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Stability results for a soil model with singular hysteretic hydrology

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Abstract

We consider a differential equation describing the mass balance in a soil hydrology model with noninvertible Preisach-type hysteresis. We approximate the singular Preisach operator by regular ones and show, as main result, that the solutions of the regularized problem converge to a solution of the original one as the regularization parameter tends to zero. For monotone right hand sides, we prove that the solution is unique. If in addition the external water sources are time periodic, then we find sufficient conditions for the existence, uniqueness, and asymptotic stability of periodic solutions.

Introduction

Large highly structured systems often admit a large number of locally stable equilibrium configurations. This is in particular the case of solid mechanics, where dislocations at the crystal level lead to a multitude of different equilibria for the same stress distribution. The instantaneous state at some time $t_0 > 0$ thus may depend on the previous history $t \leq t_0$ of the process. Similar phenomena occur in the complex dynamics of electro-magneto-mechanical processes in ferromagnetic, piezoelectric, and magnetostrictive materials.

In engineering applications, the goal of modelling is to predict the behaviour of a system at the *macroscopic* level, where the knowledge of the exact complex time and space distribution of the microstates is of minor importance with respect to the necessity of having a reliable and robust numerical method for a global simulation.

The mathematical theory of hysteresis operators, introduced by M. A. Krasnosel'skii and his collaborators in the 1970', see [4], seems to provide an efficient tool for such a macroscopic description of internal microstructure evolution in the situation, where the structure changes are much faster than the observer's time scale. Then the process can be considered as rate-independent which is, besides causality, the main feature of hysteresis. Among more recent publications devoted to different aspects of modelling and analysis of systems with hysteresis we may cite e. g. [1, 6, 13, 14, 15, 17, 18].

We focus here on a model of soil hydrology proposed in [2, 10, 9]. It is based on the same idea of describing the complicated mass exchange dynamics in the soil, where the microstructure is due to solid grains, pores, plant roots, animal activity etc., by an input-output hysteresis system. The spatial dependence is neglected, and the dynamics is driven by the mass balance between the soil water potential and the volumetric moisture content. The experiments described in [2] show that the hysteresis relation between the potential u and the moisture content w exhibits the so-called *return point memory* (or *wiping-out property*), that is, every minor loop returns back to its starting point, see Fig. 1. Furthermore, it is expected from experiments that periodic processes on the same potential level may take place with different amounts of the water content. Assuming that the mass balance is given by Eq. (0.1) below, this observation leads to the conclusion that all possible responses w corresponding to the same process $u(t)$ differ only by an additive constant independent of t. In the (u, w) phase plane, this means that all single closed loops with endpoints $(u_1, w_1), (u_2, w_2)$ and $(u_1, w_1'), (u_2, w_2')$ have the same shape, see Fig. 1. This property is called the *congruency* and a classical result by Mayergoyz in [14] states that every hysteresis relation with return point memory and congruent loops can be represented by the Preisach model.

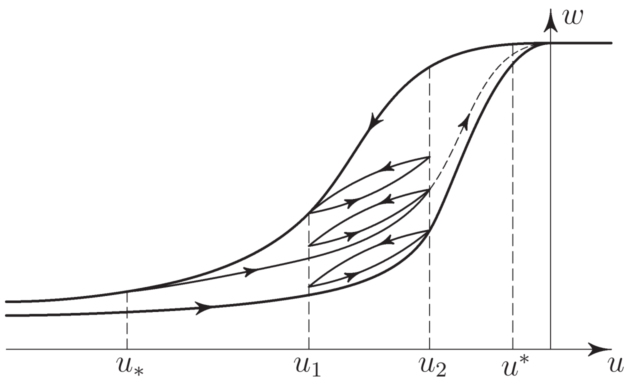

Figure 1: Return point memory and congruency.

A detailed discussion about the technical aspects of the model as well as the parameter identification is done in [2, 10]. The resulting equation governing the process is of the form

$$\left.\begin{array}{rcl} \dot{w}(t) & = & f(t, u(t)) \\ w(t) & = & \mathcal{F}[\lambda, u](t) \\ u(0) & = & u^0 \end{array}\right\} \tag{0.1}$$

for $t \geq 0$, where $u^0 \in \mathbb{R}$ is a given initial condition, $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a given function, and $\mathcal{F}[\lambda, \cdot]$ is a Preisach hysteresis operator with initial memory configuration λ, see Section 1 below.

Equation (0.1) is not autonomous, as the water-soil system is not physically closed. Water is coming in and out. The t-dependence in the right-hand side describes the external water exchange (rain, evaporation, drainage, etc.).

A derivation of Eq. (0.1) and constructive methods for its solutions can be found in [11]. Here, we pursue these investigations and develop a general existence and stability theory for this problem. In Section 3, we show that solutions u to (0.1) can be locally uniformly approximated as $\varepsilon \to 0+$ by solutions u_ε to the regularized problem

$$\left.\begin{array}{rcl} \dot{w}_\varepsilon(t) & = & f_\varepsilon(t, u_\varepsilon(t)) \\ w_\varepsilon(t) & = & \varepsilon u_\varepsilon(t) + \mathcal{F}_\varepsilon[\lambda^\varepsilon, u_\varepsilon](t) \\ u_\varepsilon(0) & = & u_\varepsilon^0\,, \end{array}\right\} \tag{0.2}$$

where $f_\varepsilon, \mathcal{F}_\varepsilon, \lambda^\varepsilon, u_\varepsilon^0$ are suitable approximations of $f, \mathcal{F}, \lambda, u^0$, respectively.

This is indeed a regularization. The mappings $\mathcal{F}_\varepsilon^* : u \mapsto \varepsilon u + \mathcal{F}_\varepsilon[\lambda^\varepsilon, u]$ are Lipschitz continuous and admit Lipschitz continuous inverses for all $\varepsilon > 0$ in the Banach space $C[0,T]$ of all continuous functions on $[0,T]$ for every $T > 0$; hence, the existence and uniqueness of solutions to (0.2) follows from the contraction principle for every $\varepsilon > 0$. The convergence result is based on the observation that small amplitude Preisach hysteresis loops are convex. We prove here in Proposition 2.3 a refined form of an integral inequality (called *convexity inequality*, or *second order energy inequality*), which has been originally established in [5] in connection with hysteresis wave propagation and then used in [8] for solving a singular oscillation problem.

The solutions to (0.1) may in general be nonunique (see Example 3.3 (ii)). On the other hand, we show that the solution u to (0.1) is unique as long as $u(t)$ does not leave an interval $[u_*, u^*]$, where f is nonincreasing. If the interval $[u_*, u^*]$ contains 0, then the solution remains in this interval for all times. If moreover f is periodic in t and decreasing in u, then every solution trajectory converges to a unique periodic solution of the problem. The argument is based in a substantial way on the Hilpert inequality established in [3].

In the hydrological context, a nonincreasing function f is relevant to a common situation where water flows into the system are driven by negative feedback loops. Examples include infiltration of rain, drainage of water under a slab of soil, transpiration from plants leaves for vegetated soil and combinations of such flows [9, 10]. The results below guarantee the global stability of the corresponding models (0.1). However, a positive feedback loop can destabilize real soil-water systems. For instance, a destabilizing effect of the runoff flow from the soil surface, leading to a bifurcation, which has strong implications for the environment, has been shown in [12]. Equations (0.1) resulting from modelling hydrological systems with both negative and positive feedbacks can have non-monotone functions f.

The following text is divided into four sections. Section 1 is a survey of mathematical properties of the Preisach operator. For our purposes, it is convenient to use the alternative description of the Preisach model as a nonlinear one-parametric combination of continuous play operators, which goes back to [5], instead of the traditional one proposed in [16] and further developed in [4, 14], which defines the model as a linear superposition of two-parametric discontinuous relay operators. The equivalence of the two definitions was proved in [5, Section 1], see also [1, Section 2.4] or [6, Section II.3]. The variational approach based on the system of play operators makes in our situation the analysis simpler and more transparent. A detailed proof of the convexity inequality for the Preisach operator is carried out in Section 2. Solutions to Problem (0.2) for $\varepsilon > 0$ are investigated in Section 3. On a fixed time interval $[0,T]$, we derive a uniform bound for u_ε in $W^{1,2q}(0,T)$ for some $q > 1$ independent of ε, which enables us to prove the convergence along a subsequence to a solution of (0.1) as $\varepsilon \to 0+$. Global existence, uniqueness, periodicity, and asymptotic stability of solutions to (0.1) are discussed in Section 4.

1 Preisach operator

Let us denote by $\mathbb{R}_+$ the interval $[0,\infty)$. We work in the space $C(\mathbb{R}_+)$ of continuous functions $u : \mathbb{R}_+ \to \mathbb{R}$ endowed with a system of seminorms

$$\|u\|_{[0,t]} \;=\; \max\{|u(\tau)|\,;\,\tau \in [0,t]\} \qquad \text{for} \;\; t \ge 0\,. \tag{1.1}$$

The metric

$$d(u,v) \;=\; \sup_{t\ge 0}\left(\frac{\|u-v\|_{[0,t]}}{1+\|u-v\|_{[0,t]}}\right) \tag{1.2}$$

transforms $C(\mathbb{R}_+)$ into a Fréchet space. We similarly denote by $AC(\mathbb{R}_+)$ the set of absolutely continuous functions $u : \mathbb{R}_+ \to \mathbb{R}$ endowed with a system of seminorms

$$\|u\|_{AC[0,t]} \;=\; |u(0)| + \int_0^t |\dot u(\tau)|\,\mathrm{d}\tau \qquad \text{for} \;\; t \ge 0 \tag{1.3}$$

and a metric analogous to (1.2).

We first introduce the Preisach state space Λ as the set of all functions $\lambda : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|\lambda(r_1) - \lambda(r_2)| \;\le\; |r_1 - r_2| \qquad \forall r_1, r_2 \in \mathbb{R}_+\,, \tag{1.4}$$

$$\exists R > 0\,:\quad \lambda(r) = 0 \quad \forall r \ge R\,. \tag{1.5}$$

The set of all $\lambda \in \Lambda$ satisfying (1.5) will be denoted Λ_R in the sequel. Note that this set is compact with respect to the sup-norm.

For each given $r > 0$ we define the *play operator* $\mathfrak{p}_r : \Lambda \times AC(\mathbb{R}_+) \to AC(\mathbb{R}_+)$: $(\lambda, u) \mapsto \xi_r$, which with each $\lambda \in \Lambda$ and $u \in AC(\mathbb{R}_+)$ associates the unique solution ξ_r of the variational inequality

$$|u(t) - \xi_r(t)| \;\le\; r \qquad \forall t \in \mathbb{R}_+\,, \tag{1.6}$$

$$\dot\xi_r(t)\,\big(u(t) - \xi_r(t) - z\big) \;\ge\; 0 \quad \text{a.e.} \qquad \forall z \in [-r, r]\,, \tag{1.7}$$

$$\xi_r(0) \;=\; \min\{u(0) + r, \max\{u(0) - r, \lambda(r)\}\}\,, \tag{1.8}$$

see Figs. 2, 3.

More about the relationship between the variational definition of the play and a constructive approach in [1, 4] can be found e.g. in [6, 17].

As an immediate consequence of the definition, we first note that for all $h > 0$ sufficiently small we have $\dot\xi_r(t)\,\big(u(t) - u(t\pm h) - \xi_r(t) + \xi_r(t\pm h)\big) \ge 0$ a.e., hence

$$\dot\xi_r(t)\,\dot u(t) \;=\; \dot\xi_r^2(t) \qquad \text{a.e.} \tag{1.9}$$

In other words, we have the implication

$$\dot\xi_r(t) \neq 0 \qquad \Rightarrow \qquad \dot\xi_r(t) = \dot u(t) = \frac{1}{r}\,|\dot\xi_r(t)|\,(u(t) - \xi(t))\,. \tag{1.10}$$

We now recall an important inequality established originally by Hilpert in [3]. For the reader's convenience, we give an elementary proof.

