)\,\rho^k \cos(k\theta)\ , \quad G_{12}(r;\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{12}(\rho)\,\rho^k \sin(k\theta)\ ,$$
$$G_{21}(r;\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{21}(\rho)\,\rho^k \sin(k\theta)\ , \quad G_{22}(r;\theta) = \frac{\rho}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{22}(\rho)\,\rho^k \cos(k\theta)\ . \tag{10}$$

Let us verify that the velocity series representation is divergence-free:

$$\operatorname{div} \mathbf{u}(\mathbf{x}) = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{u_\theta}{\partial\theta} = 0\ .$$

From (3) is follows that it is sufficient to show that

$$\frac{1}{r}\frac{\partial}{\partial r}\big[r\big(G_{11}(r;\theta) + G_{12}(r;\theta)\big)\big] + \frac{1}{r}\frac{\partial}{\partial\theta}\big[r\big(G_{21}(r;\theta) + G_{22}(r;\theta)\big)\big] = 0\ .$$

This in turn follows from the explicit series representations:

$$-\frac{\partial}{\partial r}\big[r\big(G_{11}(r;\theta)\big] = \frac{\partial}{\partial\theta}\big[r\big(G_{21}(r;\theta)\big] = \frac{1}{2\pi}\sum_{k=1}^{\infty} k(k-2)(\rho^{k-1} - \rho^{k+1})\cos(k\theta)$$

and

$$-\frac{\partial}{\partial r}\big[r\big(G_{12}(r;\theta)\big] = \frac{\partial}{\partial\theta}\big[r\big(G_{22}(r;\theta)\big] = \frac{1}{2\pi}\sum_{k=1}^{\infty} \big(k(k-2)\rho^{k-1} - k^2\rho^{k+1}\big)\sin(k\theta)\ .$$

Analogously, we can verify that the series representations for the velocity and pressure satisfy the Stokes equation (2).

3 K-L expansion of the velocity

3.1 White noise excitations

In this section we will obtain the Karhunen-Loève expansion for the velocity vector over a set of eigen-functions of the correlation operator. For this purpose we derive now some properties of the kernel functions G_{ij}.

First we note that from (9) we have the following property

$$\lambda_{11} - \lambda_{12} = \lambda_{21} + \lambda_{22} = \rho\ . \tag{11}$$

Let us consider the eigen-value problem for the integral operator with the kernel G:

$$\int_0^{2\pi} \begin{pmatrix} G_{11}(\rho;\theta-\varphi) & G_{12}(\rho;\theta-\varphi) \\ G_{21}(\rho;\theta-\varphi) & G_{22}(\rho;\theta-\varphi) \end{pmatrix} \begin{pmatrix} h_1(\varphi) \\ h_2(\varphi) \end{pmatrix} \, d\varphi = \lambda \begin{pmatrix} h_1(\theta) \\ h_2(\theta) \end{pmatrix} . \tag{12}$$

Theorem 1. *The eigen-value problem (12) has the following system of eigen-values and corresponding eigen-functions ($k = 1, 2, \dots$) :*

$$\lambda_{2k-1} = \lambda_{2k} = \rho^{k+1}, \quad \begin{pmatrix} h_{1,2k-1} \\ h_{2,2k-1} \end{pmatrix} = \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix}, \quad \begin{pmatrix} h_{1,2k} \\ h_{2,2k} \end{pmatrix} = \begin{pmatrix} \sin k\theta \\ -\cos k\theta \end{pmatrix},$$

and for $\lambda = \rho$, the column-vector $(1,1)^T$ is the corresponding eigen-function.

Proof. From the expansions (7) we find that

$$\begin{aligned}
\int_0^{2\pi} G_{11}(\rho;\theta-\varphi) \begin{pmatrix} \sin k\varphi \\ \cos k\varphi \end{pmatrix} d\varphi &= \lambda_{11}(\rho,k) \begin{pmatrix} \sin k\theta \\ \cos k\theta \end{pmatrix} , \\
\int_0^{2\pi} G_{12}(\rho;\theta-\varphi) \begin{pmatrix} \cos k\varphi \\ \sin k\varphi \end{pmatrix} d\varphi &= \lambda_{12}(\rho,k) \begin{pmatrix} \sin k\theta \\ -\cos k\theta \end{pmatrix} , \\
\int_0^{2\pi} G_{21}(\rho;\theta-\varphi) \begin{pmatrix} \sin k\varphi \\ \cos k\varphi \end{pmatrix} d\varphi &= \lambda_{21}(\rho,k) \begin{pmatrix} -\cos k\theta \\ \sin k\theta \end{pmatrix} , \\
\int_0^{2\pi} G_{22}(\rho;\theta-\varphi) \begin{pmatrix} \cos k\varphi \\ \sin k\varphi \end{pmatrix} d\varphi &= \lambda_{22}(\rho,k) \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix} .
\end{aligned} \tag{13}$$

Now, by substituting these equalities in the eigen-value problem (12) and using the property (11) we find the solution of the eigen-value problem for $k = 1, 2 \dots$. The existence of the eigen-function $(1,1)^T$ for $\lambda = \rho$ follows from the properties

$$\int_0^{2\pi} G_{11}(\rho;\theta-\varphi) \cdot 1 \, d\varphi = \rho, \quad \int_0^{2\pi} G_{22}(\rho;\theta-\varphi) \cdot 1 \, d\varphi = \rho ,$$

$$\int_0^{2\pi} G_{12}(\rho;\theta-\varphi) \cdot 1 \, d\varphi = 0, \quad \int_0^{2\pi} G_{21}(\rho;\theta-\varphi) \cdot 1 \, d\varphi = 0 .$$

The proof is complete. □

We turn now to the derivation of the correlation tensor of the solution,

$$\begin{aligned}
B_u(\rho_1,\theta_1;\rho_2,\theta_2) &= \langle \mathbf{u}(r_1,\theta_1) \otimes \mathbf{u}(r_2,\theta_2) \rangle \\
&\equiv \left\langle \begin{pmatrix} u_r(r_1,\theta_1) \\ u_\theta(r_1,\theta_1) \end{pmatrix} (u_r(r_2,\theta_2) \, , \, u_\theta(r_2,\theta_2)) \right\rangle
\end{aligned} \tag{14}$$

assuming the boundary random vector-function $\mathbf{g}$ has a Gaussian distribution specified by the zero mean and covariance tensor

$$B_g(\varphi_1,\varphi_2) = \left\langle \begin{pmatrix} g_r(\varphi_1) \\ g_\theta(\varphi_1) \end{pmatrix} \, (g_r(\varphi_2) \, , \, g_\theta(\varphi_2)) \right\rangle .$$

We use here and in what follows the following notation for $\mathbf{v} \otimes \mathbf{u}$, a tensor product of two vectors: $\mathbf{v} \otimes \mathbf{u} = \mathbf{v}\,\mathbf{u}^T$.

Substituting the representation (3) in (14) and changing the relevant product of integral expressions by double integrals, we get

$$B_u(\rho_1, \theta_1; \rho_2, \theta_2) = \int_0^{2\pi}\!\!\int_0^{2\pi} G(\rho_1; \theta_1 - \varphi')\, B_g(\varphi', \varphi'')\, G^T(\rho_2; \theta_2 - \varphi'')\, d\varphi'\, d\varphi'' \ . \tag{15}$$

The boundary vector-function g is a white noise, hence,

$$B_g(\varphi_1, \varphi_2) = \begin{pmatrix} \delta(\varphi_1 - \varphi_2) & 0 \\ 0 & \delta(\varphi_1 - \varphi_2) \end{pmatrix} \ . \tag{16}$$

Note that this property follows from the assumption that in the Cartesian coordinates, g_1 and g_2 are independent white noise processes.

Then, from (15) we obtain

$$B_u(\rho_1, \theta_1; \rho_2, \theta_2) = \int_0^{2\pi} G(\rho_1; \theta_1 - \varphi)\ G^T(\rho_2; \theta_2 - \varphi)\, d\varphi \ . \tag{17}$$

Theorem 2. *The exact Karhunen-Loève expansions for the entries of the correlation tensor (14), and the random field $(u_r, u_\theta)^T$ which solves the Stokes equation under the boundary white noise excitations with the covariance tensor (16) are given by*

$$\big(B_u(\rho_1, \theta_1; \rho_2, \theta_2)\big)_{ij} = \frac{\rho_1\, \rho_2}{2\pi}\, \delta_{ij} + \frac{1}{\pi} \sum_{k=1}^{\infty} \Lambda_{ij}(\rho_1, \rho_2; k)\, \rho_1^k \rho_2^k \sin\big[\frac{\pi}{2}\, \delta_{ij} + k(\theta_2 - \theta_1)\big],$$

for $i, j = 1, 2$, where

$$\begin{aligned} \Lambda_{11}(\rho_1, \rho_2; k) &= \lambda_{11}(\rho_1, k)\, \lambda_{11}(\rho_2, k) + \lambda_{12}(\rho_1, k)\, \lambda_{12}(\rho_2, k), \\ \Lambda_{12}(\rho_1, \rho_2; k) &= \lambda_{11}(\rho_1, k)\, \lambda_{21}(\rho_2, k) - \lambda_{12}(\rho_1, k)\, \lambda_{22}(\rho_2, k), \\ \Lambda_{21}(\rho_1, \rho_2; k) &= \lambda_{22}(\rho_1, k)\, \lambda_{12}(\rho_2, k) - \lambda_{21}(\rho_1, k)\, \lambda_{11}(\rho_2, k), \\ \Lambda_{22}(\rho_1, \rho_2; k) &= \lambda_{22}(\rho_1, k)\, \lambda_{22}(\rho_2, k) + \lambda_{21}(\rho_1, k)\, \lambda_{21}(\rho_2, k)\ , \end{aligned} \tag{18}$$

and

$$\begin{aligned} u_r(r, \theta) = \frac{\xi_0\, \rho}{\sqrt{2\pi}} &+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \lambda_{11}(\rho, k)\, \rho^k \big[\xi_k\ \cos k\theta + \eta_k\ \sin k\theta\big] \\ &+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \lambda_{12}(\rho, k)\, \rho^k \big[-\eta'_k\ \cos k\theta + \xi'_k\ \sin k\theta\big]\ , \end{aligned} \tag{19}$$

$$\begin{aligned} u_\theta(r, \theta) = \frac{\xi'_0\, \rho}{\sqrt{2\pi}} &+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \lambda_{21}(\rho, k)\, \rho^k \big[-\eta_k\ \cos k\theta + \xi_k\ \sin k\theta\big] \\ &+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \lambda_{22}(\rho, k)\, \rho^k \big[\xi'_k\ \cos k\theta + \eta'_k\ \sin k\theta\big]\ . \end{aligned} \tag{20}$$

