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# für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.


## Comparison of interacting diffusions and an application to their ergodic theory


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submitted: 15th December 1994



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Preprint No. 135
Berlin 1994



*1991 Mathematics Subject Classification.* Primary 60K35; Secondary 60J60, 60J15.
*Key words and phrases.* Interacting diffusion, interacting particle system, clustering, preservation of convexity.

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# Comparison of interacting diffusions and an application to their ergodic theory

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comp.tex November 18, 1994

**Abstract**

A general comparison argument for expectations of certain multi-time functionals of infinite systems of linearly interacting diffusions differing in the diffusion coefficient is derived. As an application we prove clustering occurs in the case when the symmetrized interaction kernel is recurrent, and the components take values in a one-sided bounded interval. The technique gives also an alternative proof in the case of compact intervals.

*AMS Subject Classification* Primary 60 K 35; Secondary 60 J 60, 60 J 15

*Keywords* interacting diffusion, interacting particle system, clustering, preservation of convexity

## 1 Introduction and results

In [9] and implicitly in [2], a comparison of general linearly interacting diffusions with interacting Fisher-Wright diffusions was a powerful tool in the study of the long-term behavior of a class of models differing in the diffusion coefficient, in particular in establishing ***universality properties.*** For this it was important that one of the models in the comparison was Fisher-Wright, since a duality argument with delayed coalescing random walks was involved.

Here we provide a general method based on the intuition that a larger diffusion coefficient leads to a process whose distribution is more "spread out". Consequently, certain functionals of the process, such as moments in the case where the state space of the components is a compact interval in $\mathbb{R}_+$, have bigger expectations. This comparison gives a useful tool for studying cluster formation in such interacting systems. At the same time it fills a gap (in an application of the integration by parts formula involving semigroups) in the proof of Proposition 4.10 (jj) of [9] concerning the comparison with an interacting *restricted* Fisher-Wright diffusion. (See also Figure 1 below.)

Using this comparison technique we are able to resolve a problem in the ergodic theory of interacting diffusions in the case where the underlying symmetrized migration term is *recurrent*, and where the state space of a component is *one-sided bounded*. We show that *clustering is universal* in the diffusion coefficient. This had been conjectured in Cox, Greven and Shiga [3] (see also Shiga [15]). On the way we obtain also a new proof for the case of components in a compact set, based on the interacting Fisher-Wright diffusion where the well-known duality is available.

Further applications will be contained in the forthcoming paper [10] on the time-space cluster formation of hierarchically interacting systems in the regime of diffusive clustering, and in [4] where the relation between finite and infinite systems is studied.

## 1.1 The model

Consider the following model (compare with [3]).

**Definition 1 (interacting diffusion $X$)** Let $X=\{X_\xi(t);\ \xi\in\Xi,\ t\geq 0\}$ denote the unique (for each specified initial state $X(0)\in\mathbb{E}$) *strong* solution of the following system of stochastic differential equations

$$\mathrm{d}X_\xi(t)=\Big(\kappa\textstyle\sum_\zeta p_{\xi,\zeta}\big[X_\zeta(t)-X_\xi(t)\big]\Big)\mathrm{d}t+\sqrt{g\big(X_\xi(t)\big)}\,\mathrm{d}w_\xi(t),\qquad \xi\in\Xi,\quad (1)$$

with values in $\mathbb{E}$.

The ingredients of this equation are as follows:

(a) **(label set)** $\Xi$ denotes a countable set and is used to label the components of the system.

(b) **(migration parameters)** $p=\{p_{\xi,\zeta}\,;\ \xi,\zeta\in\Xi\}$ is a probability transition matrix in $\Xi$, and $\kappa$ a non-negative constant. We call $p$ the *migration kernel* and $\kappa$ the *migration intensity.*

(c) **(driving Brownian motions)** $\{w_\xi;\ \xi\in\Xi\}$ is a system of independent standard Brownian motions in $\mathbb{R}$ describing the *noise* in the system.

(d) **(diffusion coefficient $g$)** The *diffusion coefficient* $g:\mathbb{R}\mapsto\mathbb{R}_+$ is assumed to satisfy the following conditions:

  **(d1)** $g$ is locally Lipschitz continuous,

  **(d2)** $g=0$ on the complement of an open interval $I$ (this "reference interval" $I$ or its complement in $\mathbb{R}$ could be empty),

  **(d3)** $\limsup_{|r|\to\infty}\frac{g(r)}{r^2}<\infty.$

(e) **(state space $\mathbb{E}$)** If $\Xi$ is finite, or the closure $\overline{I}$ of $I$ is bounded, set $\mathbb{E} := \overline{I}^{\Xi}$ and $\|z\| := \sup_{\xi\in\Xi} |z_\xi|$. Otherwise, choose a (strictly) positive, summable $\{\gamma_\xi;\ \xi \in \Xi\}$ ("reference measure") independent of $g$ satisfying

$$\textstyle\sum_\xi \gamma_\xi\, p_{\xi,\zeta} \le \Gamma\gamma_\zeta\,, \quad \zeta \in \Xi, \qquad \text{for some constant } \Gamma,$$

and put $\mathbb{E} := \{z \in \overline{I}^{\Xi};\ \|z\| < \infty\}$ where $\|z\| := \sum_\xi \gamma_\xi |z_\xi|$. The convex set $\mathbb{E}$ is endowed with the topology of componentwise convergence.