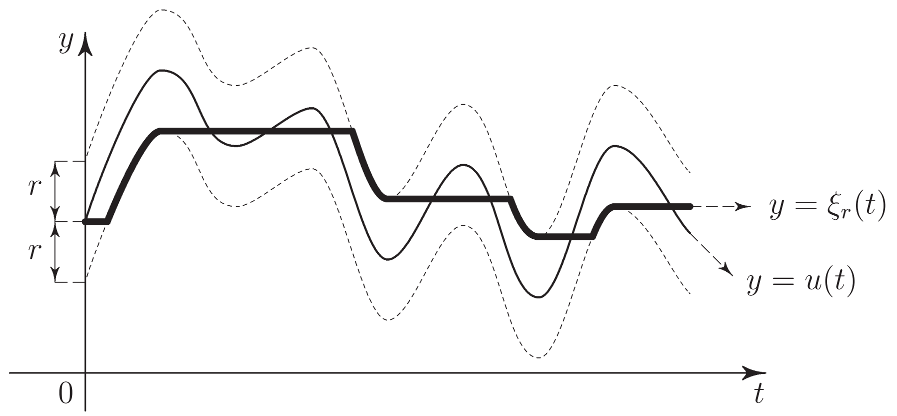

Figure 2: Time evolution of the play operator.

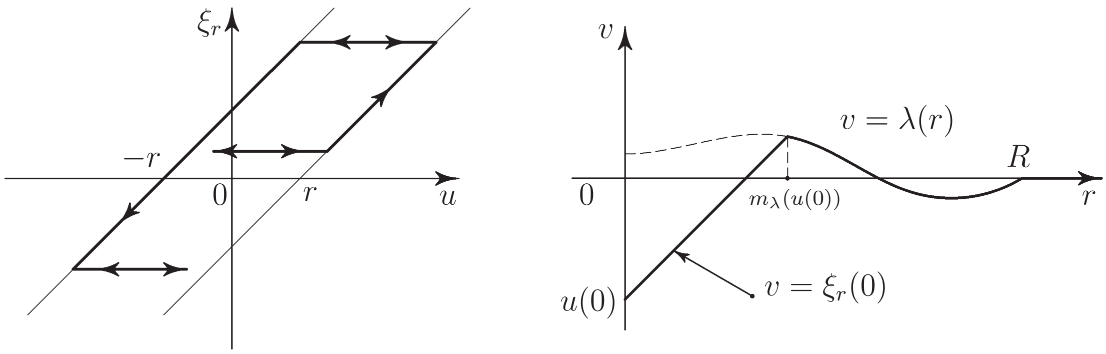

Figure 3: A phase diagram of the play and the initial memory curve $v = \xi_r(0)$.

Lemma 1.1 *For $u_1, u_2 \in AC(\mathbb{R}_+)$ and $\lambda_1, \lambda_2 \in \Lambda$ put $\xi_r^i(t) = \mathfrak{p}_r[\lambda_i, u_i](t)$, $i = 1, 2$. Then for every locally Lipschitz continuous non-decreasing function $g : \mathbb{R} \to \mathbb{R}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(\xi_r^1) - g(\xi_r^2)\right)^+(t) \ \leq \ H(u_1(t) - u_2(t))\,\frac{\mathrm{d}}{\mathrm{d}t}\left(g(\xi_r^1) - g(\xi_r^2)\right)(t) \quad a.\,e., \tag{1.11}$$

where $x^+ = \max\{x, 0\}$ for $x \in \mathbb{R}$, and H is the left continuous Heaviside function

$$H(x) = \begin{cases} 0 & \text{for } x \leq 0\,, \\ 1 & \text{for } x > 0\,. \end{cases}$$

Proof. From (1.6) – (1.7) it follows that (we omit the argument t for simplicity)

$$\begin{aligned} g'(\xi_r^1)\,\dot{\xi}_r^1\left((u_1 - \xi_r^1) - (u_2 - \xi_r^2)\right) \ &\geq \ 0 \quad \text{a. e.}\ , \\ -g'(\xi_r^2)\,\dot{\xi}_r^2\left((u_1 - \xi_r^1) - (u_2 - \xi_r^2)\right) \ &\geq \ 0 \quad \text{a. e.}\ , \end{aligned}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(\xi_r^1) - g(\xi_r^2)\right)\left((u_1 - \xi_r^1) - (u_2 - \xi_r^2)\right) \ \geq \ 0 \quad \text{a. e.} \tag{1.12}$$

Using the implication $c(a-b)\ge 0 \ \Rightarrow\ c(H(a)-H(b))\ge 0$ for $a,b,c\in\mathbb{R}$, we obtain from (1.12) that

$$H(\xi_r^1-\xi_r^2)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(g(\xi_r^1)-g(\xi_r^2)\right)\ \le\ H(u_1-u_2)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(g(\xi_r^1)-g(\xi_r^2)\right)\qquad \text{a. e.} \tag{1.13}$$

For a. e. $t>0$ we have

$$g(\xi_r^1(t))=g(\xi_r^2(t))\quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(g(\xi_r^1)-g(\xi_r^2)\right)^+(t)=\frac{\mathrm{d}}{\mathrm{d}t}(g(\xi_r^1)-g(\xi_r^2))(t)=0\,,$$

$$g(\xi_r^1(t))\ne g(\xi_r^2(t))\quad\Rightarrow\quad H(\xi_r^1(t)-\xi_r^2(t))=H(g(\xi_r^1(t))-g(\xi_r^2(t)))\,,$$

and (1.11) follows from the identity $\frac{\mathrm{d}}{\mathrm{d}t}x^+(t)=H(x(t))\dot{x}(t)$ a. e. for each $x\in AC(\mathbb{R}_+)$.
■

For completeness, we summarize the classical continuity properties of the play $\mathfrak{p}_r$. For a proof, see [1, Sect. 2.3] or [6, Proposition II.1.1].

Lemma 1.2 *Under the hypotheses of Lemma 1.1 we have*

$$\left|\dot{\xi}_r^1-\dot{\xi}_r^2\right|(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left|(u_1-u_2)-(\xi_r^1-\xi_r^2)\right|(t)\ \le\ \left|\dot{u}_1-\dot{u}_2\right|(t)\qquad \textit{a. e.}\ , \tag{1.14}$$

$$\left|\xi_r^1-\xi_r^2\right|(t)\ \le\ \max\{|\lambda_1(r)-\lambda_2(r)|\,,\,\|u_1-u_2\|_{[0,t]}\}\quad \forall t\in\mathbb{R}_+\,. \tag{1.15}$$

Lemma 1.2 states that $\mathfrak{p}_r:\Lambda\times AC(\mathbb{R}_+)\to AC(\mathbb{R}_+)$ is continuous and admits a continuous extension to $\Lambda\times C(\mathbb{R}_+)\to C(\mathbb{R}_+)$, and its restriction to any bounded interval $[0,T]$ is Lipschitz continuous in both cases.

We now continue with some finer memory properties of the play. Their proofs can be found e. g. in [1, Sect. 2.3] or [6, Sect. II.2].

Lemma 1.3 *For given $R>0$, $u\in C(\mathbb{R}_+)$, $\lambda\in\Lambda_R$, and $t\ge 0$ put*

$$\lambda_t(r)=\mathfrak{p}_r[\lambda,u](t)\quad \textit{for}\ \ r>0\,,\quad \lambda_t(0)=u(t)\,. \tag{1.16}$$

Then

(i) $\lambda_t\in\Lambda_{R(t)}$, *where* $R(t)=\max\{R,\|u\|_{[0,t]}\}$*;*

(ii) *the semigroup property*

$$\mathfrak{p}_r[\lambda,u](s+t)\ =\ \mathfrak{p}_r[\lambda_s,u(s+\cdot)](t) \tag{1.17}$$

holds for all $s,t\in\mathbb{R}_+$.

The play operator thus generates for every $t\ge 0$ a continuous *state mapping* $\Pi_t:\Lambda\times C(\mathbb{R}_+)\to\Lambda$ which with each $(\lambda,u)\in\Lambda\times C(\mathbb{R}_+)$ associates the state $\lambda_t\in\Lambda$ at time t. In other words, $\mathfrak{p}_r[\lambda,u](t)$ can be considered in a dual way: either as a function ξ_r of the time variable t for fixed r, or as a function λ_t of the memory variable r at fixed time t.

Let us recall another property of the play as a special case of [7, Lemma 3.1.2].

Lemma 1.4 *Let $u \in C(\mathbb{R}_+)$ and $t \geq 0$ be given. Set*

$$u_{\max}(t) = \sup_{\tau\in[0,t]} u(\tau)\,, \qquad u_{\min}(t) = \inf_{\tau\in[0,t]} u(\tau)\,. \tag{1.18}$$

Then for all $\lambda \in \Lambda$ and $r > 0$ we have

$$\begin{aligned}
\mathfrak{p}_r[\lambda, u](\tau) &\leq \max\{\lambda(r),\, u_{\max}(t) - r\} \quad \forall\, \tau \in [0,t]\,, & (1.19)\\
\mathfrak{p}_r[\lambda, u](\tau) &\geq \min\{\lambda(r),\, u_{\min}(t) + r\} \quad \forall\, \tau \in [0,t]\,, & (1.20)\\
\mathfrak{p}_r[\lambda, u](t) &= \lambda(r) \qquad \textit{for} \;\; r > \|m_\lambda \circ u\|_{[0,t]}\,, & (1.21)
\end{aligned}$$

where for $v \in \mathbb{R}$ we put

$$m_\lambda(v) = \inf\{r \geq 0\,;\, |\lambda(r) - v| = r\}\,. \tag{1.22}$$

The meaning of m_λ can be illustrated on Figures 3, 4. For a function u which is monotone in an interval $[t_0, t_1]$ and $u(t_0) = \lambda(0)$, the interval $[0, (m_\lambda \circ u)(t)]$ determines the active moving part of the memory curve $r \mapsto \lambda_t(r)$. More specifically, in the identity (1.9), we have in this case

$$\dot u(t) \neq 0 \quad \Longrightarrow \quad \dot\xi_r(t) = \begin{cases} \dot u(t) & \text{for } \; r < m_\lambda(u(t))\,, \\ 0 & \text{for } \; r > m_\lambda(u(t))\,. \end{cases} \tag{1.23}$$

Let us derive some consequences of Lemma 1.4. Assume that $m_\lambda(u(\cdot))$ attains at a point $\bar t \geq 0$ its maximum over $[0, \bar t]$, that is,

$$\bar r \;:=\; m_\lambda(u(\bar t)) \;=\; \|m_\lambda(u(\cdot))\|_{[0,\bar t]}\,. \tag{1.24}$$

The case $\bar r = 0$ is trivial, as it implies $u(t) = \lambda(0)$ for all $t \in [0, \bar t]$. For $\bar r > 0$ we distinguish the cases

(i) $u(\bar t) = \lambda(\bar r) + \bar r$,

(ii) $u(\bar t) = \lambda(\bar r) - \bar r$.

If (i) holds and $u(t) > u(\bar t)$ for some $t \in [0, \bar t]$, then $\lambda(\bar r) + \bar r < u(t)$, hence $m_\lambda(u(t)) > \bar r$ in contradiction with (1.24). We thus have $u(\bar t) = u_{\max}(\bar t)$. From (1.6) it follows that $\mathfrak{p}_r[\lambda, u](\bar t) \geq u(\bar t) - r$. On the other hand, (1.22) yields $u(\bar t) - r < \lambda(r) \Rightarrow m_\lambda(u(\bar t)) < r$; hence, by Lemma 1.4 we have

$$\mathfrak{p}_r[\lambda, u](\bar t) \;=\; \max\{\lambda(r), u(\bar t) - r\}\,. \tag{1.25}$$

Similarly, in the case (ii) we have $u(\bar t) = u_{\min}(\bar t)$ and

$$\mathfrak{p}_r[\lambda, u](\bar t) \;=\; \min\{\lambda(r), u(\bar t) + r\}\,. \tag{1.26}$$

The above considerations imply the following well-known result on periodic inputs, cf. [4, Chap. 1, Sect. 2.8].