Here $\{\xi_k, \eta_k\}$ and $\{\xi'_k, \eta'_k\}$, $k = 0, 1, 2, \dots$ are two independent families of standard independent gaussian random variables. Thus the random field is homogeneous with respect to the angular variable, and the respective partial spectra are: $S_{mm}(k) = \frac{1}{\pi}\Lambda_{mm}\rho_1^k\rho_2^k$, $S_{mm}(0) = \rho_1\rho_2/2\pi$, and for $n \neq m$ the spectrum is pure imaginary: $S_{mn}(k) = \mathrm{i}\frac{1}{\pi}\Lambda_{mn}\rho_1^k\rho_2^k$, $m, n = 1, 2$.

Proof. The expansion of the correlation tensor is obtained as follows: we substitute the representations (10) in (17) and use the eigen-function properties (13). This immediately yields the exact series representation given in the Theorem.

To construct the explicit simulation formula (19), (20) for our random field, we first split it into two independent Gaussian random fields:

$$\mathbf{u}(r,\theta) = \mathbf{V_1}(r,\theta) + \mathbf{V_2}(r,\theta)\ .$$

We will show now that for each of these random fields we can obtain a Karhunen-Loève expansion.

We introduce two pairs of families of single mode vector functions, the first one,

$$\mathbf{h}_{1k}(\rho,\theta) = \begin{pmatrix} \lambda_{11}(\rho,k)\ \cos k\theta \\ \lambda_{21}(\rho,k)\ \sin k\theta \end{pmatrix}\ ,\quad \tilde{\mathbf{h}}_{1k}(\rho,\theta) = \begin{pmatrix} \lambda_{11}(\rho,k)\ \sin k\theta \\ -\lambda_{21}(\rho,k)\ \cos k\theta \end{pmatrix}\ , \tag{21}$$

and the second one,

$$\mathbf{h}_{2k}(\rho,\theta) = \begin{pmatrix} -\lambda_{12}(\rho,k)\ \cos k\theta \\ \lambda_{22}(\rho,k)\ \sin k\theta \end{pmatrix}\ ,\quad \tilde{\mathbf{h}}_{2k}(\rho,\theta) = \begin{pmatrix} \lambda_{12}(\rho,k)\ \sin k\theta \\ \lambda_{22}(\rho,k)\ \cos k\theta \end{pmatrix}\ . \tag{22}$$

Here the modes are indexed by $k = 1, 2\dots$, while the subindexes $_1$ and $_2$ stand for the first and second series of eigen-functions. Each of the four systems of functions (21),(22) is orthogonal, but all the functions do not form one system of orthogonal functions. We can however construct from these basis functions two orthogonal systems since these vectors are pairwise orthogonal:

$$\int_0^1 d\rho \int_0^{2\pi} d\theta\ \mathbf{h}_{1k}\cdot\tilde{\mathbf{h}}_{1j} = 0,\qquad \int_0^1 d\rho \int_0^{2\pi} d\theta\ \mathbf{h}_{2k}\cdot\tilde{\mathbf{h}}_{2j} = 0\quad \forall j,k = 1,2\dots\ .$$

To complete the systems we add two orthogonal vectors $\mathbf{h}_0 = \begin{pmatrix} \frac{\rho}{\sqrt{2\pi}} \\ 0 \end{pmatrix}$, $\tilde{\mathbf{h}}_0 = \begin{pmatrix} 0 \\ \frac{\rho}{\sqrt{2\pi}} \end{pmatrix}$.

It is now a matter of technical evaluations to find that the correlation tensor can be represented in the form:

$$\begin{aligned} B_u(\rho_1,\theta_1;\rho_2,\theta_2) &= \mathbf{h}_0(\rho_1)\cdot\mathbf{h}_0^T(\rho_2) \qquad (23)\\ &+\frac{1}{\pi}\sum_{k=1}^{\infty}\{\mathbf{h}_{1k}(\rho_1,\theta_1)\mathbf{h}_{1k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{1k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{1k}^T(\rho_2,\theta_2)\}\rho_1^k\rho_2^k \\ +\tilde{\mathbf{h}}_0(\rho_1)\cdot\tilde{\mathbf{h}}_0^T(\rho_2) \qquad (24)\\ &+\frac{1}{\pi}\sum_{k=1}^{\infty}\{\mathbf{h}_{2k}(\rho_1,\theta_1)\mathbf{h}_{2k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{2k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{2k}^T(\rho_2,\theta_2)\}\rho_1^k\rho_2^k\ . \end{aligned}$$

This follows from the easily verified representation

$$\mathbf{h}_{1k}(\rho_1,\theta_1)\,\mathbf{h}_{1k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{1k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{1k}^T(\rho_2,\theta_2) =$$

$$\begin{pmatrix} \lambda_{11}(\rho_1,\theta_1)\,\lambda_{11}(\rho_2,\theta_2)\ \cos[k(\theta_2-\theta_1)] & \lambda_{11}(\rho_1,\theta_1)\,\lambda_{21}(\rho_2,\theta_2)\ \sin[k(\theta_2-\theta_1)] \\ -\lambda_{21}(\rho_1,\theta_1)\,\lambda_{11}(\rho_2,\theta_2)\ \sin[k(\theta_2-\theta_1)] & \lambda_{21}(\rho_1,\theta_1)\,\lambda_{22}(\rho_2,\theta_2)\ \cos[k(\theta_2-\theta_1)] \end{pmatrix}$$

and

$$\mathbf{h}_{2k}(\rho_1,\theta_1)\,\mathbf{h}_{2k}^T(\rho_2,\theta_2)+\tilde{\mathbf{h}}_{2k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{2k}^T(\rho_2,\theta_2)=$$

$$\begin{pmatrix}\lambda_{12}(\rho_1,\theta_1)\,\lambda_{12}(\rho_2,\theta_2)\,\cos[k(\theta_2-\theta_1)] & -\lambda_{12}(\rho_1,\theta_1)\,\lambda_{22}(\rho_2,\theta_2)\,\sin[k(\theta_2-\theta_1)]\\ \lambda_{22}(\rho_1,\theta_1)\,\lambda_{12}(\rho_2,\theta_2)\,\sin[k(\theta_2-\theta_1)] & \lambda_{22}(\rho_1,\theta_1)\,\lambda_{22}(\rho_2,\theta_2)\,\cos[k(\theta_2-\theta_1)]\end{pmatrix}.$$

So we can see that the first and the second pairs of lines ((23) and (24), respectively) present the covariances of the first and second vectors in our splitting, respectively:

$$B_u=\langle\mathbf{u}(r_1,\theta_1)\,\cdot\mathbf{u}^T(r_2,\theta_2)\rangle=\langle\mathbf{V_1}(r_1,\theta_1)\cdot\mathbf{V_1^T}(r_2,\theta_2)\rangle+\langle\mathbf{V_2}(r_1,\theta_1)\mathbf{V_2^T}(r_2,\theta_2)\rangle\ ,$$

thus,

$$B_{V_1}=\mathbf{h}_0(\rho_1)\cdot\mathbf{h}_0^T(\rho_2)+\frac{1}{\pi}\sum_{k=1}^{\infty}\{\mathbf{h}_{1k}(\rho_1,\theta_1)\,\mathbf{h}_{1k}^T(\rho_2,\theta_2)+\tilde{\mathbf{h}}_{1k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{1k}^T(\rho_2,\theta_2)\}\rho_1^k\rho_2^k\ ,$$

$$B_{V_2}=\tilde{\mathbf{h}}_0(\rho_1)\cdot\tilde{\mathbf{h}}_0^T(\rho_2)+\frac{1}{\pi}\sum_{k=1}^{\infty}\{\mathbf{h}_{2k}(\rho_1,\theta_1)\,\mathbf{h}_{2k}^T(\rho_2,\theta_2)+\tilde{\mathbf{h}}_{2k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{2k}^T(\rho_2,\theta_2)\}\rho_1^k\rho_2^k$$

where $B_{V_1}=\langle\mathbf{V_1}(r_1,\theta_1)\cdot\mathbf{V_1^T}(r_2,\theta_2)\rangle\quad B_{V_2}=\langle\mathbf{V_2}(r_1,\theta_1)\mathbf{V_2^T}(r_2,\theta_2)\rangle$.

