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# Regularity and uniqueness in quasilinear parabolic systems

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**Abstract**

Inspired by a problem in steel metallurgy, we prove the existence, regularity, uniqueness, and continuous data dependence of solutions to a coupled parabolic system in a smooth bounded 3D domain, with nonlinear and nonhomogeneous boundary conditions. The nonlinear coupling takes place in the diffusion coefficient. The proofs are based on anisotropic estimates in tangential and normal directions, and on a refined variant of the Gronwall lemma.

# 1 Introduction

We present here a study of the following system of parabolic equations, in the domain $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^N$:

$$\theta_t - \Delta\theta = r(\theta, c) \tag{1.1}$$

$$c_t - \operatorname{div}(D(\theta,c)\nabla c) = 0 \tag{1.2}$$

with boundary conditions on $\partial\Omega$

$$\frac{\partial\theta}{\partial\nu} + h(x, \theta, \theta_\Gamma(x,t)) = 0 \tag{1.3}$$

$$-D(\theta,c)\frac{\partial c}{\partial\nu} = b(x,t) \tag{1.4}$$

and initial conditions

$$\theta(x,0) = \theta^0(x) \tag{1.5}$$

$$c(x,0) = c^0(x)\,. \tag{1.6}$$

The properties of the nonlinearities $r, D$, and $h$, as well the hypotheses on the data $b$, $\theta_\Gamma$, $\theta^0$, and $c^0$ will be specified in the next section.

The original motivation for the study of this system comes from an industrial process, named gas carburizing. This is a heat treatment of steel with the peculiarity of adding a certain amount of carbon to the surface of the workpiece. In this method, the surface composition of the low carbon steel changes by diffusion of carbon and results in a hard outer surface with good resistance properties. In the above system, $\theta$ represents the absolute temperature and $c$ is the carbon concentration.

In the literature, there are many different approaches to model such processes. In general, the phenomenon can be described as follows: at first the steel is heated up to reach a certain temperature, high enough to allow a good diffusion of carbon into the steel, at this temperature carbon is supplied to the surface, afterwards – but still at high temperature – there is a diffusion stage for the carbon into the steel, and finally the workpiece is rapidly cooled down.

We don't intend here to go into the details of the process, but, for an accurate description of gas carburizing and its modeling, we refer to [4] and references therein.

The analysis carried out here does not cover the complete model proposed in [4], where also the evolution of phase fractions in the steel was taken into account. On the other hand, we obtain additional regularity and continuous data dependence results that are not available in [4]. Note that the system of equations considered in the present paper still describes a very general situation, including the interactions between temperature evolution and diffusion of carbon in all stages of the process. This is reflected in the carbon diffusion coefficient $D(\theta, c)$ and in the heat source term $r(\theta, c)$.

Another relevant issue for applications addressed from this model is the fact that the boundary condition for the temperature $\theta$ encompasses heat exchanges by conduction, convection and radiation. Indeed, during the diffusion period after carburizing, in principle the stage in which the desired carbon profile is achieved, the workpiece remains at a very high temperature and neglecting the thermal radiation effect could be too simplifying. This is why we require no growth restriction on $h(x, \theta, \theta_\Gamma)$, and the boundary condition (1.3) thus includes also the case

$$\frac{\partial \theta}{\partial \nu} + \alpha(x)(\theta - \theta_\Gamma) + \beta(x)(\theta^4 - \theta_\Gamma^4) = 0,$$

with coefficients $\alpha(x), \beta(x) \geq 0$, $\alpha(x) + \beta(x) \geq \alpha_0 > 0$.

The function $\theta_\Gamma$ is the external temperature of the atmosphere. The flux of carbon through the surface of the workpiece is expressed by the function $b(x,t)$. This quantity can be adjusted by an operator, therefore, from the point of view of further applications, can be seen as a control parameter for an optimal control strategy.

The main result of this paper is the proof of uniqueness, in three dimensions, of a solution to (1.1)–(1.6), and its Lipschitz continuous dependence on the data $\theta_\Gamma$, $b$, $\theta^0$, and $c_0$.

The outline of the proof is the following. First, under appropriated regularity assumptions, we prove existence of a generalized solution $(\theta, c)$ of the system (1.1)–(1.6), with

$$\theta \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad c \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))').$$

By a maximum principle and Moser iteration we also prove that the solution $\theta(x,t)$ is positive and uniformly bounded from above in $Q_T$ only assuming linear growth of $r(\theta, c)$.

Secondly we show by elementary means that $\theta$ has the additional regularity

$$\nabla\theta \in L^2(0,T;L^\infty(\Omega)).$$

To this aim, we proceed in several steps. Due to the nonlinearity in the boundary condition (1.3), we first regularize the boundary condition with a parameter $\delta > 0$ that we eventually let tend to zero. We follow the estimation technique proposed in [9] for elliptic equations with linear boundary conditions. Here, however, we obtain different estimates in tangential and normal directions and it is necessary to use an embedding theorem for anisotropic spaces that we prove in the Appendix by the methods of [3].

The last part contains the proof of uniqueness and stability for the whole system. It is based on an $L^p$-variant of the Gronwall lemma, that we believe to be of independent interest.

A similar (degenerate) system with applications in biology has recently been considered in [2] under homogeneous Neumann boundary conditions. Other applications of quasilinear parabolic systems with coupling in the diffusion coefficient can be found e. g. in [10, 11].

The paper is organized as follows: in Section 2 we present the set of assumptions and state the main results of the paper. In Section 3 we prove existence of a weak solution. In Section 4 we treat the regularity of the solution and in the Section 5 we conclude with the uniqueness and stability result for the whole system in 3D. The Appendix is devoted to the proof of an anisotropic embedding theorem.

# 2 Main results

We denote by $V$ the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$, by $V'$ its dual, and state for system (1.1)–(1.6) two sets of hypotheses: Hypothesis 2.1 for existence and its stronger version 2.2 for regularity and uniqueness. Note that we do not assume any upper bound for the growth of $h$.

**Hypothesis 2.1** *The domain $\Omega \subset \mathbb{R}^N$, $N \le 3$, has Lipschitzian boundary. We prescribe the data $b \in L^2(\partial\Omega \times (0,T))$, $c^0 \in L^2(\Omega)$, $\theta^0 \in V \cap L^\infty(\Omega)$, and assume that there exists a constant $\theta_* \ge 0$ such that $\theta^0 \ge \theta_*$ a. e. The function $h$ is measurable in $x$ and continuous in $\theta$ and $\theta_\Gamma$, with the properties*

$$\theta_\Gamma \ge \theta_* \quad \Rightarrow \quad h(x, \theta_*, \theta_\Gamma) \le 0\,,$$

$$\exists a > 0 \ \forall m > 0 \ \exists C_m > 0 : \ \theta_\Gamma \le m, \, \theta \ge 0 \quad \Rightarrow \quad h(x, \theta, \theta_\Gamma) \ge a\theta - C_m.$$

*Furthermore,*

- *$\theta_\Gamma \in L^\infty(\partial\Omega \times (0,T))$, $(\theta_\Gamma)_t \in L^2(\partial\Omega \times (0,T))$, $\theta_\Gamma \ge \theta_*$ a. e.,*
- *$r, D$ are continuous and there exist constants $d_0, d_1, r_1$ such that*

$$0 < d_0 \le D(\theta, c) \le d_1, \quad 0 \le r(\theta, c) \le r_1(|\theta| + |c|).$$

**Hypothesis 2.2** *In addition to Hypothesis 2.1, we assume that the domain $\Omega$ is of class $C^{2,1}$, that is, the outward normal vector has Lipschitz continuous derivatives. There exist connected relatively open subsets $\Gamma_j$ of $\partial\Omega$, $j = 1, \dots, n$, which are $C^{2,1}$-diffeomorphic to open bounded subsets of $\mathbb{R}^2$, and a function $h_0 \in W^{2,\infty}(\partial\Omega)$ such that $h(x, \theta, \theta_\Gamma) = h_0(x)(\theta - \theta_\Gamma)$ on $\partial\Omega \setminus \bigcup_{j=1}^n \Gamma_j$. Furthermore,*

– $\theta^0 \in W^{2,2}(\Omega)$,
– *$h$ is of class $W^{2,\infty}_{loc}$ with respect to all variables,*
– $\theta_\Gamma \in L^2(0,T;W^{2,2}(\partial\Omega))$, $(\theta_\Gamma)_t \in L^2(0,T;W^{1,2}(\partial\Omega))$,
– *$r, F, \partial_\theta F$ are globally Lipschitz continuous with respect to both variables $\theta$ and $c$, where we set*

$$F(\theta,c) = \int_0^c D(\theta,c')\mathrm{d}c' .$$

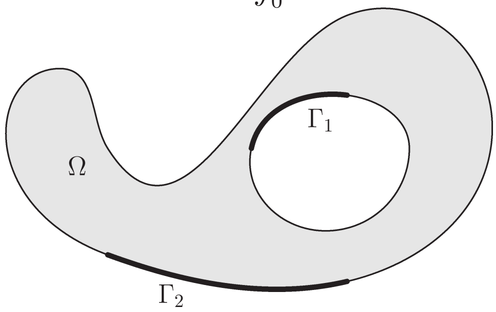


Figure 1. An admissible domain $\Omega$. Thick lines denote the nonlinear boundary regions.

