Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

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submitted: 16th December 1994

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Preprint No. 136
Berlin 1994

1991 Mathematics Subject Classification. 22E40, 22E46, 58F17, 58F18, 58F20.

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Secondary Euler characteristics of locally symmetric spaces. Results and conjectures

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0. The present paper deals with conjectural generalizations to the higher rank case of some index formulas previously found in [4] as an intrinsic part of the cohomological theory of the dynamical zeta function

$$Z_R(s) = \prod_c (1 - \exp(-s|c|))^{-1}$$

of the geodesic flow Φ_t of compact locally symmetric spaces X of rank 1. Here the product runs over the prime directed closed geodesics c in X and $|c|$ denotes length. Z_R will be called the Ruelle zeta function of Φ_t. The functional equation of Z_R relates $Z_R(s)$ to $Z_R(-s)$ and if the dimension of X is *even* then there are always two types of formulas for the multiplicities of the zeros and poles of $Z_R(s) \cdot Z_R(-s)$. In fact, the multiplicity of each singularity of the product $Z_R(s) \cdot Z_R(-s)$ can be calculated by a formula of analytical nature (analytical index) as well as by a formula only involving characteristic classes. For $s = 0$ the equality of both integers can be regarded as an index formula for a secondary type Euler characteristic of the space of *directed geodesics* in X being defined by using the hyperbolic structure of the geodesic flow Φ_t.
Here we are concerned with the problem of generalizing the index formula (theorem 1) to the space of all *directed flats* (See [7]) in a compact locally symmetric space X of arbitrary rank.
If the rank of X exceeds 1 then no relation of the index formulas to the theory of zeta functions is known.

1. Let $Y = G/K$ be a Riemannian symmetric space of the non–compact type, $\Gamma \subset G$ a uniform lattice without torsion and $X = \Gamma\backslash Y$ the compact C^∞ locally symmetric quotient space. Let $P \subset G$ be a minimal parabolic subgroup (in standard position) with Langlands decomposition $P = MAN^+$. Here M is a compact subgroup of K, $A = \exp(\mathfrak{a}_0)$ is a vector group with Lie algebra $\mathfrak{a}_0 \subset \mathfrak{p}_0$ ($\mathfrak{p}_0$ being the (-1)–eigenspace of the infinitesimal Cartan involution Θ) of dimension $rk(G/K)$ and N^+ is a nilpotent Lie group with Lie algebra $\mathfrak{n}_0^+$. Recall that the choice of P (and hence that of N^+) corresponds to the choice of an open Weyl chamber $\mathfrak{a}_0^+$ in $\mathfrak{a}_0$. In fact, $\mathfrak{a}_0^+$ determines a positive system $\Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)$ of roots of the adjoint action of $\mathfrak{a}_0$ on $\mathfrak{g}_0$ and we have

$$\mathfrak{n}_0^+ = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)} (\mathfrak{g}_0)_\alpha.$$

Let $N^- = \Theta N^+$ with Lie algebra $\mathfrak{n}_0^- = \Theta \mathfrak{n}_0^+$ given by

$$\mathfrak{n}_0^- = \bigoplus_{-\alpha \in \Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)} (\mathfrak{g}_0)_\alpha.$$

2. Let $\rho : \Gamma \to U(H)$ be a finite–dimensional unitary representation. Let

$L^2(\Gamma\backslash G,\rho)$ be the Hilbert space of sections of the vector bundle $G\times_\Gamma H\to\Gamma\backslash G$ being square–integrable (with respect to an invariant measure on $\Gamma\backslash G$). The right regular representation $R_{\Gamma,\rho}$ of G on $L^2(\Gamma\backslash G,\rho)$ splits as a direct sum

$$(1)\qquad (R_{\Gamma,\rho},L^2(\Gamma\backslash G,\rho))=\bigoplus_{\pi\in\hat{G}}N_{\Gamma,\rho}(\pi)\,(\pi,V_\pi)$$

of irreducible unitary representations (π,V_π) of G each occuring with finite multiplicity $N_{\Gamma,\rho}(\pi)$.

