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Modeling solutions with jumps for rate-independent systems on metric spaces

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Abstract

Rate-independent systems allow for solutions with jumps that need additional modeling. Here we suggest a formulation that arises as limit of viscous regularization of the solutions in the extended state space. Hence, our *parametrized metric solutions* of a rate-independent system are absolutely continuous mappings from a parameter interval into the extended state space. Jumps appear as generalized gradient flows during which the time is constant. The closely related notion of *BV solutions* is developed afterwards. Our approach is based on the abstract theory of generalized gradient flows in metric spaces, and comparison with other notions of solutions is given.

1 Introduction

This paper is concerned with the analysis of different solution notions for rate-independent evolutionary systems. The latter arise in a very broad class of mechanical problems, usually in connection with hysteretic behavior. With no claim at completeness, we may mention for instance elastoplasticity, damage, the quasistatic evolution of fractures, shape memory alloys, delamination and ferromagnetism, referring to [Mie05] for a survey of the modeling of rate-independent phenomena.

Because of their relevance in applications, the analysis of these systems has attracted some attention over the last decade, also in connection with the issue of their proper formulation. In fact, in several situations rate-independent problems may be recast in the form of a doubly nonlinear evolution equation involving two *energy functionals*, namely

$$\partial_{\dot q}\mathcal{R}(q(t),\dot q(t)) + \partial_q \mathcal{E}(t,q(t)) \ni 0 \qquad \text{in } Q' \qquad \text{for a.a. } t \in (0,T)\,, \tag{1.1}$$

where Q is a separable Banach space, $\mathcal{R} : Q \times Q \to [0,\infty]$ a *dissipation* functional and $\mathcal{E} : [0,T]\times Q \to (-\infty,\infty]$ an *energy potential*, $\partial_{\dot q}$ and ∂_q denoting their subdifferential with respect to the second variable. Rate-independence is rendered through 1-homogeneity of the functional $\mathcal{R}$ with respect to its second variable. Indeed, assuming that $\mathcal{R}(q,\gamma v) = \gamma\mathcal{R}(q,v)$ for all $\gamma \geq 0$ and $(q,v) \in Q \times Q$, one has that equation (1.1) is invariant for time-rescalings. This captures the main feature of this kind of processes, which are driven by an external loading set on a time scale much slower than the time scale intrinsic to the system, but still fast enough to prevent equilibrium. Typically, this quasistatic behavior originates in the limit of systems with a viscous, rate-dependent dissipation.

The formulation of rate-independent problems in terms of the subdifferential inclusion (1.1) has been thoroughly analyzed in [MiT04], in the case of a reflexive Banach space.

Existence of solutions to the Cauchy problem for (1.1) is proved through approximation by time discretization and solution of incremental minimization problems. However, in many applications the energy $\mathcal{E}(t,\cdot)$ is neither smooth nor convex, and the state space Q is often neither reflexive nor the dual of a separable Banach space (see e.g. [KrZ07] for energies having linear growth at infinity). Furthermore, Q may even lack a linear structure (in these cases we will denote it by the calligraphic letter $\mathcal{Q}$), like for finite-strain elastoplasticity [Mie03, MaM08] or for quasistatic evolution of fractures [DFT05].

In such situations, the differential formulation (1.1) cannot be used. In [MiT99, MTL02], the concept of *energetic solution* for general *rate-independent energetic systems* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ has been introduced, by replacing the infinitesimal metric $\mathcal{R}$ of the subdifferential formulation (1.1) by a global *dissipation distance* $\mathcal{D}:\mathcal{Q}\times\mathcal{Q}\to[0,\infty]$. This formulation, see Section 5.1, is *derivative-free*, and thus applies to solutions with jumps and can be used in very general frameworks, like, for example, in a topological space $\mathcal{Q}$, with $\mathcal{E}$ and $\mathcal{D}$ lower semi-continuous only, see [MaM05, Mie05, FrM06]. Energetic solutions are very flexible and allow for a quite general existence theory; however, the global stability condition, asking that $q(t)$ globally minimizes the map $\widetilde{q}\mapsto\mathcal{E}(t,\widetilde{q})+\mathcal{D}(q(t),\widetilde{q})$, implies that solutions jump earlier as physically expected, since they are forced to leave a locally stable state, see e.g. [Mie03, Ex. 6.1] or [KMZ07, Ex. 6.3], and Example 7.1 below. Moreover, existence of energetic solutions is proved via time discretization and incremental *global minimization*, but, as discussed in [Mie03, Sec. 6], *local minimization* would be more appropriate both from the perspective of modeling and of numerical algorithms.

In response to these issues, in [EfM06] a *vanishing viscosity approach* was proposed to derive new solution types for rate-independent systems $(\mathcal{Q},\mathcal{R},\mathcal{E})$. There, $\mathcal{Q}$ is assumed to be a *finite-dimensional Hilbert space* Q and $\mathcal{E}\in\mathrm{C}^1([0,T]\times Q)$. The natural viscous approximation of (1.1) is obtained by adding a quadratic term to the dissipation potential, viz. $\mathcal{R}_\varepsilon(q,v)=\mathcal{R}(q,v)+\frac{\varepsilon}{2}\|v\|^2$, and leads to the doubly nonlinear equation

$$\varepsilon\dot{q}(t)+\partial_{\dot{q}}\mathcal{R}(q(t),\dot{q}(t))+\partial_q\mathcal{E}(t,q(t))\ni 0 \qquad \text{for a.a. } t\in(0,T)\,. \tag{1.2}$$

Using $\dim Q<\infty$, the existence of solutions $q_\varepsilon\in\mathrm{H}^1([0,T];Q)$ is obvious and passing to the limit $\varepsilon\searrow 0$ in (1.2) leads to new solutions and to a finer description of the jumps, which occur later than for energetic solutions. The key idea (see Section 2) is that the limiting solution at jumps shall follow a path which somehow keeps track of the viscous approximation. To exploit this additional information, one has to go over to an extended state space: reparametrizing the approximating viscous solutions q_ε of (1.2) by their arclength τ_ε, and introducing the rescalings $\widehat{t}_\varepsilon=\tau_\varepsilon^{-1}$ and $\widehat{q}_\varepsilon=q_\varepsilon\circ\widehat{t}_\varepsilon$, one studies the limiting behavior of the sequence $\{(\widehat{t}_\varepsilon,\widehat{q}_\varepsilon)\}_\varepsilon$ as $\varepsilon\downarrow 0$. Hence, in [EfM06] it was proved that (up to a subsequence), $\{(\widehat{t}_\varepsilon,\widehat{q}_\varepsilon)\}_\varepsilon$ converges to a pair $(\widehat{t},\widehat{q})$, whose evolution encompasses both dry friction effects and, when the system jumps, the influence of rate-dependent dissipation. In fact, the jump path may be completely described by a gradient flow equation, which leads to this interpretation: jumps are fast (with respect to the slow external time scale) transitions between two metastable states, during which the system switches to a viscous regime. Furthermore, solutions of the limiting rate-independent problem can be constructed by means of a time-discretization scheme featuring local, rather than global, minimization.

This paper provides the first step of the generalization of these ideas to the much more general metric framework using the concept of *curves of maximal slope*, which dates back to the pioneering paper [DGMT80]. We also refer to the recent monograph [AGS05], the references therein, and to [RMS08]. The general setup starts with a

$$\text{complete metric space } (\mathcal{X}, d)$$

and introduces the *metric velocity*

$$|q'| := \lim_{h\searrow 0} \frac{d(q(t), q(t+h))}{h} = \lim_{h\searrow 0} \frac{d(q(t-h), q(t))}{h}\,, \tag{1.3}$$

which is defined a.e. along an absolutely continuous curve $q : [0,T] \to \mathcal{X}$.

This replaces the norm of the derivative q' in the smooth setting, and, in the same way, the norm of the (Gâteaux)-derivative or the subdifferential of a functional $\mathcal{F} : \mathcal{X} \to (-\infty, \infty]$ is replaced by the *local slope* of $\mathcal{F}$ in $q \in \mathrm{dom}(\Psi)$, which is defined by

$$|\partial_q \mathcal{F}|(q) := \limsup_{v\to q} \frac{(\mathcal{F}(q) - \mathcal{F}(v))^+}{d(q,v)}\,, \tag{1.4}$$

where $(\cdot)^+$ denotes the positive part. With these concepts, the viscous problem (1.2) has the equivalent metric formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t\mathcal{E}(t,q(t)) \le -(|q'|(t) + \frac{\varepsilon}{2}|q'|^2(t)) - \frac{1}{2\varepsilon}\left(\left(|\partial_q\mathcal{E}|\,(t,q(t)) - 1\right)^+\right)^2, \tag{1.5}$$

for a.a. $t \in (0,T)$, see Section 3.1 for further details. It was proved in [RMS08, Thm. 3.5] that, under suitable assumptions on $\mathcal{E}$, for every $\varepsilon > 0$ the related Cauchy problem has at least one solution $q_\varepsilon \in \mathrm{AC}([0,T];\mathcal{X})$.

Following the approach of [EfM06], for every $\varepsilon > 0$ we now consider the (arclength) rescalings $(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon)$ associated with q_ε, which in turn fulfill a rescaled version of (1.5), cf. (3.12). Under suitable assumptions, in Theorem 3.8 we shall show that, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon)\}$ converges to a limit curve $(\widehat{t}, \widehat{q}) \in \mathrm{AC}([0,S];[0,T]\times\mathcal{X})$ such that

$$\begin{aligned} &\widehat{t} : [0,S] \to [0,T] \text{ is nondecreasing,} \\ &\widehat{t}'(s) + |\widehat{q}'|(s) > 0 \quad \text{for a.a. } s \in [0,S], \end{aligned} \tag{1.6a}$$

$$\left.\begin{aligned} \widehat{t}'(s) > 0 \quad &\Longrightarrow \quad |\partial_q\mathcal{E}|\left(\widehat{t}(s),\widehat{q}(s)\right) \le 1, \\ |\widehat{q}'|(s) > 0 \quad &\Longrightarrow \quad |\partial_q\mathcal{E}|\left(\widehat{t}(s),\widehat{q}(s)\right) \ge 1, \end{aligned}\right\} \text{ for a.a. } s \in [0,S]\,, \tag{1.6b}$$

and the *energy identity*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{q}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{q}(s))\,\widehat{t}'(s) &= \langle \mathrm{D}_q\mathcal{E}(\widehat{t}(s),\widehat{q}(s)), \widehat{q}'(s)\rangle \\ &= -|\widehat{q}'|(s)\,|\partial_q\mathcal{E}|\left(\widehat{t}(s),\widehat{q}(s)\right) \quad \text{for a.a. } s \in (0,S)\,, \end{aligned} \tag{1.6c}$$

holds. A pair $(\widehat{t},\widehat{q}) : [s_0, s_1] \to [0,T] \times \mathcal{X}$ satisfying (1.6) (with $[0,S]$ replaced by $[s_0, s_1]$) is called *parametrized metric solution* of the rate-independent system $(\mathcal{X}, d, \mathcal{E})$.

Indeed, the very focus of this paper is on getting insight into the properties of parametrized metric solutions and comparing them with the other solution notions for rate-independent evolutions. That is why, in order to avoid technicalities and to highlight, rather, the features of our approach, throughout the next sections we shall work in a technically simpler setup, in which the state space $\mathcal{X}$ is a finite-dimensional manifold, endowed with a (Finsler) distance d associated with a 1-homogeneous dissipation functional $\mathcal{R} : \mathrm{T}\mathcal{Q} \to [0,\infty)$, and an energy $\mathcal{E} \in \mathrm{C}^1([0,T] \times \mathcal{Q})$. The fully general metric framework is postponed to the forthcoming paper [MRS08], see also Section 6.

The notion of parametrized metric solution $(\widehat{t},\widehat{q})$ generalizes the outcome of the finite-dimensional vanishing viscosity analysis of [EfM06] and hence allows for the same mechanical interpretation, see Remark 3.5. Namely, according to whether either of the derivatives $\widehat{t}'$ or $|\widehat{q}'|$ is null or strictly positive, one distinguishes in (1.6b) three regimes: sticking, rate-independent evolution, and switching to a viscous regime (in correspondence to jumps of the system from one metastable state to another). In this metric setup as well, we show that the behavior of the system along a jump path is described by a generalized gradient flow. This can be seen more clearly when considering the non-parametrized solution q corresponding to the pair $(\widehat{t},\widehat{q})$. The latter functions are called *BV solutions* of $(\mathcal{Q}, d, \mathcal{E})$ and are *pointwise limits of the un-rescaled* vanishing viscosity approximations q_ε, see Definition 4.3 and Section 4 for an analysis of their properties. In particular, we shall show how to pass, by means of a suitable transformation, from a (truly jumping) BV solution q to a ("virtually" jumping) parametrized solution $(\widehat{t},\widehat{q})$, and conversely.

In Section 5, we compare the notion of BV solutions with other solutions concepts, namely with the energetic solutions of [MiT99, MTL02], and with the approximable and local solutions of [KMZ07, ToZ06, Cag08] (suitably rephrased in the metric setting, see Definitions 5.1, 5.5, and 5.6). In Section 5.3 we review the notion of Φ-*minimal solutions* of a rate-independent evolutionary system, proposed in [Vis01] using a *global variational principle* in terms of a suitably defined partial order relation between trajectories. First, we conclude that the notion of local solution is the most general concept, including energetic and BV solutions, whereas BV solutions encompass approximable and Φ-minimal solutions. Moreover, our notion of BV solutions has "more structure", which makes it robust with respect to data perturbations (cf. Remark 3.10), whereas neither approximable nor Φ-minimal solutions are upper-semicontinuous with respect to data perturbations.

Further insight into the comparison between the various solution notions is provided by the examples presented in Section 7, which are one or two-dimensional, such that the set of all solutions can be discussed easily. The one-dimensional case in fact relates to crack growth (under the assumption of a prescribed crack path), which was treated in [ToZ06, Cag08, NeO07, KMZ07]. The solution concepts developed there are also based on the vanishing viscosity method. In the latter case, the solution type in fact coincides with our notion of BV solution. We postpone the more difficult PDE applications to [MRS08], where we are going to combine the notions of this paper with the methods of [RMS08] to develop the present ideas in the infinite-dimensional or fully metric setting. Related ideas using the vanishing viscosity method for PDEs are found in [DD*07], for a model for elastoplasticity problems with softening, and in [MiZ08], for general parabolic PDEs with rate-independent dissipation terms.

2 Setup and mechanical motivation

We consider a manifold $\mathcal{Q}$ that contains the states of our system. The energy $\mathcal{E}$ of the system depends on the time $t \in [0,T]$ and the state $q \in \mathcal{Q}$. Throughout the paper, we shall assume that $\mathcal{E} \in \mathrm{C}^1(\mathcal{Q}_T)$, where $\mathcal{Q}_T = [0,T] \times \mathcal{Q}$ denotes the extended state space. The evolution of the system is governed by a balance between the potential restoring force $-\mathrm{D}_q\mathcal{E}(t,q)$ and a frictional force f. The latter is given by a continuous dissipation potential $\mathcal{R} :\ \mathrm{T}\mathcal{Q} \to [0,\infty)$, in the form $f \in \partial_{\dot q}\mathcal{R}(q,\dot q)$. We generally assume that $\mathcal{R}(q,\cdot) :\ \mathrm{T}_q\mathcal{Q} \to [0,\infty)$ is convex and $\partial_{\dot q}\mathcal{R}(q,\dot q) \subset \mathrm{T}^*_q\mathcal{Q}$ is the set-valued subdifferential. Hence, the system is governed by the differential inclusion

$$0 \in \partial_{\dot q}\mathcal{R}(q(t),\varepsilon \dot q(t)) + \mathrm{D}_q\mathcal{E}(t,q(t)) \subset \mathrm{T}^*_q\mathcal{Q}\,, \qquad t \in (0,T)\,, \tag{2.1}$$

in which we have introduced a small parameter $\varepsilon > 0$ to indicate that we are on a very slow time scale.

Further, we suppose that $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, where $\mathcal{R}_1 :\ \mathrm{T}\mathcal{Q} \to [0,\infty)$ and $\mathcal{R}_2 :\ \mathrm{T}\mathcal{Q} \to [0,\infty)$ are such that for every $q \in \mathcal{Q}$

$$\begin{aligned} &\mathcal{R}_1(q,\cdot) \text{ is convex and homogeneous of degree 1,}\\ &\mathcal{R}_2(q,\cdot) \text{ is convex and homogeneous of degree 2.} \end{aligned}$$

Note that $\mathcal{R}_j(q,\gamma v) = \gamma^j \mathcal{R}(q,v)$ implies $\partial\mathcal{R}_j(q,\gamma v) = \gamma^{j-1}\partial\mathcal{R}_j(q,v)$ for all $\gamma \geq 0$ and $(q,v) \in \mathrm{T}\mathcal{Q}$. Hence, (2.1) takes the form

$$0 \in \partial_{\dot q}\mathcal{R}_1(q(t),\dot q(t)) + \varepsilon\partial_{\dot q}\mathcal{R}_2(q(t),\dot q(t)) + \mathrm{D}_q\mathcal{E}(t,q(t))\,, \qquad t \in (0,T). \tag{2.2}$$

We call $\mathcal{R}_1$ the potential of rate-independent friction and $\mathcal{R}_2$ the potential of viscous friction.

Remark 2.1 A prototype of the mechanical situation we aim to model arises in connection with a system of k particles ($k \geq 1$) moving in $\mathbb{R}^d$, hence with state-space $\mathcal{Q} = \left\{ q = (q_1,\ldots,q_k) \,:\, q_i \in \mathbb{R}^d \right\} = \mathbb{R}^{kd}$. We impose rate-independent friction $\mathcal{R}_1$ and viscous friction $\mathcal{R}_2$ via

$$\mathcal{R}_1(q,\dot q) = \sum_{j=1}^k \mu(q_j)|\dot q_j| \ \text{ and } \ \mathcal{R}_2(q,\dot q) = \sum_{j=1}^k \frac{\nu(q_j)}{2}|\dot q_j|^2 \quad \text{for } (q,\dot q) \in \mathbb{R}^{kd} \times \mathbb{R}^{kd}\,,$$

where $|\dot q_j|$ is the Euclidean norm of the j-th particle velocity and $\mu,\, \nu :\mathbb{R}^d \to [0,\infty)$ are given continuous functions. For $k = 1$ the potentials $\mathcal{R}_j$ are related by $\mathcal{R}_2(q,\dot q) = \frac{\nu(q)}{2\mu^2(q)}\,\mathcal{R}_1^2(q,\dot q)$, while for $k = 2$ their interplay is more complex.

Our aim is to understand the limiting behavior of the solutions to (2.2) for $\varepsilon \to 0$. In fact, we expect that for $\varepsilon \to 0$ the rate-independent friction dominates, but the solution $q^\varepsilon : [0,T] \to \mathcal{Q}$ may develop sharp transition layers, with $\dot q$ of order $1/\varepsilon$. In the limit we obtain a jump, but in order to characterize the jump path the viscous potential is crucial.