We deal with the following weak formulation of (1.1)–(1.4).

$$\int_\Omega (\theta_t\,\varphi + \nabla\theta\cdot\nabla\varphi - r(\theta,c)\,\varphi)\,\mathrm{d}x + \int_{\partial\Omega} (h(x,\theta,\theta_\Gamma(x,t))\varphi\,\mathrm{d}S \;=\; 0 \qquad (2.1)$$

$$\int_\Omega (c_t\,\psi + D(\theta,c)\nabla c\cdot\nabla\psi)\,\mathrm{d}x + \int_{\partial\Omega} b(x,t)\psi\,\mathrm{d}S \;=\; 0 \qquad (2.2)$$

for every test functions $\varphi, \psi \in V$.

**Theorem 2.3** (Existence) *Let Hypothesis 2.1 hold. Then there exists $K_0 > 0$ and a solution $(\theta, c)$ to the system (2.1)–(2.2) with initial conditions (1.5)–(1.6), with the regularity $c \in L^2(0,T;V)$, $c_t \in L^2(0,T;V')$, $\theta_t \in L^2(Q_T)$, $\nabla\theta \in L^\infty(0,T;L^2(\Omega))$, and such that $\theta_* \le \theta(x,t) \le K_0$ a. e.*

**Theorem 2.4** (Regularity) *Let Hypothesis 2.2 hold. Then every solution $(\theta,c)$ to (2.1)–(2.2) from Theorem 2.3 has the additional regularity $\nabla\theta \in L^2(0,T;L^\infty(\Omega))$.*

To simplify the notation, we introduce the symbol

$$|w(t)|_p = \left(\int_\Omega |w(x,t)|^p \mathrm{d}x\right)^{1/p} \qquad \text{for } t \in (0,T)\,, \qquad (2.3)$$

to denote the partial $L^p(\Omega)$-norm of a generic function $w: Q_T \to \mathbb{R}^d$, $d \ge 1$, with an obvious modification for $p = \infty$.

The main goal of this paper is the following uniqueness and continuous dependence result. It will be based on the partial Kirchhoff transform

$$u = F(\theta, c) \qquad (2.4)$$

with $F$ from Hypothesis 2.2.

**Theorem 2.5** (Uniqueness and continuous data dependence) *Let Hypothesis 2.2 hold, and let* $(\theta_1, c_1), (\theta_2, c_2)$ *be two solutions with the regularity from Theorem 2.4 corresponding to the same* $h(x,\cdot)$, *and to different data* $\theta_i^0, c_i^0, \theta_{\Gamma i}, b_i$, $i = 1,2$, *satisfying Hypothesis 2.2. Let* $u_i = F(\theta_i, c_i)$, $i = 1,2$, *be defined by the Kirchhoff transform (2.4). Set* $\bar\theta = \theta_1 - \theta_2$, $\bar u = u_1 - u_2$, $\bar c^0 = c_1^0 - c_2^0$, $\bar\theta^0 = \theta_1^0 - \theta_2^0$, $\bar\theta_\Gamma = \theta_{\Gamma 1} - \theta_{\Gamma 2}$, $\bar b = b_1 - b_2$. *Then there exists a constant* $M > 0$ *such that the inequality*

$$|\bar\theta(t)|_2^2 + \left|\nabla \int_0^t \bar u(\tau)\,\mathrm{d}\tau\right|_2^2 + \int_0^t \left(|\nabla\bar\theta|_2^2 + |\bar u|_2^2\right)(\tau)\,\mathrm{d}\tau \;\le\; M\,\alpha(t)\,, \qquad (2.5)$$

*holds for every* $t \in [0,T]$, *where we set*

$$\alpha(t) = |\bar\theta^0|_2^2 + |\bar c^0|_2^2 + \int_0^t \int_{\partial\Omega} |\bar\theta_\Gamma(x,\tau)|^2 \mathrm{d}S\,\mathrm{d}\tau + \int_0^t \int_{\partial\Omega} |\bar b(x,\tau)|^2 \mathrm{d}S\,\mathrm{d}\tau\,. \qquad (2.6)$$

Note that the method of [2] yields a slightly different estimate, where $|\nabla \int_0^t \bar u(\tau)\mathrm{d}\tau|_2$ is replaced by $|\bar c(t)|_{V'}$.

# 3 Proof of existence

We fix some $K > 0$ that will be specified later, and set

$$h_K(x,\theta,\theta_\Gamma) = h(x, \max\{\theta_*, \min\{\theta, K\}\}, \theta_\Gamma)\,.$$

Instead of (1.1)–(1.4), we consider the decoupled and truncated problem

$$\int_\Omega (\theta_t\,\varphi + \nabla\theta\cdot\nabla\varphi - r(\theta,\hat c)\,\varphi)\,\mathrm{d}x + \int_{\partial\Omega} h_K(x,\theta,\theta_\Gamma(x,t))\,\varphi\,\mathrm{d}S \;=\; 0 \qquad (3.1)$$

$$\int_\Omega \left(c_t\,\psi + D(\hat\theta,\hat c)\nabla c\cdot\nabla\psi\right)\mathrm{d}x + \int_{\partial\Omega} b(x,t)\psi\,\mathrm{d}S \;=\; 0 \qquad (3.2)$$

for every test functions $\varphi, \psi \in V$, with given functions $\hat\theta, \hat c \in L^2(Q_T)$, and with initial conditions (1.5)–(1.6). We now use the Schauder fixed point theorem. For $m_c, m_\theta > 0$, we fix the set

$$Z(m_\theta, m_c) = \left\{(\theta, c) \in L^2(Q_T)\times L^2(Q_T) : \int_0^T |\theta(t)|_2^2\,\mathrm{d}t \le m_\theta\,,\ \int_0^T |c(t)|_2^2\,\mathrm{d}t \le m_c\right\}. \qquad (3.3)$$

Choosing successively $\varphi = \theta$ and $\varphi = \theta_t$ in (3.1), and $\psi = c$ in (3.2), we obtain the bounds

$$\int_0^T (|c_t(t)|_{V'}^2 + |\nabla c(t)|_2^2)\,\mathrm{d}t + \sup_{t\in(0,T)} |c(t)|_2^2 \le C\,, \qquad (3.4)$$

$$\int_0^T (|\theta_t(t)|_2^2 + |\Delta\theta(t)|_2^2)\,\mathrm{d}t + \sup_{t\in(0,T)} |\nabla\theta(t)|_2^2 \le C\left(1 + \int_0^T |\hat c(t)|_2^2\,\mathrm{d}t\right), \qquad (3.5)$$

where $C$ is a constant independent of $\hat{\theta}$ and $\hat{c}$. We now easily find $m_\theta$ and $m_c$ such that the solution $(\theta, c)$ belongs to $Z(m_\theta, m_c)$ whenever $(\hat{\theta}, \hat{c}) \in Z(m_\theta, m_c)$. The solution mapping associated with (3.1)–(3.2) is compact in $L^2(Q_T) \times L^2(Q_T)$, hence it admits a fixed point, which is a solution to (1.5)–(2.2) with $h$ replaced by $h_K$.

It remains to find uniform bounds $\theta_* \le \theta \le K_0$ independent of $K$. Choosing $K > K_0$, we eventually obtain the assertion.

To do so, we first choose in (3.1) $\varphi = -(\theta_* - \theta)^+$, where $z^+$ denotes the positive part of an element $z \in \mathbb{R}$. We obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |(\theta_* - \theta)^+|^2 \,\mathrm{d}x + \int_\Omega |\nabla(\theta_* - \theta)^+|^2 \,\mathrm{d}x \le 0\,,$$

hence $\theta(x,t) \ge \theta_*$ a. e.

The upper bound is obtained by Moser iterations similarly as in [6]. Set $f(x,t) = r(\theta(x,t), c(x,t))$ and $\theta_K = \min\{\theta, K\}$. Estimates (3.4)–(3.5), Sobolev embeddings, and interpolations in Lebesgue spaces yield $f \in L^2(0,T;L^6(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \subset L^q(Q_T)$ for $q = 10/3$. The function $\theta$ is a solution of the equation

$$\int_\Omega (\theta_t\,\varphi + \nabla\theta\cdot\nabla\varphi - f(x,t)\,\varphi)\,\mathrm{d}x + \int_{\partial\Omega} h(x, \theta_K, \theta_\Gamma(x,t))\,\varphi\,\mathrm{d}S \;=\; 0 \qquad (3.6)$$

for every $\varphi \in V$. We may choose in particular $\varphi = p\theta_K^{p-1}$ for $p > 1$, with the intention to let $p$ tend to $\infty$. In the remaining part of this section, we denote by $C$ any constant independent of $K$ and $p$. Setting $v_{Kp} = \theta_K^{p/2}$, we obtain from (3.6) after integration with respect to $t$ that

$$\begin{aligned} &|v_{Kp}(t)|_2^2 + \int_0^t |\nabla v_{Kp}(\tau)|_2^2\,\mathrm{d}\tau + ap\int_0^t\int_{\partial\Omega} |v_{Kp}(x,\tau)|^2\,\mathrm{d}S\,\mathrm{d}\tau \qquad (3.7)\\ &\quad\le\; C^p + Cp\left(\int_0^t\int_\Omega |f|\,|v_{Kp}|^{2/p'}\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_{\partial\Omega} |v_{Kp}(x,\tau)|^{2/p'}\,\mathrm{d}S\,\mathrm{d}\tau\right), \end{aligned}$$

where prime denotes here and in the sequel the conjugate exponent. Using Hölder's inequality, we eliminate the boundary integrals and obtain

$$\begin{aligned} |v_{Kp}(t)|_2^2 + \int_0^t |\nabla v_{Kp}(\tau)|_2^2\,\mathrm{d}\tau \;&\le\; C^p + Cp\int_0^t\int_\Omega |f|\,|v_{Kp}|^{2/p'}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \qquad (3.8)\\ &\le\; C^p + Cp\|f\|_q\|v_{Kp}\|_{2q'}^{2/p'}\,, \end{aligned}$$