3. Let $\mathfrak{g}=(\mathfrak{g}_0)_{\mathbb{C}}$, $\mathfrak{n}^\pm=(\mathfrak{n}_0^\pm)_{\mathbb{C}}$ and denote by $H^*(\mathfrak{n}^\pm,V)$ the $\mathfrak{n}^\pm$–cohomology of the $\mathfrak{g}$–module V. If V is the Harish–Chandra module ($(\mathfrak{g},K)$–module) of K–finite vectors of a (global) representation (π,V_π) of G then we write $V=V_{\pi,0}$. $V_{\pi,0}$ consists of smooth vectors. The cohomology groups $H^*(\mathfrak{n}^\pm,V_{\pi,0})$ are finite dimensional MA–modules ([2]).

4. Now we define the *secondary Euler characteristic* of (Γ,ρ) as the integer

(2)

$$ch(\Gamma,\rho)\ \stackrel{\text{def}}{=}\ \sum_{p,q}(-1)^{p+q}\left(\sum_{\pi\in\hat{G}}N_{\Gamma,\rho}(\pi)\dim_{\mathbb{C}}(H^p(\mathfrak{n}^-,V_{\pi,0})\otimes\Lambda^q(\mathfrak{n}^+)^*)^{MA}\right).$$

Here the exterior powers $\Lambda^*(\mathfrak{n}^+)^*$ are considered as MA–modules (with respect to the coadjoint action).
The sum (2) is finite since for each choice of p and q there are only finitely many representations $\pi\in\hat{G}$ with a nontrivial contribution to the inner sum. The notation $ch(\Gamma,\rho)$ does not reflect the choices of $\mathfrak{n}_0^-$ and $\mathfrak{n}_0^+$ since the sum on the right hand side of (2) is *independent* of such choices. In fact, the Weyl group $W=W(\mathfrak{g}_0,\mathfrak{a}_0)=M'/M$ operates simply transitive on the set of Weyl chambers in $\mathfrak{a}_0$. This implies that all pairs $(\mathfrak{n}_0^-,\mathfrak{n}_0^+)$ of opposite algebras can be obtained from a fixed one by the action of M'/M via the adjoint action. Since the group MA is stable under conjugation by Weyl group elements $\in M'/M$ this shows that the definition of $ch(\Gamma,\rho)$ is independent of the choices of $\mathfrak{n}_0^-$ and $\mathfrak{n}_0^+$.

5. For $\rho=1$ there is a *formal interpretation* of the integer $ch(\Gamma)=ch(\Gamma,1)$ as the *Euler characteristic* of the space of Γ–orbits on the space Y_{DF} of all directed flats in Y.
Here a flat is a totally geodesic flat submanifold of maximal dimension ($=rk(Y)$). Each geodesic in Y is contained in at least one flat. A geodesic which uniquely determines the flat containing it is called regular. A regular directed geodesic determines a flat together with a distinguished class of asymptotic Weyl chambers (directed flat). As a G–space the space of all directed flats in Y is isomorphic to G/MA. If the rank of Y is 1 then directed flats are directed geodesics and the space $Y_{DF}=G/MA$ is the space of directed geodesics in Y. See [7].
The space $X_{DF}=\Gamma\backslash Y_{DF}$ of Γ–orbits on Y_{DF} is, however, far from being a manifold!
The action of A on the space $\Gamma\backslash G/M$ of all Weyl chambers in $X=\Gamma\backslash Y$ is ergodic (see [10]) and the closed orbits of this action are dense in the space of

all orbits (see [9]). These results are generalizations of well–known results on the geodesic flow of compact locally symmetric spaces of rank one.
Now the definition of the secondary Euler characteristic $ch(\Gamma)$ is formally analogous to the following formula for the Euler characteristic of a locally homogeneous complex manifold $M = \Gamma\backslash G/H$. Here G and Γ are as before and H is a *compact* Cartan subgroup of G. The Euler characteristic $\chi(M)$ of M can be written in the form