The key idea is to study the trajectories $\mathcal{T}_\varepsilon = \{\, (t, q^\varepsilon(t)) \mid t \in [0,T] \,\}$ in the extended state space $\mathcal{Q}_T$. The point is that the limit of trajectories $\mathcal{T}_\varepsilon$ may no longer be the graph of a function. To study the limits via differential inclusions, we may reparametrize the trajectories $\mathcal{T}_\varepsilon$ in the form

$$\mathcal{T}_\varepsilon = \{\, (\widehat{t}_\varepsilon(s), \widehat{q}_\varepsilon(s)) \mid s \in [0, S_\varepsilon] \,\},$$

where $\widehat{t}_\varepsilon$ is supposed to be nondecreasing and absolutely continuous.

For passing to the limit it is now helpful to select a family of parametrizations via $m_\varepsilon \in \mathrm{L}^1_{\mathrm{loc}}((0,\infty))$ converging to m in $L^1_{\mathrm{loc}}((0,\infty))$, with $m(s), m_\varepsilon(s) > 0$ for a.a. $s \in (0,\infty)$, and to assume

$$\widehat{t}'_\varepsilon(s) + \sqrt{2\mathcal{R}_2(\widehat{q}_\varepsilon(s), \widehat{q}'_\varepsilon(s))} = m_\varepsilon(s) \qquad \text{for a.a. } s \in (0, S_\varepsilon)\,. \tag{2.3}$$

Note that this can always be achieved. In particular, when

$$\mathcal{Q} = \mathbb{R}^d\,, \quad \mathcal{R}_2(q, \dot q) = \frac{1}{2}|\dot q|^2 \ \ \forall\, (q, \dot q) \in \mathbb{R}^d \times \mathbb{R}^d \quad \text{and } m = m_\varepsilon \equiv 1\,, \tag{2.4}$$

relation (2.3) leads to the arclength parametrization of $\mathcal{T}_\varepsilon$, which was considered in [EfM06]. The total length is $S_\varepsilon := T + \int_0^T \sqrt{2\mathcal{R}_2(q_\varepsilon(t), q'_\varepsilon(t))}\, dt$. Since $S_\varepsilon \to S$ up to a subsequence (thanks to standard energy estimates), it is not restrictive to assume that S_ε is independent of ε by the simple linear rescaling $\tilde m_\varepsilon(s) = m_\varepsilon(sS_\varepsilon/S)$.

By the chain rule and the j-homogeneity we have

$$\partial_{\dot q}\mathcal{R}_j(q_\varepsilon(t), \dot q_\varepsilon(t))|_{t=\widehat{t}^\varepsilon(s)} = \left(\frac{1}{\widehat{t}'_\varepsilon(s)}\right)^{j-1} \partial_{\dot q}\mathcal{R}_j(\widehat{q}_\varepsilon(s), \widehat{q}'_\varepsilon(s)) \qquad \text{for a.a. } s \in (0, S_\varepsilon)\,.$$

Now, using (2.3) and defining

$$\widetilde{\mathcal{R}}(q, v) = g(\mathcal{R}_2(q, v)) \quad \text{with } g(r) = \begin{cases} \log\left(\frac{1}{1-\sqrt{2r}}\right) - \sqrt{2r} & \text{for } r \in [0, \frac{1}{2}), \\ \infty & \text{otherwise,} \end{cases}$$

easy computations (cf. [EfM06, Thm. 3.1] in the particular case of (2.4)) show that (2.2) is equivalent to

$$0 \in \partial_{\dot q}\mathcal{R}_1(\widehat{q}_\varepsilon, \widehat{q}'_\varepsilon) + \varepsilon\partial_{\dot q}\widetilde{\mathcal{R}}\Big(\widehat{q}_\varepsilon, \frac{\widehat{q}'_\varepsilon}{m_\varepsilon}\Big) + \mathrm{D}_q\mathcal{E}(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon) \quad \text{ a.e. in } (0, S_\varepsilon)\,. \tag{2.5}$$

In this formulation we may pass to the limit, and we expect to obtain the limit problem

$$\left.\begin{array}{l} 0 \in \partial_{\dot q}\widehat{\mathcal{R}}(\widehat{q}, \frac{\widehat{q}'}{m}) + \mathrm{D}_q\mathcal{E}(\widehat{t}, \widehat{q}), \\ \widehat{t}' + \sqrt{2\mathcal{R}_2(\widehat{q}, \widehat{q}')} = m, \end{array}\right\} \quad \text{a.e. in } (0, S)\,, \tag{2.6}$$

where

$$\widehat{\mathcal{R}}(q,v) = \begin{cases} \mathcal{R}_1(q,v) & \text{for } \mathcal{R}_2(q,v) \le \frac{1}{2}\,, \\ \infty & \text{for } \mathcal{R}_2(q,v) > \frac{1}{2}\,. \end{cases}$$

Formulation (2.6) is in fact a generalization of the one in [EfM06], where rigorous convergence proofs of problem (2.5) to (2.6) are derived (in the case $\mathcal{Q}$ is finite-dimensional). Although $\widehat{\mathcal{R}}$ is no longer 1-homogeneous, the limit problem is still rate-independent: upon adjusting the free function m, one sees immediately that system (2.6) is invariant under time reparametrizations.

In the present work we concentrate on the case $\mathcal{R}_2(q,v) = \frac{1}{2}\mathcal{R}_1(q,v)^2$, since it is this case which can be generalized to abstract metric spaces and hence to the infinite-dimensional setting, see [MRS08]. By introducing the dual norm of a co-vector $w \in \mathrm{T}^*_q\mathcal{Q}$

$$\mathcal{R}_{1,*}(q,w) := \sup\big\{\langle w,v\rangle : v \in \mathrm{T}_q\mathcal{Q},\ \mathcal{R}_1(q,v) \le 1\big\}, \tag{2.7}$$

the operators $\partial\mathcal{R}_1(q,\cdot)$ and $\partial\mathcal{R}_2(q,\cdot)$ can be characterized by

$$w \in \partial\mathcal{R}_1(q,v),\ v \neq 0 \iff \mathcal{R}_{1,*}(q,w) = 1,\ \langle w,v\rangle = \mathcal{R}_1(q,v) > 0\,, \tag{2.8a}$$
$$w \in \partial\mathcal{R}_1(q,0) \iff \mathcal{R}_{1,*}(q,w) \le 1\,, \tag{2.8b}$$
$$w \in \partial\mathcal{R}_2(q,v) \iff \mathcal{R}_{1,*}(q,w) = \mathcal{R}_1(q,v) = \langle w,v\rangle\,, \tag{2.8c}$$

and they satisfy $\partial\mathcal{R}_2(q,v) = \mathcal{R}_1(q,v)\,\partial\mathcal{R}_1(q,v)$ and

$$\mathcal{R}_{1,*}(q,w)\,\mathcal{R}_1(q,v) = \langle w,v\rangle \iff w \in \lambda\partial\mathcal{R}_1(q,v) \quad \text{for some } \lambda \ge 0. \tag{2.9}$$

Proposition 2.2 *In the case $\mathcal{R}_2(q,v) = \frac{1}{2}\mathcal{R}_1(q,v)^2$, a pair $(\widehat{t},\widehat{q}) \in \mathrm{AC}([0,S];[0,T]\times\mathcal{Q})$ fulfils (2.6) (for some $m \in L^1(0,S)$ with $m(s) > 0$ a.e. in $(0,S)$) if and only if there exists a function $\lambda : (0,S) \to (0,\infty)$ such that*

$$\left.\begin{array}{l} 0 \in \lambda\partial\mathcal{R}_1(\widehat{q},\widehat{q}') + \mathrm{D}_q\mathcal{E}(\widehat{t},\widehat{q}), \\ \widehat{t}' \ge 0,\ \lambda \ge 1,\ (\lambda-1)\widehat{t}' \equiv 0, \\ \widehat{t}' + \mathcal{R}_1(\widehat{q},\widehat{q}') > 0 \end{array}\right\} \quad \textit{a.e. in } (0,S)\,. \tag{2.10}$$

Proof: First, we note that, using the 1-homogeneity of $\mathcal{R}_1$, the second of (2.6) may be rewritten as

$$\frac{\widehat{t}'}{m} + \mathcal{R}_1\left(\widehat{q},\frac{\widehat{q}'}{m}\right) = 1 \qquad \text{a.e. in } (0,S)\,. \tag{2.11}$$

Now, it is not difficult to see that

$$\partial_{\dot q}\widehat{\mathcal{R}}(\widehat{q},\frac{\widehat{q}'}{m}) = \begin{cases} \partial_{\dot q}\mathcal{R}_1(\widehat{q},\frac{\widehat{q}'}{m}) & \text{if } \mathcal{R}_1(\widehat{q},\frac{\widehat{q}'}{m}) \in [0,1) \quad (\Leftrightarrow \widehat{t}' > 0)\,, \\ [1,\infty)\cdot\partial_{\dot q}\mathcal{R}_1(\widehat{q},\frac{\widehat{q}'}{m}) & \text{if } \mathcal{R}_1(\widehat{q},\frac{\widehat{q}'}{m}) = 1 \quad (\Leftrightarrow \widehat{t}' = 0)\,, \end{cases} \tag{2.12}$$

where the equivalences in parentheses follow from (2.11). Combining (2.12) with the first of (2.6) and using that $\partial\mathcal{R}_1$ is 0-homogeneous, we deduce (2.10).

Conversely, starting from (2.10), we put $m(s) := \widehat{t}'(s) + \mathcal{R}_1(\widehat{q}(s), \widehat{q}'(s))$ for a.a. $s \in (0, S)$ and note that, by the third of (2.10), $m > 0$ a.e. in $(0, S)$ and $m \in L^1(0, S)$, since $\mathcal{R}_1$ is continuous. Using (2.11) and arguing as in the above lines, one sees that, if the pair $(\widehat{t}', \lambda)$ satisfies the second of (2.10), then $\lambda \partial_{\dot{q}} \mathcal{R}_1(\widehat{q}, \widehat{q}') = \partial_{\dot{q}} \widehat{\mathcal{R}}(\widehat{q}, \frac{\widehat{q}'}{m})$, which allows us to deduce the differential inclusion in (2.6) from the one in (2.10). ∎

3 Analysis with metric space techniques

3.1 Problem reformulation in a metric setting

First of all, we complement the setup of Section 2 by specifying our assumptions on the rate-independent system $(\mathcal{Q}, \mathcal{R}_1, \mathcal{E})$, where $\mathcal{Q}$ is the ambient space, $\mathcal{R}_1$ the dissipation functional, and $\mathcal{E}$ the energy functional. The more general setup will be studied in [MRS08]. Namely, we require that

$$\mathcal{Q} \text{ is a finite-dimensional and smooth manifold,} \tag{3.Q}$$

and the energy functional satisfies

$$\mathcal{E} \in \mathrm{C}^1(\mathcal{Q}_T)\,. \tag{3.E}$$

The dissipation functional $\mathcal{R}_1 : \mathrm{T}\mathcal{Q} \to [0, \infty)$ is a *complete Finsler structure* on $\mathcal{Q}$ (see e.g. [BCS00, Ch. I.1]), namely

$$\mathcal{R}_1 \text{ is continuous on } \mathrm{T}\mathcal{Q} \quad \text{and} \quad \forall\, q \in \mathcal{Q} : \ \mathcal{R}_1(q, \cdot) \text{ is a norm on } \mathrm{T}_q\mathcal{Q}\,, \tag{3.R}$$

called *Minkowski norm* in the Finsler setting. Then, $\mathcal{R}_1$ induces the (Finsler) distance $d : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$:

$$d(q_0, q_1) := \min\Big\{ \int_0^1 \mathcal{R}_1(\widetilde{q}(s), \widetilde{q}'(s))\,\mathrm{d}s : \widetilde{q} \in \mathcal{A}(q_0, q_1) \Big\}\,,$$

where for all $q_0, q_1 \in \mathcal{Q}$ we set

$$\mathcal{A}(q_0, q_1) = \{\, y \in \mathrm{AC}([0,1], \mathcal{Q}) \mid y(0) = q_0,\ y(1) = q_1 \,\}. \tag{3.1}$$

Hence, $(\mathcal{Q}, d)$ is a metric space, which we assume to be *complete.* Like in the previous section, we let $\mathcal{R}_2 \equiv \frac{1}{2}\mathcal{R}_1^2$. For a curve $q \in \mathrm{AC}([0, T]; \mathcal{Q})$, the Finsler length of its velocity $q'(t)$ is given by

$$|q'|(t) := \mathcal{R}_1(q(t), q'(t)), \quad \text{well-defined for a.a. } t \in (0, T), \tag{3.2}$$

and satisfies

$$d(q(s), q(t)) \le \int_s^t |q'|(r)\,\mathrm{d}r \qquad \text{for all } 0 \le s < t \le T\,. \tag{3.3}$$

Using $\mathcal{R}_1$, we define the associated *local slope* $|\partial_q \mathcal{E}| : \mathcal{Q}_T \to [0,\infty]$ via

$$|\partial_q \mathcal{E}|\,(t,q) := \sup_{v \in \mathrm{T}_q\mathcal{Q}\setminus\{0\}} \frac{\langle \mathrm{D}_q\mathcal{E}(t,q), v\rangle}{\mathcal{R}_1(q,v)} = \mathcal{R}_{1,*}(q, \mathrm{D}_q\mathcal{E}(t,q))\,, \tag{3.4}$$

which is the conjugate norm with respect to the Minkowski norm $\mathcal{R}_1(q,\cdot)$ of the differential of the energy in the cotangent space $\mathrm{T}^*_q\mathcal{Q}$.

Using the smoothness of $\mathcal{E}$ we have that for every curve $(t,q) \in \mathrm{AC}([s_0,s_1];\mathcal{Q}_T)$ the map $s \mapsto \mathcal{E}(t(s),q(s))$ is absolutely continuous and the chain rule for $\mathcal{E}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(t(s),q(s)) = \partial_t\mathcal{E}(t(s),q(s))\,t'(s) + \langle \mathrm{D}_q\mathcal{E}(t(s),q(s)), q'(s)\rangle \tag{3.5}$$

for a.a. $s \in (s_0,s_1)$. On the other hand, formulae (3.2) and (3.4) yield

$$\langle \mathrm{D}_q\mathcal{E}(t,q), q')\rangle \geq -|q'|\;|\partial_q\mathcal{E}|\,(t,q)\;. \tag{3.6}$$

Therefore, every $(t,q) \in \mathrm{AC}([s_0,s_1];\mathcal{Q}_T)$ fulfills the chain rule inequality

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(t(s),q(s)) - \partial_t\mathcal{E}(t(s),q(s))t'(s) \;\geq\; -|\partial_q\mathcal{E}|\,(t(s),q(s))\;|q'|(s) \quad \text{a.e. in } (s_0,s_1). \tag{3.7}$$

The metric formulation of doubly nonlinear equations. We now see how notions (3.2) and (3.4) so far introduced come into play in the reformulation of a class of doubly nonlinear evolution equations in the metric setting $(\mathcal{Q},d)$.

Let $\psi : [0,\infty) \to [0,\infty]$ be a lower semicontinuous, nondecreasing, and convex function and $\psi^* : [0,\infty) \to [0,\infty]$ its conjugate function (Legendre–Fenchel transform), namely

$$\psi^*(\xi) = \sup\{\,\nu\xi - \psi(\nu) \mid \nu \geq 0\,\}.$$

Following [AGS05] (see also [RMS08]), a function $q \in \mathrm{AC}([0,T];\mathcal{Q})$ is called a solution of the ψ-gradient system associated with $(\mathcal{Q},d,\mathcal{E})$ if

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) \leq \partial_t\mathcal{E}(t,q(t)) - \psi(|q'|(t)) - \psi^*\big(|\partial_q\mathcal{E}|\,(t,q(t))\big) \qquad \text{a.e. in } (0,T)\,. \tag{3.8}$$

It has been proved in [RMS08, Prop. 8.2] that q fulfills (3.8) if and only if it solves the doubly nonlinear equation (also called quasi-variational evolutionary inequality)

$$0 \in \partial_{\dot q}\Psi(q(t),\dot q(t)) + \mathrm{D}_q\mathcal{E}(t,q(t)) \text{ a.e. in } (0,T)\,, \quad \text{where } \Psi(q,\dot q) := \psi(\mathcal{R}_1(q,\dot q))\,. \tag{3.9}$$

Under assumptions (3.Q), (3.R), and (3.E), the existence of absolutely continuous solutions to the Cauchy problem for (3.8) follows from [RMS08, Thm. 3.5]. We stress that the simple, but central duality inequality $\psi(\nu) + \psi^*(\xi) \geq \nu\xi$ for all ν, $\xi \in [0,\infty)$, together with the chain rule inequality (3.7), enforces equality in (3.8) (ultimately in (3.7) as well).

In the rate-independent setting, the natural choice is

$$\psi_0(\nu) \equiv \nu \quad \text{giving} \quad \psi_0^*(\xi) = I_{[0,1]}(\xi)\,,$$

where $I_{[0,1]}(\cdot)$ denotes the indicator function of $[0,1]$, i.e. $I_{[0,1]}(\xi)=0$ if $\xi\in[0,1]$, and $I_{[0,1]}(\xi)=\infty$ otherwise. However, simple one-dimensional (not strictly convex) examples show that we cannot expect existence of absolutely continuous solutions in this case, cf. Example 7.1. Hence, we proceed as in Section 2 and consider limits of viscous regularizations after suitable reparametrizations.

Before doing so, note that (3.8) is equivalent to the parametrized version on some interval (s_0,s_1), given by

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{q}(s))-\partial_t\mathcal{E}(\widehat{t}(s),\widehat{q}(s))\widehat{t}'(s)\\ &\leq -\psi\Big(\tfrac{1}{\widehat{t}'(s)}|\widehat{q}'|(s)\Big)\widehat{t}'(s)-\psi^*\Big(|\partial_q\mathcal{E}|\left(\widehat{t}(s),\widehat{q}(s)\right)\Big)\widehat{t}'(s)\quad\text{for a.a. } s\in(s_0,s_1)\,,\end{aligned}\tag{3.10}$$

where $\widehat{q}(s)=q(\widehat{t}(s))$ and $\widehat{t}'(s)>0$ a.e. in (s_0,s_1). In the rate-independent case, the right-hand side does not depend on $\widehat{t}'(s)$, because $\psi_0(\nu)=\nu$ implies $\psi_0(\alpha\nu)=\alpha\psi_0(\nu)$ and $\psi_0^*(\xi)=\alpha\psi_0^*(\xi)$ for all $\alpha>0$.