Write $P_\mu = P^g_\mu$ for the distribution of $X$ if it starts off with the law $\mu = \mathcal{L}(X(0))$, and $P_z = P^g_z$ in the special case $\mu = \delta_z$ (Dirac measure at $z \in \mathbb{E}$). The random initial state $X(0)$ is always assumed to be *independent* of the driving Brownian motions $\{w_\xi;\ \xi \in \Xi\}$. ◇

**Remark 2** Note that under the given conditions the existence of a unique strong solution living in $\mathbb{E}$ is guaranteed by Shiga and Shimizu [16, Theorem 4.1]. This solution is a *Markov process with continuous paths.* – Note also that there is some freedom in the choice of the state space $\mathbb{E}$. In the case of an unbounded reference interval $I$ and doubly stochastic $p$, see Liggett and Spitzer [12] for a construction of a reference measure $\gamma$. The integrability condition $\|z\| < \infty$ prevents the components $|z_\xi|$ from growing too rapidly as $\xi \to \infty$. ◇

**Remark 3** If a probability law $\mu$ on $\overline{I}^{\Xi}$ satisfies $\sup_\xi E|X_\xi(0)| < \infty$, then $\mu(\mathbb{E}) = 1$ and the Markov process $X$ (living in $\mathbb{E}$) with initial law $\mu$ is well-defined. ◇

**Example 4 (diffusion coefficients)** The label set $\Xi$ is often the lattice space $\mathbb{Z}^d$ or a hierarchical group (see [9]), whereas for the diffusion coefficient $g$ the following special cases have been intensively studied (see for instance [2, 3, 7, 9, 15] and references therein):

| | $I$ | $g(r)$ on $I$ |
|---|---|---|
| *Fisher-Wright* | $(0,1)$ | $c\,r(1-r)$ |
| *Ohta-Kimura* | $(0,1)$ | $c\,r^2(1-r)^2$ |
| *Feller's branching diffusion* | $(0,+\infty)$ | $c\,r$ |
| *linear random potential* | $(0,+\infty)$ | $c\,r^2$ |
| *critical Ornstein-Uhlenbeck* | $(-\infty,+\infty)$ | $c$ |

where $c$ is always a positive constant (scaling factor). ◇

## 1.2 The comparison result

Before we formulate our comparison result, we introduce the cone inducing the corresponding order relation.

**Definition 5 (function cone F)** Fix a state space $\mathbb{E}$ as introduced in Definition 1 (e). Denote by $\mathbf{F}$ the set of all functions $F : \mathbb{E} \mapsto \mathbb{R}_+$ which are bounded, Lipschitz-continuous and convex. Moreover, we require that they are either all non-decreasing, or alternatively all non-increasing. ◇

Here a map $f : \mathbb{E} \mapsto \mathbb{R}$ is called *Lipschitz-continuous* if

$$\big|f(x) - f(y)\big| \leq L(f)\,\|x - y\|, \qquad x, y \in \mathbb{E}, \tag{2}$$

for some constant $L(f)$ (with $\|\cdot\|$ from Definition 1 (e)). Of course, $F$ *convex* means that

$$F\big(\alpha x + (1-\alpha)y\big) \leq \alpha F(x) + (1-\alpha)F(y), \qquad x, y \in \mathbb{E}, \quad 0 \leq \alpha \leq 1, \tag{3}$$

and $x \leq y$ in $\mathbb{E}$ is defined as $x_\xi \leq y_\xi$, $\xi \in \Xi$. Note that $\mathbf{F}$ is closed with respect to the operation of multiplication. ◇

**Example 6 (function cone F)** We mention a typical example for both cases, a non-decreasing and a non-increasing function:

(a) (**"moment function"**) If $I$ is a bounded subinterval of $\mathbb{R}_+$, we fix natural numbers $k \geq 1$, $n_1, ..., n_k \geq 0$, and labels $\xi_1, ..., \xi_k \in \Xi$, and set

$$F(z) := z_{\xi_1}^{n_1} \cdots z_{\xi_k}^{n_k}, \qquad z \in \mathbb{E}.$$

(b) (**"Laplace function"**) If $I$ is bounded below, we fix $\lambda_1, ..., \lambda_k \geq 0$ as well as $\xi_1, ..., \xi_k \in \Xi$, and put

$$F(z) := \exp\big[-\lambda_1 z_{\xi_1} - \cdots - \lambda_k z_{\xi_k}\big], \qquad z \in \mathbb{E}. \qquad \diamond$$

Now we are ready to state our comparison argument concerning the interacting diffusion $X = \big\{X(t);\ t \geq 0\big\}$, which for typographical simplification we also write as $\{X_t\,;\ t \geq 0\}$ (as long as the labeling of components is not needed).

**Theorem 1 (comparison argument)** *Fix two diffusion coefficients* $g_1 \geq g_2$, *corresponding state spaces satisfying* $\mathbb{E}^1 \subseteq \mathbb{E}^2$, *a finite sequence* $t_1, ..., t_k \geq 0$ *and functions* $F_1, ..., F_k \in \mathbf{F}^2$ *(the function cone related to* $\mathbb{E}^2$*). Then*

$$E_z^{g_1} F_1(X_{t_1}) \cdots F_k(X_{t_k}) \;\geq\; E_z^{g_2} F_1(X_{t_1}) \cdots F_k(X_{t_k}) \tag{4}$$

*for all initial states* $z \in \mathbb{E}^1$. *In particular, for all* $F \in \mathbf{F}^2$,

$$E_z^{g_1} F(X_t) \;\geq\; E_z^{g_2} F(X_t), \qquad z \in \mathbb{E}^1. \tag{5}$$

*For the latter conclusion we need not require that $F$ is bounded and monotone.*

**Remark 7 (extensions of the comparison argument)** Theorem 1 can be extended to hold for functions which arise as limits of functions in $\mathbf{F}$ in such a way that the corresponding functionals also converge. This is of particular interest in the case $I = \mathbb{R}$. For instance, (5) is also true for the moment functions of Example 6 (a) or the Laplace functions of 6 (b) if the respective boundedness requirement on $I$ is dropped. (In fact, extend linearly from an asymptotically large argument on to get convex Lipschitz functions.) ◇

**Example 8 (comparison with restricted Fisher-Wright)** We mention an intrinsic example for a situation where the comparison theorem is applicable (and which is intensively used in [9] and [10]). Let $g$ be a diffusion coefficient with reference interval $I = (0, 1)$. Assume that $g$ is positive on $I$, and set

$$g^{\varepsilon}(r) := c^{\varepsilon}(r-\varepsilon)^{+}(1-\varepsilon-r)^{+}, \quad r \in \mathbb{R}, \qquad I^{\varepsilon} :\equiv (0,1) = I,$$

where $0 \leq \varepsilon < \frac{1}{2}$ and $c^{\varepsilon} > 0$. Such a $g^{\varepsilon}$ is called a *restricted* Fisher-Wright diffusion coefficient related to the interval $(\varepsilon, 1-\varepsilon)$. (Figure 1.) The interact-