(3)

$$\begin{aligned}\chi(M) &= \sum_{p,q}(-1)^{p+q}h^{p,q} \\ &= \sum_{p,q}(-1)^{p+q}\left(\sum_{\pi\in\hat{G}} N_{\Gamma,1}(\pi)\dim_{\mathbb{C}}(H^p(\mathfrak{n}_H^-, V_{\pi,0})\otimes\Lambda^q(\mathfrak{n}_H^+)^*)^H\right),\end{aligned}$$

where $\mathfrak{n}_H^-$ and $\mathfrak{n}_H^+ = \overline{\mathfrak{n}_H^-}$ are the (opposite) nilpotent Lie subalgebras of $\mathfrak{g}$ defined by

$$\mathfrak{n}_H^\pm = \bigoplus_{\alpha\in\Delta^\pm(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha.$$

Here $\Delta^+(\mathfrak{g},\mathfrak{h})$ is the set of positive roots of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ with respect to the choice of an order. Then $\Delta = \Delta^+\cup(-\Delta^+)$, $\Delta^- = -\Delta^+$ and

$$\mathfrak{g} = \mathfrak{n}_H^- \oplus \mathfrak{h} \oplus \mathfrak{n}_H^+.$$

Note that the situations in (2) and (3) both are extreme cases in the sense that the algebras $\mathfrak{n}^\pm$ (in (2)) are *totally real*, i.e., $\overline{\mathfrak{n}^\pm} = \mathfrak{n}^\pm$, whereas the algebras $\mathfrak{n}_H^\pm$ (in (3)) are *totally complex*, i.e., $\overline{\mathfrak{n}_H^\pm} = \mathfrak{n}_H^\mp$.
The analogy of (2) and (3) justifies it to refer to $ch(\Gamma)$ (and $ch(\Gamma,\rho)$) also as an Euler characteristic. More precisely, $ch(\Gamma)$ will be called the *horospherical Euler characteristic* of Γ by a reason which is explained in the following point.

6. From the point of view of the geometry of geodesics in Y_{DF} the minimal parabolic subgroup $P = MAN^+$ can be regarded as the subgroup of isometries (in G) operating on the family of all directed geodesics which are asymptotic to a given regular directed geodesic $c_0 = \exp(tX_0)K$, $X_0 \in \mathfrak{a}_0^+$ in $Y = G/K$. c_0 is regular since X_0 is regular in $\mathfrak{a}_0$, i.e., $\alpha(X_0) \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}_0,\mathfrak{a}_0)$. The group $L = MA$ then is the subgroup of P consisting of all isometries leaving invariant the parallel set $P(c_0)$ of c_0. Here the parallel set $P(c_0)$ of c_0 is the set of all directed geoderics c which are parallel to c_0 in the sense that the distance between c and c_0 is finite. If the rank of Y is 1 then $P(c_0) = c_0$. See ([8]). M is the subgroup of L consisting of the isometries leaving the elements of $P(c_0)$ pointwise fixed. Moreover, the group N^+ acts simply transitive on each submanifold of Y which is orthogonal to all geodesics being asymptotic to c_0. These submanifolds are the *horospheres* (of maximal dimension) and N^+ is referred to as the horospherical group of the family defined by c_0.

7. In the definition (2) the *symplectic structure* also plays an important role. In fact, the space G/MA carries a canonical invariant symplectic form since it can be identified with the G–adjoint orbit through the (regular) element $X_0 \in \mathfrak{a}_0$ (Kostant, Kirillov).
The G–invariant tangent bundles $T^\pm(G/MA) \subset T(G/MA)$ with the property

$$T^{\pm}_{\underline{e}}(G/MA) \simeq \mathfrak{n}_0^{\pm},\ \underline{e} = eMA$$

are involutive Lagrangian subbundles, i.e., both subbundles $T^{\pm}$ are (real) invariant polarizations of G/MA. In other words, the space of directed flats is bipolarized.
Thus the integer $ch(\Gamma)$ can be regarded also as being canonically associated to the Γ–action on the *bipolarized symplectic space* Y_{DF}.

