Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

A model equation for ultrashort optical pulses

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submitted: July 25, 2008

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No. 1348
Berlin 2008

2000 *Mathematics Subject Classification.* 78A60, 35Q60.
2008 *Physics and Astronomy Classification Scheme.* 42.65.Re 42.81.Dp 42.65.Tg 05.45.Yv.

Key words and phrases. Ultrashort pulses, few-cycle pulses.

This work was supported by the DFG Research Center MATHEON under Project No. D14. The authors would like to thank Professor A. Mielke for helpful discussions.

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Abstract

The nonlinear Schrödinger equation based on the Taylor approximation of the material dispersion can become invalid for ultrashort and few-cycle optical pulses. Instead, we use a rational fit to the dispersion function such that the resonances are naturally accounted for. This approach allows us to derive a simple non-envelope model for short pulses propagating in one spatial dimension. This model is further investigated numerically and analytically.

1 Introduction

The nonlinear Shrödinger equation (NSE) together with its modifications including higher-order dispersive and nonlinear terms (higher-order NSE) is a powerful tool for analysis of optical pulse propagation [1, 2]. In this model the optical field is described in terms of a complex amplitude, which is slow as compared to the carrier wave oscillations. The slowly-varying envelope approximation (SVEA) resulting from the time-scales separation requires less computational efforts and allows for a simple treatment of the nonlinear medium response. To account for the linear medium response the dispersion relation between the angular frequency ω and the propagation constant k is replaced by the Taylor expansion of $k(\omega)$ around the carrier frequency ω_c, that is why NSE has an universal form. Higher-order terms are especially important near the zero dispersion frequency (ZDF) where the second-order dispersion vanishes.

Recent achievements in ultrashort pulse generation [3, 4] require a modification of the standard SVEA. This is especially true for the extremely short few-cycle pulses whose spectral width is comparable to the carrier frequency. The complex field envelope can still be introduced in this situation, however the envelope changes as fast as the carrier field itself. In this case analogies of the NSE can be derived [5], the variety of possible models have been intensely discussed in the literature [6, 7, 8, 9, 10, 11]. All these models preserve a simple and intuitively clear envelope picture. However, due to the absence of the time-scales separation, the computational advantage of such models over the non-envelope ones is not so obvious. Actually, the modified "fast-envelope" NSE and non-envelope unidirectional models are closely related [12, 13].

The invalidity of the SVEA is not the only difficulty arising in theoretical description of ultrashort pulses. Another difficulty is that the traditional representation of the material dispersion relation as a Taylor expansion around the pulse carrier frequency can become invalid. This happens when the pulse spectral width is comparable to the width of the transparency window [14]. Indeed, in the presence of resonances the response function $\varepsilon(\omega)$ and consequently the dispersion function $k(\omega)$ always have singularity points in the complex plane. Therefore the convergence radius of