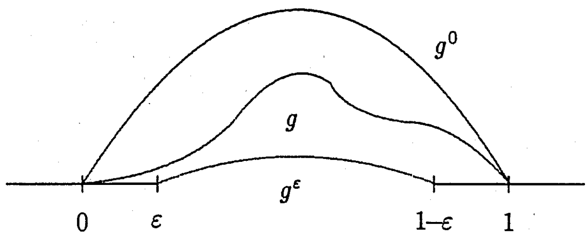


Figure 1: (restricted) Fisher-Wright bounds for $g$ with support $(0, 1)$

ing diffusion with this "reference diffusion coefficient" $g^{\varepsilon}$ can be studied using delayed coalescing random walks which are dual to interacting Fisher-Wright diffusions (Shiga [14]). In the sense of the physics literature this is therefore an explicitly solvable model. (A similar explicitly solvable case is the interacting Feller's branching diffusion of Example 4, which can also be used in comparison arguments.) Note that by our assumptions on $g$, for each $\varepsilon > 0$ sufficiently small one can always find constants $c^0, c^{\varepsilon} > 0$ such that $g^0 \geq g \geq g^{\varepsilon}$. Using "moment functions" $F$ as in Example 6 (a), Theorem 1 provides bounds of all higher and "mixed" moments of $X$ with respect to $P_z^g$ by the corresponding ones in the case of interacting (restricted) Fisher-Wright diffusions. This comparison is useful for the following reasons. First of all, statements on interacting diffusions are frequently proved by the method of moments. Second, limiting statements on the cluster formation as in Theorems 1-5 of [9], in the special case of (restricted) Fisher-Wright diffusion coefficients, do *not* depend on the scaling factor $c^{\varepsilon}$ (and are continuous in $\varepsilon$). Therefore the comparison theorem is a powerful tool for extending results from the Fisher-Wright case to general diffusion coefficients $g$ with support $(0, 1)$ (*universality*). ◇

## 1.3 An application

Our *main application* of the comparison theorem in this paper is a result on the long-time behavior of interacting diffusions in the *recurrent* case, which covers new classes of systems, and even simplifies proofs for some known cases, as for example, interacting Feller's branching diffusions (super-random walks). In the *transient* case, the long-term behavior of $X$ is relatively well understood, see

Cox and Greven [2], Deuschel [7], and Shiga [15]. To arrive at a simple form for the next theorem we require additional properties of the model.

**Assumption 9 (recurrence)** In Definition 1 we also assume:

**(a)** $\Xi$ is a (countable) *Abelian group.*

**(b)** The migration kernel $p$ is irreducible, *homogeneous* ($p_{\xi,\zeta} = p_{0,\zeta-\xi}$), and the symmetrized kernel $\widehat{p}_{\xi,\zeta} := \frac{1}{2}(p_{\xi,\zeta} + p_{\zeta,\xi})$ is *recurrent.*

**(c)** The diffusion coefficient $g$ is *positive* on the (bounded or unbounded) reference interval $I =: (a, b)$. ◇

**Assumption 10 (homogeneity)** The initial law $\mu = \mathcal{L}(X(0))$ is *homogeneous* (that is invariant with respect to the spatial shift). Moreover, suppose $E_\mu|X_0(0)|$ is finite, and set $\theta := E_\mu X_0(0)$. ◇

**Remark 11** If one wants to drop condition (a) in Assumption 9, analogs of (6) below can still be shown if instead of (b) one works with a pair of independent Markov chains which meet infinitely often almost surely (cf. Shiga [14]). ◇

The result we now want to state says in particular, that under Assumptions 9, 10 and one-sided bounded components, the interacting diffusion *clusters* for all diffusion coefficients $g$ (*universality*). Clustering means that for large times, locally, all components almost agree. In fact, with Theorem 1 and the ergodic theorem in the interacting Fisher-Wright case alone, which is easily handled via duality, we are able to derive the following result. Here $\underline{a}$ denotes the constant state $\underline{a}_\xi \equiv a$.

**Theorem 2 (clustering)** *Under Assumptions* 9 *and* 10,

$$\mathcal{L}(X_t) \underset{t\to\infty}{\Longrightarrow} \begin{cases} \dfrac{b-\theta}{b-a}\,\delta_{\underline{a}} + \dfrac{\theta-a}{b-a}\,\delta_{\underline{b}}\,, & \textit{if} \quad a, b \in \mathbb{R}, \\ \delta_{\underline{a}}\,, & \textit{if} \quad a \in \mathbb{R}, \quad b = +\infty. \end{cases} \tag{6}$$

It remains an *open problem* to prove that in the remaining case $a = -\infty$ and $b = +\infty$,

$$\mathcal{L}(X_t) \underset{t\to\infty}{\Longrightarrow} \tfrac{1}{2}(\delta_{-\underline{\infty}} + \delta_{+\underline{\infty}})$$

(in a suitable sense), which is only known for the interacting critical Ornstein-Uhlenbeck diffusion of Example 4, which is explicitly solvable using Gaussian techniques.

**Remark 12** The universality for *finite* $a$ and $b$ was known before, see Cox and Greven [2]. But for $g$ with *unbounded* support, only special cases have been handled so far. In fact, extinction behavior for interacting diffusions with linear potential had been studied in Shiga [15]; extinction properties of spatial branching models related to the interacting Feller's branching diffusion (super-random walk) of Example 4 are also well-known; cf. e.g. Dawson [5]. ◇

**Remark 13** Treating, in the case $I = (0, \infty)$, initial states with $E_\mu|X_0(0)| = \infty$ is a bit more subtle, since the limit point $\delta_\infty$ may appear; cf. Bramson et al. [1], Dawson et al. [6]. However, using the relatively well-understandable interacting Feller's branching diffusion (super-random walk), it is possible to use Theorem 1 to get results in the class of processes where $g(r)/r \to 0$ or $\infty$ as $r \to \infty$ as well. ◇

The rest of the paper is organized as follows. We prove Theorem 1 in the next section, and Theorem 2 in Section 3.

# 2 Proof of the comparison Theorem 1

The idea of the proof is to use an integration by parts formula for semigroups combined with a preservation property of the function cone **F** under the interacting diffusion semigroup. The proof is broken up in three main steps (§§ 2.1-2.3). We start by proving a fact about finite-dimensional diffusions (for simplicity again denoted by $X$) without drift, namely that convexity of functions is preserved under the corresponding diffusion semigroup.