Corollary 1.5 *Let $u \in C(\mathbb{R}_+)$ be T-periodic, that is, $u(t+T) = u(t)$ for all $t \geq 0$, with a fixed period $T > 0$. Then $\mathfrak{p}_r[\lambda, u]$ is T-periodic for $t \geq T$ for all $\lambda \in \Lambda$.*

For a function $u \in C(\mathbb{R}_+)$ which is monotone (non-decreasing or non-increasing) in an interval $[t_0, t_1]$, we easily deduce from the semigroup property (1.17) and Lemma 1.4 the representation formula

$$\mathfrak{p}_r[\lambda, u](t) \;=\; \max\{u(t) - r, \min\{u(t) + r, \lambda_{t_0}(r)\}\} \tag{1.27}$$

for $t \in [t_0, t_1]$, see Figures 2, 3. It is perhaps interesting to note that (1.27) has originally been used in [4] as alternative definition of the play on continuous piecewise monotone inputs, extended afterwards by density and continuity to the whole space of continuous functions.

The evolution of the graph of λ_t in dependence on t is depicted on Fig. 4. We now pass to the alternative definition of the Preisach operator as suggested in [5], see also [1, 6].

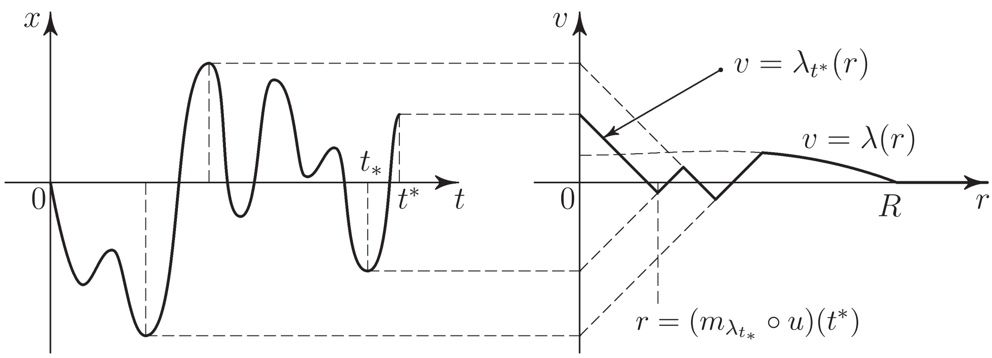

Figure 4: The memory curve $v = \lambda_t(r)$ at $t = t^*$.

Definition 1.6 *Let $R > 0$ be given. Let $\mu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be a locally bounded measurable function. For $(r, v) \in \mathbb{R}_+ \times \mathbb{R}$ put*

$$g(r, v) \;=\; \int_0^v \mu(r, z)\,\mathrm{d}z\,. \tag{1.28}$$

Then the mapping $\mathcal{F} : \Lambda_R \times C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined as

$$\mathcal{F}[\lambda, u](t) \;=\; \int_0^\infty g(r, \mathfrak{p}_r[\lambda, u](t))\,\mathrm{d}r \qquad \textit{for}\;\; (\lambda, u) \in \Lambda_R \times C(\mathbb{R}_+)\,,\; t \geq 0 \tag{1.29}$$

is called a Preisach operator.

The integral in (1.29) is finite due to the fact that $g(r, 0) = 0$ for all $r > 0$ and $\mathfrak{p}_r[\lambda, u](t) = 0$ for r sufficiently large by virtue of Lemma 1.3 (i). We now list some basic properties of the Preisach operator. For the proofs see [1, Section 2.11] or [6, Section II.3].

Hypothesis 1.7 *Let μ be as in Definition 1.6. We assume in addition that*

(i) *For every $B > 0$ there exists a constant $\gamma_B > 0$ such that*

$$0 \le \mu(r,v) \le \gamma_B \quad a.\ e. \quad in \ \{(r,v) \in \mathbb{R}_+ \times \mathbb{R}\,,\ r + |v| \le B\}\,. \tag{1.30}$$

(ii) *The function $\mu(r,\cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for almost all $r > 0$, and there exist constants $\alpha, \beta > 0$ and a continuous function $\varrho : \mathbb{R} \to (0,\infty)$ such that*

$$\left.\begin{array}{rcl} \mu(r,v) & \ge & \alpha \\ \left|\dfrac{\partial\mu}{\partial v}(r,v)\right| & \le & \beta \end{array}\right\} \quad a.\ e.\ in\ \ \mathcal{C} := \{(r,v) \in \mathbb{R}_+ \times \mathbb{R}\,, 0 < r < \varrho(v)\}\,. \tag{1.31}$$

Taking a smaller $\varrho(v) > 0$, if necessary, we may assume that

$$\alpha \ \ge\ 4\beta\,\varrho(v) \quad \forall v \in \mathbb{R}\,. \tag{1.32}$$

We will see that $\mathcal{C}$ is the *convexity domain* of the Preisach operator in the sense that *closed hysteresis loops are convex and counterclockwise oriented as long as the active memory stays in $\mathcal{C}$.* The generalized convexity inequality below in Proposition 2.3 makes use of this fact. Its energetic interpretation is discussed in detail in [6, Section II.4]. We start with Lipschitz continuity properties of the Preisach and the inverse Preisach operator.

Proposition 1.8 *Let Hypothesis 1.7(i) hold.*

(i) *Then for every $T > 0$, $B \ge R$, $\lambda_1, \lambda_2 \in \Lambda_R$ and $u_1, u_2 \in C(\mathbb{R}_+)$ such that $\|u_i\|_{[0,T]} \le B$ for $i = 1,2$ and every $t \in [0,T]$ we have*

$$|\mathcal{F}[\lambda_1,u_1] - \mathcal{F}[\lambda_2,u_2]|(t) \ \le\ \gamma_B \left(\int_0^R |\lambda_1 - \lambda_2|(r)\ \mathrm{d}r + B\ \|u_1 - u_2\|_{[0,t]}\right)\,. \tag{1.33}$$

(ii) *For every $\alpha > 0$, $\lambda \in \Lambda$, the operator $\alpha\,I + \mathcal{F}[\lambda,\cdot] : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$, where I is the identity mapping, is invertible and its inverse $\mathcal{G} = (\alpha\,I + \mathcal{F}[\lambda,\cdot])^{-1} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ satisfies for every $u_1, u_2 \in C(\mathbb{R}_+)$ and every $T > 0$ the inequality*

$$\|\mathcal{G}[u_1] - \mathcal{G}[u_2]\|_{[0,T]} \ \le\ \frac{2}{\alpha}\,\|u_1 - u_2\|_{[0,T]}\,. \tag{1.34}$$

As an easy consequence of (1.11), we obtain the Hilpert inequality for the Preisach operator (1.29) in the following form.

Proposition 1.9 *Let μ and g be as in Definition 1.6, and let $\lambda_1, \lambda_2 \in \Lambda_R$ and $u_1, u_2 \in AC(\mathbb{R}_+)$ be given. Then for a. e. $t > 0$ we have*

$$\begin{aligned} & H(u_1(t) - u_2(t)) \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathcal{F}[\lambda_1, u_1] - \mathcal{F}[\lambda_2, u_2]\right)(t) \\ & \qquad \geq \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \left(g(r, \mathfrak{p}_r[\lambda_1, u_1]) - g(r, \mathfrak{p}_r[\lambda_2, u_2])\right)^+(t)\, \mathrm{d}r\,. \end{aligned} \tag{1.35}$$

Interchanging the roles of u_1 and u_2 we obtain as a consequence of Proposition 1.9 that

$$\begin{aligned} & \operatorname{sign}(u_1(t) - u_2(t)) \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathcal{F}[\lambda_1, u_1] - \mathcal{F}[\lambda_2, u_2]\right)(t) \\ & \qquad \geq \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \left|g(r, \mathfrak{p}_r[\lambda_1, u_1]) - g(r, \mathfrak{p}_r[\lambda_2, u_2])\right|(t)\, \mathrm{d}r\,. \end{aligned} \tag{1.36}$$

In the special cases $u_1 = u,\ \lambda_1 = \lambda,\ u_2 \equiv 0,\ \lambda_2 \equiv 0$, or $u_1 \equiv 0,\ \lambda_1 \equiv 0,\ u_2 = u$, $\lambda_2 = \lambda$, we obtain for a. e. t, as a special case of inequality (1.35), that

$$H(u(t)) \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\lambda, u](t) \ \geq\ \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \left(g(r, \mathfrak{p}_r[\lambda, u](t))\right)^+ \mathrm{d}r\,, \tag{1.37}$$

$$-H(-u(t)) \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\lambda, u](t) \ \geq\ \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \left(g(r, \mathfrak{p}_r[\lambda, u](t))\right)^- \mathrm{d}r\,, \tag{1.38}$$

where we denote $x^- = \max\{-x, 0\}$ for $x \in \mathbb{R}$.

2 Generalized convexity inequality

This section is devoted to the convexity inequality as mentioned in the Introduction. We first investigate in detail the behavior of $\mathcal{F}[\lambda, u]$ on monotone inputs u.

Lemma 2.1 *Let $R > 0$, $\lambda \in \Lambda_R$, $u \in C(\mathbb{R}_+)$ and $t_0 \geq 0$ be given. Let Hypothesis 1.7 hold, and let $\mathcal{F}$ be the Preisach operator (1.29). Set*

$$\lambda_{t_0}(r) \ =\ \mathfrak{p}_r[\lambda, u](t_0) \quad \textit{for}\ \ r > 0\,, \tag{2.1}$$

$$\Phi_+(v) \ =\ \int_{\lambda_{t_0}(0)}^{v} \int_0^{m_{\lambda_{t_0}}(z)} \mu(r, z - r)\, \mathrm{d}r\, \mathrm{d}z \qquad \textit{for}\ \ v \geq \lambda_{t_0}(0)\,, \tag{2.2}$$

$$\Phi_-(v) \ =\ -\int_v^{\lambda_{t_0}(0)} \int_0^{m_{\lambda_{t_0}}(z)} \mu(r, z + r)\, \mathrm{d}r\, \mathrm{d}z \qquad \textit{for}\ \ v \leq \lambda_{t_0}(0)\,, \tag{2.3}$$

where $m_{\lambda_{t_0}}$ is as in (1.22). Let u be monotone (nondecreasing or nonincreasing) in an interval $[t_0, t_1]$. Then for all $t \in [t_0, t_1]$ we have

$$\mathcal{F}[\lambda, u](t) - \mathcal{F}[\lambda, u](t_0) = \begin{cases} \Phi_+(u(t)) & \textit{if}\ \ u(t_1) \geq u(t_0)\,, \\ \Phi_-(u(t)) & \textit{if}\ \ u(t_1) \leq u(t_0)\,. \end{cases} \tag{2.4}$$

Eq. (2.4) is an explicit formula for the *Preisach loading curves* Φ_+, Φ_- associated with monotone loadings. In particular, the *initial loading curves* are given by the equations

$$\hat{\Phi}_+(v) \;=\; \int_{\lambda(0)}^{v} \int_0^{m_\lambda(z)} \mu(r, z-r)\;\mathrm{d}r\;\mathrm{d}z \qquad \text{for} \;\; v \geq \lambda(0)\,, \tag{2.5}$$

$$\hat{\Phi}_-(v) \;=\; -\int_{v}^{\lambda(0)} \int_0^{m_\lambda(z)} \mu(r, z+r)\;\mathrm{d}r\;\mathrm{d}z \qquad \text{for} \;\; v \leq \lambda(0)\,. \tag{2.6}$$

Condition (1.32) implies that the function ϱ is bounded and we denote by $\hat{\varrho}$ its smallest upper bound. We further introduce the notation

$$\varrho_+ = \int_{\lambda(0)+\hat{\varrho}}^{\infty} \varrho(v)\;\mathrm{d}v \;\in\; (0,+\infty]\,, \qquad \varrho_- = \int_{-\infty}^{\lambda(0)-\hat{\varrho}} \varrho(v)\;\mathrm{d}v \;\in\; (0,+\infty]\,. \tag{2.7}$$

We easily check that under Hypothesis 1.7, we have

$$\lim_{v\to\infty} \hat{\Phi}_+(v) \geq \alpha\varrho_+\,, \qquad \lim_{v\to-\infty} \hat{\Phi}_-(v) \leq -\alpha\varrho_-\,. \tag{2.8}$$

Indeed, an elementary application of Fubini's Theorem yields

$$\begin{aligned}
\hat{\Phi}_+(v) \;&=\; \int_{\lambda(0)}^{v} \int_0^{m_\lambda(z)} \mu(r, z-r)\;\mathrm{d}r\;\mathrm{d}z = \int_0^{m_\lambda(v)} \int_{\lambda(r)}^{v-r} \mu(r,z)\;\mathrm{d}z\;\mathrm{d}r \\
&\geq\; \int_0^{(v-\lambda(0))/2} \int_{\lambda(0)+r}^{v-r} \mu(r,z)\;\mathrm{d}z\;\mathrm{d}r = \int_{\lambda(0)}^{v} \int_0^{\min\{v-z, z-\lambda(0)\}} \mu(r,z)\;\mathrm{d}r\;\mathrm{d}z \\
&\geq\; \alpha \int_{\lambda(0)}^{v} \min\{v-z, z-\lambda(0), \varrho(z)\}\;\mathrm{d}z\;.
\end{aligned}$$

The computation for $\hat{\Phi}_-$ is similar, and (2.8) follows.