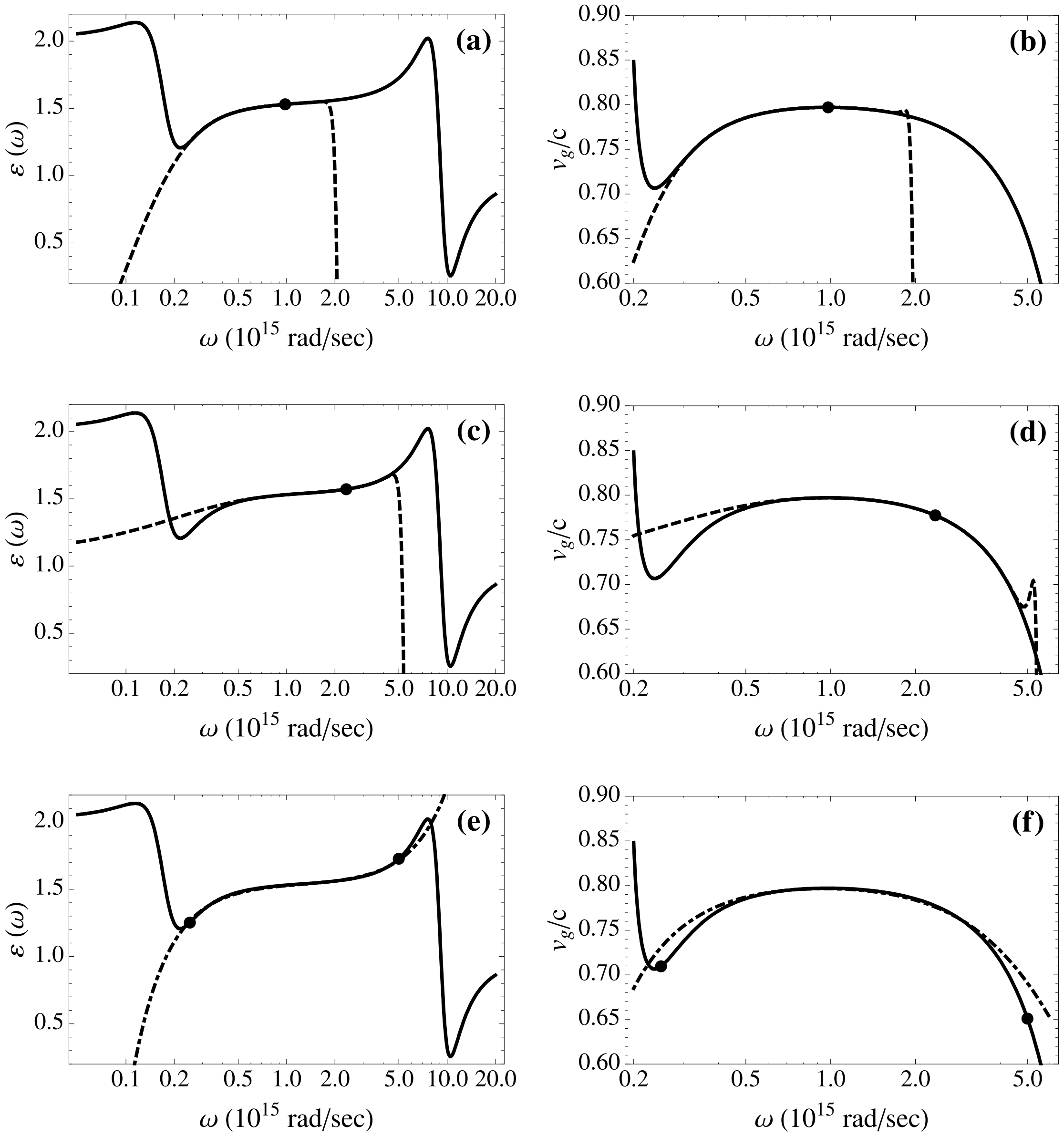

Figure 1: Real part of the response function (a,c,e, solid lines) and group velocity (b,d,f, solid lines) versus sixteenth-order Taylor expansions of $\varepsilon(\omega)$ (a,c, dotted lines) and $k(\omega)$ (b,d, dotted lines) around the carrier frequency and rational fit (4) (e,f, dotted lines) for the bulk fluoride glass. The pulse currier frequency equals either ZDF (a,b, thick point) or 800 nm (c,d, thick point). The fitting interval (5) is shown for the rational fit (e,f, thick points).

any Taylor expansion is finite and determined by the singularity nearest to ω_c. As it is seen from Fig. 1a–d, the Taylor expansion and the actual dispersion function are drastically different outside the convergence range. This discrepancy can not be reduced by increasing the approximation order.

For instance, in Fig. 1a the convergence of the Taylor expansion breaks up at $\omega \approx 2\omega_c$ so that neither the second nor the higher harmonics of the carrier wave can be described by NSE. On the other hand, the contribution of the higher harmonics is important for the few-cycle pulse dynamics. Higher harmonics can be generated intrinsically in the course of pulse evolution, e.g., during supercontinuum generation when an optical field with extremely wide spectrum is produced [15]. To overcome the limitations of the Taylor expansion we consider a rational approximation of the dispersion function instead of the polynomial one. The rational approximation ab initio accounts for the singularity points and therefore is able to describe dispersion in a whole transparency window. Besides, it can reduce the stiffness of the numerical routine.

In this paper a rational fitting function is used to construct a simple non-envelope model for the pulse electric field, which is then used to describe the dynamics of the few-cycle pulses. The paper is organized as follows. In the next section an example of the rational approximation of the response function is given. Sec. III deals with the derivation of the model equation which is then investigated in Sec. IV. The results of our studies are summarized in Sec. V.

2 Response function

As a specific example let us consider a one-dimensional pulse propagation in a bulk fluoride glass. The response function in the optical transparency window can be presented by a double resonance Lorentz dispersion model

$$\varepsilon(\omega) = 1 - \frac{b_1^2}{\omega^2 + 2i\delta_1\omega - \omega_1^2} - \frac{b_2^2}{\omega^2 + 2i\delta_2\omega - \omega_2^2}, \tag{1}$$

where $\omega_1 = 174.12\,\mathrm{THz}$ and $\omega_2 = 9144.8\,\mathrm{THz}$ are undamped resonance frequencies, $b_1 = 121.55\,\mathrm{THz}$ and $b_2 = 6719.8\,\mathrm{THz}$ are plasma frequencies, and $\delta_1 = 49.55\,\mathrm{THz}$ and $\delta_2 = 1434.1\,\mathrm{THz}$ are phenomenological damping constants [14]. To illustrate the behavior of the Taylor expansion inside and outside the convergence domain the real part of $\varepsilon(\omega)$ is shown in Fig. 1a,c together with the sixteenth-order Taylor expansion for two typical values of the carrier frequency.

The dispersion function $k(\omega)$ corresponding to Eq. (1) is derived from the dispersion relation

$$\omega^2\varepsilon(\omega) = k^2c^2. \tag{2}$$

The pulse group velocity $v_{\mathrm{gr}} = \mathrm{Re}(\partial\omega/\partial k)$ is compared to that derived from the Taylor approximation of $k(\omega)$ in Fig. 1b,d. We see that $k(\omega)$ is poorly approximated near $\omega \approx 2\omega_c$.

In what follows we use a more general rational approximation

$$\varepsilon(\omega) = \cdots + \frac{\eta_{-4}}{\omega^4} + \frac{\eta_{-2}}{\omega^2} + \eta_0 + \eta_2\omega^2 + \eta_4\omega^4 + \cdots \,, \tag{3}$$

where $\eta_{0,\pm2,\pm4...}$ are empirical dispersion constants of the medium [16]. More specifically, we will use a simple truncation of Eq. (3)

$$\varepsilon(\omega) \approx \bar{\varepsilon}\left(1 - \mu^2\frac{\omega_0^2}{\omega^2} + \nu^2\frac{\omega^2}{\omega_0^2}\right) , \tag{4}$$

where ω_0 is a suitable reference frequency; $\bar{\varepsilon}$, μ, and ν are dimensionless fit parameters.