## 2.1 Preservation of convexity for diffusions in $\mathbb{R}^m$

**Definition 14 (finite-dimensional diffusion)** Fix an open convex subset $C$ of $\mathbb{R}^m$, and an $m \times m$-matrix-valued *continuous* function $x \mapsto \sigma(x)$ defined on the closure $\overline{C}$ of $C$, with the following properties:

- The entries $x \mapsto \sigma_{i,j}(x)$ grow *at most linearly* (as $|x| \to \infty$, in the case of an unbounded $C$).
- $x \mapsto \sigma_{i,j}(x)$ satisfies a *Lipschitz* condition on
$$C^{\delta,K} := \Big\{x \in C;\ \|x\| \le K,\ \|x-y\| \ge \delta,\ y \in \partial C\Big\},$$
for each $\delta, K > 0$.
- The matrix $\sigma(x)$ is *invertible*, for each $x \in C$.
- $\sigma(x) \equiv 0$ on the boundary $\partial C$ of $C$.

Let $W$ be a standard Brownian motion in $\mathbb{R}^m$, and denote by $X$ that *unique strong* solution of the stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}W_t\,, \quad X_0 = z \in \overline{C}, \tag{7}$$

(living in $\overline{C}$) which has the following property: Once $X_t$ hits $\partial C$ then it is absorbed. Let $U = \{U_t;\ t \ge 0\}$ denote the *semigroup* related to this time-homogeneous Markov process: $U_t f(z) := E_z f(X_t)$. ◇

Note that this existence and uniqueness result follows for example from Stroock and Varadhan [17] by combining their Theorems 5.1.1, 5.1.5, and 10.2.2 in connection with Theorem 8.1.1.

**Proposition 15 (preservation of convexity)** *Let $F$ belong to the set $\mathrm{C}^2(\overline{C})$ of (real-valued) continuous functions defined on $\overline{C}$ which are twice continuously differentiable in $C$. Then for each fixed $t > 0$, the function $x \mapsto U_tF(x)$ is also convex on $\overline{C}$.*

**Proof** The idea of the proof is to use the fact that in this case of *invertible* diffusion matrices $\sigma(x)$, the noise of the basic process $W$ works in *every* direction of the space preserving convexity via the martingale property.

Fix $t > 0$ and $F$ as in the proposition. We will prove first that the *Hessian* of $U_tF$ *is non-negative definite* in each point $z \in C$. This proof is broken up in four steps. Step 5° then treats $z \in \overline{C}$.

*Step* 1° Fix an $m \times m$-matrix $A$, and define a matrix-valued function $[s,x] \mapsto \sigma_A(s,x)$ by

$$\sigma_A(s,x) := \begin{cases} \sigma(x) & \text{if } 0 \le s < t, \\ A\,\sigma(x) & \text{if } s \ge t, \end{cases} \qquad x \in \overline{C}. \tag{8}$$

Replacing $\sigma$ by $\sigma_A$ in Definition 14 (note that the new inhomogeneity is of no harm), we get an inhomogeneous diffusion process $X^A$ living on $\overline{C}$. Denote by $\{V_{r,s}\,;\ 0 \le r \le s\}$ the semigroup of this Markov process.

*Step* 2° Since $F$ is convex, and $X$ is without drift, the process $s \mapsto F(X_s^A)$ is a submartingale. Therefore we can conclude that $s \mapsto V_{0,s}F$ satisfies

$$\lim_{h\downarrow 0} \frac{V_{0,t+h}F(x) - V_{0,t}F(x)}{h} = \frac{\partial^+ V_{0,s}F}{\partial s}(x)\bigg|_{s=t} \ge 0, \qquad x \in \overline{C}. \tag{9}$$

On the other hand, by the semigroup formula, and since $V_{0,t} = U_t$ for our fixed $t$,

$$\frac{\partial^+ V_{0,s}F}{\partial s}(x)\bigg|_{s=t} = G_tU_tF(x), \qquad x \in \overline{C}, \tag{10}$$

where the partial differential operators $\{G_s\,;\ s \ge 0\}$ are defined by

$$G_sH(x) := \frac{1}{2}\sum_{i,j}(\sigma_A\sigma_A^{\top})(s,x)\,\frac{\partial^2 H}{\partial x_i \partial x_j}(x), \qquad H \in \mathrm{C}^2(\overline{C}), \quad x \in \overline{C}. \tag{11}$$

*Step* 3° For a given vector $\lambda \in \mathbf{R}^m$, define the matrix $\Lambda$ by $\Lambda_{i,j} := \lambda_i\lambda_j$. For a *fixed* $z \in C$, choose now $A$ as follows. First write the symmetric non-negative definite matrix $\Lambda$ in the form $\Lambda = BB^{\top}$, and then set $A := B\sigma^{-1}(z)$. With this choice of $A$, from (11) and (8) we get

$$G_tH(z) := \frac{1}{2}\sum_{i,j}\lambda_i\lambda_j\,\frac{\partial^2 H}{\partial x_i \partial x_j}(z). \tag{12}$$

*Step* 4° Finally choose $H := U_tF$, combine (12), (10) and (9) to obtain

$$\sum_{i,j} \frac{\partial^2 U_t F}{\partial x_i \partial x_j}(z)\, \lambda_i \lambda_j \;\geq\; 0.$$

*Step* 5° Since $z \in C$ and $\lambda \in \mathbb{R}^m$ were arbitrary, this shows that $U_t F$ is convex on $C$. The convexity on the closure $\overline{C}$ of $C$ then follows from the continuity of $U_t F$ on $\overline{C}$ by taking limits, finishing the proof. □

## 2.2 Preservation of the function cone F for $g$ with support $I$

The proof of the comparison Theorem 1 is based on a preservation property (Proposition 16 below) of the function cone **F** (introduced in Definition 5) under the semigroup of our interacting diffusion $X$ which we will prove in this section.