Note that each part, i.e., B_{V_1} and B_{V_2}, is represented as an orthogonal-mode expansion. Therefore, we can construct a K-L-expansion for our random fields $\mathbf{V}_1$ and $\mathbf{V}_2$.

We have not yet normalized the eigen-functions. We can do it through dividing the angular modes by $\sqrt{\pi}$, and the radial modes by $\Delta_1(k)=\int_0^1(\lambda_{11}^2+\lambda_{21}^2)\,d\rho$ for the first family of eigen-functions (21), and by $\Delta_2(k)=\int_0^1(\lambda_{12}^2+\lambda_{22}^2)\,d\rho$ for the second family of eigen-functions (22). We then collect the orthonormal eigen-modes in two families:

$$\mathcal{H}_{2k-1}^{(1)}=\frac{1}{\sqrt{\Delta_1(k)\,\pi}}\,\mathbf{h}_{1k}(\rho,\theta),\quad \mathcal{H}_{2k}^{(1)}=\frac{1}{\sqrt{\Delta_1(k)\,\pi}}\,\tilde{\mathbf{h}}_{1k}(\rho,\theta),\quad k=1,2,\ldots$$

and

$$\mathcal{H}_{2k-1}^{(2)}=\frac{1}{\sqrt{\Delta_2(k)\,\pi}}\,\mathbf{h}_{2k}(\rho,\theta),\quad \mathcal{H}_{2k}^{(2)}=\frac{1}{\sqrt{\Delta_2(k)\,\pi}}\,\tilde{\mathbf{h}}_{2k}(\rho,\theta),\quad k=1,2,\ldots$$

Then, the orthonormal functions $\mathcal{H}_k^{(1)}$ and $\mathcal{H}_k^{(2)}$ are eigen-functions of the covariance tensors B_{V_1} and B_{V_2}, respectively, with the corresponding eigen-values $\Delta_1(k)$ and $\Delta_2(k)$:

$$\int_0^1\int_0^{2\pi}B_{V_m}\cdot\mathcal{H}_k^{(m)}(\rho_2,\theta_2)\,d\rho_2\,d\theta_2=\Delta_m(k)\,\mathcal{H}_k^{(m)}(\rho_1,\theta_1)\ ,\quad m=1,2\ .$$

We thus end up with the K-L-expansion for the random field $\mathbf{V_1}(r,\theta)$ in the form

$$\mathbf{V_1}(r,\theta)=\sum_{k=1}^{\infty}\zeta_k\mathcal{H}_k^{(1)}(\rho,\theta)$$

where ζ_k are gaussian random variables such that

$$\langle\zeta_k\zeta_j\rangle=\Delta_1(k)\,\delta_{jk}\ ,$$

and the same for $\mathbf{V_2}(r,\theta)$.

Putting these expansions together we conclude with the desired representation (19), (20).

Finally note that the spectra given in the theorem are obtained immediately from the expansion of the correlation function. This completes the proof of Theorem 2. $\square$

3.2 General case of homogeneous excitations.

So far we studied the case where g_1 and g_2 were two independent white noise processes. We will show now that the general situation when g_1 and g_2 are some arbitrary possibly dependent homogeneous processes, is basically derived from the white noise case.

Assume we are given two homogeneous zero mean processes g_1 and g_2 with the correlation tensor $B_g(\varphi_2 - \varphi_2)$ (with the entries $B_{g,ij}, \quad i,j = 1,2$) defined on the boundary. As shown above, the correlation tensor B_u is related to B_g by the double integral representation (15). Changing the integration variable φ'' to a new one, ψ by $\varphi'' - \varphi' = \psi$, for $\mathbf{u} = (u_\rho, u_\theta)^T$ we obtain from (15)

$$B_u(\rho_1, \theta_1; \rho_2, \theta_2) = \int_0^{2\pi}\int_0^{2\pi} G(\rho_1; \theta_1 - \varphi')\, B_g(\psi)\, G^T(\rho_2; \theta_2 - \psi - \varphi')\, d\varphi'\, d\psi\ . \tag{25}$$

The inner integral with respect to φ' can be evaluated explicitly using the series expansions for the kernel $G(\rho, \theta)$ given above in (7). It is convenient to rewrite the relation (25) in a different form. We take now the entries of the correlation tensor B_u, and define a column-vector function $\hat{\mathbf{B}}_u$ as follows $\hat{\mathbf{B}}_u = (B_{u,11}, B_{u,12}, B_{u,21}, B_{u,22})^T$. Analogously, we use the notation $\hat{\mathbf{B}}_g$ for the column-vector $\hat{\mathbf{B}}_g = (B_{g,11}, B_{g,12}, B_{g,21}, B_{g,22})^T$.

Theorem 3. *Under the assumptions that the boundary random functions g_1 and g_2 are statistically homogeneous, with the given correlation tensor $B_g(\varphi)$, the solution of the Stokes problem is a random field partially homogeneous with respect to the angular coordinate, and the correlation functions $\hat{\mathbf{B}}_u$ and $\hat{\mathbf{B}}_g(\varphi)$ are related by the convolution*

$$\hat{\mathbf{B}}_u(\rho_1, \rho_2; \theta) = \int_0^{2\pi} A(\rho_1, \rho_2; \theta - \psi)\, \hat{\mathbf{B}}_g(\psi)\, d\psi \tag{26}$$

where the matrix A is explicitly given by its entries (29)-(32).

Proof. In the introduced notation, the equality (25) takes the form

$$\hat{\mathbf{B}}_u(\rho_1, \theta_1; \rho_2, \theta_2) = \int_0^{2\pi}\int_0^{2\pi} G(\rho_1; \theta_1 - \varphi') \otimes G(\rho_2; \theta_2 - \psi - \varphi')\, \hat{\mathbf{B}}_g(\psi)\, d\varphi'\, d\psi\ . \tag{27}$$

Here we denote by $\otimes$ a tensor product of two matrices which is defined as a 4×4 matrix, represented as a 2×2-block matrix each block being a 2×2 matrix of the form $G_{ij}(\rho_1; \theta_1 - \varphi')G(\rho_2; \theta_2 - \psi - \varphi'), \;\; i,j = 1,2.$

The entries a_{ij}, $i,j = 1,2,3,4$ of the matrix

$$A = \int_0^{2\pi} G(\rho_1; \theta_1 - \varphi') \otimes G(\rho_2; \theta_2 - \psi - \varphi')\, d\varphi' \tag{28}$$

can be found explicitly. Indeed, substituting the series representation of the matrix G given by

(7) in (28) we obtain

$$
\begin{aligned}
a_{11} &= \frac{\rho_1\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{11}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{12} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{11}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{13} &= -\frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{12}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{14} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{12}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right]
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
a_{21} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{11}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{22} &= \frac{\rho_1\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{11}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{23} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{12}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{24} &= -\frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{12}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right]
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
a_{31} &= -\frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{21}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{32} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{21}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{33} &= \frac{\rho_1\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{22}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{34} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{22}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right]
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
a_{41} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{21}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{42} &= -\frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{21}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{43} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{22}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k \sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{44} &= \frac{\rho_1\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{22}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k \cos\left[k(\theta_2-\theta_1-\psi)\right]\ .
\end{aligned} \tag{32}
$$

Thus we see from these formulae that the entries of the matrix A depend on the difference $\theta = \theta_2 - \theta_1$, hence the correlation tensor B_u also depends on $\theta = \theta_2 - \theta_1$, and from (27), (28) we arrive at the desired convolution representation (26). The proof of Theorem 3 is complete. $\square$

Remark 2. *Note that if the boundary correlation tensor B_g is given by its spectral expansion, we can express the correlation tensor of the solution through the spectra. For instance, assuming the spectral tensor is real-valued, so that*

$$B_{g,ij}(\varphi'' - \varphi) = \frac{f_{ij}(0)}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} f_{ij}(k)\, \cos k(\varphi'' - \varphi')\ , \quad i,j = 1,2\ ,$$

we can derive a general formula for the covariance tensor by substituting this expansion in (25). After routine evaluations we obtain the general formulae

$$B_{11} = \frac{f_{11}(0)\,\rho_1\,\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{11}^c\, \cos[k(\theta_2-\theta_1)] + \Lambda_{11}^s\, \sin[k(\theta_2-\theta_1)]\right)\ ,$$

$$B_{12} = \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{12}^c\, \cos[k(\theta_2-\theta_1)] + \Lambda_{12}^s\, \sin[k(\theta_2-\theta_1)]\right)\ ,$$

$$B_{21} = \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{21}^c\, \cos[k(\theta_2-\theta_1)] + \Lambda_{21}^s\, \sin[k(\theta_2-\theta_1)]\right)\ ,$$

$$B_{22} = \frac{f_{22}(0)\,\rho_1\,\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{22}^c\, \cos[k(\theta_2-\theta_1)] + \Lambda_{22}^s\, \sin[k(\theta_2-\theta_1)]\right)\ ,$$

where

$$\begin{aligned}
\Lambda_{11}^c &= f_{11}\lambda_{11}^1\lambda_{11}^2 + f_{22}\lambda_{12}^1\lambda_{12}^2\ , & \Lambda_{11}^s &= f_{21}\lambda_{11}^1\lambda_{12}^2 - f_{12}\lambda_{12}^1\lambda_{11}^2\ ,\\
\Lambda_{12}^c &= f_{21}\lambda_{12}^1\lambda_{21}^2 + f_{12}\lambda_{11}^1\lambda_{22}^2\ , & \Lambda_{12}^s &= f_{11}\lambda_{11}^1\lambda_{21}^2 - f_{22}\lambda_{12}^1\lambda_{22}^2\ ,\\
\Lambda_{21}^c &= f_{21}\lambda_{22}^1\lambda_{11}^2 + f_{12}\lambda_{21}^1\lambda_{12}^2\ , & \Lambda_{21}^s &= f_{22}\lambda_{22}^1\lambda_{12}^2 - f_{11}\lambda_{21}^1\lambda_{11}^2\ ,\\
\Lambda_{22}^c &= f_{22}\lambda_{22}^1\lambda_{22}^2 + f_{11}\lambda_{21}^1\lambda_{21}^2\ , & \Lambda_{22}^s &= f_{21}\lambda_{22}^1\lambda_{21}^2 - f_{12}\lambda_{21}^1\lambda_{22}^2\ .
\end{aligned}$$

Here we use the notations $\lambda_{ij}^m = \lambda_{ij}(\rho_m, k)$, $m = 1, 2$.