An exemplary fit of the response function (1) by the expression (4) is shown in Fig. 1e,f. We consider the spectral interval

$$250\,\mathrm{THz} < \omega < 5\,\mathrm{PHz} \tag{5}$$

with $\omega_0 = 1\,\mathrm{PHz}$ and obtain the following values of the parameters:

$$\bar{\varepsilon} = 1.5369, \quad \mu^2 = 0.01115, \quad \nu^2 = 0.004676, \tag{6}$$

for the best fit of $\varepsilon(\omega)$ within the interval (5). The parameter values (6) will be later used for numerical solutions of the field equations.

Let us consider the dispersion relation (2) with the response function (4), i.e.,

$$\omega^2 - \mu^2\omega_0^2 + \frac{\nu^2}{\omega_0^2}\omega^4 = k^2 v_{\mathrm{ph}}^2, \tag{7}$$

where $v_{\mathrm{ph}} = c/\sqrt{\bar{\varepsilon}}$ is a typical value of the phase velocity for the frequency range of interest.

Equation (7) gives two solutions for ω^2, positive and negative. It is helpful to simplify these solutions using the smallness of μ^2 and ν^2 in Eq. (6). Since in the frequency range of interest $\omega^2 \approx k^2 v_{\mathrm{ph}}^2$, the positive solution is given by the relation

$$\omega^2 = k^2 v_{\mathrm{ph}}^2 + \mu^2\omega_0^2 - \frac{\nu^2}{\omega_0^2}k^4 v_{\mathrm{ph}}^4 + \mathrm{h.o.t.}$$

The second (negative) solution

$$\omega^2 = -\frac{\omega_0^2}{\nu^2} - k^2 v_{\mathrm{ph}}^2 + \mathrm{h.o.t.}$$

formally corresponds to exponentially growing oscillations. These oscillations are unphysical because their spectrum is located outside the frequency domain (5) where approximation (4) is valid. They will produce a numerical instability when solving (in time domain) the wave equation with the response function (4). To avoid the instability we regularize Eq. (7) by replacing ω^4 with $\omega^2 \cdot k^2 v_{\mathrm{ph}}^2$. The resulting dispersion relation reads

$$\omega^2 = \frac{k^2 v_{\mathrm{ph}}^2 + \mu^2 \omega_0^2}{1 + \nu^2 k^2 v_{\mathrm{ph}}^2 / \omega_0^2} > 0. \tag{8}$$

One can easily check that Eqs. (7) and (8) are practically identical for the parameter values (6) and frequency domain (5).

Equation (8) can be related to Eq. (2) by defining the following regularized counterpart of Eq. (4)

$$\varepsilon(\omega, k) = \bar{\varepsilon} \left(1 - \mu^2 \frac{\omega_0^2}{\omega^2} + \nu^2 \frac{k^2}{k_0^2} \right), \tag{9}$$

where $k_0 = \omega_0 / v_{\mathrm{ph}}$. In what follows we apply Eq. (9) to calculate the linear part of the electric displacement vector.

3 Model derivation

In this section we derive a reduced model equation for the few-cycle pulses. We consider one-dimensional pulse propagation in a bulk medium and assume that the radial dependence of the electric field $\mathbf{E} = (E(z,t), 0, 0)$ is negligible. The field dynamics can then be described by a $(1+1)$ dimensional equation

$$\partial_t^2 D - c^2 \partial_z^2 E = 0. \tag{10}$$

The electric displacement vector $D(E)$ contains both linear and nonlinear parts. In the spectral domain the linear part $D^{\mathrm{lin}}(z,t)$ is given as

$$D_{\omega k}^{\mathrm{lin}} = \varepsilon(\omega, k) E_{\omega k},$$

where $\varepsilon(\omega, k)$ is defined by Eq. (9). In the physical space this corresponds to the following material relation

$$\partial_t^2 D^{\mathrm{lin}} = \bar{\varepsilon} \left(\partial_t^2 E + \mu^2 \omega_0^2 E - \frac{\nu^2}{k_0^2} \partial_t^2 \partial_z^2 E \right).$$

Inserting $D = D^{\mathrm{lin}} + D^{\mathrm{nl}}$ into Eq. (10) we obtain the following evolution equation

$$\partial_t^2 E - v_{\mathrm{ph}}^2 \partial_z^2 E + \mu^2 \omega_0^2 E - \frac{\nu^2}{k_0^2} \partial_t^2 \partial_z^2 E + \frac{1}{\bar{\varepsilon}} \partial_t^2 D^{\mathrm{nl}} = 0.$$

Introducing normalized coordinates $\bar{t} = \omega_0 t$ and $\bar{z} = k_0 z$, we rewrite the evolution equation in the form

$$\partial_{\bar{t}}^2 E - \partial_{\bar{z}}^2 E + \mu^2 E - \nu^2 \partial_{\bar{t}}^2 \partial_{\bar{z}}^2 E + \frac{1}{\bar{\varepsilon}} \partial_{\bar{t}}^2 D^{\mathrm{nl}} = 0. \tag{11}$$

Further, we assume an instantaneous self-focusing Kerr nonlinearity $D^{\rm nl} = 4\pi\chi^{(3)}E^3$ with positive frequency independent $\chi^{(3)}$. Finally introducing a normalized electric field

$$u = \sqrt{\frac{4\pi\chi^{(3)}}{\bar{\varepsilon}}}E,$$

we arrive to the dimensionless model equation

$$u_{tt} - u_{zz} + \mu^2 u - \nu^2 u_{ttzz} + \partial_t^2(u^3) = 0, \tag{12}$$

where derivatives are denoted by indices and the bars over t and z are omitted.