where we use for simplicity the notation

$$\|v\|_r = \left(\int_0^T\int_\Omega |v(x,t)|^r\,\mathrm{d}x\,\mathrm{d}t\right)^{1/r}$$

for $v \in L^r(\Omega\times(0,T))$ and $r \ge 1$. Set $q_0 = (N/2)+1$. Then $q_0 < q$, and we define $\varrho > 0$ by the formula $q_0' = (1+\varrho)q'$. From the Gagliardo-Nirenberg inequality we obtain the estimate

$$\|v_{Kp}\|_{2q_0'}^2 \le C\left(\sup_{t\in(0,T)} |v_{Kp}(t)|_2^2 + \int_0^t |\nabla v_{Kp}(\tau)|_2^2\,\mathrm{d}\tau\right),$$

hence, by virtue of (3.8) and Young's inequality, we obtain

$$\|v_{Kp}\|_{2q_0'}^2 \leq Cp \max\left\{1, C^p, \|v_{Kp}\|_{2q'}^2\right\}, \tag{3.9}$$

that is,

$$\|\theta_K\|_{pq_0'} \leq (Cp)^{1/p} \max\left\{C, \|\theta_K\|_{pq'}\right\}, \tag{3.10}$$

with a constant $C$ independent of $K$ and $p$. We now set $p_j = (1+\varrho)^j$, $z_j = \|\theta_K\|_{p_j q_0'}$, and $y_j = \max\{C, z_j\}$ for $j = 0, 1, 2, \dots$. Then (3.10) has the form

$$y_j \leq (Cp_j)^{1/p_j}\, y_{j-1} \quad \text{for } j \in \mathbb{N}. \tag{3.11}$$

This can be rewritten as

$$\log y_j \leq C\,(1+\varrho)^{-j}(1+j) + \log y_{j-1} \quad \text{for } j \in \mathbb{N}, \tag{3.12}$$

hence the sequence $y_j$ is bounded by a constant $C$ independent of $K$. Consequently, there exists $K_0$ such that

$$\|\theta_K\|_p \leq K_0 \tag{3.13}$$

independently of $p$ and $K$, which is the desired estimate that enables us to complete the proof of Theorem 2.3. ■

# 4 Proof of regularity

We give here a straightforward proof of Theorem 2.4, which will follow from a regularity result for linear heat equation with nonlinear boundary condition

$$\int_\Omega (v_t\,\varphi + A(x)\nabla v \cdot \nabla\varphi + B(x,t)\cdot\nabla\varphi - f(x,t)\,\varphi)\,\mathrm{d}x + \int_{\partial\Omega} h(x, v, v_\Gamma(x,t))\,\varphi\,\mathrm{d}S \;=\; 0 \tag{4.1}$$

for every test function $\varphi \in W^{1,2}(\Omega)$, with initial condition $v(x,0) = v^0(x)$, where $A = (A_{ij})_{i,j=1}^N : \Omega \to \mathbb{R}^{N\times N}_{\mathrm{sym}}$ is a symmetric matrix function such that there exists $\kappa > 0$ with the property

$$\forall \xi \in \mathbb{R}^N: \; A(x)\xi\cdot\xi \geq \kappa\,|\xi|^2 \quad \text{a.e.}, \tag{4.2}$$

and $f : Q_T \to \mathbb{R}$, $B : Q_T \to \mathbb{R}^N$, $h : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$, and $v_\Gamma : \partial\Omega \times (0,T) \to \mathbb{R}$ are given functions.

The reasons for introducing the functions $A(x)$ and $B(x,t)$, which do not appear in (2.1), are purely technical. They arise as a result of deformations of the domain and partition of unity.

Consider a set $\Omega \subset \mathbb{R}^N$ of the form

$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1}\times\mathbb{R}: \; x_N > g(x')\}, \tag{4.3}$$

with a given function $g$. The regularity results read as follows.

**Theorem 4.1** *Let $\Omega$ be as in (4.3), and let $g \in W^{2,\infty}(\mathbb{R}^{N-1})$. We make the following assumptions:*

- *$h$ is a globally Lipschitz continuous function in all variables; furthermore, with $v, v_\Gamma \in \mathbb{R}$ fixed, the functions $h(\cdot, v, v_\Gamma), \partial_\ell h(\cdot, v, v_\Gamma)$ belong to $L^2(\partial\Omega)$ for all $\ell = 1, \ldots, N-1$;*
- *$A \in W^{1,\infty}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$, $B \in L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^N))$, $B_t \in L^2(Q_T; \mathbb{R}^N)$;*
- *$v^0 \in W^{1,2}(\Omega)$, $f \in L^2(Q_T)$, $v_\Gamma \in L^2(0,T; W^{1,2}(\partial\Omega))$, $(v_\Gamma)_t \in L^2(0,T; L^2(\partial\Omega))$.*

*Let $v \in L^2(0,T;V)$ such that $v_t \in L^2(0,T;V')$ be a solution to (4.1). Then $v$ has the regularity $v_t \in L^2(Q_T)$, $v \in L^2(0,T; W^{2,2}(\Omega))$, and $\nabla v \in L^\infty(0,T; L^2(\Omega))$.*

**Theorem 4.2** *Let $\Omega$ be as in (4.3), and let $g \in W^{3,\infty}(\mathbb{R}^{N-1})$. We make the following assumptions:*

- *$h$ is of class $W^{2,\infty}$ with respect to all variables; furthermore, with $v, v_\Gamma \in \mathbb{R}$ fixed, the functions $h(\cdot, v, v_\Gamma)$, $\partial_\ell h(\cdot, v, v_\Gamma)$, $\partial_\ell \partial_m h(\cdot, v, v_\Gamma)$ belong to $L^2(\partial\Omega)$ for all $\ell, m = 1, \ldots, N-1$;*
- *$A \in W^{2,\infty}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$, $B \in L^2(0,T; W^{2,2}(\Omega; \mathbb{R}^N))$, $B_t \in L^2(Q_T; \mathbb{R}^N)$;*
- *$v^0 \in W^{2,2}(\Omega)$, $f \in L^2(0,T; W^{1,2}(\Omega))$, $v_\Gamma \in L^2(0,T; W^{2,2}(\partial\Omega))$, $(v_\Gamma)_t \in L^2(0,T; W^{1,2}(\partial\Omega))$.*

*Let $v \in L^2(0,T;V)$ such that $v_t \in L^2(0,T;V')$ be a solution to (4.1). If $N \le 3$, then $v$ has the regularity $v_t \in L^2(Q_T)$, $v \in L^2(0,T; W^{2,2}(\Omega))$, and $\nabla v \in L^2(0,T; C(\bar{\Omega}))$.*

The authors do not know of any reference for Theorem 4.2. It is true that the boundary nonlinearity is quite weak, but it cannot be easily removed, because the trace of $v_t$ would come into play, for which no estimate is available. An extension of the general parabolic regularity theory from [5, 6, 8] might possibly work here, but we propose instead an elementary proof based on the method from [9] designed originally for elliptic equations with linear boundary conditions.

We first consider the case that $\Omega$ is a half-space of the form

$$\Omega = \mathbb{R}^N_+ := \{(y', y_N) : y' \in \mathbb{R}^{N-1}, y_N > 0\}. \tag{4.4}$$

For a general function $w \in W^{1,2}(\mathbb{R}^N_+)$, we have the identity

$$w^2(y', y_N) - w^2(y', 0) = 2 \int_0^{y_N} w(y', z)\, \partial_N w(y', z) \,\mathrm{d}z,$$

hence, for every $M > 0$, integrating w.r.t. $y_N$, we have by Fubini's Theorem that

$$w^2(y', 0) \le \frac{1}{M} \int_0^M w^2(y', y_N) \,\mathrm{d}y_N + 2 \int_0^M |w(y', z)|\, |\partial_N w(y', z)| \,\mathrm{d}z.$$

Letting $M$ tend to $\infty$, we obtain the trace interpolation formula

$$|w(\cdot,0)|^2_{L^2(\mathbb{R}^{N-1})} \le 2|w|_2\,|\partial_N w|_2\,, \tag{4.5}$$

or, as a consequence,

$$\forall \varepsilon>0 \ \ \exists C_\varepsilon>0 \ \ \forall w\in W^{1,2}(\mathbb{R}^N_+)\ :\ \ |w(\cdot,0)|^2_{L^2(\mathbb{R}^{N-1})} \le C_\varepsilon\,|w|_2^2+\varepsilon\,|\nabla w|_2^2\,. \tag{4.6}$$

For domains of the form (4.3) with a Lipschitzian function $g$, this inequality reads after substitution in the integrals as

$$\forall \varepsilon>0 \ \ \exists C_\varepsilon>0 \ \ \forall w\in W^{1,2}(\Omega)\ :\ \ \int_{\partial\Omega}|w(x)|^2\,\mathrm{d}S \le C_\varepsilon\,|w|_2^2+\varepsilon\,|\nabla w|_2^2\,. \tag{4.7}$$

By a partition of unity argument, we obtain (4.7) for every Lipschitzian domain $\Omega$, see also [9].