8. Now having introduced $ch(\Gamma, \rho)$ the obvious problem is to calculate it. Here it turns out to be usefull to follow the suggestion of the analogy between (2) and (3) a bit further.
The Euler characteristic $\chi(M)$ of the complex manifold $M = \Gamma\backslash G/H$ can be calculated as the integral

$$(4) \qquad \int\limits_{\Gamma\backslash G/H} c_n(T^{(1,0)}(M)),\ n = \dim_{\mathbb{C}}(M),$$

where $c_n(T^{(1,0)}(M))$ is the top degree Chern class of the *holomorphic* tangent bundle $T^{(1,0)}(M)$ of M. The class $c_n(T^{(1,0)}(M)) \in H^{2n}(M)$ can be represented, for instance, by the differential form

$$c = c_{\alpha_1} \wedge \cdots \wedge c_{\alpha_n},\ \ \alpha_j \in \Delta^+(\mathfrak{g}, \mathfrak{h}),$$

where the 2–forms $c_{\alpha_j} \in C^\infty(\wedge^2 T^*(M))$ are induced by G–invariant 2–forms $\underline{c}_{\alpha_j}$ on G/H such that

$$\underline{c}_{\alpha_j}(X,Y)_{eH} = -\frac{i}{2\pi}\,\alpha([X,Y]),\ X,Y \in \mathfrak{g}_0.$$

In other words, the G–invariant volume form $\underline{c}$ on G/H (inducing c on $\Gamma\backslash G/H$) is given by

$$(5) \qquad \underline{c} = \det\left(\frac{i}{2\pi}\,\underline{\Omega}\right),$$

where $\underline{\Omega} \in C^\infty(\wedge^2\, T^*(G/H),\ \mathrm{End}(T^{(1,0)}(G/H))$ is the G–invariant $\mathrm{End}(T^{(1,0)}(G/H))$–valued 2–form (curvative form) given in eH by

$$\underline{\Omega}(X,Y)_{eH} = -[[X,Y]_{\mathfrak{h}}, \cdot].$$

Here the subscript $\mathfrak{h}$ denotes the $\mathfrak{h}$–component of elements in $\mathfrak{g}$ with respect to the decomposition

$$\mathfrak{g} = \mathfrak{n}_H^- \oplus \mathfrak{h} \oplus \mathfrak{n}_H^+.$$

The analogy therefore suggests to look for a formula of a similar type for $ch(\Gamma)$. But besides the formal analogy there is, at least at first sight, not much evidence for such a result. In fact, the non–ellipticity of the invariant complex on G/MA being responsible for the definition of $ch(\Gamma)$ rather suggest *not to expect* a result of such a type. In fact, the $\mathfrak{n}^-$–cohomology of Harish–Chandra modules is well–known to be a much more subtle subject than the $\mathfrak{n}_H^-$–cohomology (see ([1])!

9. However, in contrast to these arguments, we have the following result.

Theorem 1. *(Index formula). Let the rank of Y be 1 and assume that there exists a compact Cartan subgroup in G. Then*

$$ch(\Gamma,\rho) = \dim(\rho) \int\limits_{\Gamma\backslash G/M} \det\left(\frac{i}{2\pi}\pi^*(\underline{\Omega}^\pm)\right) \wedge \alpha^\pm, \tag{6}$$

$\pi : G/M \to G/MA$, where $\underline{\Omega}^\pm$ is the G–invariant End($T^\pm(G/MA)$)–valued 2–form (curvative form) on G/MA which is given in $\underline{e} = eMA$ by

$$\underline{\Omega}^\pm(X,Y)_{\underline{e}} = -[[X,Y]_{\mathfrak{m}_0\oplus\mathfrak{a}_0},\cdot],\ X,Y \in \mathfrak{g}_0,$$

where the subscript $\mathfrak{m}_0 \oplus \mathfrak{a}_0$ denotes the $\mathfrak{m}_0 \oplus \mathfrak{a}_0$– component of elements in $\mathfrak{g}_0$ with respect to the decomposition

$$\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus (\mathfrak{m}_0 \oplus \mathfrak{a}_0) \oplus \mathfrak{n}_0^+.$$

$\alpha^\pm$ is an A–invariant 1–form which is canonically determined by $\pi^*(\Omega^\pm)$.
The integer $ch(\Gamma) = ch(\Gamma,1)$ coincides with the multipicity of the singularity of the Ruelle zeta function Z_R in $s = 0$.