3.2 Rate-independent limit of viscous metric flows

We now consider the case of small viscosity added to the rate-independent dissipation, namely

$$\psi_\varepsilon(\nu)=\nu+\frac{\varepsilon}{2}\nu^2\qquad\forall\,\nu\in[0,\infty)\,.\tag{3.11}$$

We obtain $\psi_\varepsilon^*(\xi)=\frac{1}{2\varepsilon}((\xi-1)^+)^2$. Thus, (3.10) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{q}(s))-\partial_t\mathcal{E}(\widehat{t}(s),\widehat{q}(s))\,\widehat{t}'(s)\leq -M_\varepsilon\big(\widehat{t}'(s),|\widehat{q}'|(s),|\partial_q\mathcal{E}|\left(\widehat{t}(s),\widehat{q}(s)\right)\big)\tag{3.12}$$

for a.a. $s\in(s_0,s_1)$, with

$$M_\varepsilon\big(\alpha,\nu,\xi\big):=\alpha\psi_\varepsilon\Big(\frac{\nu}{\alpha}\Big)+\alpha\psi_\varepsilon^*(\xi)=\nu+\frac{\varepsilon}{2\alpha}\nu^2+\frac{\alpha}{2\varepsilon}((\xi-1)^+)^2\tag{3.13}$$

for all $(\alpha,\nu,\xi)\in(0,\infty)\times[0,\infty)^2$. Clearly, for fixed $\alpha>0$ the limit as $\varepsilon\searrow0$ of $M_\varepsilon(\alpha,\nu,\xi)$ gives $\psi_0(\nu)+\psi_0^*(\xi)$. However, our purpose is to blow-up the time parametrization whenever jumps occur. Indeed, the finite-dimensional case (see [EfM06]) suggests that jumps in the rate-independent evolution will occur at fixed rescaled time (i.e., when $\widehat{t}'=0$). Hence, we also have to consider the case $\alpha\to0$ as $\varepsilon\to0$. For this, note that when $\xi>1$ $M_\varepsilon(\cdot,\nu,\xi)$ assumes its minimum on $[0,\infty)$ for $\alpha_*^\varepsilon=\varepsilon\nu/(\xi-1)^+$, corresponding to the value $M_\varepsilon\big(\alpha_*^\varepsilon,\nu,\xi\big)=\nu+\nu(\xi-1)^+$. In any case we have

$$M_\varepsilon\big(\alpha,\nu,\xi\big)\geq M_{\inf}(\nu,\xi):=\nu+\nu(\xi-1)^+\ \text{ for all } (\alpha,\nu,\xi)\in(0,\infty)\times[0,\infty)^2\,.\tag{3.14}$$

Thus, we define $M_0:[0,\infty)^3\to[0,\infty]$ via

$$M_0\big(\alpha,\nu,\xi\big):=\begin{cases}M_{\inf}(\nu,\xi)=\nu+\nu(\xi-1)^+ & \text{for } \alpha=0,\\ M_{\sup}(\nu,\xi)=\nu+I_{[0,1]}(\xi) & \text{for } \alpha>0,\end{cases}\tag{3.15}$$

and obtain the following result.

Lemma 3.1 *Define $M_\varepsilon : [0,\infty)^3 \to [0,\infty]$ via (3.13), $M_\varepsilon(0,0,\xi) = 0$ for all ξ, and $M_\varepsilon(0,\nu,\xi) = \infty$ for all ξ and $\nu > 0$. Then, we have the following results:*

(A) M_ε Γ-converges to M_0, viz.

Γ*-liminf estimate:*

$$(\alpha_\varepsilon,\nu_\varepsilon,\xi_\varepsilon) \to (\alpha,\nu,\xi) \implies M_0(\alpha,\nu,\xi) \le \liminf_{\varepsilon\searrow 0} M_\varepsilon(\alpha_\varepsilon,\nu_\varepsilon,\xi_\varepsilon)\,, \tag{3.16a}$$

Γ*-limsup estimate:*

$$\forall\,(\alpha,\nu,\xi)\ \exists\,((\alpha_\varepsilon,\nu_\varepsilon,\xi_\varepsilon))_{\varepsilon>0} : \begin{cases} (\alpha_\varepsilon,\nu_\varepsilon,\xi_\varepsilon) \to (\alpha,\nu,\xi) \textit{ and} \\ M_0(\alpha,\nu,\xi) \ge \limsup_{\varepsilon\searrow 0} M_\varepsilon(\alpha_\varepsilon,\nu_\varepsilon,\xi_\varepsilon)\,. \end{cases} \tag{3.16b}$$

(B) If $(\alpha_\varepsilon,\nu_\varepsilon) \rightharpoonup (\widehat{\alpha},\widehat{\nu})$ in $\mathrm{L}^1((s_0,s_1))$ and $\liminf_{\varepsilon\to 0}\xi_\varepsilon(s) \ge \widehat{\xi}(s)$ a.e. in (s_0,s_1), then

$$\int_{s_0}^{s_1} M_0(\widehat{\alpha}(s),\widehat{\nu}(s),\widehat{\xi}(s))\,\mathrm{d}s \le \liminf_{\varepsilon\to 0}\int_{s_0}^{s_1} M_\varepsilon(\alpha_\varepsilon(s),\nu_\varepsilon(s),\xi_\varepsilon(s))\,\mathrm{d}s\,.$$

Proof: Estimate (3.16a) is trivial for $\alpha > 0$, as we have pointwise convergence then. If $\alpha = 0$, we employ (3.14) and use that M_{inf} is continuous.

To obtain (3.16b) in the case $\alpha > 0$ we simply take $(\alpha_\varepsilon,\nu_\varepsilon,\xi_\varepsilon) = (\alpha,\nu,\xi)$ and the result follows from pointwise convergence. If $\alpha = 0$, we let $(\alpha_\varepsilon,\nu_\varepsilon,\xi_\varepsilon) = (\alpha_*^\varepsilon,\nu,\xi)$ and the desired result follows. Thus, (A) is proved.

To show the estimate in Part (B), let us introduce the function $\bar{M} : [0,\infty)^4 \to [0,\infty]$

$$\bar{M}(\alpha,\nu;\xi,\varepsilon) := M_\varepsilon(\alpha,\nu,\xi);$$

by the previous point (A) and the fact that $\bar{M}$ is lower semicontinuous when $\varepsilon > 0$, it is immediate to check that $\bar{M}$ is lower semicontinuous in $[0,\infty)^4$. Moreover, $\bar{M}(\cdot,\cdot;\xi,\varepsilon)$ is convex in $[0,\infty)^2$ for all ξ,ε: this property can be directly checked starting from the definition of $\bar{M}$ or by observing that $M_\varepsilon(\cdot,\cdot,\xi)$ is convex when $\varepsilon > 0$ thanks to (3.13) and the convexity of the map $(\nu,\alpha) \mapsto \nu^2/\alpha$.

Assuming initially that $\xi_\varepsilon \to \widehat{\xi}$ in $L^1(s_1,s_2)$ and considering an arbitrary infinitesimal subsequence $\varepsilon_n \to 0$, we can then apply Ioffe's Theorem (see [Iof77]) to the sequence of maps $s \mapsto (\alpha_{\varepsilon_n}(s),\nu_{\varepsilon_n}(s),\xi_{\varepsilon_n}(s),\varepsilon_n)$, obtaining

$$\int_{s_0}^{s_1} \bar{M}(\widehat{\alpha}(s),\widehat{\nu}(s),\widehat{\xi}(s),0)\,\mathrm{d}s \le \liminf_{n\to\infty}\int_{s_0}^{s_1} \bar{M}(\alpha_{\varepsilon_n}(s),\nu_{\varepsilon_n}(s),\xi_{\varepsilon_n}(s),\varepsilon_n)\,\mathrm{d}s\,.$$

In the general case, we consider an arbitrary $\kappa > 0$ and we replace ξ_{ε_n} with the sequence $\xi_{\kappa,n}(s) := \min(\xi_{\varepsilon_n}(s),\widehat{\xi}(s),\kappa)$, converging to $\widehat{\xi}_k(s) := \min(\widehat{\xi}(s),\kappa)$ in $L^1(s_1,s_2)$. Since $\bar{M}$ is nondecreasing with respect to ξ, we argue as above and obtain

$$\begin{aligned}\int_{s_0}^{s_1} \bar{M}(\widehat{\alpha}(s),\widehat{\nu}(s),\widehat{\xi}_\kappa(s),0)\,\mathrm{d}s &\le \liminf_{n\to\infty}\int_{s_0}^{s_1} \bar{M}(\alpha_{\varepsilon_n}(s),\nu_{\varepsilon_n}(s),\xi_{\kappa,n}(s),\varepsilon_n)\,\mathrm{d}s \\ &\le \liminf_{n\to\infty}\int_{s_0}^{s_1} \bar{M}(\alpha_{\varepsilon_n}(s),\nu_{\varepsilon_n}(s),\xi_{\varepsilon_n}(s),\varepsilon_n)\,\mathrm{d}s\,.\end{aligned}$$

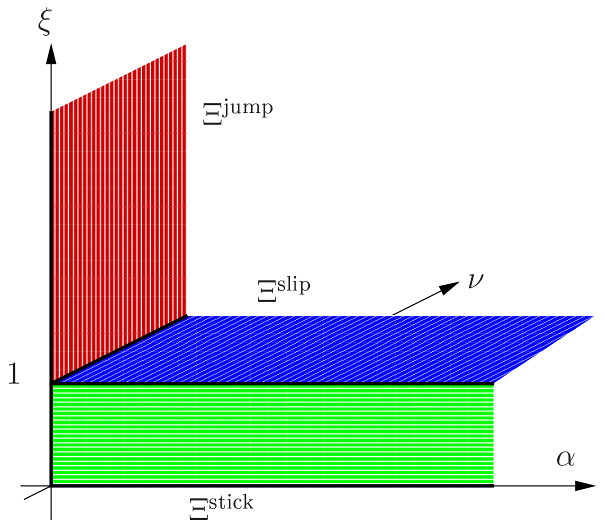

Figure 3.1: The set $\Xi = \Xi^{\text{stick}} \cup \Xi^{\text{slip}} \cup \Xi^{\text{jump}}$

Passing to the limit as $\kappa \to \infty$ and applying Fatou's Lemma we obtain the desired inequality. ∎

In fact, the specific form of M_0 is not needed in the sequel. Hence, we will consider general functions $M : [0,\infty)^3 \to [0,\infty]$ with the following properties (which are obviously satisfied by M_0):

$$M : [0,\infty)^3 \to [0,\infty] \text{ is l.s.c.,} \tag{3.17a}$$
$$M(\gamma\alpha, \gamma\nu, \xi) = \gamma M(\alpha,\nu,\xi) \quad \text{for all } \alpha,\nu,\xi,\gamma \in [0,\infty)\,, \tag{3.17b}$$
$$M(\alpha,\nu,\xi) \geq \nu\xi \quad \text{for all } \alpha,\nu,\xi \in [0,\infty)\,, \tag{3.17c}$$
$$M(\alpha,\nu,\xi) = \nu\xi \quad \Longleftrightarrow \quad (\alpha,\nu,\xi) \in \Xi\,, \tag{3.17d}$$

where the set $\Xi := \Xi^{\text{stick}} \cup \Xi^{\text{slip}} \cup \Xi^{\text{jump}}$ consists of the disjoint flat pieces (see Figure 3.1)

$$\begin{aligned} \Xi^{\text{stick}} &:= \{\, (\alpha,0,\xi) \in [0,\infty)^3 \mid \alpha \geq 0, \xi \in [0,1) \,\}, \\ \Xi^{\text{slip}} &:= \{\, (\alpha,\nu,1) \in [0,\infty)^3 \mid \alpha > 0, \nu \geq 0 \,\}, \quad \text{and} \\ \Xi^{\text{jump}} &:= \{\, (0,\nu,\xi) \in [0,\infty)^3 \mid \nu \geq 0, \xi \geq 1 \,\}. \end{aligned} \tag{3.18}$$

For instance, the function

$$\widetilde{M}(\alpha,\nu,\xi) = \nu + (\xi{-}1)^+(\alpha{+}\nu) = \max(\xi,1)\nu + (\xi{-}1)^+\alpha \tag{3.19}$$

fits in this framework. It is not difficult to check that, if M is nondecreasing with respect to ξ, then $M \leq M_0$.

The notion of parametrized metric solutions. The following notion of *parametrized metric solution of the rate-independent system* $(\mathcal{Q}, d, \mathcal{E})$ is proposed in a general form, replacing the function M_0 obtained in the vanishing viscosity limit with a generic function M satisfying (3.17). The proposed notion is fitted to the metric framework and does not need a differentiable structure. However, it strongly relies on the fact that the small viscous friction $\varepsilon|q'|(t)^2$ is given in terms of the same metric velocity as the rate-independent friction, see also the assumption $\mathcal{R}_2 = \frac{1}{2}\mathcal{R}_1^2$. We refer to [EfM06, MiZ08] for settings avoiding this assumption.

Definition 3.2 (Parametrized metric solution) *Let* $(\mathcal{Q}, d, \mathcal{E})$ *satisfy* (3.Q), (3.R), *and* (3.E) *and let* M *fulfill* (3.17). *An absolutely continuous curve* $(\widehat{t}, \widehat{q}) : (s_0, s_1) \to \mathcal{Q}_T$ *is called a* parametrized metric solution *of* $(\mathcal{Q}, d, \mathcal{E})$, *if*

$$\widehat{t} : (s_0, s_1) \to [0, T] \quad \textit{is nondecreasing,} \tag{3.20a}$$

$$\widehat{t}'(s) + |\widehat{q}'|(s) > 0 \quad \textit{for a.a. } s \in (s_0, s_1), \tag{3.20b}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{q}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{q}(s))\,\widehat{t}'(s) \\ &\leq -M\big(\widehat{t}'(s), |\widehat{q}'|(s), |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,\big) \textit{ for a.a. } s \in (s_0, s_1)\,. \end{aligned} \tag{3.20c}$$

If $(\widehat{t}, \widehat{q})$ *satisfies only* (3.20a) *and* (3.20c), *it is called a* degenerate parametrized metric solution. *If* $\widehat{t} : (s_0, s_1) \to [0, T]$ *is also surjective, i.e.* $\widehat{t}(s_0) = 0$ *and* $\widehat{t}(s_1) = T$, *then* $(\widehat{t}, \widehat{q})$ *is called a* surjective parametrized metric solution.

This solution concept has the concatenation property as well as the restriction property. The former means that if $(\widehat{t}, \widehat{q}) : (s_0, s_1) \to \mathcal{Q}_T$ and $(\widetilde{t}, \widetilde{q}) : (s_1, s_2) \to \mathcal{Q}_T$ are parametrized metric solutions with $(\widehat{t}(s_1^-), \widehat{q}(s_1^-)) = (t_1, q_1) = (\widetilde{t}(s_1^+), \widetilde{q}(s_1^+))$, then their concatenation $(t, q) : (s_0, s_2) \to \mathcal{Q}_T$ is a solution as well. We point out that, thanks to (3.17b), the notion of parametrized metric solution is rate-independent, i.e., invariant under time reparametrizations by absolutely continuous functions with strictly positive derivative a.e. (by nondecreasing absolutely continuous functions in the case of degenerate solutions). Moreover, the notion is independent of the particular choice of M, as long as M satisfies (3.17).

Remark 3.3 (Nondegeneracy and arclength reparametrization) Any degenerate parametrized metric solution admits a nondegenerate reparametrization $(\tilde{t}, \tilde{q}) : [0, \tilde{S}] \to \mathcal{Q}_T$, thus satisfying also (3.20b). This means that $\widehat{t}(s) = \tilde{t}(\sigma(s)), \widehat{q}(s) = \widehat{q}(\sigma(s))$ for some absolutely continuous, nondecreasing and surjetive map $\sigma : [s_0, s_1] \to [0, \tilde{S}]$. In particular, we can choose σ so that $\tilde{t}' + |\tilde{q}'| = 1$ a.e. in $(0, \tilde{S})$ by defining

$$\sigma(s) := \int_{s_0}^{s} \big(\widehat{t}'(s) + |\widehat{q}'|(s)\big)\,\mathrm{d}s = \widehat{t}(s) - \widehat{t}(s_0) + \int_{s_0}^{s} |\widehat{q}'|(s)\,\mathrm{d}s, \quad \tilde{S} := \sigma(s_1)$$

(cf. also Lemma 4.1). In fact, for every interval $[r_0, r_1] \subset [s_0, s_1]$ we then have

$$\sigma(r_0) = \sigma(r_1) \quad \Leftrightarrow \quad \widehat{t}(r_0) = \widehat{t}(r) = \widehat{t}(r_1), \quad \widehat{q}(r_0) = \widehat{q}(r) = \widehat{q}(r_1) \text{ for all } r \in [r_0, r_1].$$

We can then define $(\tilde{t}(\sigma), \tilde{q}(\sigma)) := (\widehat{t}(s), \widehat{q}(s))$, whenever $\sigma = \sigma(s)$. For $\sigma_0 = \sigma(r_0) < \sigma_1 = \sigma(r_1)$ we obtain

$$\tilde{t}(\sigma_1) - \tilde{t}(\sigma_0) + d(\tilde{q}(\sigma_1), \tilde{q}(\sigma_0)) \leq \int_{r_0}^{r_1} \big(\widehat{t}'(s) + |\widehat{q}'|(s)\big)\,\mathrm{d}s = \sigma_1 - \sigma_0\,,$$

giving $\tilde{t}' + |\tilde{q}'| \le 1$ a.e. in $[0, \tilde{S}]$. The nondegeneracy condition holds with $\tilde{t}' + |\tilde{q}'| = 1$ a.e. in $\tilde{S}$, which follows via the change of variable formula:

$$\begin{aligned}\tilde{S} &\ge \int_0^{\tilde{S}} \big(\tilde{t}' + |\tilde{q}'|\big)\,\mathrm{d}\sigma = \int_{s_0}^{s_1} \big(\tilde{t}'(\sigma(s)) + |\tilde{q}'|(\sigma(s))\big)\sigma'(s)\,\mathrm{d}s \\ &= \int_{s_0}^{s_1} \big(\widehat{t}'(s) + |\widehat{q}'|(s)\big)\,\mathrm{d}s = \sigma(s_1) = \tilde{S}.\end{aligned}$$

Parametrized metric solutions admit various different but equivalent metric characterizations (where we avoid to explicitly use the differential D_q of the energy).

Proposition 3.4 *Under the same assumptions of the previous Definition 3.2, an absolutely continuous curve $(\widehat{t}, \widehat{q}) : (s_0, s_1) \to \mathcal{Q}_T$ satisfying* (3.20a) *and* (3.20b) *is a parametrized metric solution of $(\mathcal{Q}, d, \mathcal{E})$ if and only if it satisfies one of the following conditions (equivalent to* (3.20c)*):*

A) *For all $s_0 \le \sigma_0 < \sigma_1 \le s_1$ we have*

$$\begin{aligned}&\mathcal{E}(\widehat{t}(\sigma_1), \widehat{q}(\sigma_1)) - \mathcal{E}(\widehat{t}(\sigma_0), \widehat{q}(\sigma_0)) - \int_{\sigma_0}^{\sigma_1} \partial_t \mathcal{E}(\widehat{t}(s), \widehat{q}(s))\,\widehat{t}'(s)\,\mathrm{d}s \\ &\qquad \le -\int_{\sigma_0}^{\sigma_1} M\big(\widehat{t}'(s), |\widehat{q}'|(s), |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,\big)\,\mathrm{d}s.\end{aligned} \tag{3.21}$$

B) *Eqn.* (3.21) *holds just for $\sigma_0 = s_0$ and $\sigma_1 = s_1$, i.e.*

$$\begin{aligned}&\mathcal{E}(\widehat{t}(s_1), \widehat{q}(s_1)) - \mathcal{E}(\widehat{t}(s_0), \widehat{q}(s_0)) - \int_{s_0}^{s_1} \partial_t \mathcal{E}(\widehat{t}(s), \widehat{q}(s))\,\widehat{t}'(s)\,\mathrm{d}s \\ &\qquad \le -\int_{s_0}^{s_1} M\big(\widehat{t}'(s), |\widehat{q}'|(s), |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,\big)\,\mathrm{d}s.\end{aligned} \tag{3.22}$$

C) *For a.a. $s \in (s_0, s_1)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{q}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{q}(s))\,\widehat{t}'(s) = -|\widehat{q}'|(s)\,|\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,, \tag{3.23}$$

and one of the following (equivalent) properties

$$M\big(\widehat{t}'(s), |\widehat{q}'|(s), |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,\big) = |\widehat{q}'|(s)\,|\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,, \tag{3.24a}$$

$$\big(\widehat{t}'(s), |\widehat{q}'|(s), |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,\big) \in \Xi\,, \tag{3.24b}$$

$$\begin{cases} \widehat{t}'(s) > 0 \implies |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big) \le 1, \\ |\widehat{q}'|(s) > 0 \implies |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big) \ge 1. \end{cases} \tag{3.24c}$$

In particular, by (3.23) *and* (3.24a)*, the following* identity *holds a.e. in (s_0, s_1):*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{q}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{q}(s))\,\widehat{t}'(s) = -M\big(\widehat{t}'(s), |\widehat{q}'|(s), |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\,\big). \tag{3.25}$$

Proof: **A)** is just the integral formulation of (3.20c).