In the context of (4.4), we rewrite Eq. (4.1) as

$$\int_{\mathbb{R}^N_+}(v_t\,\varphi+(A\nabla v+B)\cdot\nabla\varphi-f\,\varphi)\,\mathrm{d}y+\int_{\mathbb{R}^{N-1}}h(y',v(y',0,t),v_\Gamma(y',t))\,\varphi(y',0)\,\mathrm{d}y'=0. \tag{4.8}$$

We will also deal with the regularized problem

$$\begin{aligned}&\int_{\mathbb{R}^N_+}(v_t\,\varphi+(A\nabla v+B)\cdot\nabla\varphi-f\,\varphi)\,\mathrm{d}y \\ &\quad+\int_{\mathbb{R}^{N-1}}(\delta\,\nabla_{y'}v(y',0,t)\cdot\nabla_{y'}\varphi(y',0)+h(y',v(y',0,t),v_\Gamma(y',t))\,\varphi(y',0))\,\mathrm{d}y'=0\end{aligned} \tag{4.9}$$

with some $\delta\ge 0$, where $\nabla_{y'}$ denotes the partial gradient $\nabla_{y'}v=(\partial_1 v,\dots,\partial_{N-1}v)$, which has to be satisfied in the case $\delta>0$ for every test function $\varphi(y',y_N)$ from the space

$$W=\{\varphi\in W^{1,2}(\mathbb{R}^N_+)\,:\,\varphi(\cdot,0)\in W^{1,2}(\mathbb{R}^{N-1})\}\,.$$

Our goal is to derive bounds for its solution independent of $\delta$, which then imply the corresponding estimates for the solution of (4.8).

**Lemma 4.3** *Let $v^0\in W^{1,2}(\mathbb{R}^N_+)$, $f\in L^2(\mathbb{R}^N_+\times(0,T))$, $A\in W^{1,\infty}(\mathbb{R}^N_+;\mathbb{R}^{N\times N}_{\mathrm{sym}})$, $B\in L^2(0,T;W^{1,2}(\mathbb{R}^N_+;\mathbb{R}^N))$, and $v_\Gamma\in L^2(0,T;W^{1,2}(\mathbb{R}^{N-1}))$ be given. Let there exist a function $h_1\in L^2(\mathbb{R}^{N-1})\cap L^\infty(\mathbb{R}^{N-1})$ such that $h$ together with all its first derivatives is bounded above by $h_1$. Then there exists a constant $C_1>0$ independent of $\delta\ge 0$ such that the solution $v$ to (4.9) satisfies for all $t\in[0,T]$ the estimate*

$$|\partial_\ell v(t)|_2^2+\int_0^T|\nabla\partial_\ell v(t)|_2^2\,\mathrm{d}t+\delta\int_0^T\int_{\mathbb{R}^{N-1}}|\partial_\ell\partial_m v(y',0,t)|^2\,\mathrm{d}y'\,\mathrm{d}t\le C_1 \tag{4.10}$$

*for all $\ell,m=1,\dots,N-1$.*

*Proof.* A solution $v \in L^2(0,T;W)$ such that $v_t \in L^2(0,T;W')$ to (4.9) can be constructed e.g. by implicit time discretization using the Schauder fixed point theorem in each time step. The passage to the limit can be carried out using the compactness lemma in [7, Section 1.5] and the compact embedding of $W$ in the space $H = \{\varphi \in L^2(\mathbb{R}^N_+) : \varphi(\cdot,0) \in L^2(\mathbb{R}^{N-1})\}$. The solution is unique, and satisfies the bound

$$|v(T)|_2^2 + \int_0^T |\nabla v(t)|_2^2 \,\mathrm{d}t + \delta \int_0^T \int_{\mathbb{R}^{N-1}} |\nabla_{y'} v(y',0,t)|^2 \,\mathrm{d}y' \,\mathrm{d}t \le C_0 \tag{4.11}$$

by virtue of (4.6) and Gronwall's lemma, with a constant $C_0$ independent of $\delta$. To obtain higher order estimates, we denote by $e_\ell$ for $\ell = 1,\dots,N$ the $\ell$-th unit coordinate vector, and by $D_s^\ell$ for $s \ne 0$ the linear mapping

$$D_s^\ell(v)(y,t) = \frac{1}{s}\,(v(y+se_\ell,t) - v(y,t)).$$

Let $\varphi \in W$ be given. In (4.9), we choose consecutively test functions $\tilde\varphi(y) = \varphi(y)$ and $\tilde\varphi(y) = \varphi(y - se_\ell)$ for some $\ell = 1,\dots,N-1$, and subtract the two identities. This yields, after a suitable substitution, that

$$\begin{aligned}&\int_{\mathbb{R}^N_+} \big(D_s^\ell v_t\,\varphi + \big(A(y+se_\ell)\nabla(D_s^\ell v) + (D_s^\ell A)\nabla v + D_s^\ell B\big)\cdot\nabla\varphi + f(y,t)\,D_{-s}^\ell\varphi\big)\mathrm{d}y \\ &+ \int_{\mathbb{R}^{N-1}} \big(\delta\nabla_{y'} D_s^\ell v(y',0,t)\cdot\nabla_{y'}\varphi(y',0) + D_s^\ell\big(h(y',v(y',0,t),v_\Gamma(y',t))\big)\,\varphi(y',0)\big)\,\mathrm{d}y'.\end{aligned}\tag{4.12}$$

For $\varphi(y) = D_s^\ell v(y,t)$, we have in particular the estimate

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|D_s^\ell v(t)\big|_2^2 + \kappa\big|\nabla(D_s^\ell v(t))\big|_2^2 + \delta\int_{\mathbb{R}^{N-1}} |\nabla_{y'} D_s^\ell v(y',0,t)|^2\,\mathrm{d}y' \\ &\quad\le\ |\nabla A|_\infty\,|\nabla v(t)|_2\,\big|\nabla(D_s^\ell v(t))\big|_2 + (|f(t)|_2 + |\nabla B(t)|_2)\,\big|\nabla(D_s^\ell v(t))\big|_2 \\ &\qquad + \int_{\mathbb{R}^{N-1}} h_1(y')(1 + |D_s^\ell v_\Gamma(y',t)| + |D_s^\ell v(y',0,t)|)\,|D_s^\ell v(y',0,t)|\,\mathrm{d}y'.\end{aligned}\tag{4.13}$$

We can pass to the limit as $s \to 0$ and obtain from (4.11), (4.6), and Gronwall's lemma the bound

$$\begin{aligned}&|\partial_\ell v(t)|_2^2 + \int_0^T |\nabla\partial_\ell v(t)|_2^2\,\mathrm{d}t + \delta\int_0^T\int_{\mathbb{R}^{N-1}} |\nabla_{y'}\partial_\ell v(y',0,t)|^2\,\mathrm{d}y'\,\mathrm{d}t \\ &\quad\le\ C\Big(1 + C_0 + |\nabla v^0|_2^2 + |\nabla A|_\infty^2 + \int_0^T |f(t)|_2^2\,\mathrm{d}t + \int_0^T |\nabla_{y'} v_\Gamma(t)|_{2,\mathbb{R}^{N-1}}^2\mathrm{d}t\Big)\end{aligned}\tag{4.14}$$

with a constant $C$ independent of $\delta$, which we wanted to prove. ■

**Lemma 4.4** *Under the hypotheses of Lemma 4.3, assume in addition that $(v_\Gamma)_t \in L^2(\mathbb{R}^{N-1}\times(0,T))$ and $B_t \in L^2(\mathbb{R}^N_+\times(0,T);\mathbb{R}^N)$. Then there exists a constant $C_2$ independent of $\delta \ge 0$ such that the solution $v$ to (4.9) satisfies for every $t \in [0,T]$ the estimate*

$$\delta\int_{\mathbb{R}^{N-1}} |\nabla_{y'} v(y',0,t)|^2\,\mathrm{d}y' + |\nabla v(t)|_2^2 + \int_0^T \Big(|v(t)|_{W^{2,2}(\mathbb{R}^N_+)}^2 + |v_t(t)|_2^2\Big)\,\mathrm{d}t \le C_2\,. \tag{4.15}$$

*Proof.* We discretize Eq. (4.9) in time, test by the time increment of $v$, and let the time step tend to $0$. In the limit we obtain the identity

$$\int_{\mathbb{R}^N_+} \left(|v_t|^2 - f\, v_t - B_t \cdot \nabla v\right)\, \mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N_+} \left(\frac{1}{2} A(y)\nabla v + B\right) \cdot \nabla v \, \mathrm{d}y \tag{4.16}$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{N-1}} \left(\frac{\delta}{2} |\nabla_{y'} v|^2 + \hat{h}(y', v, v_\Gamma)\right) \mathrm{d}y' \; = \; \int_{\mathbb{R}^{N-1}} \partial_{v_\Gamma} \hat{h}(y', v, v_\Gamma)\, (v_\Gamma)_t \, \mathrm{d}y' \,,$$

where

$$\hat{h}(y', v, v_\Gamma) = \int_0^v h(y', u, v_\Gamma)\, \mathrm{d}u \,.$$

We have

$$\hat{h}(y', v, v_\Gamma) \le \frac{h_1}{2} v^2 + |v| \left(h_1 |v_\Gamma| + |h(y', 0, 0)|\right)$$

and

$$|\partial_{v_\Gamma} \hat{h}(y', v, v_\Gamma)| \le h_1 |v| \,.$$

This yields the estimate

$$\delta \int_{\mathbb{R}^{N-1}} |\nabla_{y'} v(y', 0, t)|^2 \, \mathrm{d}y' + \int_0^t |v_t(\tau)|_2^2 \, \mathrm{d}\tau + |\nabla v(t)|_2^2 \tag{4.17}$$
$$\le \; C\Big(1 + |\nabla v^0|_2^2 + \int_0^t |f(\tau)|_2^2 \mathrm{d}\tau + \int_{\mathbb{R}^{N-1}} |v_\Gamma(y', t)|^2 \mathrm{d}y'$$
$$+ \int_0^t \int_{\mathbb{R}^{N-1}} |(v_\Gamma)_t(y', t)|^2 \mathrm{d}y' \, \mathrm{d}\tau\Big)$$