For more details see ([4]), ([5]), where the case $\rho = 1$ is discussed. The same arguments extend to the more general situation in theorem 1.
Note that in contrast to (4) the class represented by

$$\det\left(\frac{i}{2\pi}\pi^*(\Omega^\pm)\right) \wedge \alpha^\pm$$

is of *odd degree* and should be regarded as a *secondary* characteristic class. In the case of the upper half plane $Y = H^2$ the forms under the integral in (6) represent $(2\pi)^{-2}$ times the Godbillon–Vey class of the weak–unstable (or weak–stable) foliation of $S(X)$.
The assumption of the existence of a compact Cartan subgroup is equivalent to the condition that the dimension of Y is *even*. If there is *no* compact Cartan subgroup but Y is still of rank 1 then theorem 1 is no longer true (see also 16.).

Now we formulate

Conjecture 1. *(Index formula) Let $Y = G/K$ be a Riemannian symmetric space of the non–compact type and arbitrary rank. Assume that there exists a compact Cartan subgroup in G.*
Then there is an A–invariant volume form $\alpha^\pm$ (of degree rk(Y)) on the A–orbits (being canonically determined by $\pi^*(\underline{\Omega}^\pm)$) such that

$$ch(\Gamma,\rho) = \dim(\rho) \int\limits_{\Gamma\backslash G/M} \det\left(\frac{i}{2\pi}\pi^*(\underline{\Omega}^\pm)\right) \wedge \alpha^\pm,\ \pi : G/M \to G/MA \tag{7}$$

where $\underline{\Omega}^\pm$ is the G–invariant End($T^\pm(G/MA)$)–valued (curvature) 2–form on G/MA given by

$$\underline{\Omega}^\pm(X,Y)_{eMA} = -[[X,Y]_{\mathfrak{m}_0\oplus\mathfrak{a}_0},\cdot],\ X,Y \in \mathfrak{n}_0^- \oplus \mathfrak{n}_0^+.$$

Conjecture 1 is a direct generalization of theorem 1 to the arbitrary rank case.

10. In the situation of theorem 1 there is also a proportionality formula relating $ch(\Gamma, \rho)$ to the Euler characteristic of the space Y^d_{geo} of directed geodesics in the *compact dual* space Y^d. (see [4]). More precisely, we have

Theorem 2. *(Proportionality). Let the situation be as in theorem 1. Then*

$$ch(\Gamma, \rho) = \dim(\rho) \frac{\chi(X)}{\chi(Y^d)} \chi\left(Y^d_{geo}\right). \tag{8}$$

In the case of compact quotients $\Gamma \backslash H^n_{\mathbb{H}}$ of quaternionic hyperbolic spaces $H^n_{\mathbb{H}}$ (and $\rho = 1$) a proof of theorem 2 by explicating the definition of $ch(\Gamma)$ in terms of the multiplicities $N_{\Gamma,1}(\pi)$ is given in [11].

Now theorem 2 suggests

Conjecture 2. *(Proportionality) Let the situation be as in conjecture 1. Then*

$$ch(\Gamma, \rho) = \dim(\rho) \frac{\chi(X)}{\chi(Y^d)} \chi\left(Y^d_{DF}\right), \tag{9}$$

where Y^d_{DF} is the space of directed flats in the compact dual symmetric space Y^d.

11. *Remarks.*
(*i*) The assumption of the existence of a compact Cartan subgroup in conjecture 1 is *essential.* If this condition is violated then the resulting formulas are no longer expected.
(*ii*) In the rank 1 case (theorem 1) the 1–forms $\alpha^\pm$ are closely related to the *real roots* in $\Delta^\pm(\mathfrak{g}, \mathfrak{t} \oplus \mathfrak{a})$, $T \subset M$ a maximal torus in M with Lie algebra $\mathfrak{t}_0$ (see [4], [5]). In the general case it is natural to expect that the (volume) forms $\alpha^\pm$ are canonically determined by the subsystem of real roots in $\Delta^\pm(\mathfrak{g}, \mathfrak{t} \oplus \mathfrak{a})$.
(*iii*) From the point of view of conjecture 2 a natural *normalization* condition (compatible with theorem 1 and conjecture 1) which would uniquely determine the forms $\alpha^\pm$ (up to a sign) is that the induced volume of the flat tori in Y^d is equal to 1.
(*iv*) Let $(\sigma, V_\sigma) \in \hat{M}$. Then we conjecture the following even more general results.