We note that the chain rule inequality (3.7), combined with (3.20c) and (3.17c), implies (3.23) and (3.24a). By condition (3.17d), (3.24a) is equivalent to (3.24b).

Since the set Ξ can be easily characterized via

$$(\alpha,\nu,\xi)\in\Xi \iff \Big((\alpha>0\Rightarrow\xi\le 1)\text{ and }(\nu>0\Rightarrow\xi\ge 1)\Big),$$

we ultimately find that (3.24b) can be replaced by the simple relations (3.24c).

Having obtained the equivalence between (3.20c) and **C)**, we can now show that **B)** is sufficient to characterize parametrized metric solutions (the necessity is trivial): again applying the chain rule (3.5)-(3.7), we get

$$\int_{s_0}^{s_1}\Big(M\big(\widehat{t}'(s),|\widehat{q}'|(s),|\partial_q\mathcal{E}|\,(\widehat{t}(s),\widehat{q}(s))\big)-\langle-\mathrm{D}_q\mathcal{E}(\widehat{t}(s),\widehat{q}(s)),\widehat{q}'(s)\rangle\Big)\,\mathrm{d}s\le 0$$

so that (3.17c) and (3.6) yield

$$M\big(\widehat{t}'(s),|\widehat{q}'|(s),|\partial_q\mathcal{E}|\,(\widehat{t}(s),\widehat{q}(s))\big)=\langle-\mathrm{D}_q\mathcal{E}(\widehat{t}(s),\widehat{q}(s)),\widehat{q}'(s)\rangle=|\widehat{q}'|(s)\,|\partial_q\mathcal{E}|\,\big(\widehat{t}(s),\widehat{q}(s)\big)$$

for a.a. $s\in(s_0,s_1)$. We thus get (3.23) and (3.24a). ∎

Remark 3.5 (Mechanical interpretation) The evolution described by relations (3.24c) bears the following mechanical interpretation, cf. [EfM06]. Indeed, with $(\alpha,\nu,\xi)=(\widehat{t}',|\widehat{q}'|,|\partial_q\mathcal{E}|\,(\widehat{t},\widehat{q}))$ we can use the decomposition $\Xi=\Xi^{\mathrm{stick}}\cup\Xi^{\mathrm{slip}}\cup\Xi^{\mathrm{jump}}$:

- $(\widehat{t}'>0,\ |\widehat{q}'|=0)$ leads to *sticking* ($(\alpha,\nu,\xi)\in\Xi^{\mathrm{stick}}$),
- $(\widehat{t}'>0,\ |\widehat{q}'|>0)$ leads to *rate-independent evolution* ($(\alpha,\nu,\xi)\in\Xi^{\mathrm{slip}}$),
- when $(\widehat{t}'=0,\ |\widehat{q}'|>0)$, the system has switched to a *viscous regime*, which is seen as a jump in the (slow) external time scale (the time function t is frozen and $(\alpha,\nu,\xi)\in\Xi^{\mathrm{jump}}$).

Remark 3.6 Properties (3.17c) and (3.17d) seem to be related to the notion of *bipotential* (cf. e.g. [BdV08]), which was proposed for studying non-associated constitutive laws in mechanics by convex analysis tools. We recall that, given two (topological, locally convex) spaces in duality Z and Z', a function $b:Z\times Z'\to(-\infty,\infty]$ is called a bipotential if it is convex, lower semicontinuous with respect to both arguments, and fulfills for all $(\nu,\xi)\in Z\times Z'$

$$b(\nu,\xi)\ge\langle\xi,\nu\rangle\quad\text{and}\quad\Big(\xi\in\partial_\nu b(\cdot,\xi)(\nu)\ \Leftrightarrow\ \nu\in\partial_\xi b(\nu,\cdot)(\xi)\ \Leftrightarrow\ b(\nu,\xi)=\langle\xi,\nu\rangle\Big).$$

Indeed, for every $\alpha>0$ the functions $M_\varepsilon(\alpha,\cdot,\cdot)$ given by (3.13) and $M_0(\alpha,\cdot,\cdot)$ by (3.15) are bipotentials on $[0,\infty)\times[0,\infty)$.

The next result ensures that the abstract metric evolution formulation developed here reduces to the one stated in Proposition 2.2.

Proposition 3.7 *Let* $(\mathcal{Q}, d, \mathcal{E})$ *satisfy* (3.Q), (3.R), *and* (3.E). *Then, a curve* $(\widehat{t}, \widehat{q}) \in \mathrm{AC}([s_0, s_1]; \mathcal{Q}_T)$ *is a parametrized metric solution of the rate-independent system* $(\mathcal{Q}, d, \mathcal{E})$ *if and only if there exists* $\lambda : (s_0, s_1) \to [1, \infty)$ *such that* (2.10) *holds.*

Proof: By (3.2), condition (3.20b) in Definition 3.2 coincides with the third of (2.10). Now, let us first suppose that (3.20c) holds, and set $\lambda(s) := \max\{|\partial_q \mathcal{E}| \left(\widehat{t}(s), \widehat{q}(s)\right), 1\}$. We shall prove that the triple $(\widehat{t}, \widehat{q}, \lambda)$ fulfills (2.10) on (s_0, s_1). Indeed, (3.23), (3.4), and (2.9) yield

$$-\mathrm{D}_q \mathcal{E}(\widehat{t}(s), \widehat{q}(s)) \in |\partial_q \mathcal{E}| \left(\widehat{t}(s), \widehat{q}(s)\right) \, \partial \mathcal{R}_1(\widehat{q}(s), \widehat{q}'(s)) \quad \text{for a.a. } s \in (s_0, s_1)\,. \tag{3.26}$$

Now, let us fix $\bar{s} \in (s_0, s_1)$ at which (3.26) holds: if $|\widehat{q}'|(\bar{s}) > 0$, taking into account the second of (3.24c) we find that $\lambda(\bar{s}) = |\partial_q \mathcal{E}| \left(\widehat{t}(\bar{s}), \widehat{q}(\bar{s})\right)$ and that, by (3.26), the triple $(\widehat{t}', |\widehat{q}'|, \lambda)$ satisfies the first of (2.10) at $s = \bar{s}$. On the other hand, if $|\widehat{q}'|(\bar{s}) = 0$, necessarily $\widehat{t}'(\bar{s}) > 0$ by (3.20b) and the first of (3.24c) gives that $|\partial_q \mathcal{E}| \left(\widehat{t}(\bar{s}), \widehat{q}(\bar{s})\right) \leq 1$. In this case, $\lambda(\bar{s}) = 1$ and (3.26) implies

$$-\mathrm{D}_q \mathcal{E}(\widehat{t}(\bar{s}), \widehat{q}(\bar{s})) \in \partial \mathcal{R}_1(\widehat{q}(\bar{s}), 0)\,,$$

hence we again conclude that $(\widehat{t}'(\bar{s}), |\widehat{q}'|(\bar{s}), \lambda(\bar{s}))$ fulfills $(2.10)_1$.

Conversely, from the first of (2.10) we read that for a.a. $s \in (s_0, s_1)$

$$|\partial_q \mathcal{E}| \left(\widehat{t}(s), \widehat{q}(s)\right) \, |\widehat{q}'| = \langle -\mathrm{D}_q \mathcal{E}(\widehat{t}(s), \widehat{q}(s)), \widehat{q}'(s) \rangle, \tag{3.27}$$

$$|\partial_q \mathcal{E}| \left(\widehat{t}(s), \widehat{q}(s)\right) \leq \lambda(s), \tag{3.28}$$

$$|\partial_q \mathcal{E}| \left(\widehat{t}(s), \widehat{q}(s)\right) = \lambda(s) \quad \text{if } |\widehat{q}'| > 0. \tag{3.29}$$

Then (3.20c) follows, if we check (3.24c). Indeed, if $\widehat{t}' > 0$, then the second of (2.10) yields $\lambda = 1$. Therefore the first condition of (3.24c) follows from (3.28). If $|\widehat{q}'| > 0$, combining (3.29) and the constraint $\lambda \geq 1$ of (2.10), we also get the second of (3.24c). ∎

Convergence of the viscous approximation. The main result of this section states that limits $(\widehat{t}, \widehat{q})$ of parametrized solutions $(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon)$ of the viscous system (3.10), with $\psi = \psi_\varepsilon$, are actually parametrized metric solutions of the rate-independent system $(\mathcal{Q}, d, \mathcal{E})$. By the standard energy estimates and an elementary rescaling, it is not restrictive to assume that the domain of $(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon)$ is a fixed interval $[s_0, s_1]$, independent of ε.

Theorem 3.8 (Vanishing viscosity limit) *Let* $(\mathcal{Q}, d, \mathcal{E})$ *satisfy* (3.Q), (3.R), *and* (3.E). *For every* $\varepsilon > 0$ *let* $q_\varepsilon \in \mathrm{AC}([0, T]; \mathcal{Q})$ *be a solution to* (3.8) *for* $\psi = \psi_\varepsilon$. *Choose non-decreasing surjective parametrizations* $\widehat{t}_\varepsilon \in \mathrm{AC}([s_0, s_1]; [t_{0,\varepsilon}, T])$, *and let* $\widehat{q}_\varepsilon(s) = q_\varepsilon(\widehat{t}_\varepsilon(s))$. *Suppose that there exists* $q_0 \in \mathcal{Q}$, *and* $m \in \mathrm{L}^1((0, S))$ *such that*

$$t_{0,\varepsilon} = \widehat{t}_\varepsilon(s_0) \to 0\,, \quad \widehat{q}_\varepsilon(s_0) = q_\varepsilon(t_{0,\varepsilon}) \to q_0 \quad \textit{as } \varepsilon \searrow 0, \tag{3.30}$$

$$m_\varepsilon := \widehat{t}_\varepsilon' + |\widehat{q}_\varepsilon'| \rightharpoonup m \quad \textit{in } \mathrm{L}^1(s_0, s_1) \quad \textit{as } \varepsilon \searrow 0. \tag{3.31}$$

Then, there exist a subsequence $((\widehat{t}_{\varepsilon_k}, \widehat{q}_{\varepsilon_k}))_{k\in\mathbb{N}}$ *with* $\varepsilon_k \searrow 0$ *and* $(\widehat{t}, \widehat{q}) \in \mathrm{AC}([s_0, s_1]; \mathcal{Q}_T)$ *such that* $(\widehat{t}(s_0), \widehat{q}(s_0)) = (0, q_0)$*, and, as* $k \to \infty$*,*

$$(\widehat{t}_{\varepsilon_k}, \widehat{q}_{\varepsilon_k}) \to (\widehat{t}, \widehat{q}) \quad \text{in } \mathrm{C}^0([s_0, s_1]; \mathcal{Q}_T)), \tag{3.32}$$

$$(\widehat{t}'_{\varepsilon_k}, |\widehat{q}'_{\varepsilon_k}|) \rightharpoonup (\widehat{t}', |\widehat{q}'|) \quad \text{in } \mathrm{L}^1([s_0, s_1]; \mathbb{R}^2), \qquad \int_{t_{0,\varepsilon_k}}^{T} |q'_{\varepsilon_k}|(t)\,\mathrm{d}t \to \int_{s_0}^{s_1} |\widehat{q}'|(s)\mathrm{d}s. \tag{3.33}$$

The limit $(\widehat{t}, \widehat{q})$ *is a degenerate parametrized metric solution of* $(\mathcal{Q}, d, \xi)$ *(i.e. it satisfies* (3.20a) *and* (3.20c)*), and it is nondegenerate (recall* (3.20b)*) if* $m(s) > 0$ *a.e. in* (s_0, s_1)*.*

Proof: Eqns. (3.3) and (3.31) yield

$$d(\widehat{q}_\varepsilon(r_0), \widehat{q}_\varepsilon(r_1)) \le \int_{r_0}^{r_1} |\widehat{q}'_\varepsilon|(s)\,\mathrm{d}s \le \int_{s_0}^{r_1} m_\varepsilon(s)\,\mathrm{d}s. \tag{3.34}$$

In particular, choosing $r_0 = s_0$ and using (3.30)-(3.31) we find $C > 0$ such that

$$d(q_0, \widehat{q}_\varepsilon(t)) \le C \quad \text{for all } t \in [s_0, s_1] \text{ and all } \varepsilon > 0\,. \tag{3.35}$$

Moreover, it follows from (3.31) that the sequences $\{\widehat{t}'_\varepsilon\}$ and $\{|\widehat{q}'_\varepsilon|\}$ are bounded and uniformly integrable in $\mathrm{L}^1(s_0, s_1)$. Hence, on the one hand, the Ascoli-Arzelà compactness theorem and its version for metric spaces [AGS05, Prop. 3.3.1] yield that there exists an absolutely continuous curve $(\widehat{t}, \widehat{q}) : [s_0, s_1] \to \mathcal{Q}_T$ such that, up to a subsequence, convergences (3.32) hold. On the other hand, by the Dunford-Pettis criterion (see, e.g., [DuS88, Cor. IV.8.11]), there exists $\eta \in L^1(s_0, s_1)$ such that, up to the extraction of a (not relabeled) subsequence,

$$\widehat{t}'_\varepsilon \rightharpoonup \widehat{t}', \qquad |\widehat{q}'_\varepsilon| \rightharpoonup \eta \quad \text{in } \mathrm{L}^1(s_0, s_1) \ \text{ as } \varepsilon \searrow 0. \tag{3.36}$$

Passing to the limit in (3.34) we easily get

$$|\widehat{q}'|(s) \le \eta(s) \le m(s) \qquad \text{for a.a. } s \in (s_0, s_1)\,. \tag{3.37}$$

Now, the smoothness of $\mathcal{E}$ and assumption (3.Q) yield that $\mathcal{E}$, $\partial_t\mathcal{E}$, and $|\partial_q\mathcal{E}|$ are continuous with respect to both arguments, thus we readily infer from (3.32) that

$$\begin{gathered}\mathcal{E}(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon) \to \mathcal{E}(\widehat{t}, \widehat{q}), \quad |\partial_q\mathcal{E}|\left(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon\right) \to |\partial_q\mathcal{E}|\left(\widehat{t}, \widehat{q}\right) \quad \text{and} \quad \partial_t\mathcal{E}(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon) \to \partial_t\mathcal{E}(\widehat{t}, \widehat{q}) \\ \text{uniformly in } [s_0, s_1] \text{ for } \varepsilon \searrow 0\,.\end{gathered} \tag{3.38}$$

To proceed further, we integrate (3.12) over $[s_0, s_1]$ and obtain

$$\begin{aligned}&\mathcal{E}(\widehat{t}_\varepsilon(s_0), \widehat{q}_\varepsilon(s_0)) - \mathcal{E}(\widehat{t}_\varepsilon(s_1), \widehat{q}_\varepsilon(s_1)) + \int_{s_0}^{s_1} \partial_t\mathcal{E}(\widehat{t}_\varepsilon(s), \widehat{q}_\varepsilon(s))\,\widehat{t}'_\varepsilon(s)\,\mathrm{d}s \\ &\ge \int_{s_0}^{s_1} M_\varepsilon\big(\widehat{t}'_\varepsilon(s), |\widehat{q}'_\varepsilon|(s), |\partial_q\mathcal{E}|\left(\widehat{t}_\varepsilon(s), \widehat{q}_\varepsilon(s)\right)\big)\,\mathrm{d}s\,.\end{aligned} \tag{3.39}$$

On the left-hand side we can pass to the limit $\varepsilon \to 0$ using (3.36) and (3.38), whereas for the right-hand side we use Part (B) of Lemma 3.1:

$$
\begin{aligned}
&\mathcal{E}(\widehat{t}(s_0), \widehat{q}(s_0)) - \mathcal{E}(\widehat{t}(s_1), \widehat{q}(s_1)) + \int_{s_0}^{s_1} \partial_t \mathcal{E}(\widehat{t}(r), \widehat{q}(r))\, \widehat{t}'(r)\, \mathrm{d}r \\
&= \lim_{\varepsilon \searrow 0} \Big(\mathcal{E}(\widehat{t}_\varepsilon(s_0), \widehat{q}_\varepsilon(s_0)) - \mathcal{E}(\widehat{t}_\varepsilon(s_1), \widehat{q}_\varepsilon(s_1)) + \int_{s_0}^{s_1} \partial_t \mathcal{E}(\widehat{t}_\varepsilon(r), \widehat{q}_\varepsilon(r))\, \widehat{t}_\varepsilon'(r)\, \mathrm{d}r \Big) \\
&\geq \liminf_{\varepsilon \searrow 0} \int_{s_0}^{s_1} M_\varepsilon\big(\widehat{t}_\varepsilon'(r), |\widehat{q}_\varepsilon'|(r), |\partial_q \mathcal{E}| \big(\widehat{t}_\varepsilon(r), \widehat{q}_\varepsilon(r)\big)\, \big)\, \mathrm{d}r \\
&\geq \int_{s_0}^{s_1} M_0\big(\widehat{t}'(r), \eta(r), |\partial_q \mathcal{E}| \big(\widehat{t}(r), \widehat{q}(r)\big)\, \big)\, \mathrm{d}r \\
&\geq \int_{s_0}^{s_1} M_0\big(\widehat{t}'(r), |\widehat{q}'|(r), |\partial_q \mathcal{E}| \big(\widehat{t}(r), \widehat{q}(r)\big)\, \big)\, \mathrm{d}r
\end{aligned}
$$

where we have used (3.37) and the monotonicity of $M_0(\alpha, \cdot, \xi)$ for the last estimate.