with a constant $C$ independent of $\delta$ and $t$ as a consequence of (4.6) and Gronwall's lemma. By Lemma 4.3, we have $\nabla \partial_\ell v \in L^2(\mathbb{R}^N_+ \times (0,T))$ for all $\ell = 1, \dots, N-1$. To finish the proof, we now choose in Eq. (4.9) any test function $\varphi = \varphi_0 \in L^2(0,T; W^{1,2}(\mathbb{R}^N_+))$ with a compact support in $\mathbb{R}^N_+$. We integrate by parts in all terms except for $A_{NN} \partial_N v \, \partial_N \varphi$, and obtain an identity of the form

$$\int_0^T \int_{\mathbb{R}^N_+} A_{NN}(y) \partial_N v(y,t)\, \partial_N \varphi_0(y,t)\, \mathrm{d}y \, \mathrm{d}t \; = \; \int_0^T \int_{\mathbb{R}^N_+} \Psi(y,t)\, \varphi_0(y,t)\, \mathrm{d}y \, \mathrm{d}t \tag{4.18}$$

with a function $\Psi \in L^2(\mathbb{R}^N_+ \times (0,T))$. Hence, $\partial_N(A_{NN}(y) \partial_N v(y,t))$ belongs to $L^2(\mathbb{R}^N_+ \times (0,T))$. By (4.2), we have $A_{NN}(y) \ge \kappa$, and since $A_{NN} \in W^{1,\infty}(\mathbb{R}^N_+)$, we obtain the $L^2$-bound for $\partial_N^2 v$, and the proof of Lemma 4.4 is complete. ∎

Lemmas 4.3 and 4.4 enable us to rewrite Eq. (4.9) in strong form

$$v_t - \mathrm{div}\,(A(y)\nabla v + B(y,t)) - f(y,t) \; = \; 0 \qquad \text{a. e. in } \; \mathbb{R}^N_+ \times (0,T) \,, \tag{4.19}$$
$$\sum_{j=1}^N A_{Nj} \partial_j v + B_N - \delta\, \Delta_{y'} v + h(y', v, v_\Gamma(y',t)) \; = \; 0 \qquad \text{a. e. in } \; \mathbb{R}^{N-1} \times (0,T) \,, \tag{4.20}$$

where $\Delta_{y'}$ is the Laplacian with respect to $y'$.

**Lemma 4.5** *Let $N \leq 3$ and $\delta > 0$. Under the hypotheses of Lemma 4.4, assume in addition that $v^0 \in W^{2,2}(\mathbb{R}^N_+)$, $f \in L^2(0,T;W^{1,2}(\mathbb{R}^N_+))$, $A \in W^{2,\infty}(\mathbb{R}^N_+;\mathbb{R}^{N\times N}_{\mathrm{sym}})$, $B \in L^2(0,T;W^{2,2}(\mathbb{R}^N_+;\mathbb{R}^N))$, $v_\Gamma \in L^2(0,T;W^{2,2}(\mathbb{R}^{N-1}))$, $(v_\Gamma)_t \in L^2(0,T;W^{1,2}(\mathbb{R}^{N-1}))$, and that there exists a function $h_2 \in L^2(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ such that $h$ together with all its first and second derivatives is bounded above by $h_2$. Then there exists a constant $C_3 > 0$ independent of $\delta$ such that the solution $v$ to (4.9) satisfies for all $t \in [0,T]$ the estimate*

$$|\partial_m\partial_\ell v(t)|_2^2 + \int_0^T |\nabla\partial_m\partial_\ell v(t)|_2^2 \,\mathrm{d}t + \delta\int_0^T\int_{\mathbb{R}^{N-1}} |\partial_\ell\partial_m\partial_k v(y',0,t)|^2 \,\mathrm{d}y'\,\mathrm{d}t \leq C_3 \quad (4.21)$$

*for all $\ell, m, k = 1,\dots,N-1$.*

*Proof.* Passing to the limit in (4.12) as $s \to 0$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N_+} \big(\partial_\ell v_t\,\varphi + (A\nabla\partial_\ell v + (\partial_\ell A)\nabla v + \partial_\ell B)\cdot\nabla\varphi - \partial_\ell f(y,t)\,\varphi\big)\mathrm{d}y \qquad (4.22)\\ &\quad + \int_{\mathbb{R}^{N-1}} (\delta\nabla_{y'}\partial_\ell v\cdot\nabla_{y'}\varphi + \partial_\ell(h(y',v(y',0,t),v_\Gamma(y',t)))\,\varphi(y',0)\,\mathrm{d}y' = 0\,.\end{aligned}$$

Similarly as in (4.12), we apply to (4.22) the operator $D^m_s$ with $m \in \{1,\dots,N-1\}$, and set $\varphi(y) = D^m_s\partial_\ell v(y,t)$, with the intention to proceed as in the proof of Lemma 4.3. Here, the situation is more delicate because the second derivatives of the nonlinear term $h(y',v,v_\Gamma)$ will be involved. We obtain the inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D^m_s\partial_\ell v(t)|_2^2 + \kappa\,|\nabla(D^m_s\partial_\ell v(t))|_2^2 + \delta\int_{\mathbb{R}^{N-1}} |\nabla_{y'}D^m_s\partial_\ell v(y',0,t)|^2\,\mathrm{d}y' \qquad (4.23)\\ &\quad\leq\ \gamma(t) + C\int_{\mathbb{R}^{N-1}} (|\partial_\ell v||\partial_m v| + |\partial_\ell\partial_m v|)|\partial_\ell\partial_m v|(y',0,t)\,\mathrm{d}y'\,,\end{aligned}$$

where $\gamma \in L^1(0,T)$ includes all terms that have already been estimated above, and $C$ is a constant independent of $t$ and $\delta$. The right hand side of (4.23) is in $L^1(0,T)$ by virtue of Lemmas 4.3 and 4.4 and of the interpolation inequality

$$|\psi|_{L^4(\mathbb{R}^{N-1})} \leq C(|\psi|_{L^2(\mathbb{R}^{N-1})} + |\psi|^{1/2}_{L^2(\mathbb{R}^{N-1})}\,|\nabla_{y'}\psi|^{1/2}_{L^2(\mathbb{R}^{N-1})}) \qquad (4.24)$$

for every $\psi \in W^{1,2}(\mathbb{R}^{N-1})$. Indeed, the bound still depends on $\delta$, and this dependence has to be removed. Passing to the limit in (4.23) as $s \to 0$ and using (4.24), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_m\partial_\ell v(t)|_2^2 + \kappa\,|\nabla\partial_m\partial_\ell v(t)|_2^2 + \delta\int_{\mathbb{R}^{N-1}} |\nabla_{y'}\partial_m\partial_\ell v(y',0,t)|^2\,\mathrm{d}y' \qquad (4.25)\\ &\leq\ \gamma(t) + C\int_{\mathbb{R}^{N-1}} (|\nabla_{y'}v|^4 + |\partial_\ell\partial_m v|^2)(y',0,t)\,\mathrm{d}y'\\ &\leq\ \gamma(t) + C\int_{\mathbb{R}^{N-1}} |\partial_\ell\partial_m v|^2(y',0,t)\,\mathrm{d}y'\\ &\quad + C\big(|\nabla_{y'}v(\cdot,0,t)|^4_{L^2(\mathbb{R}^{N-1})} + |\nabla_{y'}v(\cdot,0,t)|^2_{L^2(\mathbb{R}^{N-1})}\,|\Delta_{y'}v(\cdot,0,t)|^2_{L^2(\mathbb{R}^{N-1})}\big)\,,\end{aligned}$$

with a possibly different function $\gamma \in L^1(0,T)$ and different constants $C$ independent of $t$ and $\delta$. Formula (4.5) enables us to estimate the right hand side of (4.25) from above by

$$\gamma(t) + C\left(|\partial_\ell\partial_m v|_2|\partial_N\partial_\ell\partial_m v|_2 + |\nabla_{y'}v|_2^2|\partial_N\nabla_{y'}v|_2^2 + |\nabla_{y'}v|_2|\partial_N\nabla_{y'}v|_2|\Delta_{y'}v|_2|\partial_N\Delta_{y'}v|_2\right).$$

By Lemma 4.3, we have $|\nabla_{y'}v|_2 \le C_1$, and $\beta(t) := |\partial_N\nabla_{y'}v(t)|_2$ belongs to $L^2(0,T)$. Hence, by Young's inequality, we obtain from (4.25) the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\ell,m=1}^{2}|\partial_m\partial_\ell v(t)|_2^2 + \sum_{\ell,m=1}^{2}|\nabla\partial_m\partial_\ell v(t)|_2^2 \le \gamma(t) + C\sum_{\ell,m=1}^{2}\left(|\partial_\ell\partial_m v|_2^2 + \beta^2(t)|\Delta_{y'}v|_2^2\right), \tag{4.26}$$

and from Gronwall's argument we obtain (4.5). ■

We now let $\delta$ tend to 0 and prove the following step.