$$\begin{aligned} ch(\Gamma, \rho; \sigma) &\stackrel{\text{def}}{=} \sum_{p,q} (-1)^{p+q} \Big(\sum_{\pi \in \hat{G}} N_{\Gamma,\rho}(\pi) \dim_{\mathbb{C}}(H^p(\mathfrak{n}^-, V_{\pi,0}) \otimes \Lambda^q(\mathfrak{n}^+)^* \otimes V_\sigma)^{MA} \Big). \\ (10) \qquad &= \dim(\rho) \dim(\sigma) \int_{\Gamma\backslash G/M} \det\left(\frac{i}{2\pi} \pi^*(\underline{\Omega}^\pm)\right) \wedge \alpha^\pm \\ &= \dim(\rho) \dim(\sigma) \frac{\chi(X)}{\chi(Y^d)} \chi(Y^d_{DF}) \end{aligned}$$

if σ gives rise to a homogeneous vector bundle over Y^d_{DF}.
In the rank 1 case $ch(\Gamma;\sigma) \stackrel{\text{def}}{=} ch(\Gamma, 1 : \sigma)$ coincides with the multiplicity $\mathrm{ord}_0(Z_\sigma)$ of the singularity of the *twisted* Ruelle zeta function

$$(11) \qquad Z_\sigma(s) \stackrel{\text{def}}{=} \prod_c \det(1 - \sigma(m_c)\exp(-s|c|))^{-1}$$

in $s = 0$. This generalizes the formula $\mathrm{ord}_0(Z_R) = ch(\Gamma, 1)$ and can be proved also by using the same methods as in [4].

12. Let $T \subset M$ be a maximal torus with Lie algebra $\mathfrak{t}_0$. Then TA is a Cartan subgroup of G and we define

(12)

$$ch(\Gamma,\rho;TA) \stackrel{\text{def}}{=} \sum_{p,q}(-1)^{p+q}\Big(\sum_{\pi\in\hat{G}} N_{\Gamma,\rho}(\pi)\dim_{\mathbb{C}}(H^p(\mathfrak{n}^-_{TA}, V_{\pi,0}) \otimes \Lambda^q(\mathfrak{n}^+_{TA})^*)^{TA}\Big).$$

Here the complex Lie algebras $\mathfrak{n}^{\pm}_{TA} \subset \mathfrak{g}$ are defined as

$$\mathfrak{n}^{\pm}_{TA} = \bigoplus_{\alpha\in\Delta^{\pm}(\mathfrak{g},\mathfrak{t}\oplus\mathfrak{a})} \mathfrak{g}_\alpha,$$

for any choice of a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t}\oplus\mathfrak{a})$. In analogy to

$$\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus (\mathfrak{m}_0 \oplus \mathfrak{a}_0) \oplus \mathfrak{n}_0^+.$$

we have the more refined decomposition

$$\mathfrak{g} = \mathfrak{n}^-_{TA} \oplus (\mathfrak{t}\oplus\mathfrak{a}) \oplus \mathfrak{n}^+_{TA}.$$

$ch(\Gamma,\rho;TA)$ is *independent* of the choice of $\mathfrak{n}^{\pm}_{TA}$ and we have

$$(13) \qquad ch(\Gamma,\rho;TA) = \chi(M/T)\, ch(\Gamma,\rho).$$

13. In a similar way as in (12) one can define $ch(\Gamma,\rho;L)$ with respect to *any* Cartan subgroup L (with Lie algebras $\mathfrak{l}_0$) of G.
In fact, replace in (12) TA by L and $\mathfrak{n}^{\pm}_{TA}$ by $\mathfrak{n}^{\pm}_L$, where $\mathfrak{n}^{\pm}_L$ are the nilpotent Lie algebras determined by the polarization

$$\mathfrak{g} = \mathfrak{n}^-_L \oplus \mathfrak{l} \oplus \mathfrak{n}^+_L$$

of the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{l}$ given by a choice of a positive system in $\Delta(\mathfrak{g},\mathfrak{l})$. Then the L–modules $H^*(\mathfrak{n}^-_L, V_{\pi,0})$ are finite–dimensional and $ch(\Gamma,\rho;L)$ is well–defined.