We see that $(\widehat{t}, \widehat{q})$ fulfills (3.22) a.e. in (s_0, s_1) , and it is therefore a (possibly) degenerate parametrized metric solution of $(\mathcal{Q}, d, \mathcal{E})$. Moreover, comparing the last inequalities with the integrated form of (3.25), we get

$$
M_0\big(\widehat{t}'(r), \eta(r), |\partial_q \mathcal{E}| \big(\widehat{t}(r), \widehat{q}(r)\big)\, \big) = M_0\big(\widehat{t}'(r), |\widehat{q}'|(r), |\partial_q \mathcal{E}| \big(\widehat{t}(r), \widehat{q}(r)\big)\, \big) < \infty
$$

for a.a. $r \in (s_0, s_1)$. Since $M_0(\alpha, \cdot, \xi)$ is strictly monotone in its domain of finiteness, we get $\eta(r) = |\widehat{q}'|(r)$ for a.a. $r \in (s_0, s_1)$, thus obtaining (3.33). Using the first convergence in (3.36) we also find $\widehat{t}' + |\widehat{q}'| = m$ and the last assertion follows. ■

Remark 3.9 (Preservation of arclength parametrizations) If $(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon)$ are arclength parametrization (i.e. $m_\varepsilon \equiv 1$), then their limit $(\widehat{t}, \widehat{q})$ still satisfies the arclength property $\widehat{t}' + |\widehat{q}'| = 1$, thanks to (3.33). This generalizes [EfM06, Cor. 3.6].

Remark 3.10 Mimicking the argument of the proof of Proposition 3.8, under the same assumptions it is also possible to prove a result of stability with respect to initial data for parametrized metric solutions. Namely, let $\{(\widehat{t}_n, \widehat{q}_n)\}$ be a sequence of parametrized metric solutions on a time interval $[s_0, s_1]$, such that $(\widehat{t}_n(s_0), \widehat{q}_n(s_0)) = (t_0^n, q_0^n)$ for every $n \in \mathbb{N}$, with $(t_0^n, q_0^n) \to (t_0, q_0)$ and $m_n := \widehat{t}_n' + |\widehat{q}_n'| \rightharpoonup m$ in $L^1(s_0, s_1)$. Then, there exists a parametrized metric solution $(\widehat{t}_\infty, \widehat{q}_\infty)$, starting from (t_0, q_0), such that, up to the extraction of a subsequence, $(\widehat{t}_n, \widehat{q}_n) \to (\widehat{t}_\infty, \widehat{q}_\infty)$ uniformly in $[s_0, s_1]$, with $(\widehat{t}_n', |\widehat{q}_n'|) \rightharpoonup (\widehat{t}_\infty', |\widehat{q}_\infty'|)$ in $\mathrm{L}^1(s_0, s_1)$. In fact, in [MRS08] we shall prove the above result, as well as the vanishing viscosity analysis of Theorem 3.8, in the more general setting detailed in Section 6.

4 BV solutions

Before introducing the notion of BV solution to the rate-independent system driven by $\mathcal{E}$, we recall some definitions and properties of BV functions on $[0, T]$ with values in the space $(\mathcal{Q}, d)$ introduced in the previous section. Note that, however, the following notions are indeed independent of the Finsler setting (3.Q)–(3.R) and can be given for a general complete metric space.

Preliminaries on BV functions. Given a function $q : [0,T] \to \mathcal{Q}$ and an interval $I \subset [0,T]$, we define its variation on I by

$$\mathrm{Var}(q, I) := \sup \sum_{j=1}^{n} d(q(\tau_{j-1}), q(\tau_j)), \tag{4.1}$$

where sup is taken over all $n \in \mathbb{N}$ and all partitions $\tau_0 < \tau_1 \cdots < \tau_{n-1} < \tau_n$ with $\tau_0, \tau_n \in I$. We set

$$\mathrm{BV}([0,T]; \mathcal{Q}) = \{\, q : [0,T] \to \mathcal{Q} \mid \mathrm{Var}(q, [0,T]) < \infty \,\},$$

where we emphasize that functions are defined everywhere, as is common for rate-independent processes. For $q \in \mathrm{BV}([0,T]; \mathcal{Q})$ and $t \in [0,T]$ the left and right limits exist:

$$q(t^-) := \lim_{h \searrow 0} q(t-h) \quad \text{and} \quad q(t^+) := \lim_{h \searrow 0} q(t+h),$$

where we put $q(0^-) = q(0)$ and $q(T^+) = q(T)$. In general, the three values $q(t^-), q(t)$, and $q(t^+)$ may differ. We define the continuity set C_q and the jump set J_q by

$$\mathrm{C}_q = \{\, t \in [0,T] \mid q(t^-) = q(t) = q(t^+) \,\}, \qquad \mathrm{J}_q = [0,T] \setminus \mathrm{C}_q.$$

Indeed, our definition of "Var" is such that we have for all $0 \le r \le s \le t \le T$

$$\mathrm{Var}(q, [r,s]) = d(q(r), q(r+)) + \mathrm{Var}(q, (r,s)) + d(q(s-), q(s)), \tag{4.2}$$

and the additivity property

$$\mathrm{Var}(q, [r,t]) = \mathrm{Var}(q, [r,s]) + \mathrm{Var}(q, [s,t]). \tag{4.3}$$

When calculating the variation of q over an interval I, one has to be careful with (possible) jumps at the boundary of I, if I contains boundary points. Now, for a function $q \in \mathrm{BV}([0,T], \mathcal{Q})$ we introduce the nondecreasing function

$$V_q : [0,T] \to [0,\infty), \qquad V_q(t) := \mathrm{Var}(q, [0,t]).$$

The distributional derivative of V_q defines a nonnegative Radon measure μ_q such that

$$\mu_q([s,t]) = V_q(t) - V_q(s) \qquad \forall\, t,\, s \in \mathrm{C}_q\,, \tag{4.4}$$

and, more generally (see [Fed69, 2.5.17])

$$\int_0^T \zeta(t)\, \mathrm{d}V_q(t) = \int_0^T \zeta(t)\, \mu_q(\mathrm{d}t) \quad \text{for all } \zeta \in \mathrm{C}^0_{\mathrm{c}}(0,T), \tag{4.5}$$

where $\int_0^T \zeta \, \mathrm{d}V_q$ denotes the Riemann-Stieltjes integral. As usual, μ_q can be decomposed into a *continuous* (also called *diffuse*) part μ_q^{co} and a discrete part μ_q^{J}, where for a Borel set $A \subset [0,T]$ we have

$$\mu_q^{\mathrm{J}}(A) = \mu_q(A \cap \mathrm{J}_q) = \sum_{t \in A \cap \mathrm{J}_q} d(q(t^-), q(t)) + d(q(t), q(t^+)), \tag{4.6}$$

in accordance with formula (4.2) above.

In the following technical lemma (whose proof is postponed to the end of this section), we will discuss the link between a BV map $q : [0,T] \to \mathcal{Q}$ and its graph $\mathbf{q}(t) = (t, q(t))$ in the extended state space $\mathcal{Q}_T$, endowed with the distance $d_{\mathcal{Q}_T}((t_0, q_0), (t_1, q_1)) := |t_0 - t_1| + d(q_0, q_1)$. We denote by $\mathcal{L}_{[a,b]}$ the Lebesgue measure on the interval $[a,b]$, whereas $\mathcal{L}$ denotes a general one-dimensional Lebesgue measure.

Lemma 4.1 *Let $q \in \mathrm{BV}([0,T]; \mathcal{Q})$ and $\mathbf{q} \in \mathrm{BV}([0,T]; \mathcal{Q}_T)$ with $\mathbf{q}(t) := (t, q(t))$. Set*

$$\rho(t) := V_q(t) = \mathrm{Var}(q, [0,t]), \ R := \rho(T); \quad \sigma(t) := V_{\mathbf{q}}(t) = \mathrm{Var}(\mathbf{q}, [0,t]), \ S := \sigma(T),$$

with their right-continuous inverse

$$\widehat{\tau}(r) := \sup\{\, t \in [0,T] \mid V_q(t) = \rho(t) < r \,\}, \quad \widehat{t}(s) := \sup\{\, t \in [0,T] \mid V_{\mathbf{q}}(t) = \sigma(t) < s \,\}.$$

Then, the following statements hold:

A) $\sigma(t) = t + \rho(t), \quad \mathrm{J}_{\mathbf{q}} = \mathrm{J}_q, \quad \mathrm{C}_{\mathbf{q}} = \mathrm{C}_q, \quad \mu_{\mathbf{q}} = \mathcal{L} + \mu_q, \quad \mu_{\mathbf{q}}^{\mathrm{co}} = \mathcal{L} + \mu_q^{\mathrm{co}}, \quad \mu_{\mathbf{q}}^{\mathrm{J}} = \mu_q^{\mathrm{J}}.$

B) *There exist 1-Lipschitz maps $\widehat{\mathbf{q}} = (\widehat{t}, \widehat{q}) : [0,S] \to \mathcal{Q}_T$ and $\widehat{\rho} : [0,S] \to [0,R]$ such that $\mathbf{q}(t) = \widehat{\mathbf{q}}(\sigma(t))$, $\rho(t) = \widehat{\rho}(\sigma(t))$ for all $t \in [0,T]$. The map $\widehat{t}$ is uniquely determined, it is the right-continuous inverse of σ, and it is injective on $\widehat{\mathrm{C}}_q := \sigma(\mathrm{C}_q) = \widehat{t}^{-1}(\mathrm{C}_q)$. The maps $\widehat{\mathbf{q}}$ and $\widehat{\rho}$ are uniquely determined on the set $\widehat{\mathrm{C}}_q$ and satisfy $\widehat{q}(s) = q(\widehat{t}(s))$ and $\widehat{\tau}(\widehat{\rho}(s)) = \widehat{t}(s)$ for all $s \in \widehat{\mathrm{C}}_q$.*

C) *$\widehat{t}_{\#}(\mathcal{L}_{[0,S]}) = \mu_{\mathbf{q}}$ and $\widehat{\tau}_{\#}(\mathcal{L}_{[0,R]}) = \mu_q$, in the sense that for all bounded Borel function $\zeta : [0,T] \to \mathbb{R}$ and Borel set $A \subset [0,T]$*

$$\int_{\widehat{t}^{-1}(A)} \zeta(\widehat{t}(s)) \, \mathrm{d}s = \int_A \zeta(t) \, \mu_{\mathbf{q}}(\mathrm{d}t), \quad \int_{\widehat{\tau}^{-1}(A)} \zeta(\widehat{\tau}(r)) \, \mathrm{d}r = \int_A \zeta(t) \, \mu_q(\mathrm{d}t). \tag{4.7}$$

In particular, if $A \subset [0,T], B \subset \mathrm{C}_q \subset [0,S]$ are Borel sets, then

$$\mu_{\mathbf{q}}(A) = \mathcal{L}(\widehat{t}^{-1}(A)), \qquad \mathcal{L}(B) = \mu_{\mathbf{q}}(\widehat{t}(B)). \tag{4.8}$$

D) *The Lebesgue densities of the measures $\mathcal{L}, \mu_q^{\mathrm{co}} \ll \mu_{\mathbf{q}}^{\mathrm{co}}$ with respect to $\mu_{\mathbf{q}}^{\mathrm{co}}$ are expressed by the formulae*

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\mu_{\mathbf{q}}^{\mathrm{co}}} = \widehat{t}' \circ \sigma, \quad \frac{\mathrm{d}\mu_q^{\mathrm{co}}}{\mathrm{d}\mu_{\mathbf{q}}^{\mathrm{co}}} = |\widehat{q}'| \circ \sigma = \widehat{\rho}' \circ \sigma. \tag{4.9}$$

The notion of BV solution. Let us first introduce a new family of 1-homogeneous dissipation functionals $\mathcal{S}_\alpha(t;\cdot,\cdot) : \mathrm{T}\mathcal{Q} \to [0,\infty)$, depending on the two parameters $\alpha \in [0,\infty)$ and $t \in [0,T]$, defined as

$$\mathcal{S}_\alpha(t;q,v) := \max\left\{|\partial_q\mathcal{E}|\,(t,q)\,,\alpha\right\}\mathcal{R}_1(q,v) \tag{4.10}$$

Notice that for all $\alpha > 0$, $t \in [0,T]$, and $q \in \mathcal{Q}$ the functional $\mathcal{S}_\alpha(t;q,\cdot)$ is a norm on $\mathrm{T}_q\mathcal{Q}$ (possibly degenerate, when $\alpha = 0$), thus satisfying condition (3.R). As in Section 3.1, we can therefore consider the corresponding Finsler distances $S_\alpha(t;\cdot,\cdot) : \mathcal{Q}\times\mathcal{Q} \to [0,\infty)$ via

$$S_\alpha(t;q_0,q_1) := \inf\left\{\int_0^1 \mathcal{S}_\alpha(t;y(s),y'(s))\,\mathrm{d}s \,\Big|\, y \in \mathcal{A}(q_0,q_1)\right\}. \tag{4.11}$$

The functional S_0 is called *slope distance* and $S_\alpha(t;\cdot,\cdot)$ also admits the equivalent formulation in terms of the metric velocity

$$S_\alpha(t;q_0,q_1) := \inf\left\{\int_0^1 \max\left\{|\partial_q\mathcal{E}|\,(t,y(s))\,,\alpha\right\}|y'|(s)\,\mathrm{d}s \,\Big|\, y \in \mathcal{A}(q_0,q_1)\right\}. \tag{4.12}$$

For $\alpha > 0$ the infimum in (4.11) and (4.12) is attained.

A straightforward consequence of the symmetry $\mathcal{R}_1(q,-v) = \mathcal{R}_1(q,v)$ (see (3.R)) is that $S_\alpha(t;q_0,q_1) = S_\alpha(t;q_1,q_0)$. Using the chain rule inequality (3.7) we find

$$|\mathcal{E}(t,q_1)-\mathcal{E}(t,q_0)| \le S_0(t;q_0,q_1) \le S_\alpha(t;q_0,q_1) \quad \text{for all } (t,q_0,q_1) \in [0,T]\times\mathcal{Q}\times\mathcal{Q}\,. \tag{4.13}$$

The notion of BV solution to the rate-independent system $(\mathcal{Q},d,\mathcal{E})$, which we are going to introduce, relies on a version of the chain rule for BV functions with values in a metric space. In order to state it, for a general $q \in \mathrm{BV}([0,T];\mathcal{Q})$ and $0 \le t_0 \le t_1 \le T$ we define

$$\begin{aligned}\Sigma_0(q,[t_0,t_1]) := \int_{t_0}^{t_1} |\partial_q\mathcal{E}|\,(r,q(r))\ \mu_q^{\mathrm{co}}(\mathrm{d}r) + S_0(t_0;q(t_0),q(t_0^+)) + S_0(t_1;q(t_1^-),q(t_1))\\ + \sum_{t\in \mathrm{J}_q\cap(t_0,t_1)} [S_0(t;q(t^-),q(t))+S_0(t;q(t),q(t^+))]\,.\end{aligned} \tag{4.14}$$

Based on (4.13), we define a second functional Γ via

$$\begin{aligned}\Gamma(q,[t_0,t_1]) := &\int_{t_0}^{t_1} |\partial_q\mathcal{E}|\,(r,q(r))\ \mu_q^{\mathrm{co}}(\mathrm{d}r)\\ &+ |\mathcal{E}(t_0,q(t_0))-\mathcal{E}(t_0,q(t_0^+))| + |\mathcal{E}(t_1,q(t_1^-))-\mathcal{E}(t_1,q(t_1))|\\ &+ \sum_{t\in \mathrm{J}_q\cap(t_0,t_1)} \Big[\mathcal{E}(t,q(t))-\mathcal{E}(t,q(t^+))| + |\mathcal{E}(t,q(t^-))-\mathcal{E}(t,q(t))|\Big]\,.\end{aligned} \tag{4.15}$$

Obviously, we have $\Sigma_0(q,[s,t]) \ge \Gamma(q,[s,t]) \ge 0$ and both functionals $\Sigma_0(q,\cdot)$ and $\Gamma(q,\cdot)$ fulfill the additivity property (4.3), when considered as functions on intervals.

Proposition 4.2 *Under assumptions* (3.Q), (3.R), *and* (3.E), *the following chain rule inequality holds for all* $q \in \mathrm{BV}([0,T],\mathcal{Q})$ *and* $0 \le t_0 \le t_1 \le T$*:*

$$\mathcal{E}(t_1, q(t_1)) - \mathcal{E}(t_0, q(t_0)) - \int_{t_0}^{t_1} \partial_t \mathcal{E}(t, q(t))\, \mathrm{d}t \ge -\Gamma(q, [t_0, t_1]) \ge -\Sigma_0(q, [t_0, t_1])\,. \quad (4.16)$$

Proof: The function $t \mapsto E(t) := \mathcal{E}(t, q(t))$ is of bounded variation on $[0,T]$ and its jump set is contained in J_q. We denote by $\eta = \frac{\mathrm{d}}{\mathrm{d}t}E$ its distributional derivative (a bounded Radon measure on $(0,T)$) and by η^{co} its diffuse part, defined as $\eta^{\mathrm{co}}(A) := \eta(A \cap \mathrm{C}_q)$ for all Borel sets $A \subset [0,T]$. Thanks to (4.13), we have (4.16) if we show that

$$\eta^{\mathrm{co}} \ge \partial_t \mathcal{E}(\cdot, q(\cdot))\mathcal{L} - |\partial_q \mathcal{E}|\,(\cdot, q(\cdot))\, \mu_q^{\mathrm{co}}. \quad (4.17)$$

We introduce the maps σ and $\widehat{\mathbf{q}} = (\widehat{t}, \widehat{q})$ as in Lemma 4.1 and we set $\widehat{E}(s) := \mathcal{E}(\widehat{t}(s), \widehat{q}(s))$ for all $s \in [0,S]$, so that $E(t) = \widehat{E}(\sigma(t))$ for all $t \in [0,T]$. Since $\widehat{t}, \widehat{q}$ are Lipschitz continuous and $\mathcal{E}$ is of class C^1, the classical chain rule (3.5)–(3.7) yields

$$\widehat{E}'(s) \ge \partial_t \mathcal{E}(\widehat{t}(s), \widehat{q}(s))\widehat{t}'(s) - |\partial_q \mathcal{E}|\,\big(\widehat{t}(s), \widehat{q}(s)\big)\, |\widehat{q}'|(s) \quad \text{for } \mathcal{L}\text{-a.a. } s \in (0,S). \quad (4.18)$$

On the other hand, since $\widehat{E}$ is a Lipschitz map and since $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \mu_{\mathbf{q}}$, the general chain rule of [AmD90] yields

$$\eta^{\mathrm{co}} = (\widehat{E}' \circ \sigma)\, \mu_{\mathbf{q}}^{\mathrm{co}} \ge \Big(\partial_t \mathcal{E}(t, q(t))\widehat{t}' \circ \sigma - |\partial_q \mathcal{E}|\,(t, q(t))\, |\widehat{q}'| \circ \sigma\Big)\mu_{\mathbf{q}}^{\mathrm{co}}. \quad (4.19)$$

Taking into account (4.9), we conclude (notice that $\widehat{E}' \circ \sigma$ is well defined $\mu_{\mathbf{q}}^{\mathrm{co}}$-a.e., since, for every Lebesgue negligible set $N \subset \widehat{\mathrm{C}}_{\mathbf{q}} = \sigma(\mathrm{C}_{\mathbf{q}})$, (4.8) yields $\mu_{\mathbf{q}}^{\mathrm{co}}(\sigma^{-1}(N)) = 0$). ■

Now we are able to define the notion of BV solution. The formulation is more complicated than the one defining parametrized metric solutions, but it nicely reflects the different flow regimes of rate-independent flow, and the jumps. A shorter but much more implicit formulation will be given in Remark 4.4.