Remark 3. *Note that using the relation between the vectors in polar and rectangular coordinates,*

$$\begin{pmatrix} u_1(r,\theta) \\ u_2(r,\theta) \end{pmatrix} = \mathcal{R}_\theta \begin{pmatrix} u_\rho(r,\theta) \\ u_\theta(r,\theta \end{pmatrix}$$

we can easily relate the desired statistical characteristics in these two coordinate systems. For example, the covariance tensors are related as follows

$$B_{(u_1,u_2)}(\rho_1,\rho_2;\theta_1,\theta_2) = \mathcal{R}_{\theta_1}\, B_{(u_r,u_\theta)}(\rho_1,\rho_2;\theta_1,\theta_2)\, \mathcal{R}_{\theta_2}^T\ . \tag{33}$$

The K-L-expansion in the rectangular coordinates is also obtained directly from the K-L-expansion of the random field in the polar coordinates on the basis that the eigen-functions are related by $h_{rectangular} = \mathcal{R}_\theta\, h_{polar}$ and $\tilde{h}_{rectangular} = \mathcal{R}_\theta\, \tilde{h}_{polar}$.

Let us write down here the relation (33) in details. We denote the entries of the covariance matrix $B_{(u_1,u_2)}$ by B_{ij}^{rec}, and the entries of the covariance matrix $B_{(u_r,u_\theta)}$ by B_{ij}^{pol}. From (33) we obtain

$$
\begin{aligned}
B^{rec}_{11} &= \cos\theta_1\cos\theta_2\, B^{pol}_{11} - \cos\theta_1\sin\theta_2\, B^{pol}_{12} - \sin\theta_1\cos\theta_2\, B^{pol}_{21} + \sin\theta_1\sin\theta_2\, B^{pol}_{22} \ ,\\
B^{rec}_{12} &= \cos\theta_1\sin\theta_2\, B^{pol}_{11} + \cos\theta_1\cos\theta_2\, B^{pol}_{12} - \sin\theta_1\sin\theta_2\, B^{pol}_{21} - \sin\theta_1\cos\theta_2\, B^{pol}_{22} \ ,\\
B^{rec}_{21} &= \sin\theta_1\cos\theta_2\, B^{pol}_{11} - \sin\theta_1\sin\theta_2\, B^{pol}_{12} + \cos\theta_1\cos\theta_2\, B^{pol}_{21} - \cos\theta_1\sin\theta_2\, B^{pol}_{22} \ ,\\
B^{rec}_{22} &= \sin\theta_1\sin\theta_2\, B^{pol}_{11} + \sin\theta_1\cos\theta_2\, B^{pol}_{12} + \sin\theta_2\cos\theta_1\, B^{pol}_{21} + \cos\theta_1\cos\theta_2\, B^{pol}_{22} \ .
\end{aligned}
$$

4 Correlation function of the pressure

From (8), it is easy to derive an explicit representation for the correlation function of the pressure $p(r,\theta)$. Indeed, let us denote by B_p the correlation function of the pressure: $B_p(r_1,\theta_1;r_2,\theta_2) = \langle p(r_1,\theta_1)p(r_2,\theta_2)\rangle$.

Theorem 4. *Assume that the functions g_1 and g_2 prescribed on the boundary are independent Gaussian white noise random processes defined on the circle $S(0,R)$: $B_\varphi = \langle g_i(\varphi_1)\, g_j(\varphi_2)\rangle = \delta_{ij}\,\delta(\varphi_2-\varphi_1)$, $i,j=1,2$. Then $B_p(r_1,\theta_1;r_2,\theta_2)$, the correlation function of the pressure $p(r,\theta)$ has the following series representation*

$$
B_p = B_p(r_1,\theta_1;r_2,\theta_2) = \frac{8\eta^2}{\pi R^2}\sum_{k=1}^{\infty} k^2 \rho_1^{k+1}\,\rho_2^{k+1}\,\cos[(k+1)(\theta_2-\theta_1)] \ . \tag{34}
$$

Thus the correlation function (34) depends on the difference $\theta=\theta_2-\theta_1$ and the product $\rho=\rho_1\rho_2$: $B_p = B_p(\rho,\theta)$. Moreover, it can be explicitly expressed as

$$
B_p(\rho,\theta) = \frac{8\eta^2}{\pi R^2}\Big[\rho\cos(\theta)\frac{\partial^2}{\partial\theta^2}B_\Delta(\rho,\theta) + \rho^2\sin(\theta)\frac{\partial^2}{\partial\theta\partial\rho}B_\Delta(\rho,\theta)\Big] \tag{35}
$$

where $B_\Delta(\rho,\theta)$ is the correlation function for the Laplace equation under white noise boundary excitations derived in [25]:

$$
B_\Delta(\rho,\theta) = \frac{1}{2\pi}\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2} \ .
$$

Proof. Changing the product of integrals by double integrals and using the definition of the white noise we find

$$
\begin{aligned}
B_p(r_1,\theta_1;r_2,\theta_2) &= \langle p(r_1,\theta_1)p(r_2,\theta_2)\rangle\\
&= \Big[\Big\{\int_0^{2\pi} P_1(r_1;\theta_1-\varphi')g_1(\varphi')d\varphi' + \int_0^{2\pi} P_2(r_1;\theta_1-\varphi')g_2(\varphi')d\varphi'\Big\}\\
&\quad\times\Big\{\int_0^{2\pi} P_1(r_2;\theta_2-\varphi'')g_1(\varphi'')d\varphi'' + \int_0^{2\pi} P_2(r_2;\theta_2-\varphi'')g_2(\varphi'')d\varphi''\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{2\pi} \Big\{ P_1(r_1;\theta_1-\varphi')P_1(r_2;\theta_2-\varphi'')\langle g_1(\varphi')g_1(\varphi'')\rangle \\
&\qquad + P_1(r_1;\theta_1-\varphi')P_2(r_2;\theta_2-\varphi'')\langle g_1(\varphi')g_2(\varphi'')\rangle \\
&\qquad + P_2(r_1;\theta_1-\varphi')P_1(r_2;\theta_2-\varphi'')\langle g_2(\varphi')g_1(\varphi'')\rangle \\
&\qquad + P_2(r_1;\theta_1-\varphi')P_2(r_2;\theta_2-\varphi'')\langle g_2(\varphi')g_2(\varphi'')\rangle \Big\}\, d\varphi' d\varphi'' \\
&= \int_0^{2\pi} P_1(r_1;\theta_1-\varphi)P_1(r_2;\theta_2-\varphi)d\varphi + \int_0^{2\pi} P_2(r_1;\theta_1-\varphi)P_2(r_2;\theta_2-\varphi)d\varphi\,.
\end{aligned} \tag{36}
$$

Now we substitute the expansions for P_1 and P_2 obtained above in (8). This yields

$$
B_p(r_1,\theta_1;r_2,\theta_2) = \frac{8\eta^2}{\pi R^2}\sum_{k=1}^{\infty} k^2 \rho_1^{k+1}\rho_2^{k+1}\cos[(k+1)(\theta_2-\theta_1)]\ .
$$

The exact expression (35) follows from (34) and the series representation for the correlation function of the solution to the Laplace equation under white noise excitations derived in my recent paper [26]:

$$
B_\Delta = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho^k\ \cos[k(\theta_2-\theta_1)] = \frac{1}{2\pi}\cdot\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}
$$

where we recall that $\rho=\rho_1\rho_2$ and $\theta=\theta_2-\theta_1$.

The proof is complete. □

It is interesting to note that we could obtain expressions for the pressure by substituting formally a generalized representation of the boundary white noises on the circle

$$
g_1(\varphi) = \frac{\xi_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\left[\xi_k\ \cos\ k\varphi + \eta_k\ \sin\ k\varphi\right]
$$

$$
g_2(\varphi) = \frac{\xi_0'}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\left[\xi_k'\ \cos\ k\varphi + \eta_k'\ \sin\ k\varphi\right]
$$

into the Poisson formula (8) with the kernels $P_1(r;\theta)$ and $P_2(r;\theta)$. Indeed, a little algebra results in the representation for the random pressure:

$$
p(r,\theta) = \frac{4\eta}{\sqrt{2\pi}R}\sum_{k=1}^{\infty} k\rho^{k+1}\{\xi_k\cos[(k+1)\theta] + \eta_k\sin[(k+1)\theta]\}
$$

where ξ_l and η_k are families of independent standard gaussian random variables.

Note that from this representation the series expansion (34) is easily obtained by direct evaluation.

5 Vorticity and stress

The stress tensor for the Stokes flow has the form [13]

$$
\sigma_{ik} = -p\delta_{ik} + \eta\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big), \quad i,k=1,2
$$

where p is the pressure, and δ_{ik} is the Kronecker symbol.