Proof of Lemma 2.1. Assume first that u in nondecreasing in $[t_0, t_1]$. Then, by (1.25),

$$\mathfrak{p}_r[\lambda, u](t) \;=\; \max\{\lambda_{t_0}(r), u(t) - r\} \qquad \text{for} \;\; r > 0,\; t \in [t_0, t_1]\,. \tag{2.9}$$

In particular, $\mathfrak{p}_r[\lambda, u](t) = \lambda_{t_0}(r)$ for $r > m_{\lambda_{t_0}}(u(t))$. Hence, by Fubini's Theorem,

$$\begin{aligned}
\mathcal{F}[\lambda, u](t) - \mathcal{F}[\lambda, u](t_0) \;&=\; \int_0^\infty \left(g(r, \mathfrak{p}_r[\lambda, u](t)) - g(r, \lambda_{t_0}(r))\right)\mathrm{d}r \\
&=\; \int_0^{m_{\lambda_{t_0}}(u(t))} \int_{\lambda_{t_0}(r)+r}^{u(t)} \mu(r, z-r)\;\mathrm{d}z\;\mathrm{d}r \\
&=\; \int_{\lambda_{t_0}(0)}^{u(t)} \int_0^{m_{\lambda_{t_0}}(z)} \mu(r, z-r)\;\mathrm{d}r\;\mathrm{d}z \\
&=\; \Phi_+(u(t))\,.
\end{aligned} \tag{2.10}$$

Similarly, if u in nonincreasing, we have

$$\mathfrak{p}_r[\lambda,u](t) \;=\; \min\{\lambda_{t_0}(r), u(t)+r\} \quad \text{for} \;\; r>0,\; t\in[t_0,t_1]\,, \tag{2.11}$$

hence,

$$\begin{aligned}
\mathcal{F}[\lambda,u](t_0) - \mathcal{F}[\lambda,u](t) &= \int_0^\infty (g(r,\lambda_{t_0}(r)) - g(r,\mathfrak{p}_r[\lambda,u](t),))\,\mathrm{d}r \\
&= \int_0^{m_{\lambda_{t_0}}(u(t))} \int_{u(t)}^{\lambda_{t_0}(r)-r} \mu(r,z+r)\,\mathrm{d}z\,\mathrm{d}r \\
&= \int_{u(t)}^{\lambda_{t_0}(0)} \int_0^{m_{\lambda_{t_0}}(z)} \mu(r,z+r)\,\mathrm{d}r\,\mathrm{d}z \\
&= -\Phi_-(u(t))\,.
\end{aligned} \tag{2.12}$$

■

In the situation of Lemma 2.1, set for $s\geq 0$

$$\varphi_+(s) = \int_0^s \mu(r,\lambda_{t_0}(s)+s-r)\,\mathrm{d}r\,, \qquad \varphi_-(s) = \int_0^s \mu(r,\lambda_{t_0}(s)-s+r)\,\mathrm{d}r\;. \tag{2.13}$$

Then

$$\left.\begin{aligned}
\Phi'_+(v) &= \varphi_+(m_{\lambda_{t_0}}(v)) && \text{for a.e.} \;\; v>\lambda_{t_0}(0)\,, \\
\Phi'_-(v) &= \varphi_-(m_{\lambda_{t_0}}(v)) && \text{for a.e.} \;\; v<\lambda_{t_0}(0)\,.
\end{aligned}\right\} \tag{2.14}$$

Note that

$$m_{\lambda_{t_0}}(v_1) - m_{\lambda_{t_0}}(v_2) \;\geq\; \frac{1}{2}(v_1-v_2) \quad \text{for} \quad v_1>v_2>\lambda_{t_0}(0) \tag{2.15}$$

and similarly,

$$m_{\lambda_{t_0}}(v_1) - m_{\lambda_{t_0}}(v_2) \;\geq\; \frac{1}{2}(v_2-v_1) \quad \text{for} \quad v_1<v_2<\lambda_{t_0}(0). \tag{2.16}$$

Referring to the notation of Lemma 2.1, we fix some $B \geq \max\{R, \|u\|_{[0,t_1]}\}$. By Lemma 1.4, we have $|\lambda_{t_0}(s)| \leq \max\{B-s, 0\}$ for all $s\geq 0$. In particular, this yields for $0<r<s\leq B$

$$|\lambda_{t_0}(s)+s-r|+r \leq B\,, \qquad |\lambda_{t_0}(s)-s+r|+r \leq B\,.$$

Let ϱ_B be any number such that

$$0 < \varrho_B \leq \min\{\varrho(v); |v|\leq B\}\,. \tag{2.17}$$

Making use of Hypothesis 1.7 (ii), we obtain from (2.13) that

$$\left.\begin{aligned}
\varphi_\pm(s) &\geq \alpha\varrho_B && \text{for} \;\; s\in[\varrho_B, B]\,, \\
\varphi'_\pm(s) &\geq \alpha - 2\beta\varrho_B \;\geq\; \tfrac{\alpha}{2} && \text{for a.e.} \;\; s\in(0,\varrho_B)\,.
\end{aligned}\right\} \tag{2.18}$$

Combined with (2.14)–(2.16), these relations imply

$$\frac{\Phi'_+(v_1) - \Phi'_+(v_2)}{v_1 - v_2} \;\geq\; \frac{\alpha}{4} \quad \text{if } v_1 > v_2 > \lambda_{t_0}(0),\ 0 \leq m_{\lambda_{t_0}}(v_2) < m_{\lambda_{t_0}}(v_1) \leq \varrho_B, \tag{2.19}$$

$$\frac{\Phi'_-(v_2) - \Phi'_-(v_1)}{v_2 - v_1} \;\leq\; -\frac{\alpha}{4} \quad \text{if } v_1 < v_2 < \lambda_{t_0}(0),\ 0 \leq m_{\lambda_{t_0}}(v_2) < m_{\lambda_{t_0}}(v_1) \leq \varrho_B. \tag{2.20}$$

This is precisely what we had in mind when we introduced the convexity domain $\mathcal{C}$ in Hypothesis 1.7. By virtue of (2.4) and (2.19)–(2.20), Φ_+ is a convex function describing increasing branches and Φ_- is a concave function describing decreasing branches as long as the evolution takes place in $\mathcal{C}$, hence small closed hysteresis loops are counterclockwise convex.

We now state an auxiliary result, which is a basis for the generalized convexity inequality for the Preisach operator we give below.

Lemma 2.2 *Let $[a,b] \subset \mathbb{R}$, $\delta > 0$ and let $u, v \in W^{1,1}(a,b)$ be given functions such that $v(t) \geq 0$, and u is monotone (nondecreasing or nonincreasing) in $[a,b]$. Let $\psi : \mathrm{Conv}\{u(a), u(b)\} \to (0,\infty)$, where $\mathrm{Conv}\, A$ denotes the convex hull of a set A, be a function such that*

$$\frac{\psi(z_1) - \psi(z_2)}{z_1 - z_2} \;\geq\; \delta \qquad \textit{if } \ u(a) \leq z_2 < z_1 \leq u(b)\,, \tag{2.21}$$

$$\frac{\psi(z_1) - \psi(z_2)}{z_1 - z_2} \;\leq\; -\delta \quad \textit{if } \ u(b) \leq z_2 < z_1 \leq u(a)\,. \tag{2.22}$$

Then for an arbitrary $p > 0$

$$\int_a^b \frac{\dot v(t)}{\psi^p(u(t))}\,\mathrm{d}t \;\geq\; \left[\frac{v(t)}{\psi^p(u(t))}\right]_a^b + p\delta \int_a^b \frac{|\dot u(t)|\, v(t)}{\psi^{p+1}(u(t))}\,\mathrm{d}t\ . \tag{2.23}$$

Proof. We only treat the case that u is nondecreasing. Then ψ is positive and increasing; hence, $\psi \circ u$ is nondecreasing and the integrals are meaningful. We may approximate ψ by continuously differentiable functions ψ_n satisfying (2.21) and such that $\psi_n(z) \to \psi(z)$ for each $z \in [u(a), u(b)]$. Inequality (2.23) holds for each ψ_n as a direct consequence of an integration by parts. Passing to the limit in n we obtain the assertion. ∎

Consider now some $\varepsilon_0 > 0$ and a family $\{\mathcal{F}_\varepsilon\,;\, \varepsilon \in (0,\varepsilon_0)\}$ of Preisach operators of the form (1.28)–(1.29), associated with generating functions μ_ε and initial states $\lambda^\varepsilon \in \Lambda_R$ for some fixed $R > 0$, assuming that Hypothesis 1.7 is satisfied independently of ε. For $\varepsilon \in (0,\varepsilon_0)$ we define the operator $\mathcal{G}_\varepsilon : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by the formula

$$\mathcal{G}_\varepsilon \;=\; (\varepsilon I + \mathcal{F}_\varepsilon[\lambda^\varepsilon, \cdot])^{-1}\ . \tag{2.24}$$

By Proposition 1.8, $\mathcal{G}_\varepsilon$ is Lipschitz continuous with Lipschitz constant $2/\varepsilon$. As the main result of this section, we derive an estimate for $\mathcal{G}_\varepsilon$ *independent of* ε and similar to [8, Theorem 2.1] as a generalization of [6, Proposition II.4.22]. Consistently with (1.16), we denote $\lambda^\varepsilon_t(r) = \mathfrak{p}_r[\lambda^\varepsilon, u_\varepsilon](t)$ for $t \geq 0$ and $r \geq 0$.

Proposition 2.3 (Generalized convexity inequality) *Let $\mathcal{G}_\varepsilon$ be as in (2.24) and let $q > 1$ be given. Let $\{w_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ be a system of functions in $W^{2,q}_{\mathrm{loc}}(\mathbb{R}_+)$ parameterized by $\varepsilon > 0$, and set*

$$u_\varepsilon \;=\; \mathcal{G}_\varepsilon[w_\varepsilon]\,. \tag{2.25}$$

Let $B > R$ be given, and let ϱ_B, α be as in (2.17). Assume that for every $\varepsilon \in (0, \varepsilon_0)$ we have

$$\textit{either}\;\; \dot w_\varepsilon(0) = 0\,, \quad \textit{or}\;\; (m_{\lambda_0^\varepsilon} \circ u_\varepsilon)(0+) \geq \varrho_B\,. \tag{2.26}$$

Then there exists a constant $C_{q,B} > 0$ depending only on q and B such that for every $T > 0$ and every $\varepsilon \in (0, \varepsilon_0)$ with the property

$$\|u_\varepsilon\|_{[0,T]} \;<\; B \tag{2.27}$$

we have

$$\int_0^T |\dot u_\varepsilon(t)|^{2q}\,\mathrm{d}t \;\leq\; C_{q,B}\int_0^T \left(|\ddot w_\varepsilon(t)|^q + |\dot w_\varepsilon(t)|^{2q}\right)\mathrm{d}t\,. \tag{2.28}$$

Relation (2.26) is a condition of compatibility between the initial memory configuration λ^ε and the initial velocity. Note that for every monotone function u on an interval $[t_0, t_1]$, every $\lambda \in \Lambda$, and every $t \in (t_0, t_1)$, we have the implication

$$\exists \dot u(t) \neq 0 \implies (m_{\lambda_{t_0}} \circ u)(t) > 0\,. \tag{2.29}$$

Indeed, $(m_{\lambda_{t_0}} \circ u)(t) = 0$ would imply $u(t) = \lambda_{t_0}(0) = u(t_0)$, which is a contradiction. The following Example 2.4 shows that Proposition 2.3 does not hold if condition (2.26) is violated.