Equation (12) is our model equation for the ultrashort pulses that will be investigated in the reminder of the paper. Note, that when deriving Eq. (12) neither SVEA nor the unidirectional approximation was used.

4 Analysis of the model

In this section we discuss properties of the model (12) and consider its numerical solutions.

4.1 Lagrangian formulation

In the course of derivation of the model (12) all dissipative effects, e.g., the imaginary part of the response function, were neglected. As a consequence, one could expect that Eq. (12) has an intrinsic Lagrangian structure and integrals of motion which are discussed in this section.

To obtain conservation laws for Eq. (12) we introduce the following Lagrangian density

$$\mathfrak{L} = \frac{\phi_t^2}{2} - \frac{\phi_z^2}{2} - \frac{\mu^2\phi^2}{2} - \frac{\nu^2\Phi^2}{2} + \nu^2\phi_z\Phi_t + \frac{\phi_t^4}{4} \tag{13}$$

for two real scalar fields $\phi(z,t)$ and $\Phi(z,t)$. The corresponding Lagrangian equations

$$\frac{\delta}{\delta\phi}\int \mathfrak{L}\,dxdt = 0 \qquad \text{and} \qquad \frac{\delta}{\delta\Phi}\int \mathfrak{L}\,dxdt = 0$$

have the form

$$\phi_{tt} - \phi_{zz} + \mu^2\phi + \nu^2\Phi_{tz} + \partial_t(\phi_t^3) = 0,$$
$$\Phi + \phi_{tz} = 0,$$

or

$$\phi_{tt} - \phi_{zz} + \mu^2\phi - \nu^2\phi_{ttzz} + \partial_t(\phi_t^3) = 0. \tag{14}$$

Now, applying $\partial/\partial t$ to Eq. (14) and replacing ϕ_t with u we arrive at Eq. (12). Hence, we conclude that our basic Eq. (12) is a Lagrangian one.

Using the Lagrangian (13) one immediately derives an energy density

$$\mathfrak{e} = \frac{\partial \mathfrak{L}}{\partial \phi_t}\phi_t + \frac{\partial \mathfrak{L}}{\partial \Phi_t}\Phi_t - \mathfrak{L} = \frac{\phi_t^2}{2} + \frac{\phi_z^2}{2} + \frac{\mu^2\phi^2}{2} + \frac{\nu^2\Phi^2}{2} + \frac{3\phi_t^4}{4},$$

and a momentum density

$$\mathfrak{p} = \frac{\partial \mathfrak{L}}{\partial \phi_t}\phi_z + \frac{\partial \mathfrak{L}}{\partial \Phi_t}\Phi_z = \phi_t\phi_z + \nu^2\phi_z\Phi_z + \phi_t^3\phi_z.$$

Finally, the energy and the momentum are given by

$$\mathcal{E} = \int \left(\frac{\phi_t^2}{2} + \frac{\phi_z^2}{2} + \frac{\mu^2\phi^2}{2} + \frac{\nu^2\phi_{tz}^2}{2} + \frac{3\phi_t^4}{4} \right) dz,$$
$$\mathcal{P} = \int \left(\phi_t\phi_z + \nu^2\phi_{zz}\phi_{tz} + \phi_t^3\phi_z \right) dz.$$

These integrals of motion have been used to test the quality of numerical solutions. They are completely expressed in terms of the potential ϕ so that Eq. (14) is a useful reformulation of the basic model (12).

4.2 Numerical solutions

In this section we discuss numerical solutions of the model (12). The values of the parameters μ and ν are specified in Eq. (6). The solutions are calculated on the spatial domain $|z| < 100\pi$ corresponding to $152\,\mu$m for the system parameters from Fig. 1. Periodic boundary conditions are used. The initial pulse shape is given by the expression

$$u|_{t=0} = \frac{A \sin \kappa z}{\cosh(z/L)}, \tag{15}$$

where the dimensionless parameter A determines the amplitude, κ is the wave vector, and L is the duration of the pulse. The carrier frequency ω_c is given by Eq. (8) with $k = k_0\kappa$. To a good approximation ω_c is equal to $\kappa \cdot 1\,$PHz. The initial condition for the first derivative of the electric field u_t is specified in such a way that

$$(u_t + u_z)|_{t=0} = 0,$$

so that the pulse moves along z-axis. The total simulation time is 10 fs (Fig. 2 and 4, total pulse path $\approx 2.42\,$mm) and 30 fs (Fig. 3).

The carrier frequency for the initial pulse in Fig. 2 corresponds to the ZDF which is equal to 0.98 PHz for the dispersion law (1). As we see from Fig. 2d,f, the spectrum quickly becomes considerably brighter. The pulse splits in two parts corresponding to normal ($\omega >$ ZDF) and anomalous ($\omega <$ ZDF) dispersion domains. A considerable part of the pulse spectrum is located at $\omega \gtrsim 2\omega_c$. Therefore the NSE based on the Taylor expansion around ω_c can not describe pulse splitting adequately. A