Recall that in polar coordinates we use,

$$u_1 = u_r \cos\theta - u_\theta \sin\theta, \qquad u_2 = u_r \sin\theta + u_\theta \cos\theta,$$

and

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \sin\theta \frac{\partial}{r\partial\theta}, \quad \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \cos\theta \frac{\partial}{r\partial\theta}.$$

Now choosing the axis x along the vector direction defined by θ we find

$$\begin{aligned} &\frac{\partial u_1}{\partial x} = \frac{\partial u_r}{\partial r}, \quad \frac{\partial u_2}{\partial y} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\ &\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}, \end{aligned} \tag{37}$$

hence the entries of the stress tensor become

$$\begin{aligned} &\sigma_{rr} = -p + 2\eta \frac{\partial u_r}{\partial r}, \qquad \sigma_{\theta\theta} = -p + 2\eta\Big(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\Big), \\ &\sigma_{r\theta} = \eta\Big(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\Big). \end{aligned} \tag{38}$$

Using the series expansions for the kernels G_{ij} and the pressure obtained above, it is easy to derive series expansion representations for the correlation tensor for all the strain entries. Let us do it for the vorticity.

From (37) we get for the vorticity

$$\omega_{r\theta} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = \frac{1}{r}\Big[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\Big].$$

Our goal is to find the correlation tensor of vorticity.

Theorem 5. *The correlation function of the pressure, B_p, and the correlation function of the vorticity, $B_{\omega_{r\theta}}$, are related by*

$$B_{\omega_{r\theta}} = \frac{1}{2\eta^2} B_p = \frac{4}{\pi R^2} \sum_{k=1}^{\infty} k^2 \rho_1^{k+1} \rho_2^{k+1} \cos[(k+1)(\theta_2 - \theta_1)]. \tag{39}$$

Proof. We start by substituting the series expansions (10) for the kernels G_{ij} to get the expressions

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = &\frac{1}{2\pi R}\int_0^{2\pi} \sum_{k=1}^{\infty} \big[-(k-2)^2 \rho^k + k(k-2)\rho^{k+2}\big] \sin[(k(\theta-\varphi)]\, g_r(\varphi)\, d\varphi \\ &+ \frac{1}{2\pi R}\int_0^{2\pi} \sum_{k=1}^{\infty} \big[(k-2)^2 \rho^k - k^2 \rho^{k+2}\big] \cos[(k(\theta-\varphi)]\, g_\theta(\varphi)\, d\varphi \end{aligned}$$

and

$$\begin{aligned} \frac{1}{r}\frac{\partial u_r}{\partial \theta} = &\frac{1}{2\pi R}\int_0^{2\pi} \sum_{k=1}^{\infty} \big[-k^2 \rho^k + k(k-2)\rho^{k+2}\big] \sin[(k(\theta-\varphi)]\, g_r(\varphi)\, d\varphi \\ &+ \frac{1}{2\pi R}\int_0^{2\pi} \sum_{k=1}^{\infty} \big[k^2 \rho^k - k^2 \rho^{k+2}\big] \cos[(k(\theta-\varphi)]\, g_\theta(\varphi)\, d\varphi. \end{aligned}$$

This yields

$$\omega_{r\theta} = \frac{2}{\pi R} \int_0^{2\pi} \sum_{k=1}^{\infty} k\rho^{k+1} \sin[(k+1)(\theta-\varphi)]\, g_r(\varphi)\, d\varphi$$
$$-\frac{2}{\pi R} \int_0^{2\pi} \sum_{k=1}^{\infty} k\rho^{k+1} \cos[(k+1)(\theta-\varphi)]\, g_\theta(\varphi)\, d\varphi \ .$$

From this we readily find the correlation function of the vorticity:

$$B_{\omega_{r\theta}}(r_1,\theta_1;r_2,\theta_2) = \frac{4}{\pi R^2} \sum_{k=1}^{\infty} k^2 \rho_1^{k+1}\rho_2^{k+1} \cos[(k+1)(\theta_2-\theta_1)]$$

which by comparing with the expansion (34) gives the desired result (39). □

5.1 Radial stress

Analogously, the expansions for the correlations of the stress tensor can be obtained. For instance, let us consider the radial stress $\sigma_{rr} = -p + 2\eta\, \partial u_r/\partial r$ given in (38). We have obviously

$$B_{\sigma_{rr}} = \left\langle \sigma_{rr}(r_1,\theta_1)\sigma_{rr}(r_2,\theta_2)\right\rangle = B_p + 4\eta^2 \left\langle \frac{\partial u_r(r_1,\theta_1)}{\partial r}\ \frac{\partial u_r(r_2,\theta_2)}{\partial r}\right\rangle$$
$$-2\eta \left\langle p(r_1,\theta_1)\frac{\partial u_r(r_2,\theta_2)}{\partial r}\right\rangle - 2\eta \left\langle \frac{\partial u_r(r_1,\theta_1)}{\partial r} p(r_2,\theta_2)\right\rangle \qquad (40)$$

where B_p is the correlation function of the pressure derived in Theorem 4.

Taking the radial derivative we get

$$\frac{\partial u_r}{\partial r}(r,\theta) =$$
$$\frac{1}{2\pi R} \int_0^{2\pi} \left\{-\rho^2 + \sum_{k=1}^{\infty} \left[(k+1)(k-2)\rho^{k+2} - k(k-1)\rho^k\right] \cos[k(\theta-\varphi)]\right\} g_r(\varphi) d\varphi$$
$$+\frac{1}{2\pi\, R} \int_0^{2\pi} \sum_{k=1}^{\infty} \left[k(k+1)\rho^{k+2} - k(k-1)\rho^k\right] \sin[k(\theta-\varphi)]\, g_\theta(\varphi) d\varphi \ . \qquad (41)$$

For the pressure we have

$$p(r,\theta) = \frac{2\eta}{\pi\, R} \int_0^{2\pi} \Big\{ \sum_{k=1}^{\infty} k\rho^{k+1}\ \cos[(k+1)(\theta-\varphi)]\, g_r(Re^{\mathrm{i}\varphi})$$
$$\sum_{k=1}^{\infty} k\rho^{k+1}\ \sin[(k+1)(\theta-\varphi)]\ g_\theta(Re^{\mathrm{i}\varphi})\Big\} d\varphi \ . \qquad (42)$$

From (41) we find:

$$4\eta^2 \left\langle \frac{\partial u_r(r_1,\theta_1)}{\partial r}\ \frac{\partial u_r(r_2,\theta_2)}{\partial r}\right\rangle = \frac{2\eta^2\rho_1^2\rho_2^2}{\pi R^2} + \frac{4\eta^2\rho_1^2\rho_2^2}{\pi R^2}\cos(\theta_2-\theta_1)$$
$$+\frac{\eta^2}{\pi R^2} \sum_{k=1}^{\infty} \rho_1^{k+1}\rho_2^{k+1} \Lambda_{k+1}(\rho_1,\rho_2) \cos[(k+1)(\theta_2-\theta_1)] \ . \qquad (43)$$

Here

$$\Lambda_k(\rho_1, \rho_2) = \lambda_1(k, \rho_1)\lambda_1(k, \rho_2) + \lambda_2(k, \rho_1)\lambda_2(k, \rho_2)$$

where

$$\lambda_1(k, \rho) = (k+1)(k-2)\rho^2 - k(k-1), \quad \lambda_2(k, \rho) = k(k+1)\rho^2 - k(k-1) \ .$$

Finally, from (41) and (42) we find that the sum of two last terms in (40) has the form

$$-2\eta \left\langle p(r_1, \theta_1)\frac{\partial u_r(r_2, \theta_2)}{\partial r}\right\rangle - 2\eta \left\langle \frac{\partial u_r(r_1, \theta_1)}{\partial r} p(r_2, \theta_2)\right\rangle$$
$$= -\frac{2\eta^2}{\pi R^2}\sum_{k=1}^{\infty} k\, \rho_1^{k+1}\rho_2^{k+1}\hat{\Lambda}_{k+1}(\rho_1, \rho_2)\cos[(k+1)(\theta_2 - \theta_1) \tag{44}$$

where

$$\hat{\Lambda}_k(\rho_1, \rho_2) = \lambda_1(k, \rho_1) + \lambda_2(k, \rho_1) + \lambda_1(k, \rho_2) + \lambda_2(k, \rho_2) \ .$$

From the series representations (34), (43), and (44) we finally obtain

$$B_{\sigma_{rr}} = \frac{2\eta^2}{\pi R^2}\{\rho_1^2\rho_2^2 + 2\rho_1^3\rho_2^3\cos(\theta_2 - \theta_1)\}$$
$$+\frac{\eta^2}{\pi R^2}\sum_{k=1}^{\infty}\left[k^2 + \Lambda_{k+1}(\rho_1, \rho_2) - 2k\hat{\Lambda}_{k+1}(\rho_1, \rho_2)\right]\rho_1^{k+1}\rho_2^{k+1}\ \cos[(k+1)(\theta_2 - \theta_1)]\,.$$

5.2 Homogeneous random boundary excitations

We extend the result of the Theorem 4 to the case of general homogeneous random boundary excitations. So we suppose that the velocity $\mathbf{g}(\psi)$ prescribed on the circle $S(0, R)$ is a real-valued vector zero mean homogeneous random process. Let us denote the correlation tensor of $\mathbf{g}$ by $B^{(g)}(\psi)$ with the entries $B_{ij}^{(g)}(\psi) = \langle g_i(\psi_1)g_j(\psi_1 + \psi)\rangle$, $i, j = 1, 2$.