Denote by $\mathsf{S} = \mathsf{S}^g$ the *semigroup* associated with the Markov process $X$ of Definition 1,

$$\mathsf{S}^g_t h\,(z) = E^g_z h(X_t), \qquad z \in \mathbb{E},$$

acting on the set $\mathbf{C}_b = \mathbf{C}_b(\mathbb{E})$ of bounded continuous functions $h$ defined on $\mathbb{E}$, equipped with the topology of uniform convergence. This semigroup has as its *generator* $\mathsf{G} = \mathsf{G}^g$ the closure of the following operator acting on the set $\mathbf{C}^2_0 = \mathbf{C}^2_0(\mathbb{E})$ of all those functions in $\mathbf{C}_b$ which depend only on finitely many components $x_\xi$ and are at least twice continuously differentiable on $\mathbb{E} \cap I^{\Xi}$:

$$\mathsf{G}^g := \kappa \sum_{\xi,\zeta} (p_{\xi,\zeta} - \delta_{\xi,\zeta}) x_\zeta \frac{\partial}{\partial x_\xi} + \frac{1}{2} \sum_{\xi} g(x_\xi) \frac{\partial^2}{\partial x_\xi^2}, \qquad x \in \mathbb{E} \cap I^{\Xi}. \tag{13}$$

(Compare Shiga and Shimizu [16].)

Now we are ready to state the following preservation property of **F** under the semigroup $\mathsf{S}$ of $X$.

**Proposition 16 (preservation of F)** *Assume that the diffusion coefficient $g$ is positive on its reference interval $I$. For each finite sequence $F_1, ..., F_k \in$ **F** and time points $t_1, ..., t_k \geq 0$, the function*

$$z \mapsto E^g_z F_1(X_{t_1}) \cdots F_k(X_{t_k}), \qquad z \in \mathbb{E},$$

*belongs to **F**. In the case $k = 1$, the requirements that $F_1$ is bounded and monotone can be dropped.*

The key idea of the proof is to use Trotter's product formula to reduce the assertion to a system of pure diffusion, and to a system of pure migration.

To prepare for this, we first want to formulate two lemmata which allow us later to reduce the problem to models with only a finite number of components. Such finite systems are technically easier to handle. These lemmata are used in the existence and uniqueness theorem for $X$ and essentially appear in Liggett and Spitzer [12], Shiga [14]. Hence for proofs, we refer the reader to these papers.

**Lemma 17 (preservation of the Lipschitz property)** *Suppose that $f$ is a Lipschitz function on $\mathbb{E}$ with Lipschitz constant $L(f)$. Then*

$$\left|\mathrm{S}_t f(x) - \mathrm{S}_t f(y)\right| \le L(f)\, \mathrm{e}^{t\Gamma}\, \|x-y\|, \qquad t \ge 0, \quad x,y \in \mathbb{E},$$

*with the constant $\Gamma$ from Definition* 1(e).

**Lemma 18 (approximation by finite sets)** *Consider finite sets $\Xi_1 \subseteq \Xi_2 \subseteq \cdots \uparrow \Xi$. Let $\mathrm{S}^{(n)}$ denote the semigroup which belongs to the Markov process obtained from $X$ by the following modification. For $\xi \notin \Xi_n$, freeze $X_\xi$, whereas for $\xi \in \Xi_n$, restrict the summation in* (1) *to $\zeta \in \Xi_n$. Then, for $k \ge 1$, $t_1, \ldots, t_k \in \mathbb{R}_+$, and bounded Lipschitz functions $f_1, \ldots, f_k$ on $\mathbb{E}$, we have*

$$\mathrm{S}_{t_1}^{(n)} f_1 \cdots \mathrm{S}_{t_k}^{(n)} f_k\,(z) \underset{n\to\infty}{\longrightarrow} \mathrm{S}_{t_1} f_1 \cdots \mathrm{S}_{t_k} f_k\,(z), \qquad z \in \mathbb{E}. \tag{14}$$

*For $k = 1$ we need not require that $f_1$ is bounded.*

Note that the semigroup $\mathrm{S}^{(n)}$ describes essentially a system with only finitely many components. For technical purposes it is nicer to have for every $n$ a process on the same state space. For this reason we just use the freezing to achieve that only finitely many components interact, but nevertheless for all $n$ we have the common state space $\mathbb{E}$.

**Proof of Proposition 16** Since $\mathbf{F}$ is closed with respect to multiplications, without loss of generality we may assume that $0 < t_1 < \cdots < t_k$. Preservation of non-negativity and boundedness is trivial.

1° (*finite system semigroup*) We first treat the case where S is replaced by the finite system semigroup $\mathrm{S}^{(n)}$ introduced in Lemma 18.

Lipschitz continuity follows from Lemma 17 combined with the boundedness of the functions in $\mathbf{F}$. The proof of the remaining statements (monotonicity and convexity) is by induction on the number $k$ of time points. In fact, the case $k = 1$ is the hard part, the induction step itself is very simple.

Before giving the proof, we start with an observation: *We may additionally require* that $F_1, \ldots, F_k \in \mathbf{C}_0^2$. Indeed, since we are dealing with the finite system semigroup, we may restrict our attention to finitely many components. Moreover, each monotone convex Lipschitz function $f$ on a *closed* cube $C$ in Euclidean space $\mathbb{R}^m$ can be approximated in uniform convergence by monotone convex Lipschitz functions $f_1, f_2, \ldots \in \mathbf{C}^2 = \mathbf{C}^2(C)$. In fact, assume for the moment that $C = \mathbb{R}^m$. Take a non-negative $\mathbf{C}^2$-function $h$ with the unit ball $B$ as support, and with integral 1. Consider

$$f_N(x) \;:=\; N^m \int \mathrm{d}y\; h\big(N\,(y-x)\big)\, f(y), \qquad x \in \mathbb{R}^m, \quad N \ge 1.$$

These functions satisfy all requirements (in the present case $C = \mathbb{R}^m$). Indeed, differentiating with respect to $x$ shows that $f_N \in \mathbf{C}^2$ (note that, for $x$ varying in a bounded set, the domains of integration are uniformly bounded). On the other hand, from the identity $f_N(x) = N^m \int_B \mathrm{d}y\, h(Ny)\, f(y+x)$ we conclude the monotonicity, convexity, and Lipschitz property of $f_N$. Furthermore, $f_N(x) \to f(x)$ as $N \to \infty$, uniformly in $x$ (recall that $f$ is Lipschitz, and $h$ satisfies $\int \mathrm{d}y\ |y|\, h(y) < \infty$). This gives the claim in the case $C = \mathbb{R}^m$.