**Lemma 4.6** *Under the hypotheses of Lemma 4.5, there exists a constant $C_4 > 0$ such that the solution $v$ to (4.8) satisfies for all $t \in [0,T]$ the estimate*

$$|\partial_N\partial_\ell v(t)|_2^2 + \int_0^T |\nabla\partial_N\partial_\ell v(t)|_2^2\,\mathrm{d}t \le C_4. \tag{4.27}$$

*for all $\ell = 1,\dots,N-1$.*

*Proof.* From Lemma 4.5 it follows that the solution $v$ to (4.8) satisfies (4.19)–(4.21) with $\delta = 0$. Let us consider now test functions $\varphi$ with compact support in $\mathbb{R}^N_+$, and apply the operator $D^N_s$ to Eq. (4.8). As a counterpart of (4.12), we obtain

$$\int_{\mathbb{R}^N_+}\left(D^N_s v_t\,\varphi + (A(y+se_N)\nabla(D^N_s v) + (D^N_s A)\nabla v + D^N_s B)\cdot\nabla\varphi - D^N_s f(y,t)\,\varphi\right)\mathrm{d}y = 0. \tag{4.28}$$

Passing to the limit as $s \to 0$ yields

$$\int_{\mathbb{R}^N_+}\left(\partial_N v_t\,\varphi + (A(y)\nabla\partial_N v + (\partial_N A(y))\nabla v + \partial_N B(y,t))\cdot\nabla\varphi - \partial_N f(y,t)\,\varphi\right)\mathrm{d}y = 0. \tag{4.29}$$

Let $V_0$ denote the space $W^{1,2}_0(\mathbb{R}^N_+)$. We choose any $\varphi_0 \in V_0$, set $\varphi = A_{N\ell}\,\varphi_0$ in (4.22), $\varphi = A_{NN}\,\varphi_0$ in (4.29), and obtain, using the formula

$$A\nabla\partial_\ell v\cdot\nabla(A_{N\ell}\,\varphi_0) = A\nabla(A_{N\ell}\,\partial_\ell v)\cdot\nabla\varphi_0 + (A\nabla\partial_\ell v\cdot\nabla A_{N\ell})\,\varphi_0 - \partial_\ell v(A\nabla A_{N\ell}\cdot\nabla\varphi_0),$$

that

$$\begin{aligned}\int_{\mathbb{R}^N_+}\Big(A_{N\ell}\,\partial_\ell v_t\,\varphi_0 + A\nabla(A_{N\ell}\,\partial_\ell v)\cdot\nabla\varphi_0 + \underbrace{(A\nabla\partial_\ell v\cdot\nabla A_{N\ell})\,\varphi_0}_{I} - \underbrace{\partial_\ell v(A\nabla A_{N\ell}\cdot\nabla\varphi_0)}_{II}\\ + \underbrace{(\partial_\ell A\nabla v + \partial_\ell B)\cdot\nabla(A_{N\ell}\,\varphi_0)}_{III} - A_{N\ell}\,\partial_\ell f(y,t)\,\varphi_0\Big)\mathrm{d}y = 0\end{aligned} \tag{4.30}$$

for all $\ell = 1, \dots, N$. Consider now the function $w = \sum_{\ell=1}^N A_{N\ell}\partial_\ell v$. Summing up the above identities over $\ell$ and using (4.20), we see that $w$ is a solution of the nonhomogeneous Dirichlet problem

$$\int_{\mathbb{R}^N_+} \big(w_t\,\varphi_0 + A(y)\nabla w\cdot\nabla\varphi_0 - f_1(y,t)\,\varphi_0\big)\mathrm{d}y \;=\; 0 \quad \forall\varphi_0\in V_0\,, \tag{4.31}$$

with boundary condition

$$w(y',0,t) + B_N(y',0,t) + h(y', v(y',0,t), v_\Gamma(y',t)) = 0 \tag{4.32}$$

on $\mathbb{R}^{N-1}\times(0,T)$ as a consequence of (4.20) with $\delta = 0$. The function $f_1$ in (4.31) has the form

$$f_1 = \sum_{\ell=1}^N \Big(A_{N\ell}\partial_\ell f - \underbrace{A\nabla\partial_\ell v\cdot\nabla A_{N\ell}}_{I} - \underbrace{\mathrm{div}\,((\partial_\ell v)A\nabla A_{N\ell})}_{II} + \underbrace{A_{N\ell}\mathrm{div}\,((\partial_\ell A)\nabla v + \partial_\ell B)}_{III}\Big), \tag{4.33}$$

hence it belongs to $L^2(\mathbb{R}^N_+\times(0,T))$. The symbols $I, II, III$ denote corresponding terms in (4.30) and (4.33). We now fix a smooth function $\varrho$ with compact support in $\mathbb{R}_+$ and such that $\varrho(0) = 1$, and set

$$w_1(y,t) = B_N(y,t) + \varrho(y_N)\,h(y', v(y,t), v_\Gamma(y',t))\,. \tag{4.34}$$

The function $w_0 := w - w_1$ is a solution to the homogeneous Dirichlet problem for the following counterpart of (4.31)

$$\int_{\mathbb{R}^N_+} \big((w_0)_t\,\varphi_0 + A(y)\nabla w_0\cdot\nabla\varphi_0 - f_2(y,t)\,\varphi_0\big)\mathrm{d}y \;=\; 0 \quad \forall\varphi_0\in V_0\,, \tag{4.35}$$

where

$$f_2 = f_1 - (w_1)_t + \mathrm{div}\,A(y)\nabla w_1\,. \tag{4.36}$$

Let us check that $f_2 \in L^2(\mathbb{R}^N_+\times(0,T))$. By virtue of Lemmas 4.4–4.5, this will be the case provided we prove that

$$\nabla v \in L^4(\mathbb{R}^N_+\times(0,T))\,. \tag{4.37}$$

To this end, we refer to [3, Theorem 10.2], see also Remark A.3, which states that there exists a constant $C > 0$ such that for every function $\xi\in L^2(\mathbb{R}^N_+\times(0,T))$ with the regularity $\partial_\ell^2\xi, \partial_N\xi \in L^2(\mathbb{R}^N_+\times(0,T))$ for all $\ell = 1, \dots, N-1$, and for every $\sigma\in(0,1]$ we have the inequality (note that $N \le 3$!)

$$|\xi|_4 \le C\left(\sigma^{-1/2}|\xi|_2 + \sigma^{1/2}\left(|\partial_N\xi|_2 + \sum_{\ell=1}^{N-1}|\partial_\ell^2\xi|_2\right)\right)\,. \tag{4.38}$$

This can be equivalently written as

$$|\xi|_4 \le C\left(|\xi|_2 + |\xi|_2^{1/2}\left(|\partial_N\xi|_2 + \sum_{\ell=1}^{N-1}|\partial_\ell^2\xi|_2\right)^{1/2}\right)\,. \tag{4.39}$$

In (4.39), we choose $\xi = \partial_k v(t)$ for $k = 1, \dots, N$ and a.e. $t$. From Lemmas 4.4–4.5 we obtain (4.37), hence $f_2 \in L^2(\mathbb{R}^N_+ \times (0,T))$.

Similarly as in the proof of Lemma 4.3, we now apply the operator $D^\ell_s$ to Eq. (4.35) for $\ell = 1, \dots, N-1$ and test by $\varphi_0 = D^\ell_s w_0$. Using the identity $\int (D^\ell_s w_1)_t D^\ell_s w_0 \,\mathrm{d}y = \int (w_1)_t D^\ell_{-s} D^\ell_s w_0 \,\mathrm{d}y$, we may let $s$ tend to 0 and conclude that $\partial_\ell \nabla w_0$ belongs to $L^2(\mathbb{R}^N_+ \times (0,T))$ for all $\ell = 1, \dots, N-1$. By Lemma 4.5, and since $A_{NN} \geq \kappa$, we obtain that $\partial_\ell \partial^2_N v \in L^2(\mathbb{R}^N_+ \times (0,T))$ for all $\ell = 1, \dots, N-1$, and the proof is complete. ■

We now define the anisotropic spaces

$$X^{p,q} = \left\{ w \in L^1(\mathbb{R}^N_+) : \int_0^\infty \left( \int_{\mathbb{R}^{N-1}} |w(y', y_N)|^q \mathrm{d}y' \right)^{p/q} \mathrm{d}y_N < \infty \right\}.$$

We can extend the functions defined on $\mathbb{R}^N_+$ by symmetry to $\mathbb{R}^N$, and use Corollary A.2 in the Appendix to obtain the compact embedding

$$Y := \left\{ w \in L^1(\mathbb{R}^N_+) : \partial_N w \in X^{p_0,q_0}, \quad \nabla_{y'} w \in X^{p_1,q_1} \right\} \quad \subset \quad C^\alpha(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+) \tag{4.40}$$

in the space $C^\alpha(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$ of bounded $\alpha$-Hölder continuous functions for some $\alpha > 0$, provided

$$\frac{p_0'}{p_1 q_0} + \frac{1}{q_1} < \frac{1}{N-1},$$

where $p_0'$ is the conjugate exponent to $p_0$. As a direct consequence, we have

**Lemma 4.7** *Under the conditions of Lemma 4.5, we have $\nabla v \in L^2(0,T; L^\infty(\mathbb{R}^N_+))$.*

*Proof.* The functions $\partial_\ell v$ for $\ell = 1, \dots, N-1$ belong to $L^2(0,T; W^{2,2}(\mathbb{R}^N_+))$, which is embedded into $L^2(0,T; L^\infty(\mathbb{R}^N_+))$ by classical Sobolev embedding theorems, see [1, 3]. For $w(y,t) = \partial_N v(y,t)$ and a.e. $t \in (0,T)$, we have

$$\begin{aligned}
&|\partial_\ell w(t)|_{X^{6,6}} = |\partial_\ell w(t)|_6 \leq C\,(|\partial_\ell w(t)|_2 + |\nabla \partial_\ell w(t)|_2) \quad \text{for } \ell = 1, \dots, N-1, \\
&|\partial_N w(t)|_{X^{2,q}} \leq C\,(|\partial_N w(t)|_2 + |\nabla_{y'} \partial_N w(t)|_2)
\end{aligned}$$

with a constant $C > 0$ and for every $q \geq 2$. Hence, (4.40) is fulfilled with $p_0 = 2$, $q_0 = q$, $p_1 = q_1 = 6$, and it suffices to integrate over $t$. ■

This enables us to prove here Theorems 4.1 and 4.2.