Our goal is now to relate the correlation function of the pressure and the input correlation tensor $B^{(g)}$.

Theorem 6. *The correlation function of the pressure is represented through the correlation tensor of the homogeneous boundary excitations by the following convolution formula*

$$B_p(r_1, \theta_1; r_2, \theta_2) = B_p(\rho_1\rho_2, \theta_1 - \theta_2) \tag{45}$$
$$= \frac{4\eta^2}{\pi R^2}\left[\int_0^{2\pi}\sum_{k=1}^{\infty}k^2\rho_1^{k+1}\rho_2^{k+1}\cos[(k+1)(\theta_1 - \theta_2 - \psi)]\{B_{11}^{(g)}(\psi) + B_{22}^{(g)}(\psi)\}\,d\psi\right.$$
$$\left.+\int_0^{2\pi}\sum_{k=1}^{\infty}k^2\rho_1^{k+1}\rho_2^{k+1}\sin[(k+1)(\theta_1 - \theta_2 - \psi)]\{B_{21}^{(g)}(\psi) - B_{12}^{(g)}(\psi)\}\,d\psi\right],$$

hence the pressure is a random field homogeneous with respect to the angular variable.

Proof. The proof follows the same way presented above by starting with the expression (36), and making the change of variable $\psi = \varphi' - \varphi''$, where φ' is fixed (i.e., ψ is fixed), and the inner integrals are taken explicitly with respect to the variable φ''. Using the orthogonality of the functions $\sin k\varphi''$ and $\cos k\varphi''$ we arrive at (45). □

6 Interior Stokes problem

Let us now consider the interior Stokes problem in the disc D_R^- of radius R centered at the origin. On the boundary, which is the circle $S(0,R) = \partial D_R$, the velocity components g_1 , g_2 are prescribed, and the velocity vector $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$ and the pressure $p(\mathbf{x})$ are to be found from the Stokes problem:

$$\begin{aligned}
&\eta\Delta\mathbf{u}(\mathbf{x}) = \nabla p(\mathbf{x}), \quad \mathbf{x} \in D_R^- \ , \\
&\mathrm{div}\ \mathbf{u}(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in D_R^- \ , \\
&\mathbf{u}(\mathbf{x}') = \mathbf{g}(\mathbf{x}') \ , \quad \mathbf{x}' \in S(0,R)
\end{aligned} \tag{46}$$

where η is the dynamic viscosity, and the velocity components g_1 and g_2 are supposed to be zero mean random processes on the circle $S(0,R)$.

6.1 Poisson formula

We use again polar coordinates, $r \le R$ is the radial coordinate, and θ is the angular coordinate. The Poisson integral formula for the interior problem has the form analogous to (3) (e.g., see [38])

$$\begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix} = \int\limits_0^{2\pi} \begin{pmatrix} G_{11}(r;\theta-\varphi) & G_{12}(r;\theta-\varphi) \\ G_{21}(r;\theta-\varphi) & G_{22}(r;\theta-\varphi) \end{pmatrix} \begin{pmatrix} g_r(R\,e^{\mathrm{i}\,\varphi}) \\ g_\theta(R\,e^{\mathrm{i}\,\varphi}) \end{pmatrix} d\varphi$$

where the entries G_{ij} $(i,j=1,2)$ have the form

$$\begin{aligned}
G_{11}(r;\theta) &= \cos\theta\, K(r,\theta) + \frac{R^2-r^2}{2r^2}\Big\{\cos\theta\,\big(r\frac{\partial}{\partial r}\,K(r,\theta)\big) - \sin\theta\,\frac{\partial}{\partial\theta}K(r,\theta)\Big\} \\
&\qquad -\frac{R^2-r^2}{2\pi Rr} \ , \\
G_{12}(r;\theta) &= \sin\theta\, K(r,\theta) + \frac{R^2-r^2}{2r^2}\Big\{\sin\theta\,\big(r\frac{\partial}{\partial r}\,K(r,\theta)\big) + \cos\theta\,\frac{\partial}{\partial\theta}K(r,\theta)\Big\} \ , \\
G_{21}(r;\theta) &= -\sin\theta\, K(r,\theta) + \frac{R^2-r^2}{2r^2}\Big\{\sin\theta\,\Big(r\frac{\partial}{\partial r}\,K(r,\theta)\Big) + \cos\theta\,\frac{\partial}{\partial\theta}K(r,\theta)\Big\} \ , \\
G_{22}(r;\theta) &= \cos\theta\, K(r,\theta) - \frac{R^2-r^2}{2r^2}\Big\{\cos\theta\,\Big(r\frac{\partial}{\partial r}\,K(r,\theta)\Big) - \sin\theta\,\frac{\partial}{\partial\theta}K(r,\theta)\Big\} \\
&\qquad +\frac{R^2-r^2}{2\pi Rr} \ .
\end{aligned}$$

The pressure $p(r,\theta)$ is related to the velocities $u_r(R,\theta)$ and $u_\theta(R,\theta)$ on the boundary by analogous Poisson type integral formula:

$$\begin{aligned}
p(r,\theta) = -\frac{2\eta}{r}\Big\{ \int\limits_0^{2\pi} P_r(r;\theta-\varphi) u_r(Re^{\mathrm{i}\varphi})\ d\varphi + \int\limits_0^{2\pi} P_\theta(r;\theta-\varphi) u_\theta(Re^{\mathrm{i}\varphi}) d\varphi \Big\} \\
+\frac{1}{2\pi}\int\limits_0^{2\pi} p(R,\varphi)\, d\varphi
\end{aligned} \tag{47}$$

where

$$P_r(r;\theta) = \cos\theta\,\big(r\frac{\partial}{\partial r}K(r,\theta)\big) - \sin\theta\,\frac{\partial}{\partial\theta}K(r,\theta)\ ,$$

$$P_\theta(r;\theta) = \sin\theta\,\big(r\frac{\partial}{\partial r}K(r,\theta)\big) + \cos\theta\,\frac{\partial}{\partial\theta}K(r,\theta)\ .$$

The function $K(r,\theta)$ is the Poisson kernel for the harmonic equation in the circular domain of radius R:

$$K(r,\theta) = \frac{R^2 - r^2}{2\pi(R^2 + r^2 - 2Rr\cos\theta}\ , \quad 0 \le r \le R\ .$$

We define here $\rho = r/R$, so that $\rho < 1$, and in the notation

$$\begin{aligned}
\lambda_{11} &= \frac{1}{\rho} + \frac{k(1-\rho^2)}{2\rho}\ , \quad \lambda_{12} = -\frac{k(1-\rho^2)}{2\rho}\ ,\\
\lambda_{21} &= -\frac{(k+2)(1-\rho^2)}{2\rho}\ , \quad \lambda_{22} = \rho - \frac{k(1-\rho^2)}{2\rho}\ , \quad k = 1,2\ldots,
\end{aligned} \tag{48}$$

we obtain the series expansions in the form

$$G_{11}(r;\theta) = \frac{\rho}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{11}(\rho)\rho^k\cos(k\theta)\ , \quad G_{12}(r;\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{12}(\rho)\,\rho^k\sin(k\theta)\ ,$$
$$G_{21}(r;\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{21}(\rho)\,\rho^k\sin(k\theta)\ , \quad G_{22}(r;\theta) = \frac{\rho}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{22}(\rho)\,\rho^k\cos(k\theta)\ .$$

Now we note that all the statements presented for the exterior problem hold true for the interior problem, up to the section 4, we just have to replace the coefficients λ_{ij} in the functions G_{ij} by the expressions (48). The pressure however should be analyzed separately.

Indeed, the Poisson formula (47) for the interior problem has a structure different from that of the exterior problem, compare with (5).

After substituting the series expansions for the kernels we find that

$$p(r,\theta) = -\frac{\eta}{\pi R}\Bigg\{\int_0^{2\pi}\Big[1 + \sum_{k=1}^{\infty}(k+1)\rho^k\cos\left[(k(\theta-\varphi)\right]\Big]u_r(\varphi)d\varphi$$
$$+\int_0^{2\pi}\Big[\sum_{k=1}^{\infty}(k+1)\rho^k\sin\left[(k(\theta-\varphi)\right]\Big]u_\theta(\varphi)d\varphi\Bigg\} + \frac{1}{2\pi}\int_0^{2\pi}p(R,\varphi)d\varphi. \tag{49}$$

Integrating both parts of this equality from $\theta = 0$ to 2π we conclude that

$$\int_0^{2\pi} u_r(R,\theta)d\theta = 0 \tag{50}$$

which is just the consistency condition for the interior Dirichlet problem of the Stokes equation

$$\int_{S(0,R)} \mathbf{u}\cdot\mathbf{n}\,ds = 0$$

which does not appear in the case of the exterior Dirichlet problem.

From this we see that the random process on the boundary $p(R, \theta)$ is homogeneous with respect to the angular coordinate, with a constant mean $p_0 = \bar{p}(R, \cdot)$, and moreover, by (49), $\bar{p}(r, \theta) = p_0$ inside the disc.