Example 2.4 Consider the case $\lambda^\varepsilon(r) = \lambda(r) \equiv 0$, $\mu_\varepsilon(r,v) = \mu(r,v) \equiv 1$, $w_\varepsilon(0) = w(0) = 0$. In particular, $m_\lambda(v) = |v|$ for all $v \in \mathbb{R}$. By (1.8) and (1.29), we have

$$\mathcal{F}[\lambda, u_\varepsilon](0) \;=\; \int_0^\infty g(r, \xi_r^\varepsilon(0))\,\mathrm{d}r$$

with

$$\xi_r^\varepsilon(0) \;=\; \min\{u_\varepsilon(0) + r, \max\{u_\varepsilon(0) - r, 0\}\}\,,$$

hence $\mathcal{F}[\lambda, u_\varepsilon](0) = \frac{1}{2}|u_\varepsilon(0)|u_\varepsilon(0)$. From (2.25) it follows that $\varepsilon u_\varepsilon(0) + \frac{1}{2}|u_\varepsilon(0)|u_\varepsilon(0) = 0$, hence $u_\varepsilon(0) = 0$. Let now $w_\varepsilon = w$ be increasing in $[0,T]$. Then the functions u_ε are increasing as well by virtue of (1.9), and from Lemma 2.1 we obtain $\mathcal{F}[\lambda, u_\varepsilon](t) = \frac{1}{2}u_\varepsilon^2(t)$, hence $u_\varepsilon(t)$ is the solution of the equation

$$\varepsilon u_\varepsilon(t) + \frac{1}{2}u_\varepsilon^2(t) = w(t)\,.$$

If, for example, $w(t) = t$ for $t \in [0,T]$, then $u_\varepsilon(t) = \sqrt{\varepsilon^2 + 2t} - \varepsilon$. We see that $\int_0^T |\dot u_\varepsilon(t)|^{2q}\,dt$ is not bounded independently of ε for any $q \geq 1$, although the right hand side of (2.28) is bounded.

Proof of Proposition 2.3. We fix $T > 0$ and $\varepsilon > 0$ such that (2.27) holds, and assume first that w_ε has only finitely many monotonicity intervals in $[0, T]$. Let $[a, b] \subset [0, T]$ be a monotonicity interval of w_ε. Set

$$c \;=\; \sup\{t \in [a, b]; (m_{\lambda_a^\varepsilon} \circ u_\varepsilon)(t+) \le \varrho_B\} \tag{2.30}$$

with the convention $c = a$ if $(m_{\lambda_a^\varepsilon} \circ u_\varepsilon)(a+) \ge \varrho_B$. The function $t \mapsto m_{\lambda_a^\varepsilon}(u_\varepsilon(t))$ is left-continuous and nondecreasing in $[a, b]$. By (2.12)–(2.18) we have

$$|\dot w_\varepsilon(t)| \;\ge\; (\varepsilon + \varrho_B \alpha)\, |\dot u_\varepsilon(t)| \qquad \text{for a. e. } \; t \in (c, b)\,. \tag{2.31}$$

In (a, c), we distinguish the cases that w_ε increases or decreases. Assume first that $\dot w_\varepsilon(t) \ge 0$ for all $t \in (a, c)$, and let Φ_+ be the loading curve on $[u_\varepsilon(a), u_\varepsilon(c)]$ associated with $\mathcal{F}_\varepsilon$ as in Lemma 2.1. Then

$$\dot w_\varepsilon(t) \;=\; \big(\varepsilon + \Phi_+'(u_\varepsilon(t))\big)\, \dot u_\varepsilon(t) \qquad \text{for a. e. } \; t \in (a, c)\,. \tag{2.32}$$

Set $p = 2q - 2$. Then

$$\ddot w_\varepsilon(t)\, \dot u_\varepsilon^p(t) \;=\; \frac{\frac{\mathrm{d}}{\mathrm{d}t}\,(\dot w_\varepsilon^{p+1}(t))}{(p+1)\,(\varepsilon + \Phi_+'(u_\varepsilon(t)))^p} \qquad \text{for a. e. } \; t \in (a, c)\,. \tag{2.33}$$

By (2.19), we may use Lemma 2.2 with $v(t) = \dot w_\varepsilon^{p+1}(t)$, $\psi(z) = \varepsilon + \Phi_+'(z)$, $u(t) = u_\varepsilon(t)$, and obtain from (2.32) that

$$\begin{aligned}
\int_a^c \ddot w_\varepsilon(t)\, \dot u_\varepsilon^p(t)\,\mathrm{d}t \;&\ge\; -\frac{\dot w_\varepsilon^{p+1}(a)}{(p+1)\,(\varepsilon + \Phi_+'(u_\varepsilon(a)))^p} + \frac{\alpha p}{4(p+1)} \int_a^c \frac{\dot u_\varepsilon(t)\, \dot w_\varepsilon^{p+1}(t)}{(\varepsilon + \Phi_+'(u_\varepsilon(t)))^{p+1}}\,\mathrm{d}t \\
&=\; -\frac{1}{p+1} \dot u_\varepsilon^p(a)\, \dot w_\varepsilon(a) + \frac{\alpha p}{4(p+1)} \int_a^c \dot u_\varepsilon^{p+2}(t)\;\mathrm{d}t \\
&=\; -\frac{1}{p+1} \dot u_\varepsilon^p(a)\, \dot w_\varepsilon(a) + \frac{\alpha p}{4(p+1)} \int_a^c \dot u_\varepsilon^{2q}(t)\;\mathrm{d}t\;.
\end{aligned} \tag{2.34}$$

Hence, by Hölder's inequality,

$$\begin{aligned}
\frac{\alpha p}{4(p+1)} \int_a^c \dot u_\varepsilon^{2q}(t)\;\mathrm{d}t \;&\le\; \frac{1}{p+1} \dot u_\varepsilon^p(a)\, \dot w_\varepsilon(a) \\
&\quad + \left(\int_a^c |\ddot w_\varepsilon(t)|^q\;\mathrm{d}t\right)^{2/(2+p)} \left(\int_a^c \dot u_\varepsilon^{2q}(t)\;\mathrm{d}t\right)^{p/(2+p)}
\end{aligned} \tag{2.35}$$

We see that there exists a constant $C > 0$ depending only on p (or, equivalently, on q) and α such that

$$\int_a^c \dot u_\varepsilon^{2q}(t)\;\mathrm{d}t \;\le\; C\left(\dot u_\varepsilon^p(a)\, \dot w_\varepsilon(a) + \int_a^c |\ddot w_\varepsilon(t)|^q\;\mathrm{d}t\right). \tag{2.36}$$

Taking into account inequality (2.31), we thus have, with a possibly different constant depending only on q, ϱ_B, and α,

$$\int_a^b \dot u_\varepsilon^{2q}(t)\;\mathrm{d}t \;\le\; C\left(\dot u_\varepsilon^p(a)\, \dot w_\varepsilon(a) + \int_a^c |\ddot w_\varepsilon(t)|^q\;\mathrm{d}t + \int_c^b |\dot w_\varepsilon(t)|^{2q}\;\mathrm{d}t\right). \tag{2.37}$$

Let now $\dot{w}_\varepsilon(t) \le 0$ in (a,b), hence also $\dot{u}_\varepsilon(t) \le 0$ a. e. in (a,b). Then (2.31) still holds, and, as a counterpart of (2.32)–(2.33), we have

$$\dot{w}_\varepsilon(t) \;=\; \big(\varepsilon + \Phi'_-(u_\varepsilon(t))\big)\, \dot{u}_\varepsilon(t) \qquad \text{for a. e.} \quad t \in (a,c)\,, \tag{2.38}$$

and

$$-\ddot{w}_\varepsilon(t)\, |\dot{u}_\varepsilon(t)|^p \;=\; \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left((-\dot{w}_\varepsilon)^{p+1}(t)\right)}{(p+1)\,(\varepsilon + \Phi'_-(u_\varepsilon(t)))^p} \qquad \text{for a. e.} \quad t \in (a,c)\,. \tag{2.39}$$

We now argue as above using Lemma 2.2, and obtain again

$$\int_a^b |\dot{u}_\varepsilon|^{2q}(t)\ \mathrm{d}t \;\le\; C\left(|\dot{u}_\varepsilon(a)|^p\, |\dot{w}_\varepsilon(a)| + \int_a^c |\ddot{w}_\varepsilon(t)|^q\ \mathrm{d}t + \int_c^b |\dot{w}_\varepsilon(t)|^{2q}\ \mathrm{d}t\right). \tag{2.40}$$

We conclude that there exists $C_{q,B}$ with the desired properties such that in every monotonicity interval $[a,b] \subset [0,T]$ we have

$$\int_a^b |\dot{u}_\varepsilon|^{2q}(t)\ \mathrm{d}t \;\le\; C_{q,B}\left(|\dot{u}_\varepsilon(a)|^p\, |\dot{w}_\varepsilon(a)| + \int_a^b \left(|\ddot{w}_\varepsilon(t)|^q + |\dot{w}_\varepsilon(t)|^{2q}\right) \mathrm{d}t\right). \tag{2.41}$$

Let now $0 = t_0 < t_1 < \cdots < t_m = T$ be a minimal partition of $[0,T]$ such that w_ε is monotone in each interval $[t_{k-1}, t_k]$, $k = 1, \dots, m$. We now apply formula (2.41) with $a = t_{k-1}$, $b = t_k$. For all $k = 2, \dots, m$ we have $\dot{w}_\varepsilon(t_{k-1}) = 0$ (note that w_ε is continuously differentiable). Moreover, in the case $a = 0$, $b = t_1$, we have by hypothesis (2.26) that either $\dot{w}_\varepsilon(0) = 0$, or $|\dot{w}_\varepsilon(t)| \ge (\varepsilon + \varrho_B \alpha)\, |\dot{u}_\varepsilon(t)|$ for a. e. $t \in (0, t_1)$. In both cases we conclude that

$$\int_{t_{k-1}}^{t_k} |\dot{u}_\varepsilon|^{2q}(t)\ \mathrm{d}t \;\le\; C_{q,B} \int_{t_{k-1}}^{t_k} \left(|\ddot{w}_\varepsilon(t)|^q + |\dot{w}_\varepsilon(t)|^{2q}\right) \mathrm{d}t \tag{2.42}$$

for every $k = 1, \dots, m$. Summing up over k we obtain the assertion for every piecewise monotone function w_ε. Let now $w_\varepsilon \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}_+)$ and $T > 0$ be arbitrary. We approximate w_ε on $[0,T]$ by piecewise monotone functions $w_\varepsilon^{(k)}$ converging strongly to w_ε in $W^{2,q}(0,T)$. In particular, $w_\varepsilon^{(k)}$ converge to w_ε uniformly in $C[0,T]$, hence, by continuity of $\mathcal{G}_\varepsilon$, $u_\varepsilon^{(k)} = \mathcal{G}_\varepsilon[w_\varepsilon^{(k)}]$ converge uniformly to $u_\varepsilon = \mathcal{G}_\varepsilon[w_\varepsilon]$. For sufficiently large k, condition (2.27) is preserved, and by the above considerations, we have

$$\int_0^T |\dot{u}_\varepsilon^{(k)}|^{2q}(t)\ \mathrm{d}t \;\le\; C_{q,B} \int_0^T \left(|\ddot{w}_\varepsilon^{(k)}(t)|^q + |\dot{w}_\varepsilon^{(k)}(t)|^{2q}\right) \mathrm{d}t\,,$$

hence, $\{\dot{u}_\varepsilon^{(k)}\}$ is a bounded sequence in $L^{2q}(0,T)$. Consequently, $\dot{u}_\varepsilon^{(k)}$ converge weakly to $\dot{u}_\varepsilon$ in $L^{2q}(0,T)$, and the assertion follows. ■

3 Regularization

In this section, we show that solutions to Eq. (0.2) converge locally uniformly as $\varepsilon \to 0+$ to solutions to (0.1) under the following hypotheses.