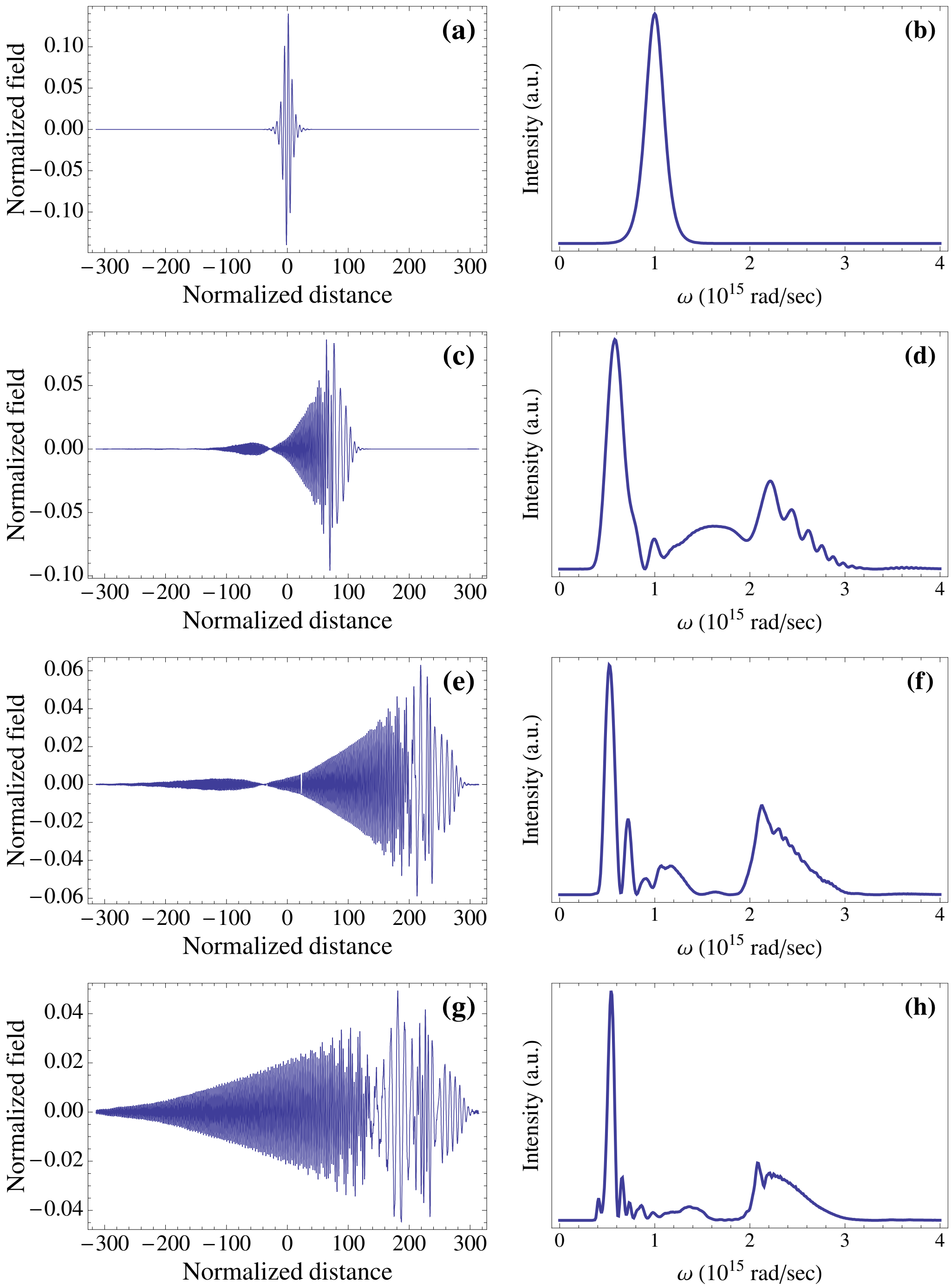

Figure 2: Numerical solution of Eq. (12) with the parameter values (6) and initial pulse shape (15). $A = 0.15$, $\kappa = 1$, and $L = 5$. Left: pulse shape in a suitably shifted coordinate frame. Right: pulse spectrum. Solutions are shown at times: 0 fs (a,b); 2 fs (c,d); 5 fs (e,f); 10 fs (g,h).