From the representation (49) we can find the correlation function of the pressure. The last term $\bar{P} = \frac{1}{2\pi} \int\limits_0^{2\pi} p(R, \varphi) d\varphi$ in this expression is just a random variable which is normally distributed, so it does not affect the angular behaviour of the correlation function. Note also that due to the property (50) we can assume without loss of generality that the mean of the pressure is zero. Thus it is sufficient to find the correlation function

$$B_p = \langle (p(r_1, \theta_1) - \bar{P})(p(r_2, \theta_2) - \bar{P}) \rangle \ .$$

Direct evaluation yields by (49)

$$B_p = \frac{2\eta^2}{\pi R^2} \sum_{k=1}^{\infty} (k+1)^2 \rho_1^k \rho_2^k \cos[k(\theta_2 - \theta_1)] \ .$$

7 Numerical results

We present in this section the results of numerical experiments for the Stokes problem. The angular and radial behaviour of the correlations for the velocity, pressure, stress and vorticity is analyzed. We also compare these functions for interior and exterior problems.

Let us start with the velocity correlation tensor for the exterior Stokes problem. In Figure 1 we present the angular behaviour of the correlation functions B_{11} (left panel) and B_{22} (right panel), for different values of ρ_1 and ρ_2. An interesting feature of the correlation function B_{22} is that unlike B_{11}, there is a peak around the angle $\theta = 15°$ for a certain range of the radial values of ρ_1 and ρ_2. In Figure 2, we present these correlations versus the angle θ, for fixed and equal values of ρ_1 and ρ_2. It is clearly seen that as the distance to the boundary increases (ρ decreases), the fluctuation intensity decreases while the correlation length increases. In Figure 3 the cross-correlations B_{12} and B_{21} are shown. Here we check the property $B_{ij}(\theta, \rho_1, \rho_2) = B_{ji}(\theta, \rho_2, \rho_1)$ (left panel). In the right panel of Figure 3 we show the function B_{21} for different values of ρ_1, ρ_2.

Let us turn to the pressure correlation function. To illustrate the rate of convergence of the K-L expansion, we compare in Figure 4 the pressure correlation $B_p(\rho_1, \rho_2; \theta)$ for different values of the retained terms n in the K-L series representation, with the exact result. It is seen that a few number of terms is sufficient for a good approximation even near the boundary where the decrease of correlations is very fast (see the right panel). We can compare this fast behaviour from the results presented in the left and right panels of Figure 5. Here it is also seen that the larger the distance from the boundary, the smaller the intensity fluctuations and the larger the correlation length.

The rate of convergence of the K-L expansion is closely related to the smoothness of the correlation kernel and L, the correlation length of the process. For example, in [16] is reported that for the particular case $B(x_1, x_2) = \sigma \, e^{-|x_2 - x_1|/L}$, an upper bound for the relative error in variance ε of the process represented by its K-L expansion is given by $\varepsilon \leq \frac{4}{\pi^2} \frac{1}{n} \frac{1}{L}$ where n is the number of retained terms.

The correlation function of the radial stress, $B_{\sigma_{rr}}$, is presented in Figure 6, for different values of ρ_1 and ρ_2: we fix the value $\rho_1 = 0.99$, and vary the value of ρ_2 from $\rho_2 = 0.1$ (small values

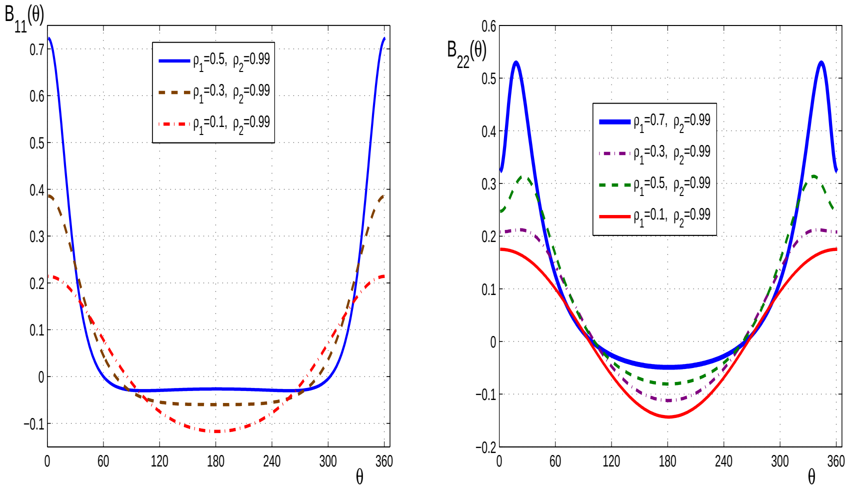

Figure 1: Angular correlation function for B_{11} (left panel) and B_{22} (right panel), for different values of ρ_1 and ρ_2.

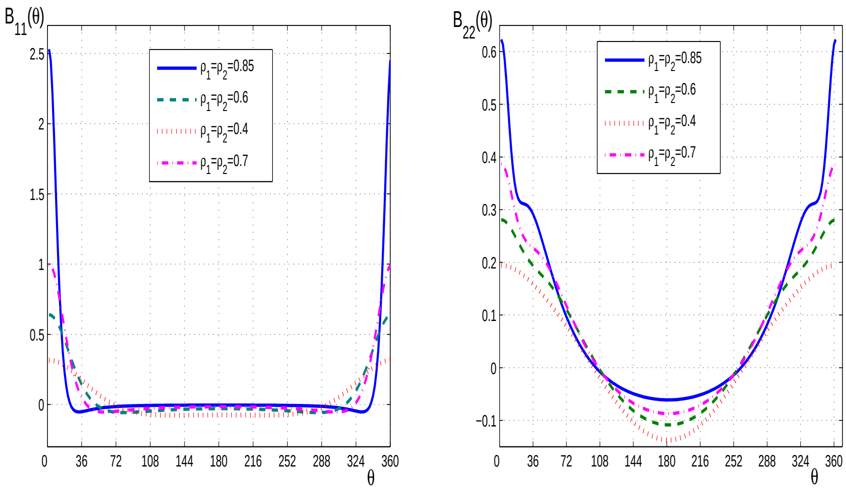

Figure 2: Angular correlation function for B_{11} (left panel) and B_{22} (right panel), for equal values of ρ_1 and ρ_2.

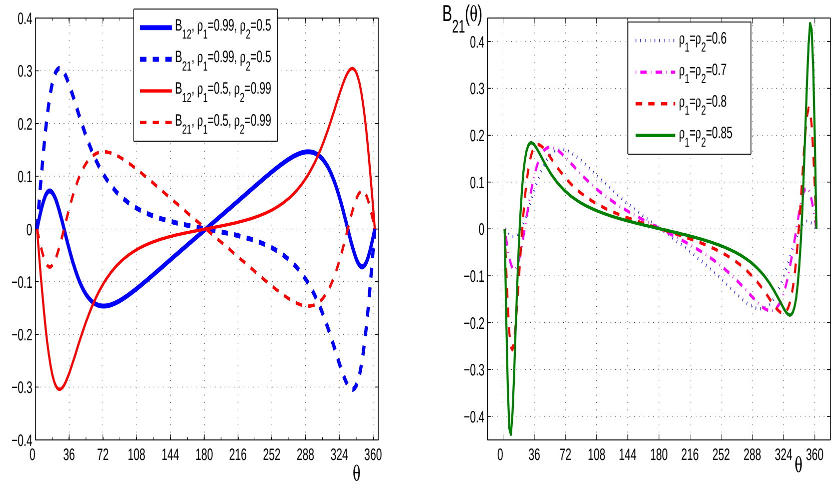

Figure 3: Angular correlation functions B_{12} and B_{21}, for different values of ρ_1 and ρ_2 (left panel), and B_{21}, for equal values of ρ_1 and ρ_2 (right panel).

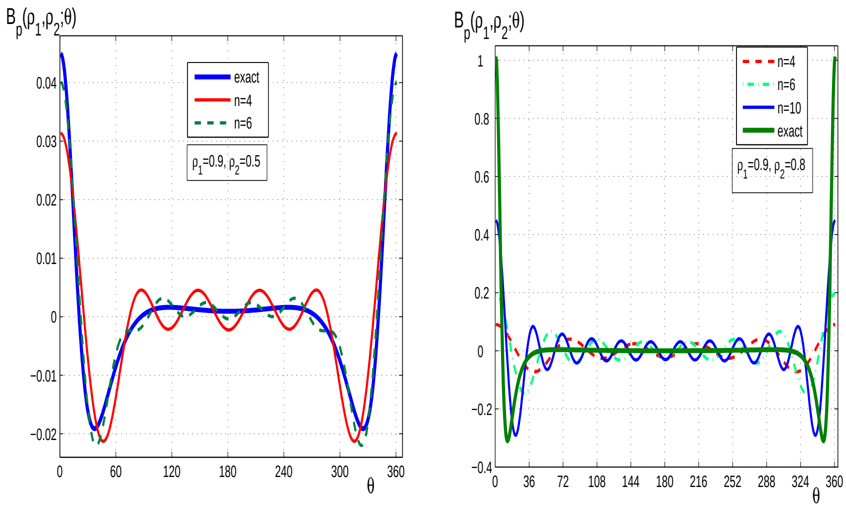

Figure 4: Comparison of the K-L-approximation with the exact result: angular correlation function of the pressure $B_p(\rho_1, \rho_2; \theta)$, for different values of ρ_1 and ρ_2, and for different number of terms in the K-L expansion.

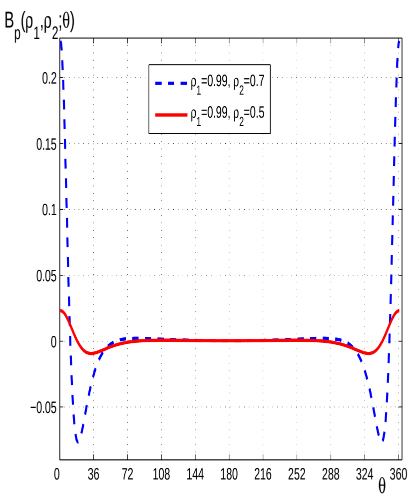

Figure 5: Angular correlation function of the pressure $B_p(\rho_1, \rho_2; \theta)$, for different values of ρ_1 and ρ_2.