Hypothesis 3.1 *Let $q > 1$, $\varepsilon_0 > 0$, $R > 0$ be fixed, let $f_\varepsilon : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous functions in both variables, and let $\mathcal{F}_\varepsilon$ be Preisach operators of the form (1.28)–(1.29), associated with generating functions μ_ε and initial states $\lambda^\varepsilon \in \Lambda_R$ for $\varepsilon \in (0, \varepsilon_0)$. The following conditions are assumed to hold.*

(i) *Hypothesis 1.7 holds for each μ_ε independently of $\varepsilon \in (0, \varepsilon_0)$.*

(ii) *There exists $f^0 \in L^{2q}_{\mathrm{loc}}(\mathbb{R}_+)$ such that $f^0(t) \geq 0$, and*

$$v\, f_\varepsilon(t, v) \leq f^0(t)|v| \tag{3.1}$$

for all $(\varepsilon, t, v) \in (0, \varepsilon_0) \times \mathbb{R}_+ \times \mathbb{R}$.

(iii) *For every $B > 0$ there exist functions $\ell_B \in L^{2q}_{\mathrm{loc}}(\mathbb{R}_+)$, $f^1_B \in L^q_{\mathrm{loc}}(\mathbb{R}_+)$, such that for almost all $(t, v) \in \mathbb{R}_+ \times (-B, B)$ and all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\left|\frac{\partial f_\varepsilon}{\partial t}(t, v)\right| \leq f^1_B(t)\,, \tag{3.2}$$

$$\left|\frac{\partial f_\varepsilon}{\partial v}(t, v)\right| \leq \ell_B(t)\,. \tag{3.3}$$

(iv) *There exists $\varrho_0 > 0$ such that the following initial compatibility conditions hold for every $\varepsilon \in (0, \varepsilon_0)$:*

$$\begin{aligned} f_\varepsilon(0, u^0_\varepsilon) > 0 &\implies m_{\lambda^\varepsilon_0}(u^0_\varepsilon +) \geq \varrho_0\,, \\ f_\varepsilon(0, u^0_\varepsilon) < 0 &\implies m_{\lambda^\varepsilon_0}(u^0_\varepsilon -) \geq \varrho_0\,, \end{aligned}$$

where $m_{\lambda^\varepsilon_0}$ is the mapping defined in (1.22).

(v) *There exists $U_0 \geq R$ such that $|u^0_\varepsilon| \leq U_0$ for every $\varepsilon \in (0, \varepsilon_0)$.*

(vi) *There exist $u^0 \in \mathbb{R}$, a locally bounded measurable function $\mu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$, a continuous function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, and a function $\lambda \in \Lambda_R$ such that*

$$\begin{aligned} &\lim_{\varepsilon \to 0+} u^0_\varepsilon = u^0\,, \\ &\lim_{\varepsilon \to 0+} f_\varepsilon(t, v) = f(t, v) \quad \text{for all } (t, v) \in \mathbb{R}_+ \times \mathbb{R}\,, \\ &\lim_{\varepsilon \to 0+} \int_\Omega (\mu_\varepsilon - \mu)(r, v)\, \mathrm{d}r\, \mathrm{d}v = 0 \quad \text{for every open bounded set } \Omega \subset \mathbb{R}_+ \times \mathbb{R}\,, \\ &\lim_{\varepsilon \to 0+} \lambda^\varepsilon(r) = \lambda(r) \quad \text{for all } r \geq 0\,. \end{aligned}$$

We denote by $\hat{\Phi}^\varepsilon_\pm$ the initial loading curves associated with the Preisach operator $\mathcal{F}_\varepsilon$ according to formulas (2.5)–(2.6). A similar computation as in (2.8) shows, as a consequence of Hypotheses 1.7 and 3.1, that they have the property

$$\hat{\Phi}^\varepsilon_+(v) \geq \int_{U_0 + \hat{\varrho}}^{v - \hat{\varrho}} \varrho(z)\, \mathrm{d}z \qquad \text{for } v > U_0 + 2\hat{\varrho}\,, \tag{3.4}$$

$$\hat{\Phi}^\varepsilon_-(v) \leq -\int_{v + \hat{\varrho}}^{U_0 - \hat{\varrho}} \varrho(z)\, \mathrm{d}z \qquad \text{for } v < U_0 - 2\hat{\varrho}\,. \tag{3.5}$$

We prove the following result.

Theorem 3.2 *Let Hypothesis 3.1 hold. Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists a unique solution $u_\varepsilon \in W^{1,2q}_{\mathrm{loc}}(\mathbb{R}_+)$ to (0.2). If moreover $u_1^\varepsilon, u_2^\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ are two solutions to (0.2) corresponding to two initial conditions $u_\varepsilon^{01} \le u_\varepsilon^{02}$ and $\lambda_1^\varepsilon(r) \le \lambda_2^\varepsilon(r)$ for all $r \ge 0$, then $u_1^\varepsilon(t) \le u_2^\varepsilon(t)$ for every $t \ge 0$. If, in addition, there exist $B > R$ and $T > 0$ such that*

$$|u_\varepsilon(t)| \;\le\; B \quad \forall \varepsilon \in (0, \varepsilon_0) \;\; \forall t \in [0, T]\,, \tag{3.6}$$

then there exists a constant $C_B > 0$ independent of ε such that

$$\int_0^T |\dot u_\varepsilon(t)|^{2q}\, \mathrm{d}t \;\le\; C_B \int_0^T \left(|f_B^1(t)|^q + |f^0(t)|^{2q} + |\ell_B(t)|^{2q}\right) \mathrm{d}t\,, \tag{3.7}$$

and a sequence $\varepsilon_k \to 0+$ such that u_{ε_k} converge uniformly in $C[0,T]$ and weakly in $W^{1,2q}(0,T)$ to a solution u of the limit system (0.1).

Proof. For the sake of completeness, we give an existence proof for solutions to (0.2) for every $\varepsilon \in (0, \varepsilon_0)$, which is fairly standard. Notice first that there is a one-to-one correspondence between the initial conditions for u_ε and w_ε in (0.2), which has the form

$$w_\varepsilon(0) = w_\varepsilon^0 := \varepsilon u_\varepsilon^0 + \int_0^\infty g(r, \xi_r^0)\, \mathrm{d}r\,, \quad \xi_r^0 = \min\left\{u_\varepsilon^0 + r, \max\{u_\varepsilon^0 - r, \lambda(r)\}\right\} \tag{3.8}$$

by virtue of (1.8). Keeping $\varepsilon \in (0, \varepsilon_0)$ and $T > 0$ fixed, we define

$$\hat w_\varepsilon = \varepsilon |u_\varepsilon^0| + \int_0^\infty |g(r, \xi_r^0)|\, \mathrm{d}r\,, \quad B^* \;=\; \frac{1}{\varepsilon}\left(\hat w_\varepsilon + \int_0^T f^0(t)\, \mathrm{d}t\right). \tag{3.9}$$

Condition (3.1) implies no lower bound for f_ε if $v > 0$ and no upper bound if $v < 0$. We therefore put

$$f_\varepsilon^*(t, v) \;=\; \begin{cases} f_\varepsilon(t, v) & \text{for } (t, v) \in (0, T) \times [-B^*, B^*]\,, \\ f_\varepsilon(t, B^*) & \text{for } (t, v) \in (0, T) \times (B^*, +\infty)\,, \\ f_\varepsilon(t, -B^*) & \text{for } (t, v) \in (0, T) \times (-\infty, -B^*)\,. \end{cases} \tag{3.10}$$

By Hypotheses 3.1 (ii), (iii) we have $|f_\varepsilon(t, 0)| \le f^0(t)$ for all t, hence

$$|f_\varepsilon^*(t, v)| \;\le\; f^0(t) + B^* \ell_{B^*}(t) \quad \forall (t, v) \in (0, T) \times \mathbb{R}\,. \tag{3.11}$$

We introduce the set

$$K \;=\; \left\{\tilde u \in C[0, T] : \tilde u(0) = u_\varepsilon^0,\; \|\tilde u\|_{[0,T]} \le B^* + \frac{B^*}{\varepsilon} \int_0^T \ell_{B^*}(t)\, \mathrm{d}t\right\}. \tag{3.12}$$

We now check that the mapping $\mathcal{S} : K \to C[0, T]$, which with each $\tilde u \in K$ associates the solution u of the problem

$$\left.\begin{aligned} \dot w(t) &= f_\varepsilon^*(t, \tilde u(t)) \\ w(t) &= \varepsilon u(t) + \mathcal{F}_\varepsilon[\lambda^\varepsilon, u](t) \\ u(0) &= u_\varepsilon^0\,, \end{aligned}\right\} \tag{3.13}$$

is a contraction on K, endowed with the norm

$$\|\tilde{u}\|_* = \max\left\{\left|\mathrm{e}^{-L^*(t)}\tilde{u}(t)\right| : t \in [0,T]\right\} ,$$

with

$$L^*(t) = \frac{1}{\varepsilon}\int_0^t \ell_{B^*}(\tau)\,\mathrm{d}\tau .$$

The existence and uniqueness of the solution u to (3.13) follows immediately from the Lipschitz continuity property in Proposition 1.8 (ii). To see that $u \in K$, we test Eq. (3.13) consecutively by $H(u(t))$ and $-H(-u(t))$, which yields, by virtue of (1.37)–(1.38), that

$$\varepsilon|u(t)| \;\le\; \hat{w}_\varepsilon + \int_0^t |f^*_\varepsilon(\tau,\tilde{u}(\tau))|\,\mathrm{d}\tau ,$$

and it suffices to use inequality (3.11) to conclude that $u \in K$. The contracting property of $\mathcal{S}$ can again be proved via Hilpert's inequality (1.36). For $\tilde{u}_1, \tilde{u}_2 \in K$, we test the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varepsilon(u_1(t)-u_2(t)) + \mathcal{F}_\varepsilon[\lambda^\varepsilon,u_1](t) - \mathcal{F}_\varepsilon[\lambda^\varepsilon,u_2](t)\big) = f^*_\varepsilon(t,\tilde{u}_1(t)) - f^*_\varepsilon(t,\tilde{u}_2(t))$$

by $\operatorname{sign}(u_1-u_2)$ to obtain, using also Hypothesis (3.3), that

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}|u_1(t)-u_2(t)| \le \ell_{B^*}(t)|\tilde{u}_1(t)-\tilde{u}_2(t)| .$$

For $t \in [0,T]$ we then have

$$\mathrm{e}^{-L^*(t)}|u_1(t)-u_2(t)| \le \frac{1}{\varepsilon}\|\tilde{u}_1-\tilde{u}_2\|_*\int_0^t \mathrm{e}^{L^*(\tau)-L^*(t)}\ell_{B^*}(\tau)\,\mathrm{d}\tau \le \left(1-\mathrm{e}^{-L^*(T)}\right)\|\tilde{u}_1-\tilde{u}_2\|_*,$$

which yields the desired contraction. To see that the fixed point u of the mapping $\mathcal{S}$ satisfies Eq. (0.2), we test the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varepsilon u(t) + \mathcal{F}_\varepsilon[\lambda^\varepsilon,u](t)\big) = f^*_\varepsilon(t,u(t))$$

by $\operatorname{sign}(u(t))$ and obtain by (1.37), (1.38), and Hypothesis 3.1 (ii) that

$$\varepsilon|u(t)| \;\le\; \hat{w}_\varepsilon + \int_0^t f^0(\tau)\,\mathrm{d}\tau .$$

We see that $\|u\|_{[0,T]} \le B^*$, hence $f^*_\varepsilon(t,u(t)) = f_\varepsilon(t,u(t))$, and the existence proof is complete.