If $C \neq \mathbb{R}^m$, take a monotone convex Lipschitz extension of $f$ to $\mathbb{R}^m$ and apply the above construction, to get the desired approximation on $C$. Such an extension can be obtained as follows. For every point $z \in \mathbb{R}^m$ there exists a unique point $z^* \in C$ such that

$$\|z^* - z\| := \min_{x \in C} \|x - z\|.$$

Without loss of generality, we may assume that $f$ is non-decreasing. Define

$$f(z) := f(z^*) + L(f) \textstyle\sum_{i=1}^m (z_i - z_i^*)^+, \qquad z \in \mathbb{R}^m,$$

(With $L(f)$ a Lipschitz constant). That is, we extend in increasing directions of a component linearly with a "maximal" slope, and in decreasing ones with the slope 0. This function satisfies all requirements.

Note also that if $f$ is bounded then all the $f_N$ are (uniformly) bounded.

(a) (*first step of induction*) For $k = 1$ drop the index 1 in notation, that is, look at

$$H_t(z) := E_z^g F(X_t), \qquad z \in \mathbb{E}, \tag{15}$$

for a fixed $t > 0$ and $F \in \mathbf{F}$. Then the idea of the proof is to show that the claimed property is true for systems with only migration or with only diffusion, and later to use Trotter's product formula to get the full result (for $k = 1$ and the finite system semigroup case).

(a.1) (*only migration*) First, simplify the model by setting the diffusion term $g$ to 0 in (1). Then $X$ degenerates to a *deterministic* process. In this case we can explicitly solve the linear system (1) (see, for instance, the expectation formula (2.59) in [9]):

$$X_\xi(t) = \begin{cases} \sum_{\zeta \in \Xi_n} p_n(t,\xi,\zeta)\, z_\zeta & \text{if } \xi \in \Xi_n\,, \\ z_\xi & \text{if } \xi \notin \Xi_n\,. \end{cases} \tag{16}$$

Here $z$ is the initial state $X(0)$, and $p_n(t,\xi,\zeta)$ are the transition probabilities of the continuous-time Markov chain in $\Xi_n$ related to the finite system semigroup $S^{(n)}$. Hence, in this pure migration case, (15) can be written as

$$H_t(z) = F\big(X_\xi(t)\big) \quad \text{with } X_\xi(t) \text{ from (16).}$$

This expression is obviously both monotone and convex in $z$. In fact, use the definition (3) of convexity of $F$, as well as $p_n \geq 0$.

(a.2) (*only diffusion*) Now we simplify the model in the opposite direction by omitting the migration term in (1), that is, by putting $\kappa = 0$. We may restrict our attention to $\Xi_n$. Then $X$ degenerates to a finite system of *independent* diffusions each with the same diffusion coefficient $g$. Now convexity is obtained using the preservation of convexity Proposition 15.

The *monotonicity* part works as follows. Assume first that the label set consists of a single point. Then $X$ is a one-dimensional diffusion in $\mathbb{E} = \overline{I}$ with diffusion coefficient $g$. The semigroup of $X$ preserves monotone functions $F$. Indeed, this results from the following coupling argument. Let $X^1$ and $X^2$ be two versions of $X$, defined as solutions of the stochastic one-dimensional equation (1) but with the *same* driving Brownian motion. If now $X_0^1 \geq X_0^2$ then $X_t^1 \geq X_t^2$ a.s. (cf. [13, Theorem 9.3.7]). Then by the monotonicity of $F$, and taking expectations, we get the preservation of monotonicity.

Monotonicity in the case of a finite number of diffusing components follows again via the above construction since all components evolve independently.

(a.3) (*general case*) Now we decompose the interval $[0, t]$ into small pieces of length $\frac{t}{m}$ and apply alternately (a.1) and (a.2) with $t$ replaced by $\frac{t}{m}$. More specifically, consider

$$H_{m,t}(z) := \left[\mathrm{S}^1_{t/m}\mathrm{S}^2_{t/m}\right]^m F(z), \qquad z \in \mathbb{E}, \quad m \geq 1, \tag{17}$$

where $\mathrm{S}^1$ refers to the degenerate semigroup related to (a.1), and $\mathrm{S}^2$ to the semigroup of independent diffusions of (a.2). Since each successive step results into a function in $\mathbf{F}$, we end up in $\mathbf{F}$ with the whole chain of operations in (17). That is, $H_{m,t} \in \mathbf{F}$ for each $m$ (and the fixed $t$ and $F$). Finally, by *Trotter's product formula* (see for instance Corollary 1.6.7 of [8]), we get pointwise $\lim_{m\to\infty} H_{m,t} = \mathrm{S}_t F = H_t$. (Note that in the present model this is a consequence of the usual construction of the solution by freezing the drift and diffusion coefficient to constants during time intervals of length $\frac{t}{m}$.) Hence, the limiting $H_t$ of (15) is certainly a monotone and convex function.

This finishes the proof in the case $k = 1$ for the finite system semigroup.

(b) (*induction step*) Now assume that $k > 1$. Then by the Markov property the expression under consideration can be written as the following product of two functions, one with a single time point and one with $k - 1$ time points:

$$E_z^g F_1(X_{t_1}) E_{X_{t_1}}^g F_2(X_{t_2 - t_1}) \cdots F_k(X_{t_k - t_1}).$$

Since $\mathbf{F}$ is closed under multiplication, the proof for the finite system semigroups can be completed by induction.

2° (*general* S) Approximate $\Xi$ by finite sets and use the approximation of the infinite system as in Lemma 18. Then we can reduce the general case to the result of Step 1°, by Lemma 18. Namely, we know monotonicity and convexity for the terms at the l.h.s. of (14) (with the $t_i$ and $f_i$ appropriately replaced), so their limit is convex and monotone.

*Step* 3° (*extension for* $k = 1$) An analysis of the arguments so far shows that the requirements of boundedness and monotonicity were related to the closeness of $\mathbf{F}$ with respect to *multiplications* which we do not need in the case $k=1$. This completes the proof. □

## 2.3 Completion of the proof of the comparison theorem

Fix $\mathbb{E}^1 \subseteq \mathbb{E}^2$, $g_1 \geq g_2$, $t_1, ..., t_k \geq 0$ and $F_1, ..., F_k \in \mathbf{F}^2$. The proof proceeds in three steps, first a special case is treated, and then the general result is derived from the special one in two subsequent steps.