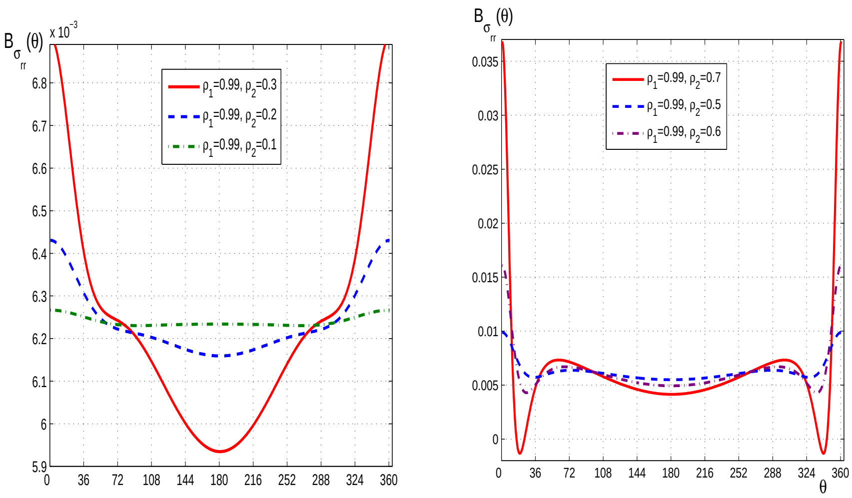

Figure 6: Angular behaviour of the correlation function of the radial stress $B_{\sigma_{rr}}$, for different values of ρ_1 and ρ_2 with ρ_1 fixed near the boundary, and varying second point: far from the boundary (small values of ρ_2, left panel), and closer to the boundary (larger values of ρ_2, right panel).

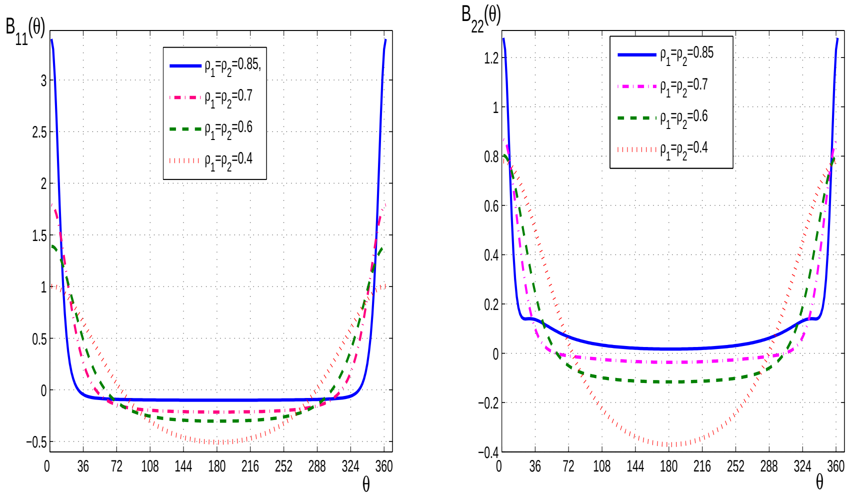

Figure 7: The velocity correlation functions B_{11} (left panel), and B_{22} (right panel), for equal values of ρ_1 and ρ_2. Interior Stokes problem.

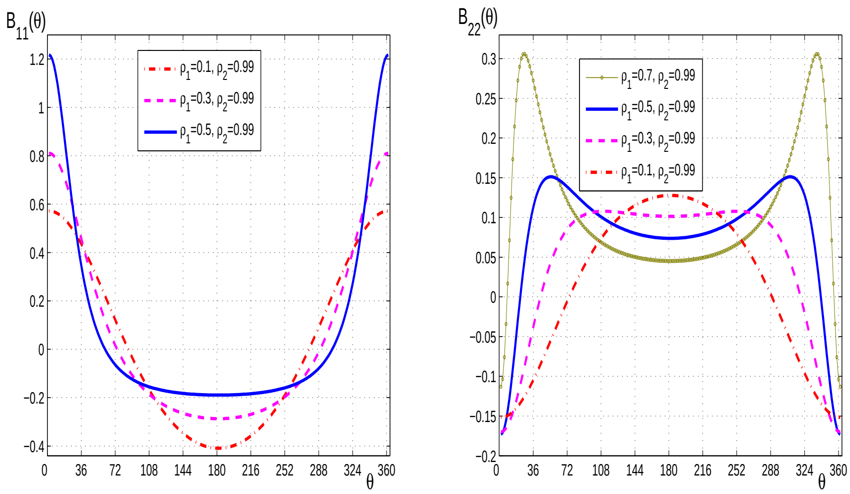

Figure 8: The velocity correlation functions B_{11} (left panel), and B_{22} (right panel), for different values of ρ_1 and ρ_2. Interior Stokes problem.

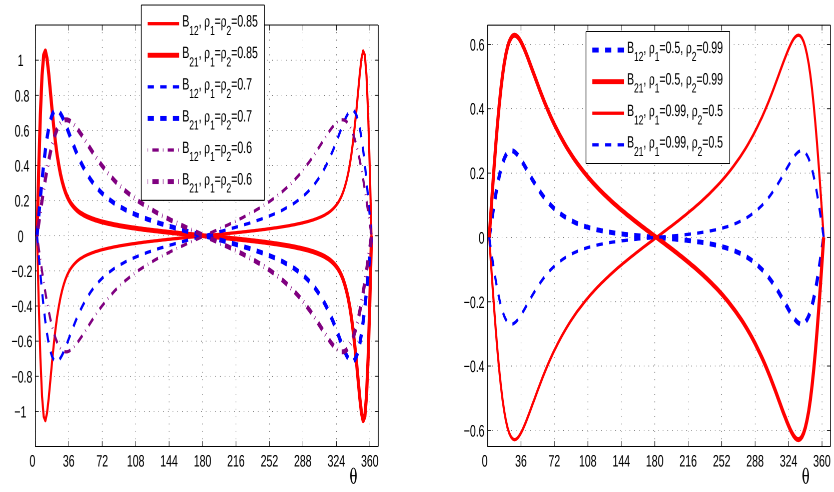

Figure 9: The velocity cross-correlation functions B_{12} and B_{21}, for equal values of ρ_1 and ρ_2 (left panel), and different values of ρ_1 and ρ_2 (right panel). Interior Stokes problem.

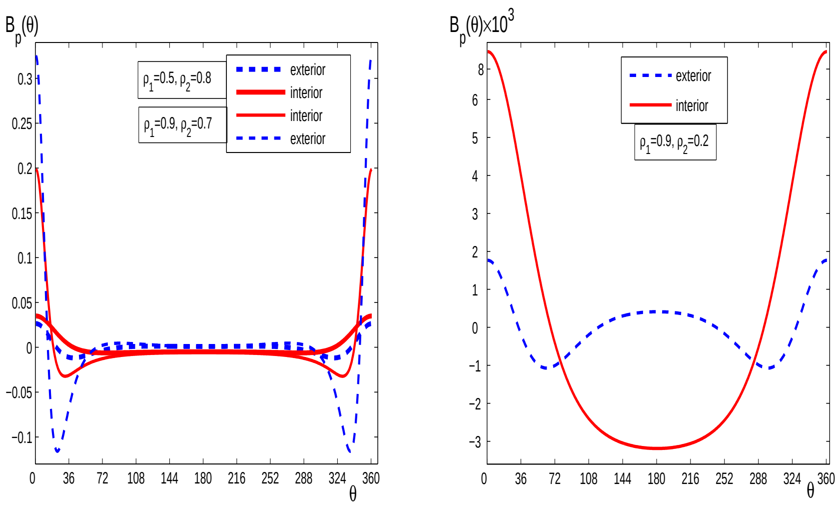

Figure 10: Comparison of the pressure correlation functions for the exterior and interior Stokes problems, for different values of ρ_1 and ρ_2.

in the left panel), to $\rho_2 = 0.7$ (large values in the right panel). Here we can see how strong the correlation function is changed when varying the radial coordinate.

Let us consider the interior Stokes problem. The correlation function of the velocity is presented in figures 7 and 8. In Figure 7 we present the velocity correlation functions B_{11} (left panel) and B_{22} (right panel), for equal values of the radial coordinate, where $\rho = \rho_1 = \rho_2$ is varied from 0.4 to 0.85. The same curves are shown in Figure 8, but for different values of the radial coordinates: we fix $\rho_2 = 0.99$, and vary ρ_1 from 0.1 to 0.7. Comparing the results for these two cases, we note that there is no qualitative difference between the correlations B_{11} in Figure 7 and 8, while the functions B_{22} do differ dramatically. This illustrates how complex might be the behaviour of inhomogeneous random fields.

This is also illustrated by the results presented in Figure 9 where we show the cross-correlations B_{12} and B_{21} of equal (left panel) and different (right panel) values of ρ_1, ρ_2.

It is also interesting to compare the correlation functions for the interior and exterior problems, which can be done by a comparative analysis of results presented in Figures 1 - 6 and that shown in Figures 7-9. We present here only the pressure correlation functions, see Figure 10, for different values of ρ_1, ρ_2.

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