Definition 4.3 (BV solution) *Let* $(\mathcal{Q}, d, \mathcal{E})$ *satisfy* (3.Q), (3.R), (3.E). *A function* $q \in \mathrm{BV}([0,T];\mathcal{Q})$ *is called a* BV solution *of the rate-independent system* $(\mathcal{Q}, d, \mathcal{E})$, *if the following four conditions hold:*

$$\begin{aligned} \mathcal{E}(t_1, q(t_1)) - \mathcal{E}(t_0, q(t_0)) - &\int_{t_0}^{t_1} \partial_t \mathcal{E}(t, q(t))\, \mathrm{d}t \\ &\le -\Sigma_0(q, [t_0, t_1]) \ \textit{for } 0 \le t_0 < t_1 \le T; \end{aligned} \quad (4.20\mathrm{a})$$

$$|\partial_q \mathcal{E}|\,(t, q(t)) \le 1 \quad \textit{for } t \in [0,T] \setminus \mathrm{J}_q; \quad (4.20\mathrm{b})$$

$$|\partial_q \mathcal{E}|\,(t, q(t)) \ge 1 \quad \textit{for } t \in \mathrm{sppt}(\mu_q); \quad (4.20\mathrm{c})$$

$$\begin{aligned} &\textit{for } t \in \mathrm{J}_q \quad \textit{there exist } y^t \in \mathcal{A}(q(t^-), q(t^+)) \textit{ and } \theta^t \in [0,1] \textit{ such that} \\ &(\alpha) \quad y^t(\theta^t) = q(t), \\ &(\beta) \quad |\partial_q \mathcal{E}|\,\big(t, y^t(\theta)\big) \ge 1 \quad \textit{for all } \theta \in [0,1], \\ &(\gamma) \quad \mathcal{E}(t, q(t^+)) - \mathcal{E}(t, q(t^-)) = -\textstyle\int_0^1 |\partial_q \mathcal{E}|\,(t, y^t(\theta))\, |y'|(\theta)\, \mathrm{d}\theta. \end{aligned} \quad (4.20\mathrm{d})$$

Again we point out that, due to the chain rule inequality (4.16), relation (4.20a) holds as an equality, which is the energy balance. Using this energy identity on the intervals $[t-h, t]$ and $[t, t+h]$ and letting $h \searrow 0$ leads to the first two of the following jump relations, which will be used later (recall the definition (4.11) of S_1):

$$\begin{aligned}
&\mathcal{E}(t, q(t^-))-\mathcal{E}(t, q(t)) = S_0(t; q(t^-), q(t)) = S_1(t; q(t^-), q(t)),\\
&\mathcal{E}(t, q(t))-\mathcal{E}(t, q(t^+)) = S_0(t; q(t), q(t^+)) = S_1(t; q(t), q(t^+)),\\
&\mathcal{E}(t, q(t^-))-\mathcal{E}(t, q(t^+)) = S_0(t; q(t^-), q(t^+)) = S_1(t; q(t^-), q(t^+)),
\end{aligned} \tag{4.21}$$

for each $t \in \mathrm{J}_q$. The third relation follows from (4.20d). By the definition of the slope distance S_0, these jump relations already include the existence of a connecting gradient-flow curve $y \in \mathcal{A}(q(t^-), q(t^+))$, i.e. ($\alpha$), ($\beta$), and ($\gamma$) of (4.20d) follow.

The above formulation of BV solutions looks quite lengthy compared to the more elegant forms of gradient-like flows, which can be characterized by one inequality, cf. e.g., (3.8) or (3.20c). However, this formulation reflects the mechanical interpretation of the three different flow types quite well, namely sticking, slipping and jumping. The following result presents a more compact form, which is however less tractable for further analysis.

Proposition 4.4 *In the setting of* (3.Q), (3.R), (3.E), *let* $\Sigma_1(\cdot, [t_0, t_1])$ *be the functional defined on* $\mathrm{BV}([0, T], \mathcal{Q})$ *via*

$$\begin{aligned}
\Sigma_1(q, [t_0, t_1]) := &\int_{t_0}^{t_1} \max\left\{|\partial_q\mathcal{E}|\,(t, q(t))\,, 1\right\}\mu_q^{\mathrm{co}}(\mathrm{d}t) + \int_{t_0}^{t_1} \Big(|\partial_q\mathcal{E}|\,(t, q(t)) - 1\Big)^+ \mathrm{d}t\\
&+ S_1(t_0; q(t_0), q(t_0^+)) + S_1(t_1; q(t_1^-), q(t_1))\\
&+ \sum_{t\in \mathrm{J}_q\cap(t_0,t_1)} [S_1(t, q(t^-), q(t)) + S_1(t, q(t), q(t^+))].
\end{aligned}$$

Then, $q \in \mathrm{BV}([0, T]; \mathcal{Q})$ *is a BV solution if and only if*

$$\mathcal{E}(t_1, q(t_1))-\mathcal{E}(t_0, q(t_0))-\int_{t_0}^{t_1} \partial_t\mathcal{E}(t, q(t))\,\mathrm{d}t \leq -\Sigma_1(q, [t_0, t_1]) \text{ for } 0 \leq t_0 < t_1 \leq T\,. \tag{4.22}$$

Proof: It is clear that under conditions (4.20b), (4.20c), and (4.20d) a BV solution q satisfies

$$\Sigma_0(q; [t_0, t_1]) = \Sigma_1(q; [t_0, t_1]) \quad \text{for } 0 \leq t_0 < t_1 \leq T, \tag{4.23}$$

so that (4.20a) yields (4.22).

Conversely, if $q \in \mathrm{BV}([0, T]; \mathcal{Q})$ satisfies (4.22), the chain rule (4.16) and the inequality $\Sigma_0(\cdot; [t_0, t_1]) \leq \Sigma_1(\cdot; [t_0, t_1])$ yield (4.20a) and (4.23). Choosing e.g. $t_0 = 0, t_1 = T$ we get

$$\begin{aligned}
0 = &\int_0^T \Big(\max\left\{|\partial_q\mathcal{E}|\,(t, q(t))\,, 1\right\} - |\partial_q\mathcal{E}|\,(t, q(t))\Big)\mu_q^{\mathrm{co}}(\mathrm{d}t) + \int_0^T \Big(|\partial_q\mathcal{E}|\,(t, q(t)) - 1\Big)^+ \mathrm{d}t\\
&+ \Big(S_1(0; q(0), q(0^+)) - S_0(0; q(0), q(0^+))\Big) + \Big(S_1(T; q(T^-), q(T)) - S_0(T; q(T^-), q(T))\Big)\\
&+ \sum_{t\in \mathrm{J}_q} \Big[S_1(t; q(t^-), q(t)) - S_0(t; q(t^-), q(t)) + S_1(t; q(t), q(t^+)) - S_0(t; q(t), q(t^+))\Big].
\end{aligned}$$

Since each addendum is nonnegative, we easily find (4.20b) and (4.20c) (recalling that $|\partial\mathcal{E}|$ is continuous). Moreover, passing to the limit in (4.22) as $t_0 \nearrow t$, $t_1 \searrow t$, with $t \in \mathrm{J}_q$, we conclude

$$\mathcal{E}(t, q(t^+)) - \mathcal{E}(t, q(t^-)) \leq -S_1(t; q(t^-), q(t)) - S_1(t; q(t), q(t^+)).$$

Recalling (4.11) and the chain rule, for every $t \in \mathrm{J}_q$ we find a curve y^t satisfying condition (4.20d). ■

BV and parametrized metric solutions. We claim that the notion of BV solution is essentially the same as that of parametrized metric solution. Intuitively, (4.20a) corresponds to (3.23). Further, (4.20b) and (4.20c) are the analog in the BV setting of the first of (3.24c), which encompasses both sticking and rate-independent evolution (recall Remark 3.5). The jumping regime is accounted for by condition (4.20d): at jump times, the system switches to a viscous, rate-dependent behavior, following a path described by a *generalized gradient flow*, see (γ) in (4.20d).

In order to formalize these considerations, we return to the trajectories in $\mathcal{Q}_T$. Indeed, we may associate with each BV solution q_{BV} a trajectory, by filling the jumps of the graph $\{ (t, q_{\mathrm{BV}}(t)) \mid t \in [0,T]) \}$ with the curves $y^t \in \mathrm{AC}([0,1], \mathcal{Q})$, for $t \in J_{q_{\mathrm{BV}}}$. Thus, we obtain

$$\mathcal{T} = \{ (t, q_{\mathrm{BV}}(t)) \mid t \in [0,T] \} \cup \bigcup_{t \in J_{q_{\mathrm{BV}}}} \{ (t, y^t(\theta)) \mid \theta \in [0,1] \}.$$

By construction, $\mathcal{T}$ is a connected curve that has exactly the length $\mathrm{Var}(q, [0,T]) + T$ if we use the extended metric $d_{\mathcal{Q}_T}((t_0, q_0), (t_1, q_1)) := |t_0 - t_1| + d(q_0, q_1)$ on $\mathcal{Q}_T$. Hence, there exists an absolutely continuous parametrization of $\mathcal{T}$, and it can be shown that this parametrized curve is a parametrized metric solution. Indeed, in Example 7.4 we shall show, that to a given BV solution, there may correspond infinitely many distinct parametrized metric solutions.

On the other hand, we can pass from parametrized metric solutions $(\widehat{t}, \widehat{q})$ defined in $[s_0, s_1]$ to BV solutions by choosing

$$\sigma(t) \in \{ s \in [s_0, s_1] \mid \widehat{t}(s) = t \} \quad \text{and defining} \quad q(t) := \widehat{q}(\sigma(t)) \,. \tag{4.24}$$

Hence, $\mathrm{J}_q = \{ t \in [0,T] \mid \sigma(t^+) > \sigma(t^-) \}$, and we see that $q(t)$ is uniquely determined from $\widehat{q}$ for $t \in [0,T] \setminus \mathrm{J}_q$. At the jump times t we can in fact choose any point $q(t) = \widehat{q}(s)$ with $s \in [\sigma(t^-), \sigma(t^+)]$. Note that

$$y^t(\theta) := \widehat{q}\big(\sigma(t^-) + \theta[\sigma(t^+) - \sigma(t^-)]\big), \quad \theta \in [0,1], \quad t \in \mathrm{J}_q \tag{4.25}$$

defines a connecting jump path as desired in (4.20d). We collect these remarks in the next proposition, whose proof easily follows from Lemma 4.1 (see also the Remark 3.3).

Proposition 4.5 *In the setting of* (3.Q), (3.R), (3.E), *let* $q_{\mathrm{BV}} \in \mathrm{BV}([0,T];\mathcal{Q})$ *be a BV solution of the rate-independent system* $(\mathcal{Q}, d, \mathcal{E})$ *and let* $\widehat{\mathbf{q}} = (\widehat{t}, \widehat{q})$ *be a map as in Lemma 4.1. Then, setting*

$$\widehat{z}(s) := \begin{cases} \widehat{q}(s) & \text{if } s \in \widehat{\mathrm{C}}_q\,, \\ y^t(\theta) & \text{if } s \in \widehat{\mathrm{J}}_q,\ \widehat{t}(s) = t,\ s = (1-\theta)\sigma(t^-) + \theta\sigma(t^+) \text{ for } \theta \in [0,1]\,, \end{cases} \tag{4.26}$$

the map $(\widehat{t}, \widehat{z}) : [0,S] \to \mathcal{Q}_T$ *is a parametrized metric solution of* $(\mathcal{Q}, d, \mathcal{E})$ *according to Definition 3.2.*

Conversely, if $(\widehat{t}, \widehat{q}) : [s_0, s_1] \to \mathcal{Q}_T$ *is a surjective parametrized metric solution (i.e.* $\widehat{t}(s_0) = 0$ *and* $\widehat{t}(S_1) = t$*), then any map* q *defined as in* (4.24) *is a BV solution.*

The next result shows that BV solutions can be directly obtained as a vanishing viscosity limit, as in Theorem 3.8, but now rescaling is not needed. The imposed *a priori* bound on the total variation for the viscosity solutions q_ε can be easily obtained from the energy inequality (3.8) under general assumptions on $\mathcal{E}$, see e.g. (6.6).

Corollary 4.6 (Vanishing viscosity limit (II)) *Let* $(\mathcal{Q}, d, \mathcal{E})$ *satisfy* (3.Q), (3.R), *and* (3.E). *For every* $\varepsilon > 0$ *let* $q_\varepsilon \in \mathrm{AC}([0,T];\mathcal{Q})$ *be a solution to* (3.8) *for* $\psi = \psi_\varepsilon$. *Assume that* $q_\varepsilon(0) \to q_0$ *as* $\varepsilon \searrow 0$ *and* $\mathrm{Var}(q_\varepsilon, [0,T]) \leq C$ *for all* $\varepsilon > 0$ *with a constant* C *independent of* ε. *Then, there exist a subsequence* q_{ε_k} *with* $\varepsilon_k \searrow 0$ *and a BV solution* q *for* $(\mathcal{Q}, d, \mathcal{E})$ *such that* $q_{\varepsilon_k}(t) \to q(t)$ *as* $k \to \infty$ *for all* $t \in [0,T]$.

Proof: Let us consider the functions σ_ε as in Lemma 4.1. By Helly's selection theorem we can find subsequences $(q_{\varepsilon_k})_k, (\sigma_{\varepsilon_k})_k$ converging pointwise in $[0,T]$. Let us consider the corresponding parametrized metric solutions $(\widehat{t}_\varepsilon, \widehat{q}_\varepsilon)$ introduced in Proposition 4.5. Since σ_ε is absolutely continuous with $\sigma_\varepsilon' \geq 1$, differentiating the identity $\sigma_\varepsilon(t) = \widehat{t}_\varepsilon(\sigma_\varepsilon(t)) + \widehat{\rho}_\varepsilon(\sigma_\varepsilon(t))$ we obtain $m_\varepsilon := \widehat{t}_\varepsilon' + |\widehat{q}_\varepsilon'| = 1$ a.e. in $(0, S_\varepsilon)$ and $S_\varepsilon := \sigma_\varepsilon(T) \geq T$. Since S_{ε_k} converges to $S \geq T > 0$, up to a further linear rescaling it is not restrictive to assume that $S_{\varepsilon_k} = S$ and $m_{\varepsilon_k} = S_{\varepsilon_k}/S \to 1$.

Applying Theorem 3.8 we can find suitable subsequences (still labelled ε_k) such that $(\widehat{t}_{\varepsilon_k}, \widehat{q}_{\varepsilon_k}) \to (\widehat{t}, \widehat{q})$ in $\mathrm{C}^0([0,S];\mathcal{Q}_T))$. Since $q_\varepsilon(t) = \widehat{q}_\varepsilon(\sigma_\varepsilon(t))$ and $t = \widehat{t}_\varepsilon(\sigma_\varepsilon(t))$, we easily get $q_{\varepsilon_k}(t) \to \widehat{q}(\sigma(t))$ and $\widehat{t}(\sigma(t)) = t$, so that q is a BV solution induced by $(\widehat{t}, \widehat{q})$ as in (4.24). ∎

We conclude the section with the **Proof of Lemma 4.1:**
Part **A)** is immediate. Part **B)** is an obvious extension of [Fed69, 2.5.16], since each couple of points in $\mathcal{Q}$ can be connected by a geodesic. Notice that $\widehat{\tau}(V_q(t)) = t$ and therefore $\widehat{\tau}(\widehat{\rho}(\sigma(t))) = t$ if $t \in \mathrm{C}_q$. We thus get $\widehat{\tau} \circ \widehat{\rho} = \widehat{t}$ in $\widehat{\mathrm{C}}_q$.

In order to prove **C)** it is not restrictive to assume $A = [0,T]$ and $\zeta \in \mathrm{C}^0_\mathrm{c}([0,T])$. Then, (4.7) follows from (4.5) and [Fed69, 2.5.18(3)], observing that

$$\mathcal{L}\big(\{\, s \in [0,S] \mid \widehat{t}(s) < t \,\}\big) = V_q(t) \text{ for all } t \in \mathrm{C}_q, \quad \mathcal{L}\big(\{\, r \in [0,R] \mid \widehat{\tau}(r) < t \,\}\big) = V_q(t).$$

Let us now prove the first identity of (4.9): setting $\widehat{\mathrm{J}}_q := \widehat{t}^{-1}(\mathrm{J}_q) = [0,S] \setminus \widehat{\mathrm{C}}_q$, we observe that $\widehat{t}'(s) = 0$ for $\mathcal{L}$-a.e. $s \in \widehat{\mathrm{J}}_q$. Since $\widehat{t}$ is Lipschitz continuous and monotone, the change of variable formula and (4.7) yield, for every continuous function ζ with compact support in $(0,T)$,

$$\int_0^T \zeta(t)\,\mathrm{d}t = \int_0^S \zeta(\widehat{t}(s))\widehat{t}'(s)\,\mathrm{d}s = \int_{\widehat{\mathrm{C}}_q} \zeta(\widehat{t}(s))\widehat{t}'(s)\,\mathrm{d}s = \int_{\mathrm{C}_\mathrm{q}} \zeta(t)\,\widehat{t}' \circ \sigma(t)\,\mu_{\mathbf{q}}^{\mathrm{co}}(\mathrm{d}t).$$

The second identity of (4.9) follows by a similar argument:

$$\begin{aligned}\int_0^T \zeta(t)\,\mu_q^{\mathrm{co}}(\mathrm{d}t) &= \int_{\mathrm{C}_q} \zeta(t)\,\mu_q(\mathrm{d}t) = \int_{\widehat{\tau}^{-1}(\mathrm{C}_q)} \zeta(\widehat{\tau}(r))\,\mathrm{d}r = \int_{\widehat{\rho}^{-1}(\widehat{\tau}^{-1}(\mathrm{C}_q))} \zeta(\widehat{\tau}(\widehat{\rho}(s)))\widehat{\rho}'(s)\,\mathrm{d}s \\ &= \int_{\widehat{\mathrm{C}}_q} \zeta(\widehat{t}(s))\widehat{\rho}'(s)\,\mathrm{d}s = \int_{\mathrm{C}_q} \zeta(t)\widehat{\rho}' \circ \sigma(t)\,\mu_x(\mathrm{d}t) = \int_0^T \zeta(t)\widehat{\rho}' \circ \sigma(t)\,\mu_{\mathbf{q}}^{\mathrm{co}}(\mathrm{d}t).\end{aligned}$$

The identity $\widehat{\rho}' \circ \sigma = |\widehat{q}'| \circ \sigma$ follows from the property $V_q(t) = V_{\widehat{q}}(\sigma(t))$ for all $t \in \mathrm{C}_q$, so that $V_{\widehat{q}}(s) = \widehat{\rho}(s)$ for all $s \in \widehat{\mathrm{C}}_q$. ∎

5 Other solution concepts

Here we discuss other notions of solutions for rate-independent systems $(\mathcal{Q}, d, \mathcal{E})$, namely energetic solutions, local and approximable solutions, and Φ-minimal solutions.