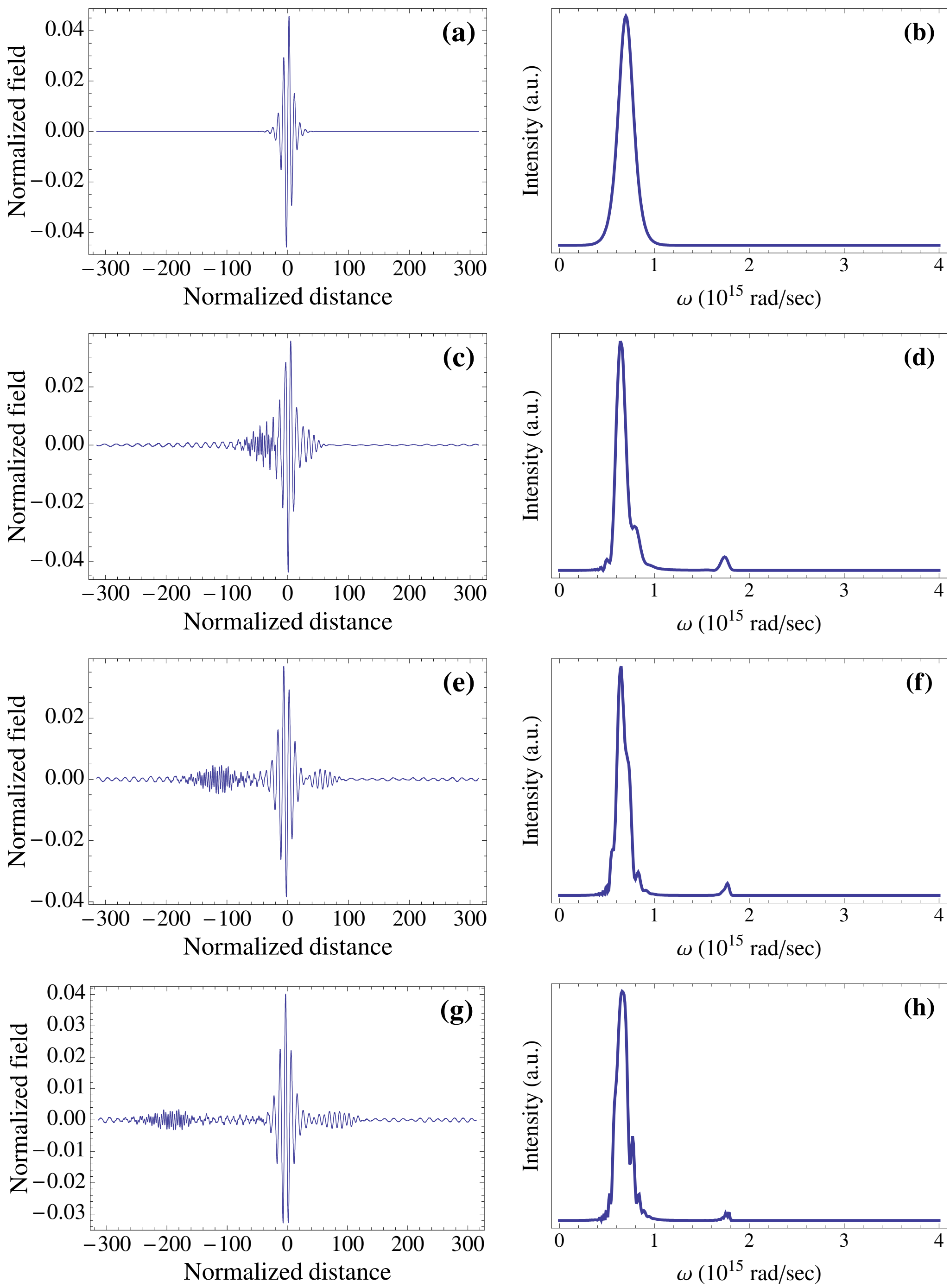

Figure 3: Numerical solution of Eq. (12) with the parameter values (6) and initial pulse shape (15). $A = 0.05$, $\kappa = 0.7$, and $L = 6$. Left: pulse shape in a suitably shifted coordinate frame. Right: pulse spectrum. Solutions are shown at times: 0 fs (a,b); 10 fs (c,d); 20 fs (e,f); 30 fs (g,h).

similar splitting of the pulse spectrum was also observed in simulations of the full Maxwell system coupled with the instantaneous Kerr nonlinearity (see, e.g., [17]).

Figure 3 shows the pulse evolution in the anomalous dispersion regime where ordinary envelope solitons exist [1, 2]. The parameters A and L in Eq. (15) are chosen in such a way that the pulse envelope $A/\cosh(z/L)$ coincides with one of the exact NSE solitary solutions. The pulse has a clear tendency to survive during the evolution. Nevertheless, it splits in three parts (Fig. 3c). A tail containing oscillations in the normal dispersion domain still appears. This tail and the corresponding localized solution moving behind the main pulse (Fig. 3e,g) can not be described by NSE.

Figure 4 illustrates propagation of a short pulse in the normal dispersion regime when ordinary envelope solitons do not exist. The dominating dynamical process is permanent pulse broadening. Note, that the pulse envelope evolves smoothly but a sudden jump at the pulse front. This jump can be interpreted as an optical shock wave.

4.3 Traveling-wave solutions

In this section we demonstrate that Eq. (12) does not have localized traveling-wave solutions of the form $u = u(z - st)$ where a constant parameter s determines soliton velocity. To this end we apply a variational approach based on the potential formulation (14). Introducing an ansatz

$$\phi(z,t) = \frac{1}{s^2} f(\xi) \qquad \text{with} \qquad \xi = z - st,$$

we reduce Eq. (14) to an ordinary differential equation for $f(\xi)$

$$(s^2-1)f'' + \mu^2 f - s^2\nu^2 f'''' + (f'^3)' = 0,$$

where the derivative with respect to ξ is denoted by a prime. This equation can be reformulated as an extremum condition $\delta I[f]/\delta f = 0$, where the functional

$$I = \int \left[(1-s^2)(f')^2 + \mu^2 f^2 - (s\nu f'')^2 - \frac{(f')^4}{2} \right] d\xi$$

is related with the Lagrangian (13).