Let now $u_1^\varepsilon, u_2^\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ be two solutions corresponding to two different initial conditions $u_\varepsilon^{01} \le u_\varepsilon^{02}$ and $\lambda_1^\varepsilon(r) \le \lambda_2^\varepsilon(r)$ for all $r \ge 0$. We fix some $T > 0$ and $B_T > \max\{R, \|u_1^\varepsilon\|_{[0,T]}, \|u_2^\varepsilon\|_{[0,T]}\}$. Then for all $t \in [0,T]$ and $r > 0$ we have by Lemma 1.4 that $|\mathfrak{p}_r[\lambda_i^\varepsilon, u_i^\varepsilon](t)| \le B_T$, and $\mathfrak{p}_r[\lambda_i^\varepsilon, u_i^\varepsilon](t) = 0$ for $r > B_T$, $i = 1,2$. Testing the difference of the equations for $u_1^\varepsilon, u_2^\varepsilon$ by $H(u_1^\varepsilon - u_2^\varepsilon)$ and using Proposition 1.9, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon(u_1^\varepsilon(t) - u_2^\varepsilon(t))^+ + \int_0^{B_T} (g(r, \mathfrak{p}_r[\lambda_1^\varepsilon, u_1^\varepsilon]) - g(r, \mathfrak{p}_r[\lambda_2^\varepsilon, u_2^\varepsilon]))^+(t)\, \mathrm{d}r\right)$$
$$\leq \ell_{B_T}(t)(u_1^\varepsilon(t) - u_2^\varepsilon(t))^+ \quad \text{a.e.}$$

We have $\mathfrak{p}_r[\lambda_1^\varepsilon, u_1^\varepsilon](0) \leq \mathfrak{p}_r[\lambda_2^\varepsilon, u_2^\varepsilon](0)$, and using the classical Gronwall argument we obtain $u_1^\varepsilon(t) \leq u_2^\varepsilon(t)$ for all $t \in [0,T]$. Since $T > 0$ has been chosen arbitrarily, the solution exists and is unique globally in time for all $\varepsilon \in (0, \varepsilon_0)$.

To derive the upper bound (3.7) independent of ε, we use Proposition 2.3 in a substantial way. It follows from Hypothesis 3.1 (iv) that (2.26) holds for all $\varepsilon \in (0, \varepsilon_0)$, provided we set $\varrho_B = \min\{\varrho_0, \min\{\varrho(v); |v| \leq B\}\}$. Proposition 2.3, Hypotheses 3.1 (ii)–(iii), and Hölder's inequality then yield

$$\begin{aligned}
\int_0^T |\dot u_\varepsilon(t)|^{2q}\, \mathrm{d}t \;&\leq\; C\int_0^T \left(\left(f_B^1(t) + \ell_B(t)\,|\dot u_\varepsilon(t)|\right)^q + |f^0(t)|^{2q}\right) \mathrm{d}t \\
&\leq\; \hat C\int_0^T \left(|f_B^1(t)|^q + |f^0(t)|^{2q}\right) \mathrm{d}t \\
&\quad + \hat C\left(\int_0^T |\ell_B(t)|^{2q}\, \mathrm{d}t\right)^{1/2}\left(\int_0^T |\dot u_\varepsilon(t)|^{2q}\, \mathrm{d}t\right)^{1/2}
\end{aligned} \tag{3.14}$$

with a constant $\hat C$ independent of ε; hence, (3.7) holds. In particular, the functions $\dot u_\varepsilon$ are uniformly bounded in $L^{2q}(0,T)$. By compact embedding, there exists a uniformly convergent subsequence $u_{\varepsilon_k} \to u$ as $\varepsilon_k \to 0+$. We have $\mathfrak{p}_r[\lambda^\varepsilon, u_\varepsilon](t) = 0$ for all $r \geq B$ and $\varepsilon \in (0, \varepsilon_0)$ by virtue of Lemma 1.3. Hence,

$$\lim_{\varepsilon\to 0+} (\mathcal{F}_\varepsilon[\lambda^\varepsilon, u_\varepsilon](t) - \mathcal{F}_\varepsilon[\lambda, u](t)) = 0$$

as a consequence of (1.33), and the formula

$$\mathcal{F}_\varepsilon[\lambda, u](t) - \mathcal{F}[\lambda, u](t) = \int_0^B \int_0^{\mathfrak{p}_r[\lambda,u](t)} (\mu_\varepsilon - \mu)(r, v)\, \mathrm{d}v\, \mathrm{d}r$$

together with Hypothesis 3.1 (vi) enable us to pass to the limit as $\varepsilon \to 0+$ in (0.2) and check that u is a solution to the limit problem (0.1). ■

Example 3.3 Consider again the case $\lambda^\varepsilon(r) = \lambda(r) \equiv 0$, $\mu_\varepsilon(r,v) = \mu(r,v) \equiv 1$, and $u_\varepsilon^0 = u^0 = 0$. The computation in Example 2.4 shows that $\mathcal{F}[\lambda, u](t) = \frac{1}{2}u^2(t)$ if u increases, and $\mathcal{F}[\lambda, u](t) = -\frac{1}{2}u^2(t)$ if u decreases in $[0,T]$.

(i) To show that Hypothesis 3.1 (iv) is necessary, set $f_\varepsilon(t,v) = f(t,v) \equiv 1$. We are in the situation of Example 2.4, and u_ε converge uniformly to the function $u(t) = \sqrt{2t}$, which does not belong to $W^{1,2q}(0,T)$ for any $q \geq 1$.

(ii) Uniqueness of solutions to the limit problem (0.1) cannot be expected in general even if Hypothesis 3.1 (iv) holds. For the same λ, μ, u^0 as in (i), and for $f_\varepsilon(t,v) = f(t,v) \equiv v$, all hypotheses of Theorem 3.2 are fulfilled, but $u_1(t) = 0$, $u_2(t) = t$, $u_3(t) = -t$ are three distinct solutions of (0.1). However, the ε-approximations u_ε in Theorem 3.2 all converge to u_1.

The following theorem ensures uniqueness of a solution to problem (0.1) with a non-increasing f.

Theorem 3.4 *Let Hypothesis 3.1 hold, and let there exist $T > 0$ and $u_* < u^*$ such that*

$$\frac{\partial f}{\partial v}(t,v) \le 0 \quad \text{a. e. in } \ (0,T) \times (u_*, u^*).$$

Let there exist two solutions $u_1, u_2 : [0,T] \to [u_*, u^*]$ to (0.1) corresponding to two initial conditions $u_1^0 \le u_2^0$ and $\lambda_1, \lambda_2 \in \Lambda_R$, $\lambda_1(r) \le \lambda_2(r)$ for all $r \ge 0$. Then we have $u_1(t) \le u_2(t)$ for every $t \in [0,T]$. In particular, the solution to (0.1) is unique for every initial condition $u^0 \in [u_*, u^*]$.

Proof. Set $B = \max\{R, |u_*|, |u^*|\}$. For all $t \in [0,T]$ and $r > 0$ we have by Lemma 1.4 that $|\mathfrak{p}_r[\lambda_i, u_i](t)| \le B$, and $\mathfrak{p}_r[\lambda_i, u_i](t) = 0$ for $r > B$, $i = 1,2$. Proposition 1.9 yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^B (g(r, \mathfrak{p}_r[\lambda_1, u_1]) - g(r, \mathfrak{p}_r[\lambda_2, u_2]))^+(t) \, \mathrm{d}r \le 0 \ \text{ a.e. in } \ (0,T).$$

We have $\mathfrak{p}_r[\lambda_1, u_1](0) \le \mathfrak{p}_r[\lambda_2, u_2](0)$, hence $g(r, \mathfrak{p}_r[\lambda_1, u_1](t)) \le g(r, \mathfrak{p}_r[\lambda_2, u_2](t))$ for all $t \in [0,T]$ and $r \in (0,B)$. By Hypothesis (1.31), there exists $\varrho_B > 0$ such that for $r < \varrho_B$ we have $g(r, v_1) - g(r, v_2) \ge \alpha(v_1 - v_2)$ for all $B > v_1 > v_2 > -B$. Consequently,

$$\mathfrak{p}_r[\lambda_1, u_1](t) \le \mathfrak{p}_r[\lambda_2, u_2](t) \text{ for } \ t \in [0,T] \ \text{ and } \ r \in (0, \varrho_B).$$

Since $|\mathfrak{p}_r[\lambda_i, u_i](t) - u_i(t)| \le r$ by definition of the play, we may let r tend to zero and obtain $u_1(t) \le u_2(t)$ for all $t \ge 0$. ∎

We now state a sufficient condition for the estimate (3.6).

Proposition 3.5 *Let Hypothesis 3.1 hold. Let there exist U_0 such that $|u_\varepsilon^0| \le U_0$ for every $\varepsilon \in (0, \varepsilon_0)$, and let $T > 0$ be such that*

$$-\int_{-\infty}^{U_0 - \hat\varrho} \varrho(z) \, \mathrm{d}z < \int_0^T f^0(t) \, \mathrm{d}t < \int_{U_0 + \hat\varrho}^{\infty} \varrho(z) \, \mathrm{d}z, \tag{3.15}$$

where $\hat\varrho$ is the least upper bound of the function ϱ introduced in (1.31). Then there exists a positive constant $B > R$ such that (3.6) holds.

Proof. Consider the equation

$$\left.\begin{array}{rcl} \dot{w}_\varepsilon^\sharp(t) & = & f^0(t) \\ w_\varepsilon^\sharp(t) & = & \varepsilon u_\varepsilon^\sharp(t) + \mathcal{F}_\varepsilon[\lambda^\varepsilon, u_\varepsilon^\sharp](t) \\ u_\varepsilon^\sharp(0) & = & (u_\varepsilon^0)^+ . \end{array}\right\} \tag{3.16}$$

We have $\dot{u}_\varepsilon^\sharp \geq 0$ a. e. and $u_\varepsilon^\sharp(0) \geq 0$, hence $u_\varepsilon^\sharp$ is positive and nondecreasing, and we have the representation formula

$$\mathcal{F}_\varepsilon[\lambda^\varepsilon, u_\varepsilon^\sharp](t) = \mathcal{F}_\varepsilon[\lambda^\varepsilon, u_\varepsilon^\sharp](0) + \hat{\Phi}_+^\varepsilon(u_\varepsilon^\sharp(t))$$

for $t \geq 0$, where $\hat{\Phi}_+^\varepsilon$ is the positive initial loading curve associated with $\mathcal{F}_\varepsilon$. We integrate the first equation in (3.16) in time and use inequality (3.4) to obtain

$$\int_0^t f^0(\tau)\, \mathrm{d}\tau \geq \varepsilon \left(u_\varepsilon^\sharp(t) - u_\varepsilon^\sharp(0)\right) + \int_{U_0+\hat{\varrho}}^{u_\varepsilon^\sharp(t)-\hat{\varrho}} \varrho(z)\, \mathrm{d}z\ .$$

Condition (3.15) entails that $u_\varepsilon^\sharp(t)$ are uniformly bounded from above in $[0,T]$ by some $B > 0$. We test the difference of Eqs. (0.1) and (3.16) by $H(u_\varepsilon(t) - u_\varepsilon^\sharp(t))$, and obtain from Proposition 1.9 and Hypothesis 3.1 (ii) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon(u_\varepsilon - u_\varepsilon^\sharp)^+ + \int_0^\infty (g_\varepsilon(r, \mathfrak{p}_r[\lambda^\varepsilon, u_\varepsilon]) - g_\varepsilon(r, \mathfrak{p}_r[\lambda^\varepsilon, u_\varepsilon^\sharp]))^+\, \mathrm{d}r\right)(t) \leq 0 \quad \text{a. e.}\,,$$

hence $u_\varepsilon(t) \leq u_\varepsilon^\sharp(t) \leq B$ for all $t \in [0,T]$. The lower bound is obtained similarly by considering the equation

$$\left.\begin{array}{rcl} \dot{w}_\varepsilon^\flat(t) & = & -f^0(t) \\ w_\varepsilon^\flat(t) & = & \varepsilon u_\varepsilon^\flat(t) + \mathcal{F}_\varepsilon[\lambda^\varepsilon, u_\varepsilon^\flat](t) \\ u_\varepsilon^\flat(0) & = & -(u_\varepsilon^0)^- , \end{array}\right\} \tag{3.17}$$

and the assertion follows. ■

Theorem 3.2 and Proposition 3.5 imply local existence of solutions to problem (0.1) if Hypothesis 3.1 and relation (3.15) hold.