*Step* 1° We consider first the case where $g_1 > 0$ on $I^1$ and $\mathbb{E}^1 = \mathbb{E}^2$. Again without loss of generality we may assume that $0 < t_1 < \cdots < t_k$ and that $F_1, ..., F_k \in \mathbf{C}_0^2$ (compare the beginning of the proof of Proposition 16).

The proof of (4) is by induction over $k$, the number of time points considered. Start with $k=1$, and drop in this special case the assumption that $F_1$ is bounded and monotone. We have to show that

$$\mathsf{S}_{t_1}^{g_1} F_1(z) = E_z^{g_1} F_1(X_{t_1}) \geq E_z^{g_2} F_1(X_{t_1}) = \mathsf{S}_{t_1}^{g_2} F_1(z), \qquad z \in \mathbb{E}^1. \tag{18}$$

Without loss of generality, we may assume that the l.h.s. in (18) is finite (recall that the functions in $\mathbf{F}$ are non-negative). By the *integration by parts formula*

$$\mathsf{S}_{t_1}^{g_1} - \mathsf{S}_{t_1}^{g_2} = \int_0^{t_1} \mathrm{d}s\, \mathsf{S}_{t_1-s}^{g_2} (\mathsf{G}^{g_1} - \mathsf{G}^{g_2}) \mathsf{S}_s^{g_1} \tag{19}$$

(see for instance p. 367 in [11]), it suffices to demonstrate that

$$(\mathsf{G}^{g_1} - \mathsf{G}^{g_2}) \mathsf{S}_s^{g_1} F_1 \geq 0, \qquad 0 \leq s \leq t_1, \tag{20}$$

on $\mathbb{E}^1$. (Note that $\mathsf{S}_s^{g_1} F_1$ is in the domain of $\mathsf{G}^{g_i}$.) By the form of the generators (recall (13)),

$$\mathsf{G}^{g_1} - \mathsf{G}^{g_2} = \frac{1}{2} \sum_{\xi} \Big( g_1(x_\xi) - g_2(x_\xi) \Big) \frac{\partial^2}{\partial x_\xi^2}.$$

Since $g_1 \geq g_2$ by assumption, for the proof of (20) it therefore suffices to show that for fixed $s$

$$\mathsf{S}_s^{g_1} F_1(z) \quad \text{is } \textit{convex} \text{ in each component } z_\zeta,\ \zeta \in \Xi, \text{ of } z \in \mathbb{E}^1. \tag{21}$$

But this follows from the preservation Proposition 16, since we assumed $g_1 > 0$ on $I^1$. Consequently, (18), hence (4) is true in the case $k=1$.

Consider $k \geq 2$. For a preparation of the induction step, we first rewrite (4) in a more convenient form. Namely, using the Markov property at time $t_1$ and time-homogeneity of $X$, we see that (4) becomes

$$\mathsf{S}_{t_1}^{g_1} F^1(z) = E_z^{g_1} F^1(X_{t_1}) \geq E_z^{g_2} F^2(X_{t_1}) = \mathsf{S}_{t_1}^{g_2} F^2(z), \qquad z \in \mathbb{E}^1, \tag{22}$$

by setting

$$F^j(x) := F_1(x) E_x^{g_j} F_2(X_{t_2-t_1}) \cdots F_k(X_{t_k-t_1}), \qquad x \in \mathbb{E}^j, \quad j = 1, 2. \tag{23}$$

Assume now that (4), respectively (22), is valid for some $k-1 \geq 1$ (*induction hypothesis*). Then, by (4) and the non-negativity of $F_1$, from the definition (23) we immediately get $F^1 \geq F^2$ on $\mathbb{E}^1 = \mathbb{E}^2$. Then the relation (22) for $k \geq 2$, hence (4) for $k \geq 2$, will follow from the positivity of the semigroups $S^{g_i}$, once we prove (18) with $F_1$ replaced by $F^1$. As in the case $k = 1$, for this we need to know (21), with $F_1$ replaced by $F^1$. By the definition (23) of $F^1$, we may return to the original expression:

$$S_s^{g_1} F^1(z) = E_z^{g_1} F_1(X_s) F_2(X_{t_2}) \cdots F_k(X_{t_k}). \tag{24}$$

Again by the preservation Proposition 16, the needed convexity property holds since we assumed $g_1 > 0$ on $I^1$. This finishes the induction step, hence the proof in the case $g_1 > 0$ on $I^1$ and $\mathbb{E}^1 = \mathbb{E}^2$.

*Step* 2° Now we deal with the case $g_1 > 0$ on $I^1$ but assuming only $\mathbb{E}^1 \subseteq \mathbb{E}^2$. Choose a sequence $g_{2,n}$ of diffusion coefficients satisfying $g_1 \geq g_{2,1} \geq g_{2,2} \geq \cdots \geq g_2$ with $g_{2,n} \downarrow g_2$ as $n \to \infty$, but $g_{2,n} > 0$ on $I^1$. Hence we may assume $\mathbb{E}^1 = \mathbb{E}^{2,n}$, for all $n$ (note that we are interested in (4) only on $\mathbb{E}^1$). Then by the previous step of proof,

$$E^{g_1}\pi \geq E^{g_{2,1}}\pi \geq E^{g_{2,2}}\pi \geq \cdots$$

where $\pi := F_1(X_{t_1}) \cdots F_k(X_{t_k})$. Hence it suffices to prove that $E^{g_{2,n}}\pi \downarrow E^{g_2}\pi$. That is, using a representation as in (23), we have to prove monotone convergence with respect to the parameter $g$.

Consider first the case $k = 1$, and drop for this again the boundedness and monotonicity requirements on $F_1$, but assume that $E^{g_2}\pi$ is finite. For every fixed $t$ and $f \in \mathbf{F}$, by (an equivalent form of) the integration by parts formula (19),

$$S_t^{g_{2,n}} f - S_t^{g_2} f = \int_0^t \mathrm{d}s\, S_{t-s}^{g_2}\Big[\frac{1}{2}\sum_\xi (g_{2,n} - g_2)(\cdot)\frac{\partial^2}{\partial(\cdot)_\xi^2}\Big] S_s^{g_{2,n}} f \quad \text{on } \mathbb{E}^1. \tag{25}$$

But $\frac{\partial^2}{\partial x_\xi^2} S_s^{g_{2,n}} f(x) \geq 0$ by the preservation Proposition 16. Hence, the integrand is non-negative. On the other hand, it is bounded from above by

$$S_{t-s}^{g_2}\Big[\frac{1}{2}\sum_\xi (g_1 - g_2)(\cdot)\frac{\partial^2}{\partial(\cdot)_\xi^2}\Big] S_s^{g_1} f \;\geq\; 0$$

which integral equals $S_t^{g_1} f - S_t^{g_2} f$, hence is finite. Then by monotone convergence, the integral in (25) converges to 0. This completes the proof of $E^{g_1}\pi \geq E^{g_2}\pi$ for $k = 1$. Using the Markov property, the case $k \geq 2$ follows by induction as in Step 1°.