Assume that $f = h(\xi)$ is a localized solitary solution in question. Inserting into $I[f]$ a scaled test function $f = h(\sigma\xi)/\sqrt{\sigma}$ with a free positive scaling parameter σ, we rewrite the result as

$$I = \int \left[(1-s^2)(h')^2 + \frac{\mu^2 h^2}{\sigma^2} - (s\nu\sigma h'')^2 - \frac{\sigma(h')^4}{2} \right] d\xi.$$

Since for $\sigma = 1$ the test function is equal to the solitary solution, $I(\sigma)$ should have an extremum at this point. However, this contradicts to the fact that

$$\left.\frac{dI(\sigma)}{d\sigma}\right|_{\sigma=1} = -\int \left[2\mu^2 h^2 + 2(s\nu h'')^2 + \frac{(h')^4}{2} \right] d\xi < 0.$$

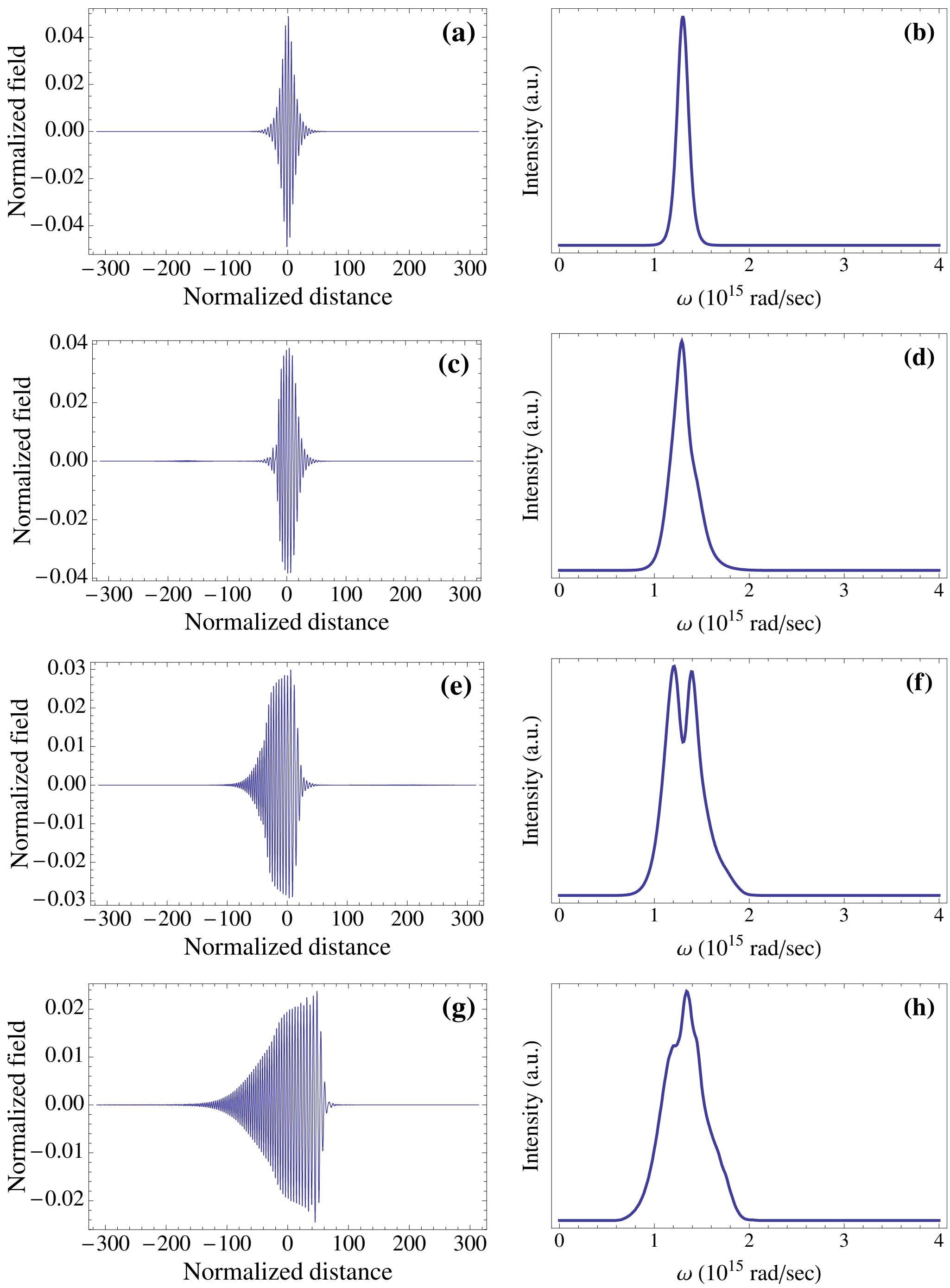

Figure 4: Numerical solution of Eq. (12) with the parameter values (6) and initial pulse shape (15). $A = 0.05$, $\kappa = 1.3$, and $L = 8$. Left: pulse shape in a suitably shifted coordinate frame. Right: pulse spectrum. Solutions are shown at times: 0 fs (a,b); 2 fs (c,d); 5 fs (e,f); 10 fs (g,h).

Hence, the localized solitary solution $h(\xi)$ can not exist. In other words, all solitary solutions of Eq. (12) are non-stationary in the comoving frame of reference. Such solutions are usually referred to as breathers. At present, explicit expressions for breathers are known only for simple special cases of Eq. (12) (see [18, 19, 20, 21] and also the next section).