5.1 Energetic solutions

The concept of energetic solutions provides the most general setting, in the sense that it does not even rely on a differentiability structure like the Finsler metric $\mathcal{R}$, but only uses the distance d. In such a framework it is even possible to consider quasi-metrics (i.e. unsymmetric and allowed to take the value ∞), cf. [Mie05].

Definition 5.1 *A mapping $q : [0,T] \to \mathcal{Q}$ is called* energetic solution *for the rate-independent system* $(\mathcal{Q}, d, \mathcal{E})$ *if for all $t \in [0,T]$ the global stability* (S) *and the energy balance* (E) *hold:*

$$\begin{aligned} &\text{(S)} \quad \forall \widetilde{q} \in \mathcal{Q}: \qquad \mathcal{E}(t, q(t)) \le \mathcal{E}(t, \widetilde{q}) + d(q(t), \widetilde{q}); \\ &\text{(E)} \quad \mathcal{E}(t, q(t)) + \mathrm{Var}(q, [0,t]) = \mathcal{E}(0, q(0)) + \textstyle\int_0^t \partial_s \mathcal{E}(s, q(s))\,\mathrm{d}s\,. \end{aligned}$$

We refer to [MiT99, MTL02] for the origins of this theory and to [Mie05] for a survey. In analogy with (4.21) we have the jump relations

$$\begin{aligned} \mathcal{E}(t, q(t^-)) - \mathcal{E}(t, q(t)) &= d(q(t^-), q(t)), \\ \mathcal{E}(t, q(t)) - \mathcal{E}(t, q(t^+)) &= d(q(t), q(t^+)), \\ \mathcal{E}(t, q(t^-)) - \mathcal{E}(t, q(t^+)) &= d(q(t^-), q(t^+)), \end{aligned} \tag{5.1}$$

for all $t \in \mathrm{J}_q$. Here they are easily obtained by considering the energy identity $\mathcal{E}(s,q(s))+\mathrm{Var}(q,[r,s]) = \mathcal{E}(r,q(r)) + \int_r^s \partial_\tau \mathcal{E}(\tau,q(\tau))\,\mathrm{d}\tau$, which follows immediately from (E), for the intervals $[t{-}h,t]$, $[t,t{+}h]$, and $[t{-}h,t{+}h]$, respectively, and letting $h \searrow 0$.

To compare energetic and BV solutions, we introduce the *global slope* $\mathcal{G}[\mathcal{E}(t,\cdot)] : \mathcal{Q} \to [0,\infty]$ via

$$\mathcal{G}[\mathcal{E}(t,\cdot)](q) := \sup_{\widetilde{q}\neq q} \frac{\big(\mathcal{E}(t,q)-\mathcal{E}(t,\widetilde{q})\big)^+}{d(q,\widetilde{q})} \quad \text{for all } q \in D\,.$$

Using this definition, the global stability (S) can obviously be rephrased as $\mathcal{G}[\mathcal{E}(t,\cdot)](q(t)) \leq 1$. We also have

$$|\partial_q \mathcal{E}|\,(t,q) \leq \mathcal{G}[\mathcal{E}(t,\cdot)](q) \quad \text{for all } (t,q) \in \mathcal{Q}_T\,. \tag{5.2}$$

Indeed, choosing a local coordinate system in $\mathcal{Q}$, one can check (cf. [BCS00, Ch.VI.2]) that

$$|\partial_q \mathcal{E}|\,(t,q) = \sup_{v \in \mathrm{T}_q\mathcal{Q}\setminus\{0\}} \frac{\langle \mathrm{D}_q\mathcal{E}(t,q), v\rangle}{\mathcal{R}_1(q,v)} = \limsup_{\widetilde{q}\to q} \frac{(\mathcal{E}(t,q)-\mathcal{E}(t,\widetilde{q}))^+}{d(q,\widetilde{q})}\,, \tag{5.3}$$

whence (5.2).

Remark 5.2 It is well known that (S) implies the lower energy estimate

$$\mathcal{E}(s,q(s)) - \mathcal{E}(r,q(r)) - \int_r^s \partial_t \mathcal{E}(t,q(t))\,\mathrm{d}t \geq -\,\mathrm{Var}(q,[r,s])$$

for $0 \leq r < s \leq T$, cf. [MTL02, Thm. 2.5] and [Mie05, Prop. 5.7]. In the present setting this is in fact an easy consequence of the chain rule inequality (4.16) and of the observation that $\mathcal{G}[\mathcal{E}(t,\cdot)](q(t)) \leq C$ implies $\Gamma(q,[r,s]) \leq C\,\mathrm{Var}(q,[r,s])$.

Moreover, it is possible to derive a gradient-flow like inequality of the type given in (3.8), (3.20c), or (4.22). For this, define the functional $\Gamma_*(\cdot,[r,s])$ on $\mathrm{BV}([0,T],\mathcal{Q})$ via

$$\begin{aligned}\Gamma_*(q,[r,s]) := {} & \int_r^s \max\big\{\mathcal{G}[\mathcal{E}(t,\cdot)](q(t)),1\big\}\mu_q^{\mathrm{co}}(\mathrm{d}t) + \int_r^s \big(\mathcal{G}[\mathcal{E}(t,\cdot)](q(t))-1\big)^+\,\mathrm{d}t \\ & + \max\{|\mathcal{E}(r,q(r)){-}\mathcal{E}(r,q(r^+))|, d(q(r),q(r^+))\} \\ & + \max\{|\mathcal{E}(s,q(s^-)){-}\mathcal{E}(s,q(s))|, d(q(s^-),q(s))\} \\ & + \sum_{t\in(r,s)\cap \mathrm{J}_q} \big[\max\{|\mathcal{E}(t,q(t^-)){-}\mathcal{E}(t,q(t))|, d(q(t^-),q(t))\} \\ & \qquad + \max\{|\mathcal{E}(t,q(t)){-}\mathcal{E}(t,q(t^+))|, d(q(t),q(t^+))\}\big]\,.\end{aligned}$$

Then, $q : [0,T] \to \mathcal{Q}$ is an energetic solution for $(\mathcal{Q},d,\mathcal{E})$ if and only if $\mathcal{G}[\mathcal{E}(0,\cdot)](q(0)) \leq 1$ and

$$\mathcal{E}(s,q(s)) - \mathcal{E}(r,q(r)) - \int_r^s \partial_t \mathcal{E}(t,q(t))\,\mathrm{d}t \leq -\Gamma_*(q,[r,s]) \text{ for } 0 \leq r < s \leq T\,. \tag{5.4}$$

The following result essentially states that every energetic solution q is a BV solution outside its jump set. Moreover, if the jump relations (4.21) and (5.1) are both satisfied, then an energetic solution is also a BV solution. Conversely, if a BV solution additionally satisfies $\mathcal{G}[\mathcal{E}(t,\cdot)](q(t)) \leq 1$, then it is an energetic solution as well.

Proposition 5.3 (Comparison between energetic and BV solutions) *Assume that* $(\mathcal{Q}, d, \mathcal{E})$ *satisfies* (3.Q), (3.R), *and* (3.E).

(A) If $q \in \mathrm{BV}([0,T];\mathcal{Q})$ *is an energetic solution, then* q *is also a BV solution if and only if* (4.20d) *holds additionally.*

(B) If $q \in \mathrm{BV}([0,T];\mathcal{Q})$ *is a BV solution with* $\mathcal{G}[\mathcal{E}(t,\cdot)](q(t)) \leq \max\{1, |\partial_q\mathcal{E}|\,(t,q(t))\}$ *for all* $t \in [0,T]$ *and* $\mathcal{G}[\mathcal{E}(0,\cdot)](q(0)) \leq 1$, *then* q *is also an energetic solution.*

Proof: To prove (A), we first note that the necessity of (4.20d) is obvious at it is part of the definition of BV solutions. To establish the sufficiency in (A), we observe that (4.20d) yields (4.21), so that the dissipation term Σ_1 of Proposition 4.4 satisfies

$$\Sigma_1(q,[t_0,t_1]) \leq \Gamma_*(q,[t_0,t_1]) \quad \text{for all } 0 \leq t_0 < t_1 \leq T.$$

Hence, (4.22) follows from (5.4). Thus, statement (A) is established.

The necessity of the additional condition on $\mathcal{G}[\mathcal{E}(t,\cdot)](q(t))$ is obvious, since energetic solutions have to satisfy the stronger stability condition (S). To show the sufficiency we observe that the additional condition yields

$$\Gamma_*(q,[t_0,t_1]) \leq \Sigma_1(q,[t_0,t_1]) \quad \text{for all } 0 \leq t_0 < t_1 \leq T,$$

so that (5.4) follows from (4.22). ∎

Remark 5.4 The additional condition in Proposition 5.3(B) is implied by the general condition

$$\mathcal{G}[\mathcal{E}(t,\cdot)](\widehat{q}) = |\partial_q\mathcal{E}|\,(t,\widehat{q}) \qquad \text{for all } (t,\widehat{q}) \in \mathcal{Q}_T\,. \tag{5.5}$$

If this condition holds, then the notions of energetic solutions and BV solutions coincide under the additional assumption that the initial state q_0 is stable, i.e. $\mathcal{G}[\mathcal{E}(0,\cdot)](q_0) \leq 1$. One condition guaranteeing (5.5) is a metric version of convexity for $\mathcal{E}(t,\cdot)$, see [AGS05, Def. 2.4.3]. Here, we say that $\mathcal{F} : \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$ is convex on $(\mathcal{Q}, d)$, if

$$\begin{aligned} \forall\, q_0, q_1 \in \mathcal{Q},\ \theta \in [0,1]\ \exists\, q_\theta \in \mathcal{Q} : d(q_0,q_\theta) = \theta d(q_0,q_1),\ d(q_\theta,q_1) = (1{-}\theta)d(q_0,q_1),& \\ \mathcal{F}(q_\theta) \leq (1{-}\theta)\,\mathcal{F}(q_0) + \theta\,\mathcal{F}(q_1).& \end{aligned} \tag{5.6}$$

To establish (5.5) for $\mathcal{F}$ note that for each $\widehat{q}$ and $\varepsilon > 0$ we have $\mathcal{G}[\mathcal{F}](\widehat{q}) \geq \frac{\mathcal{F}(\widehat{q})-\mathcal{F}(\widetilde{q})}{d(\widehat{q},\widetilde{q})} - \varepsilon$ for some $\widetilde{q}$. Moreover, for each $n \in \mathbb{N}$, choosing $\theta = 1/n$ we find q_n with $d(\widehat{q},q_n) = \frac{1}{n}d(\widehat{q},\widetilde{q})$ and $d(q_n,\widetilde{q}) = \frac{n-1}{n}d(\widehat{q},\widetilde{q})$. Applying (5.6) we obtain

$$\begin{aligned} |\partial_q\mathcal{F}|(\widehat{q}) &\geq \limsup_{n\to\infty} \frac{\mathcal{F}(\widehat{q}) - \mathcal{F}(q_n)}{d(\widehat{q},q_n)} \geq \limsup_{n\to\infty} \frac{\mathcal{F}(\widehat{q}) - \frac{n-1}{n}\mathcal{F}(\widehat{q}) - \frac{1}{n}\mathcal{F}(\widetilde{q})}{\frac{1}{n}\,d(\widehat{q},\widetilde{q})} \\ &= \frac{\mathcal{F}(\widehat{q}) - \mathcal{F}(\widetilde{q})}{d(\widehat{q},\widetilde{q})} \geq \mathcal{G}[\mathcal{F}](\widehat{q}) - \varepsilon\,. \end{aligned}$$

In this way, part (B) of Proposition 5.3 is a generalization to the metric setting of Theorem 3.5 in [MiT04], which states that for a Banach space $\mathcal{Q}$, a convex energy functional $\mathcal{E}$, and a translation invariant metric d the subdifferential formulation and the energetic formulation for $(\mathcal{Q}, d, \mathcal{E})$ are equivalent.

5.2 Local and approximable solutions

As we have already mentioned in the introduction, energetic solutions have the disadvantage that the stability condition (S) is global, so that solutions tend to jump earlier than expected, see Example 7.1. To avoid these early jumps, the vanishing viscosity method was employed in [EfM06, DD*07, ToZ06, KMZ07]. When avoiding parametrization and studying the limits of the viscous solutions $q_\varepsilon : [0,T] \to \mathcal{Q}$ directly, one obtains an *energy inequality* and a *local stability* condition. Hence, we next introduce the notions of *local* solution and of *approximable* solution, generalizing the definitions given in [ToZ06] to the metric setting.

Definition 5.5 *A mapping* $q : [0,T] \to \mathcal{Q}$ *is called* local solution*, if* (a1) *and* (a2) *hold:*

(a1) $|\partial_q \mathcal{E}|\,(t, q(t)) \le 1$ *for a.a.* $t \in [0,T]$;

(a2) *for all* $r, s \in [0,T]$ *with* $r < s$ *we have*
$\mathcal{E}(s, q(s)) + \mathrm{Var}(q, [r,s]) \le \mathcal{E}(r, q(r)) + \int_r^s \partial_t \mathcal{E}(\tau, q(\tau))\,\mathrm{d}\tau.$

We will see in the examples of Section 7 that the notion of local solution is very general. Using (5.2), it is clear that all energetic solutions are local solutions. Similarly, all BV solutions are local solutions. To see this, we use (4.20c) and (4.20d) to obtain (a1), since there are at most a countable number of jump points, and we conclude $\Sigma_0(q, [r,s]) \ge \mathrm{Var}(q, [r,s])$ for $0 \le r \le s \le T$, which gives (a2).

On the other hand, note that, unlike the case of energetic solutions, the combination of the local stability condition with the energy inequality does not provide full information on the solution q. In particular, the behavior of the solution at jumps is poorly described by relations (a1) and (a2). This also highlights the role of the term $\Sigma_0(q, \cdot)$, here missing, in the energy identity for BV solutions. As a consequence there are many more local solutions, see also Example 7.1.

Indeed, the vanishing viscosity method turns out to provide a selection criterion for local solutions. Among local solutions, we thus distinguish the following ones:

Definition 5.6 *A mapping* $q : [0,T] \to \mathcal{Q}$ *is called* approximable solution*, if there exists a sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *with* $\varepsilon_k \searrow 0$ *and solutions* $q_{\varepsilon_k} \in \mathrm{AC}([0,T], \mathcal{Q})$ *of* (3.8) *with* $\psi = \psi_{\varepsilon_k}$ *such that for all* $t \in [0,T]$ *we have* $q_{\varepsilon_k}(t) \to q(t)$.

It follows from Corollary 4.6 that, under the assumptions (3.Q), (3.R), and (3.E), any approximable solution is a BV solution as well.

The notion of approximable solutions suffers from two drawbacks. First of all, there is no direct characterization of the limits in terms of a subdifferential inclusion or variational inequality, unlike for parametrized/BV solutions, recall Proposition 3.7. Secondly, since the solution set is defined through a limit procedure, it is not upper semicontinuous with respect to small perturbations, as shown in Example 7.3. This is in contrast with the stability properties of the set of parametrized/BV solutions, see Remark 3.10.

5.3 Visintin's Φ-minimal solutions

In [Vis01, Vis06] a new minimality principle was introduced. Here, we present the adaptation to rate-independent evolutions proposed in [Mie05, Sect.5.4], in the current metric setting. Again, we use parametrized curves, as it is essential to have continuous paths. For simplicity, we restrict to arclength parametrization, i.e.,

$$t'(s) + |q'|(s) = 1 \quad \text{for a.a. } s \in [s_0, s_1]. \tag{5.7}$$

In the framework of (3.Q), (3.R), (3.E), for a given initial pair (t_0, q_0) we introduce the space of arclength-parametrized paths (on some interval $[0, S]$) starting in (t_0, q_0) via

$$\begin{aligned} \mathcal{A}_S(t_0, q_0) := \; & \{\, (t, q) \in \mathrm{C}^0([0, S], \mathcal{Q}_T) \mid t(0) = t_0,\ t \text{ nondecreasing}, \\ & \quad q(0) = q_0,\ t(s) + \mathrm{Var}(q, [0, s]) = s \quad \text{for all } s \in [0, S] \,\}. \end{aligned}$$

On this set we define the function $\Phi : \mathcal{A}_S(t_0, q_0) \to \mathrm{L}^\infty([0, S])$ via

$$\Phi[t, q](s) = \mathcal{E}(t(s), q(s)) + \mathrm{Var}(q, [0, s]) - \int_0^s \partial_t \mathcal{E}(t(r), q(r)) t'(r) \, \mathrm{d}r\,.$$

Between paths in $\mathcal{A}_S(t_0, q_0)$ we introduce an order relation $\preceq$ as follows. For $(t, q), (\tau, p) \in \mathcal{A}_S(t_0, q_0)$, define the "arclength of equality" via

$$\mathbf{S}[(t, q), (\tau, p)] = \inf\{\, s \in [0, S] \mid (t(s), q(s)) \neq (\tau(s), p(s)) \,\}.$$

Then, the order relation is given by

$$(t, q) \preceq (\tau, p) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \forall\, s > \mathbf{S}[(t, q), (\tau, p)] \ \ \exists\, s_* \in (\mathbf{S}[(t, q), (\tau, p)], s) : \\ \qquad \Phi[(t, q)](s_*) \leq \Phi[(\tau, p)](s_*). \end{array} \right.$$

Definition 5.7 *An arclength-parametrized function* $(t, q) : [s_0, s_1] \to \mathcal{Q}_T$ *is called a* Φ-minimal solution *for* $(\mathcal{Q}, d, \mathcal{E})$*, if for all* $(\tau, p) \in \mathcal{A}_{s_1 - s_0}(t(s_0), q(s_0))$ *we have* $(t, q) \preceq (\tau, p)$.

Like for energetic solutions, this solution notion appears particularly suitable to handle nonsmooth energy functionals, since no derivatives/slopes of $\mathcal{E}$ with respect to the variable q occur in the definition of the functional Φ.

We now show that, in a smooth setting, Φ-minimal solutions are parametrized metric solutions. Using suitably chosen test functions, it can be shown that a necessary condition for Φ-minimality is the local condition

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi[(t, q)](s) \leq \mathcal{N}(t(s), q(s)) \quad \text{for a.a. } s \in (s_0, s_1),$$

where $\mathcal{N} : \mathcal{Q}_T \to \mathbb{R}$ is defined via

$$\mathcal{N}(t, q) = \liminf_{\varepsilon \to 0} \Big(\frac{1}{\varepsilon} \inf\{\, \mathcal{E}(t, \widetilde{q}) - \mathcal{E}(t, q) + d(q, \widetilde{q}) \mid d(q, \widetilde{q}) \leq \varepsilon \,\} \Big)\,.$$

A simple calculation gives

$$\mathcal{N}(t,q) = \begin{cases} 0 & \text{for } |\partial_q\mathcal{E}|\,(t,q) \le 1, \\ 1 - |\partial_q\mathcal{E}|\,(t,q) & \text{for } |\partial_q\mathcal{E}|\,(t,q) \ge 1 \end{cases} \qquad \forall\,(t,q) \in \mathcal{Q}_T\,.$$

Since $t' + |q'| = 1$ a.e. and

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi[(t,q)](s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(t(s),q(s)) + |q'|(s) - \partial_t\mathcal{E}(t(s),q(s))t'(s) \quad \text{for a.a. } s \in (s_0,s_1),$$

we conclude that all Φ-minimal solutions satisfy a.e. in (s_0,s_1)

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(t(s),q(s)) - \partial_t\mathcal{E}(t(s),q(s))t'(s) \le -\widetilde{M}\big(t'(s),|q'|(s),|\partial_q\mathcal{E}|\,(t(s),q(s))\,\big),$$

together with the constraint $t'(s) + |q'|(s) = 1$, where $\widetilde{M}(\alpha,\nu,\xi) = \nu + (\xi-1)^+$. We have thus proved that any Φ-minimal solution (t,q) on $[s_0,s_1]$ is a parametrized metric solution, and hence a BV solution (up to a parametrization). The opposite is in general not true, see Example 7.2. Further, Example 7.3 shows that the set of Φ-minimal solutions is not stable with respect to perturbations.