Conjecture 3. *Let $Y = G/K$ be a Riemannian symmetric space of the non–compact type and arbitrary rank. Then*

$$(14) \qquad ch(\Gamma,\rho;TA) = ch(\Gamma,\rho;L_{fund}).$$

where L_{fund} is a fundamental Cartan subgroup of G (being uniquely determined up to conjugation).

Conjecture 3 relates the integers $ch(\Gamma,\rho;L)$ for the both *extreme* types of Cartan subgroups: the maximal compact one (L_{fund}) and the maximal non–compact one (TA). If the fundamental Cartan subgroup L_{fund} is *compact*

then the right hand side of (14) coincides with the usual Euler characteristic $\chi(\Gamma\backslash G/L_{fund})$ of the complex manifold $\Gamma\backslash G/L_{fund}$.
Since on the other hand by conjecture 2 and (13)

$$\begin{aligned} ch(\Gamma,\rho;TA) &= ch(\Gamma,\rho)\,\chi(M/T) \\ &= \dim(\rho)\,\frac{\chi(X)}{\chi(Y^d)}\,\chi(Y^d_{flat})\,\chi(M/T) \\ &= \dim(\rho)\,\chi\,(\Gamma\backslash G/L_{fund}) \end{aligned}$$

we see that the conjecture 2, in fact, implies conjecture 3 if there exists a compact Cartan subgroup.
If the fundamental Cartan subgroup is *not compact* then the nature of the integer $ch(\Gamma,\rho;L_{fund})$ is still mysterious! In any case one should regard it as a *secondary Euler characteristic* of (Γ,ρ).

Conjecture 4. *Let the situation be as in conjecture 3. Then $ch(\Gamma,\rho;L_{fund})$ only depends on the group cohomology $H^*(\Gamma;\rho)$. In particular, $H^*(\Gamma;\rho)=0$ implies*
$ch(\Gamma,\rho;L_{fund})=0$.

However, we still have no general conjecture concerning an explicit description of the relation $ch(\Gamma,\rho;L_{fund})$ and $H^*(\Gamma;\rho)$.
If L_{fund} *has real rank 1* then there is also a relation between $ch(\Gamma,\rho;L_{fund})$ and the multiplicity of the singularity of a dynamical zeta function in $s=0$ (see [4]). Thus the vanishing property in conjecture 4 implies the regularity of the corresponding zeta function in $s=0$.

15. Example. Let $Y=H^{2n+1}_{\mathbb{R}}$ be an odd–dimensional real hyperbolic space and X= $\Gamma\backslash Y$ a compact quotient. Then

$$\begin{aligned} ch(\Gamma,\rho) = &\; 2\left((-1)^n\,b_{n+1}\,(X;\rho)+\cdots+(-1)^{2n}(n+1)b_{2n+1}(X;\rho)\right), \\ = &\; (\sum_p(-1)^{p+1}p\;b_p(X;\rho))+\sum_{n+1\le p\le 2n+1}(-1)^{p+1}b_p(X;\rho), \end{aligned}$$

where

$$b_p(X;\rho)=\dim(H^p(X;\rho))=\dim(H^p(\Gamma;\rho))$$

is the p^{th} ρ–twisted Betti number and $H^*(X;\rho)$ denotes the cohomology of differential forms on X with values in the vector bundle $Y\times_\Gamma V_\rho\to X$. – $ch(\Gamma,\rho)$ coincides with the multiplicity of the singularity of the Ruelle zeta function

$$Z_{R,\rho}(s)=\prod_c\det(1-\rho(c)\exp(-s|c|))^{-1}$$

of X in $s=0$.

16. We conjecture also that it is always possible to replace the abstract representation theoretical definition of $ch(\Gamma)$ by a definition only involving

differential forms with distributional coefficients on $\Gamma\backslash G/M$. In the case $Y = H_{\mathbb{R}}^{2n}$ this is discussed in detail in [6], but the general case is not yet understood!

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