*Proof of Theorems 4.1 and 4.2.* We substitute in (4.1) new variables $y' = x'$, $y_N = x_N - g(x')$, and obtain for the new unknown function $\tilde{v}(y', y_N) = v(y', y_N + g(y'))$ the equation

$$\int_{\mathbb{R}^N_+} \left( \tilde{v}_t\, \varphi + (\tilde{A} \nabla \tilde{v} + \tilde{B}) \cdot \nabla \varphi - \tilde{f}\, \varphi \right) \mathrm{d}y + \int_{\mathbb{R}^{N-1}} \tilde{h}(y', \tilde{v}(y', 0, t), v_\Gamma(y', t)) \varphi(y', 0) \,\mathrm{d}y' = 0 \tag{4.41}$$

for every $\varphi \in W^{1,2}(\mathbb{R}^N_+)$, where

$$\begin{aligned}
\tilde f(y', y_N, t) &= f(y', y_N + g(y'), t),\\
\tilde v_\Gamma(y', t) &= v_\Gamma(y', g(y'), t)\,,\\
\tilde h(y', v, v_\Gamma) &= h(y', g(y'), v, v_\Gamma)\sqrt{1 + |\nabla_{y'} g(y')|^2}\,,\\
\tilde A(y', y_N) &= L^T(y')\, A(y', y_N + g(y'))\, L(y'),\\
\tilde B(y', y_N, t) &= L^T(y')\, B(y', y_N + g(y'), t),
\end{aligned}$$

and where the matrix $L$ has the form

$$L = \begin{pmatrix} 1 & 0 & \dots & 0 & -\partial_1 g \\ 0 & 1 & \dots & 0 & -\partial_2 g \\ & & \dots & & \\ 0 & 0 & \dots & 1 & -\partial_{N-1} g \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Theorem 4.1 now follows from Lemma 4.4, Theorem 4.2 is a consequence of Lemma 4.7. ■

We are now ready to prove Theorem 2.4.

*Proof of Theorem 2.4.* The nonlinear boundary condition is active only on the subsets $\Gamma_j$ of $\partial\Omega$ for $j = 1, \dots, n$. We choose a covering $\bar\Omega \subset \bigcup_{j=1}^n \Omega_j$ of $\Omega$ with the property that $\Gamma_j \subset \Omega_j$ and $\Gamma_i \cap \bar\Omega_j = \emptyset$ for $i \neq j$. We now find a smooth partition of unity $1 = \sum_{j=1}^n \lambda_j(x)$ on $\bar\Omega$ such that $\operatorname{supp}\lambda_j \subset \Omega_j$, and set $v_j = \theta\,\lambda_j$, $f(x,t) = r(\theta(x,t), c(x,t))$. After suitable deformations and rotations, we may assume that each set $\Omega_j$ can be extended to a domain $\tilde\Omega_j$ of the form (4.3). To derive the equation for $v_j$, we test the equation

$$\int_\Omega (\theta_t\,\varphi + \nabla\theta\cdot\nabla\varphi - f(x,t)\,\varphi)\,\mathrm{d}x + \int_{\partial\Omega} h(x, \theta, \theta_\Gamma(x,t))\,\varphi\,\mathrm{d}S \;=\; 0 \tag{4.42}$$

by $\varphi = \lambda_j \tilde\varphi$, and obtain

$$\int_{\tilde\Omega_j} ((v_j)_t\,\tilde\varphi + \nabla v_j\cdot\nabla\tilde\varphi + B_j\cdot\nabla\tilde\varphi - f_j(x,t)\,\tilde\varphi)\,\mathrm{d}x + \int_{\partial\tilde\Omega_j} h(x, v_j, v_{\Gamma j}(x,t))\,\tilde\varphi\,\mathrm{d}S \;=\; 0\,, \tag{4.43}$$

with $B_j = -\theta\nabla\lambda_j$, $f_j = f\lambda_j - \nabla\theta\cdot\nabla\lambda_j$, $v_{\Gamma j} = \theta_\Gamma\lambda_j$. Here we have used the fact that $\lambda_j = 1$ on $\Gamma_j$, and that $h$ is linear on $\partial\tilde\Omega_j \setminus \Gamma_j$.

The assumptions of Theorem 4.1 are satisfied; hence, each $v_j$ has the regularity $(v_j)_t \in L^2(\Omega_j \times (0,T))$, $v_j \in L^2(0,T; W^{2,2}(\Omega_j))$. From the formula $\theta = \sum_{j=1}^n v_j$ it follows that $\theta_t \in L^2(\Omega \times (0,T))$, $\theta \in L^2(0,T; W^{2,2}(\Omega))$. Consequently, we may use Theorem 4.2 and obtain $\nabla v_j \in L^2(0,T; L^\infty(\Omega_j))$ for each $j$, hence $\nabla\theta \in L^2(0,T; L^\infty(\Omega))$. ■

## 5 Proof of continuous data dependence

Let the hypotheses of Theorem 2.5 hold. In terms of $(\theta_i, u_i)$, Eqs. (2.1)–(2.2) have the form

$$\int_\Omega ((\theta_i)_t\,\varphi + \nabla\theta_i\cdot\nabla\varphi - R(\theta_i,u_i)\,\varphi)\,\mathrm{d}x + \int_{\partial\Omega} h(x,\theta_i,\theta_{\Gamma i}(x,t))\,\varphi\,\mathrm{d}S \;=\; 0 \qquad (5.1)$$

$$\int_\Omega (G(\theta_i,u_i)_t\,\psi + \nabla u_i\cdot\nabla\psi - H(\theta_i,u_i)\nabla\theta_i\cdot\nabla\psi)\,\mathrm{d}x + \int_{\partial\Omega} b_i(x,t)\psi\,\mathrm{d}S \;=\; 0 \qquad (5.2)$$

for every test functions $\varphi,\psi \in V$, where $G$, $H$, and $R$ are defined by the identities

$$F(\theta, G(\theta,u)) = u\,, \quad H(\theta,u) = \frac{\partial F}{\partial\theta}(\theta, G(\theta,u))\,, \quad R(\theta,u) = r(\theta, G(\theta,u))\,. \qquad (5.3)$$

Hypothesis 2.2 implies that $G, H, R$ are Lipschitz continuous in both variables, $1/d_1 \le \partial_u G \le 1/d_0$.

Set $U_i(x,t) = \int_0^t u_i(x,\tau)\,\mathrm{d}\tau$, $u_i^0 = F(\theta_i^0, c_i^0)$, $\bar U = U_1 - U_2$. We consider the difference of the equations (5.1) for $i=1$ and $i=2$, tested by $\varphi = \bar\theta$, integrate the difference of the equations (5.2) for $i=1$ and $i=2$ from 0 to $t$, and test by $\psi = \bar U_t$. We denote by $C$ any constant independent of the solutions, and by $\varepsilon$ a small parameter, which will be suitably chosen. Since $\theta_i$ and $\theta_{\Gamma i}$ are uniformly bounded, we may assume that $h$ is Lipschitz continuous in $\theta$ and $\theta_\Gamma$. Hence, using (4.7) for an appropriate $\varepsilon$, we obtain

$$\begin{aligned}\int_\Omega \left(\bar\theta_t\,\bar\theta + |\nabla\bar\theta|^2\right)\mathrm{d}x \;&\le\; \int_\Omega (R(\theta_1,u_1) - R(\theta_2,u_2))\bar\theta\,\mathrm{d}x + C\int_{\partial\Omega}(|\bar\theta_\Gamma| + |\bar\theta|)\bar\theta\,\mathrm{d}S\\ &\le\; C\int_\Omega (|\bar\theta| + |\bar U_t|)|\bar\theta|\,\mathrm{d}x + C\int_{\partial\Omega}|\bar\theta_\Gamma|^2\,\mathrm{d}S\,, \end{aligned} \qquad (5.4)$$

$$\begin{aligned}&\int_\Omega (G(\theta_1,U_{1t}) - G(\theta_2,U_{2t}))\,\bar U_t(x,t)\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\bar U|^2(x,t)\,\mathrm{d}x \qquad (5.5)\\
&= \int_\Omega \left(\int_0^t \big(H(\theta_1,U_{1t})\nabla\theta_1 - H(\theta_2,U_{2t})\nabla\theta_2\big)(x,\tau)\,\mathrm{d}\tau\right)\cdot\nabla\bar U_t(x,t)\,\mathrm{d}x\\
&\quad - \int_{\partial\Omega}\left(\int_0^t \bar b(x,\tau)\,\mathrm{d}\tau\right)\bar U_t(x,t)\,\mathrm{d}S + \int_\Omega (G(\theta_1^0,u_1^0) - G(\theta_2^0,u_2^0))\,\bar U_t(x,t)\,\mathrm{d}x\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\int_0^t \big(H(\theta_1,U_{1t})\nabla\theta_1 - H(\theta_2,U_{2t})\nabla\theta_2\big)(x,\tau)\,\mathrm{d}\tau\right)\cdot\nabla\bar U(x,t)\,\mathrm{d}x\\
&\quad - \int_\Omega \big(H(\theta_1,U_{1t})\nabla\theta_1 - H(\theta_2,U_{2t})\nabla\theta_2\big)\cdot\nabla\bar U(x,t)\,\mathrm{d}x\\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\left(\int_0^t \bar b(x,\tau)\,\mathrm{d}\tau\right)\bar U(x,t)\,\mathrm{d}S + \int_{\partial\Omega}\bar b(x,t)\,\bar U(x,t)\,\mathrm{d}S\\
&\quad + \int_\Omega (c_1^0 - c_2^0)\,\bar U_t(x,t)\,\mathrm{d}x\,.\end{aligned}$$