6 Outlook to the analysis in metric spaces

In [MRS08] we shall analyze rate-independent evolutions in

$$\text{a complete metric space } (\mathcal{X},d)\,. \tag{6.1}$$

In fact, using the results in [RMS08] we shall be able to handle the case in which d is a quasi-distance on $\mathcal{X}$, i.e. possibly unsymmetric and possibly taking the value ∞.

In this framework, the *metric velocity* (1.3) of a curve $q \in \mathrm{AC}([0,T];\mathcal{X})$ is defined by

$$|q'|(t) := \lim_{h\searrow 0}\frac{1}{h}d(q(t),q(t{+}h)) = \lim_{h\searrow 0}\frac{1}{h}d(q(t{-}h),q(t)) \quad \text{for a.a. } t \in (0,T)\,. \tag{6.2}$$

In the Finsler setting (3.Q)–(3.R) of Section 3.1, one indeed has $|q'|(t) = \mathcal{R}_1(q(t),q'(t))$ for a.a. $t \in (0,T)$ (see [BCS00, Chap.VI.2]). Further, given a functional

$$\mathcal{E}:\ [0,T]\times\mathcal{X} \to \mathbb{R}\cup\{\infty\}, \quad \text{with domain } \mathrm{dom}(\mathcal{E}) = [0,T]\times D \tag{6.3}$$

for some $D \subset \mathcal{X}$, the *local slope* is defined via

$$|\partial_q\mathcal{E}|\,(t,q) = \limsup_{\widetilde{q}\to q}\frac{1}{d(q,\widetilde{q})}(\mathcal{E}(t,q) - \mathcal{E}(t,\widetilde{q}))^+\,, \tag{6.4}$$

which in the Finsler setting $\mathcal{X} = \mathcal{Q}$ coincides with (3.4). With these tools, (3.8) is the purely metric formulation of doubly nonlinear equations of the type (3.9), provided the local slope fulfils the following *chain rule inequality*.

Definition 6.1 *We say that the triple $(\mathcal{X}, d, \mathcal{E})$ satisfies the chain rule inequality if for every absolutely continuous curve $(t, q) : [s_0, s_1] \to [0, T] \times \mathcal{X}$ such that $t' \geq 0$ a.e. in (s_0, s_1), $q(s) \in D$ for every $s \in [s_0, s_1]$, and*

$$\int_{s_0}^{s_1} |\partial_q \mathcal{E}|\,(t(s), q(s))\; |q'|(s) + |\partial_t \mathcal{E}(t(s), q(s))|\, t'(s)\, \mathrm{d}s < \infty\,,$$

the map $s \mapsto \mathcal{E}(t(s), q(s))$ is absolutely continuous on $[s_0, s_1]$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(t(s), q(s)) - \partial_t \mathcal{E}(t(s), q(s)) t'(s) \geq -|\partial_q \mathcal{E}|\,(t(s), q(s))\, |q'|(s) \;\; \text{a.e. in } (s_0, s_1). \tag{6.5}$$

Unlike in Section 3.1, within this abstract setting the chain rule inequality is no longer granted, but has to be imposed instead. We refer to [RMS08] for a discussion on some sufficient conditions for (6.5) to hold.

If the above chain rule holds, the (parametrized) metric formulation (3.12) is again the starting point for the vanishing viscosity analysis, which was developed in Section 3.2 in a Finsler setting for smooth $\mathcal{E}$. In this general framework, $\mathcal{E}$ has to satisfy some coercivity and (lower semi-) continuity properties:

$$\begin{aligned}
&\text{the functionals } \mathcal{E}(t,\cdot) \text{ are lower semicontinuous and}\\
&\text{uniformly bounded from below with } K_0 := \inf_{t \in [0,T],\, q \in D} \mathcal{E}(t, q) > -\infty\,;\\
&\forall\, t \in [0,T] :\; \mathcal{E}(t,\cdot) \text{ has compact sublevels in } \mathcal{X}\,;\\
&\exists\, K_1 > 0\; \forall\, q \in D :\; \mathcal{E}(\cdot, q) \in \mathrm{C}^1([0,T]) \text{ and}\\
&\qquad\qquad\qquad\qquad |\partial_t \mathcal{E}(t,q)| \leq K_1(\mathcal{E}(t,q)+1) \text{ for all } t \in [0,T]\,;\\
&\forall\, ((t_n, q_n))_{n \in \mathbb{N}} \subset [0,T] \times \mathcal{X} \text{ with } (t_n, q_n) \to (t, q) :\\
&\partial_t \mathcal{E}(t,q) = \lim_{n\to\infty} \partial_t \mathcal{E}(t_n, q_n) \quad \text{and} \quad |\partial_q \mathcal{E}|\,(t,q) \leq \liminf_{n\to\infty} |\partial_q \mathcal{E}|\,(t_n, q_n)\,.
\end{aligned} \tag{6.6}$$

In [MRS08], under assumptions (6.1) and (6.3)–(6.6) on $\mathcal{E}$, we shall perform the vanishing viscosity analysis of Theorem 3.8, leading to the notion of parametrized metric solution of the rate-independent system $(\mathcal{X}, d, \mathcal{E})$. In this general setting, it is obviously still possible to consider the notion of BV solution, and our remarks on the comparison between BV (parametrized) and local/approximable/Φ-minimal solutions carry over.

Indeed, in [MRS08] we shall discuss BV solutions with more detail, in particular proving existence through approximation by time discretization and solution of incremental (local) minimization problems.

7 Examples

Many of the differences between the various solution concepts discussed above manifest themselves already in the case in which the state space is the real line. Hence, we discuss the very simple model with

$$\mathcal{Q} = \mathbb{R}, \quad d(q_0, q_1) = |q_0 - q_1|\,, \quad \mathcal{E}(t, q) = U(q) - \ell(t)q\,, \tag{7.1}$$

where the function ℓ will be specified in the different examples. The potential U is the nonconvex function given via

$$U(q) = \begin{cases} \frac{1}{2}(q{+}4)^2 & \text{for } q \le -2, \\ 4{-}\frac{1}{2}q^2 & \text{for } |q| \le 2, \\ \frac{1}{2}(q{-}4)^2 & \text{for } q \ge 2. \end{cases} \tag{7.2}$$

As initial datum we shall take

$$q_0 = -5. \tag{7.3}$$

Example 7.1 *We let $\ell(t) = t$ for all $t \ge 0$. We claim that the approximable, the Φ-minimal, the parametrized, and the BV solutions on $[0, \infty)$ are essentially unique and coincide. However, the unique energetic solution is different. Moreover, we show that there is an uncountable family of different local solutions. With direct calculations, one sees that the energetic solution takes the form*

$$q(t) = t{-}5 \quad \text{for } t \in [0, 1) \quad \text{and} \quad q(t) = t{+}3 \quad \text{for } t > 1.$$

Choose any $t_* \in [1, 3]$ and any $q_* \in \big[3{+}t_*, 3{+}t_*{+}\min\{2, 4\sqrt{t_*{-}1}\}\big]$. Then,

$$q(t) = \begin{cases} t{-}5 & \text{for } t \in [0, t_*), \\ q_* & \text{for } t \in (t_*, q_*{-}3], \\ t{+}3 & \text{for } t \ge q_*{-}3, \end{cases}$$

is a local solution. Note that the starting point of the jump at $q(t_*{-}) = t_*{-}5$ can be chosen in a full interval. Moreover, for a fixed $t_* > 1$ we still have the possibility to choose the ending point $q_* = q(t_*{+})$ of the jump in a full interval.

All the other solution types essentially lead (up to definition in one point) to the same solution. Without time parametrization it reads

$$q(t) = \begin{cases} t{-}5 & \text{for } t \in [0, 3), \\ q_* & \text{for } t = 3, \\ t{+}3 & \text{for } t > 3, \end{cases}$$

where $q_* \in [-2, 6]$ is arbitrary. The associated arclength-parametrized solution takes the form

$$\big(\widehat{t}(s), \widehat{q}(s)\big) = \begin{cases} \big(\frac{s}{2}, \frac{s}{2}{-}5\big) & \text{for } s \in [0, 6], \\ (3, s{-}8) & \text{for } s \in [6, 14], \\ \big(\frac{s}{2}{-}4, \frac{s}{2}{-}1\big) & \text{for } s \ge 14. \end{cases}$$

Example 7.2 *In this example we show that, in general, approximable solutions and Φ-minimal solutions are different. In particular, recalling the discussions in Sections 5.2 and 5.3, this shows that the set of BV solutions (or parametrized metric solutions) is strictly bigger then any of the other solution sets.*

In the setting of (7.1)–(7.3), we now choose the function $\ell(t) := \min\{t, 6-t\}$ for all $t \geq 0$, i.e., the loading reduces exactly when the solution reaches the jump point. It is easy to see that there are two different BV solutions: q_1, which jumps at $t = 3$, and q_2, which does not jump. We have

$$q_1(t) = \begin{cases} t-5 & \text{for } t \in [0,3), \\ 6 & \text{for } t \in (3,5], \\ 11-t & \text{for } t \in [5,9), \\ 3-t & \text{for } t \geq 9; \end{cases} \qquad q_2(t) = \begin{cases} t-5 & \text{for } t \in [0,3], \\ -2 & \text{for } t \in [3,5], \\ 3-t & \text{for } t \geq 5. \end{cases}$$

For $\varepsilon > 0$ the viscous solution q^ε of the differential inclusion

$$0 \in \mathrm{Sign}(\dot{q}) + \varepsilon\dot{q} + U'(q) - \ell(t), \quad q(0) = -5,$$

is unique and can be calculated by matching solutions of linear ODEs. We find

$$q^\varepsilon(t) = \begin{cases} t-5+\varepsilon(\mathrm{e}^{-t/\varepsilon}-1) & \text{for } t \in [0,3], \\ q^\varepsilon_* & \text{for } t \in [3, t^\varepsilon_*], \\ 3-t+\varepsilon(\mathrm{e}^{-(t-t^\varepsilon_*)/\varepsilon}-1) & \text{for } t \geq t^\varepsilon_*, \end{cases}$$

where $q^\varepsilon_* = q^\varepsilon(3-) \lesssim -2$ and $t^\varepsilon_* = 3 - q^\varepsilon_* \gtrsim 5$. Thus, we have $q^\varepsilon(t) \to q_2(t)$ for every $t \geq 0$ as $\varepsilon \downarrow 0$, and q_2 turns out to be approximable, whereas q_1 is not. As a general principle, one may conjecture that viscosity slows down solutions, and thus approximable solutions tend to avoid jumps if there is a choice.

For Φ-minimal solutions this seems to be opposite. We claim that q_1 is (up to a reparametrization) Φ-minimal but q_2 is not. For this, we use the arclength parametrizations

$$(\widehat{t}_1, \widehat{q}_1)(s) = \begin{cases} \left(\frac{s}{2}, \frac{s}{2}-5\right) & \text{for } s \in [0,6], \\ (3, s-8) & \text{for } s \in [6,14], \\ (s-11, 6) & \text{for } s \in [14,16]; \end{cases} \qquad (\widehat{t}_2, \widehat{q}_2)(s) = \begin{cases} \left(\frac{s}{2}, \frac{s}{2}-5\right) & \text{for } s \in [0,6], \\ (s-3, -2) & \text{for } s \in [6,8], \\ \left(\frac{s}{2}+1, 2-\frac{s}{2}\right) & \text{for } s \geq 8. \end{cases}$$

The functionals $\varphi_j(s) = \Phi[(\widehat{t}_j, \widehat{q}_j)](s)$ for all $s \geq 0$, $j = 1, 2$, can be calculated explicitly: indeed, one checks that

$$\varphi_1(s) = \begin{cases} \frac{1}{2} & \text{for } s \in [0,6], \\ \frac{1}{2} - \frac{1}{2}(s-6)^2 & \text{for } s \in [6,10], \end{cases} \qquad \text{while} \qquad \varphi_2(s) = \frac{1}{2} \text{ for } s \geq 0\,, \tag{7.4}$$

which clearly shows that $(\widehat{t}_2, \widehat{q}_2)$ is not Φ-minimal for $s \in [0,7]$.

To prove Φ-minimality of $(\widehat{t}_1, \widehat{q}_1)$ we point out that chain rule inequality (3.7) gives

$$\Phi(\tau(s), p(s)) \geq \frac{1}{2} + \mathrm{Var}(p, [0,s]) - \int_0^s |\partial_q \mathcal{E}|\,(\tau(\sigma), p(\sigma))\,|p'|(\sigma)\,\mathrm{d}\sigma \tag{7.5}$$

for all $s \in [0,T]$ and all $(\tau,p) \in \mathcal{A}_T(0,q_0)$. Equality holds in (7.5) if and only if (τ,p) is a parametrized metric solution $(\widehat{t},\widehat{q})$ on $[0,T]$. In that case, in view of (3.24a) one further has, for all $s \in [0,T]$,

$$\Phi(\widehat{t}(s),\widehat{q}(s)) = \frac{1}{2} + \mathrm{Var}(\widehat{q},[0,s]) - \int_0^s M(|\widehat{t}'|(\sigma),|\widehat{q}'|(\sigma),|\partial_q\mathcal{E}|\left(\widehat{t}(\sigma),\widehat{q}(\sigma)\right)|\widehat{q}'|(\sigma)\,\mathrm{d}\sigma\,.$$

Therefore, in order to check that $(\widehat{t}_1,\widehat{q}_1)$ is Φ-minimal, it is sufficient to prove that $(\widehat{t}_1,\widehat{q}_1) \preceq (\widehat{t},\widehat{q})$ for all parametrized metric solutions $(\widehat{t},\widehat{q})$, and this, for all the arclength parametrizations $(\widehat{t},\widehat{q})$ corresponding to the (not jumping) BV solution q_2, follows from the previous discussion on $(\widehat{t}_2,\widehat{q}_2)$. Now, the above energy balance states a general fact about parametrized metric solutions: $\Phi(\widehat{t},\widehat{q})$ is constant as long as no jumps occur, i.e. $|\partial_q\mathcal{E}|\left(\widehat{t},\widehat{q}\right) \le 1$ holds. If jumps with $|\partial_q\mathcal{E}|\left(\widehat{t},\widehat{q}\right) > 1$ occur, then Φ will strictly decrease. Thus, if there is a choice between one solution with a fast jump and another without jumps, then the solution without jumps cannot be Φ-minimal.

Example 7.3 *Here, we study the parameter dependence of solutions under the loading*

$$\ell_\delta(t) = \min\{t, 6{+}2\delta{-}t\} \quad \text{for } t \ge 0,$$

where δ is a small parameter. In the case $\delta = 0$ we have two BV solutions q_1 and q_2 (or similarly parametrized metric solutions), as was discussed in Example 7.2. For $-1 < \delta < 0$ there is only one solution, namely

$$q^\delta(t) = \begin{cases} t{-}5 & \text{for } t \in [0,3{+}\delta], \\ \delta{-}2 & \text{for } t \in [3{+}\delta,5{+}\delta], \\ 3{+}2\delta{-}t & \text{for } t \ge 5{+}\delta. \end{cases}$$

The corresponding parametrized solution is the unique Φ-minimal solution. Now, for $\delta \nearrow 0$ we find $q^\delta(t) \to q_2(t)$ for every $t \ge 0$. Hence, the set of Φ-minimal solutions is not closed (or "not stable" or "not upper semicontinuous") under pointwise convergence. Similarly, we may consider $\delta > 0$ to obtain a unique BV solution q^δ that jumps at time $t = 3$ before the unloading starts at $t = 3{+}\delta > 3$. Clearly, these solutions are approximable and converge pointwise to q_1, which is not approximable. Thus, the set of approximable solutions is not upper semicontinuous.

Example 7.4 *We provide an example where one BV solution corresponds to many different parametrized metric solutions. The BV solution has exactly one jump, and there are infinitely many distinct connecting orbits y in* (iv) *of Definition 4.3, giving rise to infinitely many distinct parametrized metric solutions. We consider*

$$\mathcal{Q} = \mathbb{R}^2, \quad \text{and} \quad d(q,\widetilde{q}) = \frac{1}{2}\left(|q_1 - \widetilde{q}_1| + |q_2 - \widetilde{q}_2|\right).$$

With $q = (q_1,q_2) \in \mathcal{Q} = \mathbb{R}^2$ the potential takes the form

$$\mathcal{E}(t,q) = U\left(\frac{q_1+q_2}{2}\right) + W(q_1-q_2) - t\left(\frac{q_1+q_2}{2}\right),$$

where U is defined in (7.2) and $W : \mathbb{R} \to [0,\infty)$ by $W(\rho) = 0$ for $|\rho| < 1$ and $W(\rho) = (|\rho|-1)^2$ else. Starting from $q(0) = (-5,-5)$, we have $q(t) = (\widetilde{q}(t), \widetilde{q}(t))$, $\widetilde{q}$ being the BV solution of Example 7.1. Hence, the (unique) jump occurs at $t = 3$, starting in $(-2,-2)$ and ending in $(6,6)$. However, the set of connecting paths y is infinite. Indeed, for every connecting path there holds for a.a. $s \in (s_0, s_1)$

$$|y'|(s) = \frac{1}{2}(|y_1'(s)| + |y_2'(s)|)\,, \quad |\partial_q \mathcal{E}(t,\cdot)|(q(s)) = \frac{1}{2}\left| U'\left(\frac{y_1(s)+y_2(s)}{2}\right) - t\right|$$

if $|y_1(s) - y_2(s)| \le 1$. Now, for a given curve $\gamma : [0,1] \to [0,\infty)$ let us set $y_\gamma := (\widetilde{q}-\gamma, \widetilde{q}+\gamma)$. Indeed, $|y_\gamma'| = 1/2(|\widetilde{q}' - \gamma'| + |\widetilde{q}' + \gamma'|) = |\widetilde{q}'|$ whenever $|\gamma'| \le |\widetilde{q}'|$. Therefore,

$$\int_0^1 |\partial_q \mathcal{E}(t,\cdot)|(y_\gamma(s))|y_\gamma'|(s)\,\mathrm{d}s = \int_0^1 |\partial_q \mathcal{E}(t,\cdot)|(y(s))|y'|(s)\,\mathrm{d}s$$

for all curves γ with $\gamma(0) = \gamma(1) = 0$ and $|\gamma'|(s) \le |\widetilde{q}'|(s)$ for a.a. $s \in (s_0, s_1)$, and for such γ's y_γ is an optimal connecting curve.

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