4 Global and periodic solutions

In this section we give sufficient conditions for the existence, uniqueness, and asymptotic stability of global and time periodic solutions to (0.1).

Hypothesis 4.1 *An interval $[u_*, u^*]$, an initial condition $u^0 \in [u_*, u^*]$, and a continuous function $f : \mathbb{R}_+ \times [u_*, u^*]$ are given such that*

(i) $f(t, u^*) \leq 0$, $f(t, u_*) \geq 0$ *for all* $t \geq 0$*;*

(ii) *There exist functions* $\ell \in L^{2q}_{\text{loc}}(\mathbb{R}_+)$, $f^1 \in L^q_{\text{loc}}(\mathbb{R}_+)$, *such that for almost all* $(t,v) \in \mathbb{R}_+ \times (u_*, u^*)$ *we have*

$$\left|\frac{\partial f}{\partial t}(t,v)\right| \leq f^1(t)\,, \tag{4.1}$$

$$\left|\frac{\partial f}{\partial v}(t,v)\right| \leq \ell(t)\,. \tag{4.2}$$

(iii) *There exists* $\varrho_0 > 0$ *such that the following initial compatibility conditions hold:*

$$\begin{array}{l} f(0,u^0) > 0 \implies m_{\lambda_0}(u^0+) \geq \varrho_0\,, \\ f(0,u^0) < 0 \implies m_{\lambda_0}(u^0-) \geq \varrho_0\,. \end{array}$$

The main results of this section read as follows.

Theorem 4.2 *Let Hypotheses 1.7 and 4.1 hold, and let* $R > \max\{|u_*|, |u^*|\}$ *and* $\lambda \in \Lambda_R$ *be given. Then there exists a solution* $u \in W^{1,2q}_{\text{loc}}(\mathbb{R}_+)$ *to Problem (0.1) such that* $u(t) \in [u_*, u^*]$ *for all* $t \geq 0$. *If moreover* $f(t,\cdot)$ *is nonincreasing in* $[u_*, u^*]$, *and* $u_1, u_2 : \mathbb{R}_+ \to [u_*, u^*]$ *are two solutions to (0.1) corresponding to two initial conditions* $u_1^0 \leq u_2^0$, *then* $u_1(t) \leq u_2(t)$ *for every* $t \geq 0$. *In particular, the solution to (0.1) is unique.*

Theorem 4.3 *Let the hypotheses of Theorem 4.2 hold, let* $f(t,\cdot)$ *be decreasing in* $[u_*, u^*]$, *and let there exist* $T > 0$ *such that*

$$f(t,v) = f(t+T, v) \tag{4.3}$$

for all $t \geq 0$ *and* $v \in [u_*, u^*]$. *Let* u *be a solution to Problem (0.1). Then there exists* $\lambda^T \in \Lambda_R$ *and a solution* $u^T \in W^{1,2q}_{\text{loc}}(\mathbb{R}_+)$ *to the problem*

$$\left.\begin{array}{rcl} \dot{w}^T(t) &=& f(t, u^T(t)) \\ w^T(t) &=& \mathcal{F}[\lambda^T, u^T](t) \\ u^T(t) &=& u^T(t+T) \end{array}\right\} \quad \forall t > 0 \tag{4.4}$$

such that $\lim_{t\to\infty} |u(t) - u^T(t)| = 0$. *Moreover, if* (λ_1^T, u_1^T), (λ_2^T, u_2^T) *are two solutions of (4.4), then* $u_1^T = u_2^T$.

In the last statement of Theorem 4.3, the case $\lambda_1^T \neq \lambda_2^T$ and $w_1^T - w_2^T \equiv const \neq 0$ cannot be excluded.

Proof of Theorem 4.2. We define the extended function

$$\tilde{f}(t,v) = \begin{cases} f(t,v) & \text{for } v \in [u_*, u^*]\,, \\ f(t,u_*) & \text{for } v < u_*\,, \\ f(t,u^*) & \text{for } v > u^*\,, \end{cases} \tag{4.5}$$

and consider for $\varepsilon > 0$ the problem

$$\left.\begin{array}{rcl} \dot{w}_\varepsilon(t) & = & \tilde{f}(t, u_\varepsilon(t)) \\ w_\varepsilon(t) & = & \varepsilon u_\varepsilon(t) + \mathcal{F}[\lambda, u_\varepsilon](t) \\ u_\varepsilon(0) & = & u^0 \,. \end{array}\right\} \tag{4.6}$$

We first prove that the solution u_ε to (4.6) satisfies the bounds $u_\varepsilon(t) \in [u_*, u^*]$ for all $\varepsilon > 0$ and $t \geq 0$. We test the first equation of (4.6) by $H(u_\varepsilon(t) - u^*)$ and obtain

$$\dot{w}_\varepsilon(t)\, H(u_\varepsilon(t) - u^*) \leq 0$$

for all $t > 0$. Using the fact that $\mathcal{F}[\lambda, u^*]$ is a constant, we may apply Proposition 1.9, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon(u_\varepsilon - u^*)^+ + \int_0^\infty (g(r, \mathfrak{p}_r[\lambda, u_\varepsilon]) - g(r, \mathfrak{p}_r[\lambda, u^*]))^+ \,\mathrm{d}r\right)(t) \leq 0 \quad \text{a. e.}$$

Hence, $u_\varepsilon(t) \leq u^*$ for all $\varepsilon > 0$ and $t \geq 0$. The lower bound $u_\varepsilon(t) \geq u_*$ is derived similarly by testing with $H(u_* - u_\varepsilon(t))$. The existence of a global solution to (0.1) now follows from Theorem 3.2, and the comparison result for two solutions u_1, u_2 is a direct consequence of Theorem 3.4. ∎

We now pass to the periodic case.

Proof of Theorem 4.3. Let u be the solution to (0.1) from Theorem 4.2. For $n \in \mathbb{N}$ put $\lambda^{(n)}(r) = \mathfrak{p}_r[\lambda, u](nT)$, $u^{(n)}(t) = u(t + nT)$, $w^{(n)}(t) = w(t + nT)$ for $t \geq 0$. By Lemma 1.3 (ii) and (4.3), we have $w^{(n)}(t) = \mathcal{F}[\lambda^{(n)}, u^{(n)}](t)$ for all $t \in [0, T]$.

We define an auxiliary function

$$D(u_1, u_2, \lambda_1, \lambda_2)(t) = \int_0^\infty |g(r, \mathfrak{p}_r[\lambda_1, u_1]) - g(r, \mathfrak{p}_r[\lambda_2, u_2])|(t) \,\mathrm{d}r \tag{4.7}$$

for $u_1, u_2 \in AC(\mathbb{R}_+)$, $\lambda_1, \lambda_2 \in \Lambda_R$, and $t \geq 0$. Whenever (u_i, w_i), $w_i = \mathcal{F}[\lambda_i, u_i]$ for $i = 1, 2$ are two solutions of (0.1), then, as a consequence of (1.36), we have

$$\frac{d}{dt} D(u_1, u_2, \lambda_1, \lambda_2)(t) + |f(t, u_1(t)) - f(t, u_2(t))| \;\leq\; 0 \quad \text{a. e.}\ , \tag{4.8}$$

hence, by virtue of Lemma 1.3 (ii),

$$\frac{d}{dt} D(u^{(n)}, u^{(m)}, \lambda^{(n)}, \lambda^{(m)})(t) + |f(t, u^{(n)}(t)) - f(t, u^{(m)}(t))| \;\leq\; 0 \quad \text{a. e.} \quad \forall n, m \in \mathbb{N}\,. \tag{4.9}$$

We have in particular $\frac{d}{dt} D(u^{(1)}, u, \lambda^{(1)}, \lambda)(t) \leq 0$, hence there exists the limit

$$\Delta \;=\; \lim_{t\to\infty} D(u^{(1)}, u, \lambda^{(1)}, \lambda)(t) \;\geq\; 0\,. \tag{4.10}$$

The set Λ_R is compact with respect to the sup-norm, hence we may find a sequence $\{n_k\}$ in $\mathbb{N}$ such that

$$u^{(n_k)}(0) \to \bar{u}_0 \;\in\; [u_*, u^*]\,, \quad \lambda^{(n_k)} \to \bar{\lambda} \;\in\; \Lambda_R \;\; \text{uniformly}\,. \tag{4.11}$$

By Theorem 3.2, the sequence $\{u^{(n)}\}$ is bounded in $W^{1,2q}(0,T)$. Hence, by compact embedding of $W^{1,2q}(0,T)$ in $C[0,T]$, n_k can be chosen in such a way that

$$u^{(n_k)} \to \bar{u} \ \in \ C[0,T] \ \text{ uniformly} \ . \tag{4.12}$$

Set $\bar{w} = \mathcal{F}[\bar{\lambda}, \bar{u}]$. By continuity of $\mathcal{F}$ we have that $(\bar{u}, \bar{w})$ is a solution of (0.1). To see that $\bar{u}$ is T-periodic, put $\hat{u}(t) = \bar{u}(t+T)$ for $t \geq 0$, $\hat{\lambda}(r) = \mathcal{F}[\bar{\lambda}, \bar{u}](T)$. Using the identity $\mathcal{F}[\lambda^{(n_k)}, u^{(n_k)}](t+T) = \mathfrak{p}_r[\lambda^{(1)}, u^{(1)}](t + n_k T)$ we obtain from (4.10) that

$$D(\hat{u}, \bar{u}, \hat{\lambda}, \bar{\lambda})(t) = \lim_{k\to\infty} D(u^{(1)}, u, \lambda^{(1)}, \lambda)(t + n_k T) \ = \ \Delta \qquad \forall t \geq 0\,, \tag{4.13}$$

hence $\frac{d}{dt} D(\hat{u}, \bar{u}, \hat{\lambda}, \bar{\lambda})(t) \equiv 0$ and (4.8) yields that $f(t, \hat{u}(t)) = f(t, \bar{u}(t))$ a.e. Since f is decreasing, we have $\hat{u}(t) = \bar{u}(t)$ for all $t \geq 0$. This enables us to use (4.8) and conclude for all $n \in \mathbb{N}$ that

$$D(u^{(n+1)}, \bar{u}, \lambda^{(n+1)}, \bar{\lambda})(0) = D(u^{(n)}, \bar{u}, \lambda^{(n)}, \bar{\lambda})(T) \leq D(u^{(n)}, \bar{u}, \lambda^{(n)}, \bar{\lambda})(0)\,. \tag{4.14}$$

We see that $\{D(u^{(n)}, \bar{u}, \lambda^{(n)}, \bar{\lambda})(0)\}$ is a nonincreasing sequence with a subsequence converging to 0. Hence, it converges to 0, and by virtue of (4.8) we have

$$\lim_{n\to\infty} \int_0^T |f(t, u^{(n)}(t)) - f(t, \bar{u}(t))| \, \mathrm{d}t \ = \ 0\,.$$

Using again the compactness of the sequence $\{u^{(n)}\}$ in $C[0,T]$ and the fact that f is injective, we conclude that $u^{(n)}$ converge to $\bar{u}$ uniformly, and the assertion follows. The uniqueness of $\bar{u}$ is also a direct consequence of (4.8). ∎

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