*Step* 3° Finally, for general $g_1 \geq g_2$ we may assume that $I^1 \supseteq I^2$. This time we approximate $g_1$ from above by a sequence of diffusion coefficients $g_{1,n}$ which are positive on $I^1$. As in the previous step, by "monotone continuity" in $g$ and induction on $k$, the claims follow. □

# 3 Proof of the clustering Theorem 2

First note that under Assumption 10 by Remark 3 we have $\mu(\mathbb{E}) = 1$, and the process $X$ living in $\mathbb{E}$ exists.

Because of [2], we may restrict our attention to the second convergence claim in (6), even though we shall outline a new proof of the first statement in step 2° below.

The idea of the proof is to bound the diffusion coefficient $g$ of the interacting diffusion $X$ below by appropriate Fisher-Wright diffusion coefficients on large intervals, and exploit the comparison argument for suitable functionals.

1° (*Proof of the second convergence statement*) Without loss of generality, we may put $a = 0$. Take an $\varepsilon_0 \in (0,1)$. First consider an initial distribution $\mu$ which besides Assumption 10 additionally satisfies

$$\mu\Big(\varepsilon_0 \leq z_\xi \leq \varepsilon_0^{-1},\ \xi \in \Xi\Big) = 1. \tag{26}$$

It then suffices to show that for the Laplace functional of $X_t$

$$\liminf_{t\to\infty} E^g_\mu \exp - \langle \lambda, X_t \rangle \geq 1$$

for each $\lambda \in \mathbb{R}^{\Xi}_{+}$ with $\lambda_\xi \neq 0$ for only finitely many $\xi$. In other words, we may consider a function $F$ as written in Example 6 (b), and we have to estimate $E^g_\mu F(X_t)$ from below appropriately.

For each $\varepsilon \in (0, \varepsilon_0)$ sufficiently small, we find a constant $c^\varepsilon > 0$ such that $g \geq g^\varepsilon$ with $g^\varepsilon$ defined by

$$g^\varepsilon(r) := c^\varepsilon (r-\varepsilon)^+ (\varepsilon^{-1} - r)^+, \qquad r \in \mathbb{R}.$$

By the comparison Theorem 1, for $F$ as given in Example 6 (b), we get

$$E^g_\mu F(X_t) \;\geq\; E^{g^\varepsilon}_\mu F(X_t). \tag{27}$$

However by the first convergence statement in (6), $X_t$ with respect to $P^{g^\varepsilon}_\mu$ has a limit in law as $t \to \infty$ denoted by $X_\infty$. Applying this to the continuous bounded function $F$ of Example 6 (b) (recall that $z \geq 0$ by the assumption $a = 0$) yields

$$E^{g^\varepsilon}_\mu F(X_t) \underset{t\to\infty}{\longrightarrow} EF(X_\infty) \geq \frac{\varepsilon^{-1} - \theta}{\varepsilon^{-1} - \varepsilon} \exp\big[-(\lambda_1 + \cdots + \lambda_k)\,\varepsilon\big].$$

But the latter term converges to 1 as $\varepsilon \downarrow 0$.

Consequently, $\mathcal{L}(X_t) \Rightarrow \delta_{\underline{0}}$ as $t \to \infty$ which proves the second claim in (6) in the case of a $\mu$ satisfying the restriction (26).

In order to treat a general initial distribution $\mu$, for $\varepsilon \in (0,1)$ let $\Gamma^\varepsilon$ denote the image law on $\mathbb{E} \times \mathbb{E}$ of $\mu$ under the mapping

$$z_\xi \mapsto \big[z_\xi\,,\, \varepsilon \vee z_\xi \wedge \varepsilon^{-1}\big], \qquad \xi \in \Xi.$$

Note that the first marginal law of $\Gamma^{\varepsilon}$ is $\mu$, whereas the second, *truncated* one, again satisfies Assumption 10. Now it is easy to show that if we construct a bivariate process $[X, X^{\varepsilon}]$ starting with law $\Gamma^{\varepsilon}$ and such that $X$ and $X^{\varepsilon}$ satisfies (1) but using the *same* driving Wiener processes for both (*coupling*), then

$$E^{g}_{\Gamma^{\varepsilon}}\left|X_{\xi}(t) - X^{\varepsilon}_{\xi}(t)\right| \le E^{g}_{\Gamma^{\varepsilon}}\left|X_{0}(0) - X^{\varepsilon}_{0}(0)\right|, \qquad \xi \in \Xi, \quad t \ge 0,$$

(see [9, Proof of Lemma 4.6]). But the r.h.s. converges to 0 as $\varepsilon \downarrow 0$. Hence, the claim holds for general $\mu$.

2° (*outline of a Proof of the first convergence statement*) Without loss of generality, we may put $a = 0$ and $b = 1$. Since $E^{g}_{\mu} X_{\xi}(t) \equiv \theta$, it suffices to show that for $0 < \varepsilon < \frac{1}{2}$, and all $\xi, \zeta \in \Xi$,

$$P^{g}_{\mu}\Big(X_{\xi}(t) \in [\varepsilon, 1-\varepsilon]\Big) \;\vee\; P^{g}_{\mu}\Big(\left|X_{\xi}(t) - X_{\zeta}(t)\right| \ge \varepsilon\Big) \underset{t\to\infty}{\longrightarrow} 0.$$

This result is known for interacting Fisher-Wright diffusions using duality (Shiga [14]). Now proceed as in step 1°, but with replacing Laplace functions by the moment of $X_{\xi}(t) X_{\zeta}(t) - X_{\xi}(t)$. We leave the details to the reader (cf. [2]). □

*Acknowledgment:* We thank Donald Dawson who suggested the use of Trotter's product formula.

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