4.4 Limiting cases of Eq. (12)

In this section we investigate relations between Eq. (12) and some other reduced models derived earlier to describe propagation of short optical pulses. We recall that Eq. (12) was obtained without any assumptions on pulse duration. When the pulse spectrum is sufficiently narrow, Eq. (12) can be reduced to the envelope equation, which is nothing else but NSE. To derive it we insert a standard plane-wave ansatz $u \sim e^{i(\kappa z-\Omega t)}$ into the linear part of Eq. (12) and obtain the dispersion relation

$$\Omega^2 - \kappa^2 - \mu^2 + \nu^2\Omega^2\kappa^2 = 0 \tag{16}$$

which coincides with Eq. (8) after the back-transformation to the dimensional $\omega = \Omega\omega_0$ and $k = \kappa k_0$ [see Eqs. (4) and (9)]. We now introduce the SVEA by writing

$$u(z,t) = \frac{1}{2}\psi(z,t)e^{i(\kappa z-\Omega t)} + \text{c.c.}, \tag{17}$$

where Ω and κ obey Eq. (16) and the complex amplitude $\psi(z,t)$ is slow. Inserting Eq. (17) into Eq. (12) in the first order of the perturbation theory we obtain

$$\psi_t + \bar{v}_{\text{gr}}\psi_z = 0,$$

where

$$\bar{v}_{\text{gr}} = \frac{\partial\Omega}{\partial\kappa} = 1 - \frac{\mu^2}{2\kappa^2} - \frac{3\nu^2\kappa^2}{2}$$

is the dimensional group velocity $v_{\text{gr}} = \partial\omega/\partial k$ normalized by $v_{\text{ph}} = \omega_0/k_0 = c/\sqrt{\bar{\varepsilon}}$. In the second order order of the perturbation theory one obtains a classical NSE

$$i(\psi_z + \bar{v}_{\text{gr}}^{-1}\psi_t) + \frac{\mu^2 - 3\nu^2\kappa^4}{2\kappa^3}\psi_{tt} + \frac{3\kappa}{8}|\psi|^2\psi = 0, \tag{18}$$

where the factor in front of ψ_{tt} is proportional to $\partial\bar{v}_{\text{gr}}/\partial\kappa$ and changes its sign at the ZDF point $\kappa^2 = \mu/(\sqrt{3}\nu)$. A solitary solution of Eq. (18) for parameter values (6) and $\kappa = 0.7$ was used as an initial condition in the numerical solution shown in Fig. 3.

It is also of interest to compare Eq. (12) with the simplified unidirectional models reported in the literature. In the case when the fit parameters μ and ν are small, we first introduce the scaling parameter $\epsilon \ll 1$ such that Eq. (12) becomes

$$u_{tt} - u_{zz} + \epsilon^2(a^2u - b^2u_{ttzz}) + \partial_t^2(u^3) = 0, \tag{19}$$

where both $a = \mu/\epsilon$ and $b = \nu/\epsilon$ are of order 1. Next, we introduce a suitable unidirectional scaling

$$u(z,t) = \epsilon U(\zeta,\tau) = \epsilon U(\epsilon^2 z,\ t - z)$$

and evaluate

$$\begin{aligned} u_{tt} - u_{zz} &= 2\epsilon^3 U_{\tau\zeta} + O(\epsilon^5), \\ \epsilon^2 u_{ttzz} &= \epsilon^3 U_{\tau\tau\tau\tau} + O(\epsilon^5), \\ \partial_t^2(u^3) &= \epsilon^3 \partial_\tau^2(U^3). \end{aligned}$$

Finally, keeping terms of the leading (cubic) order in ϵ, we transform Eq. (19) into

$$2U_{\zeta\tau} + a^2 U - b^2 U_{\tau\tau\tau\tau} + \partial_\tau^2(U^3) = 0, \tag{20}$$

which is identical to the unidirectional model first derived in [22].

Even more simple unidirectional equations can be obtained when the pulse spectrum is situated entirely either on the left or on the right side from the ZDF. Let us start with the pulse propagation in the normal dispersion regime where the μ^2 term in Eq. (12) can be neglected. Without loss of generality one can take $a = 0$ and $b = 1$ and rewrite Eq. (20) as a modified Korteweg de-Vries equation [23, 24, 25, 20]

$$2U_\zeta - U_{\tau\tau\tau} + 3U^2 U_\tau = 0. \tag{21}$$

Equation (21) is completely integrable by the inverse scattering technique and has localized solutions in the form of breathers [26].

Another limiting case corresponds to the pulse propagation in the anomalous dispersion regime where the ν^2 term in Eq. (12) can be neglected. Taking $a = 1$ and $b = 0$ we transform Eq. (20) into the so-called short pulse equation [27, 28]

$$2U_{\zeta\tau} - U + \partial_\tau^2(U^3) = 0. \tag{22}$$

Similar to Eq. (21), Eq. (22) is completely integrable and has localized solutions in the form of breathers [29, 30]. However, unlike Eq. (21), pulse evolution within the framework of Eq. (22) can lead to shock formation. Thereafter the higher-order derivative term can not be ignored and the more general Eq. (20) should be used instead of Eq. (22).

5 Conclusions

Theoretical analysis of ultrashort pulse propagation requires a modification of standard envelope models not only because of the invalidity of the SVEA. When the pulse spectral width is comparable to that of the transparency window, the usual dispersion description in terms of Taylor expansion of the dispersion relation $k(\omega)$

becomes invalid as well. The reason is a resonant nature of the medium response function $\varepsilon(\omega)$ which contains singularity points in the complex plane.

In this paper we replace a polynomial dispersion operator with a rational one and derive a simple non-envelope model Eq. (12) for the electric field. This model does not assume unidirectional propagation and in this sense it generalizes several previously reported unidirectional equations. We reveal the Lagrangian structure of the model and derive the integrals of motion. Using these results we demonstrate that all solitary solutions of Eq. (12) are breathers oscillating in the comoving frame of reference. Numerical analysis of the model equation reveals various dynamical effects, such as spectral broadening, pulse splitting near the zero-dispersion frequency, and formation of optical shocks.

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