Integrating Eq. (5.5)–(5.4) with respect to $t$ and using the hypotheses on the data, we obtain

$$\frac{1}{2}\int_\Omega |\bar\theta|^2(x,t)\,dx + \int_0^t\int_\Omega |\nabla\bar\theta(x,\tau)|^2 \mathrm{d}x\,\mathrm{d}\tau \tag{5.6}$$
$$\leq\; C\int_0^t\int_\Omega (|\bar\theta| + |\bar U_t|)|\bar\theta|(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + C\int_0^t\int_{\partial\Omega} |\bar\theta_\Gamma|^2 \mathrm{d}S\,\mathrm{d}\tau + \frac{1}{2}\int_\Omega |\bar\theta^0|^2(x)\,dx\,,$$

$$\frac{1}{d_1}\int_0^t\int_\Omega |\bar U_t(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_\Omega |\nabla\bar U|^2(x,t)\,\mathrm{d}x \tag{5.7}$$
$$\leq\; C\int_\Omega\left(\int_0^t \big((|\bar\theta| + |\bar U_t|)|\nabla\theta_1| + |\nabla\bar\theta|\big)(x,\tau)\,\mathrm{d}\tau\right)|\nabla\bar U(x,t)|\,\mathrm{d}x$$
$$+\,C\int_0^t\int_\Omega \big((|\bar\theta| + |\bar U_t|)|\nabla\theta_1| + |\nabla\bar\theta|\big)(x,\tau)\,|\nabla\bar U(x,\tau)|\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_{\partial\Omega}\left(\int_0^t |\bar b(x,\tau)|\,\mathrm{d}\tau\right)|\bar U(x,t)|\,\mathrm{d}S + \int_0^t\int_{\partial\Omega} |\bar b(x,\tau)|\,|\bar U(x,\tau)|\,\mathrm{d}S\,\mathrm{d}\tau$$
$$+\,C\int_0^t\int_\Omega |\bar\theta(x,\tau)|\,|\bar U_t(x,\tau)|\,\mathrm{d}x\,\mathrm{d}\tau + C\int_\Omega |\bar c^0|\,|\bar U(x,t)|\,\mathrm{d}x\,.$$

Using Hölder's and Young's inequalities, we may rewrite (5.6)–(5.7) as

$$|\bar\theta(t)|_2^2 + \int_0^t |\nabla\bar\theta(\tau)|_2^2\,\mathrm{d}\tau \tag{5.8}$$
$$\leq\; C\left(\alpha(t) + \int_0^t |\bar\theta(\tau)|_2^2\,\mathrm{d}\tau\right) + \varepsilon\int_0^t |\bar U_t(\tau)|_2^2 \mathrm{d}\tau\,,$$

$$\int_0^t |\bar U_t(\tau)|_2^2\,\mathrm{d}\tau + |\nabla\bar U(t)|_2^2 \tag{5.9}$$
$$\leq\; C\int_\Omega\left(\int_0^t \big((|\bar\theta| + |\bar U_t|)|\nabla\theta_1| + |\nabla\bar\theta|\big)(x,\tau)\,\mathrm{d}\tau\right)^2\,\mathrm{d}x$$
$$+\,C\int_0^t \big((|\bar\theta|_2 + |\bar U_t|_2)|\nabla\theta_1|_\infty + |\nabla\bar\theta|_2\big)(\tau)\,|\nabla\bar U(\tau)|_2\,\mathrm{d}\tau$$
$$+\,C\left(\alpha(t) + \int_0^t |\bar\theta(\tau)|_2^2 \mathrm{d}\tau\right)$$
$$+\,\varepsilon\left(|\bar U(t)|_2^2 + \int_0^t\int_{\partial\Omega} |\bar U(x,\tau)|^2\,\mathrm{d}S\,\mathrm{d}\tau + \int_{\partial\Omega} |\bar U(x,t)|^2\,\mathrm{d}S\right),$$

with $\alpha(t)$ defined by (2.6). The first two integrals on the right hand side of (5.9) will be estimated using Minkowski's inequality

$$\left(\int_\Omega\left(\int_0^t \big((|\bar\theta| + |\bar U_t|)|\nabla\theta_1| + |\nabla\bar\theta|\big)(x,\tau)\,\mathrm{d}\tau\right)^2\,\mathrm{d}x\right)^{1/2} \tag{5.10}$$
$$\leq\; \int_0^t\left(\int_\Omega \big((|\bar\theta| + |\bar U_t|)|\nabla\theta_1| + |\nabla\bar\theta|\big)^2(x,\tau)\,\mathrm{d}x\right)^{1/2}\,\mathrm{d}\tau$$
$$\leq\; \int_0^t \big((|\bar\theta|_2 + |\bar U_t|_2)|\nabla\theta_1|_\infty + |\nabla\bar\theta|_2\big)(\tau)\,\mathrm{d}\tau\,,$$

and Hölder's and Young's inequalities

$$
\begin{aligned}
C\int_0^t \left((|\bar\theta|_2+|\bar U_t|_2)|\nabla\theta_1|_\infty+|\nabla\bar\theta|_2\right)(\tau)\,|\nabla\bar U(\tau)|_2\,\mathrm{d}\tau \qquad (5.11)\\
\le\ C\int_0^t \left(1+|\nabla\theta_1(\tau)|_\infty^2\right)|\nabla\bar U(\tau)|_2^2\,\mathrm{d}\tau+\varepsilon\int_0^t\left(|\bar\theta|_2^2+|\bar U_t|_2^2+|\nabla\bar\theta|_2^2\right)(\tau)\,\mathrm{d}\tau\,.
\end{aligned}
$$

respectively. Using the inequality $\frac{\mathrm{d}}{\mathrm{d}t}|\bar U(t)|_2\le|\bar U_t(t)|_2$ a. e., we have in (5.9)

$$
|\bar U(t)|_2^2\le\left(\int_0^t|\bar U_t(\tau)|_2\,\mathrm{d}\tau\right)^2 .
$$

For the boundary terms in (5.9), we refer to the trace embedding (4.7). We thus obtain from (5.8)–(5.10) the inequality

$$
\begin{aligned}
|\bar\theta(t)|_2^2+|\nabla\bar U(t)|_2^2+\int_0^t\left(|\nabla\bar\theta|_2^2+|\bar U_t|_2^2\right)(\tau)\,\mathrm{d}\tau \qquad (5.12)\\
\le\ C\left(\alpha(t)+\int_0^t(1+|\nabla\theta_1|_\infty)^2\left(|\bar\theta|_2^2+|\nabla\bar U|_2^2\right)(\tau)\,\mathrm{d}\tau\right)\\
+C\left(\int_0^t(1+|\nabla\theta_1|_\infty)\left(|\nabla\bar\theta|_2^2+|\bar U_t|_2^2\right)^{1/2}(\tau)\,\mathrm{d}\tau\right)^2 .
\end{aligned}
$$

Inequality (5.12) is of the form

$$
v(t)+\int_0^t s^2(\tau)\,\mathrm{d}\tau\le C\left(\alpha(t)+\int_0^t\beta^2(\tau)\,v(\tau)\,\mathrm{d}\tau+\left(\int_0^t\beta(\tau)\,s(\tau)\,\mathrm{d}\tau\right)^2\right), \qquad (5.13)
$$

with

$$
\beta=1+|\nabla\theta_1|_\infty\in L^2(0,T)\,,\quad v(t)=|\bar\theta(t)|_2^2+|\nabla\bar U(t)|_2^2\,,\quad s^2(t)=|\nabla\bar\theta(t)|_2^2+|\bar U_t(t)|_2^2. \qquad (5.14)
$$

To estimate $v(t)$ and $s(t)$, we derive below in Lemma 5.2 a refined variant of the Gronwall lemma. Recall first the classical Gronwall estimate.

**Lemma 5.1** *Let $\alpha\in L^\infty(0,T)$ and $\gamma\in L^1(0,T)$ be given nonnegative functions, and let a nonnegative function $v\in L^\infty(0,T)$ satisfy for a. e. $t\in(0,T)$ the inequality*

$$
v(t)\ \le\ \alpha(t)+\int_0^t\gamma(\tau)\,v(\tau)\,\mathrm{d}\tau\,.
$$

*Then for a. e. $t\in(0,T)$ we have*

$$
v(t)\ \le\ \alpha(t)+\int_0^t\alpha(\tau)\,\gamma(\tau)\,\mathrm{e}^{\int_\tau^t\gamma(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}\tau\ \le\ \sup_{0<\tau<t}\operatorname{ess}\alpha(\tau)\,\mathrm{e}^{\int_0^t\gamma(\sigma)\,\mathrm{d}\sigma}\,.
$$

*Sketch of the proof.* The assertion follows directly by integrating the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\int_0^t \gamma(\sigma)\,\mathrm{d}\sigma}\int_0^t \gamma(\tau)\,v(\tau)\,\mathrm{d}\tau\right) \;\le\; \mathrm{e}^{-\int_0^t \gamma(\sigma)\,\mathrm{d}\sigma}\,\alpha(t)\,\gamma(t)\,.$$

■

Lemma 5.1 can be viewed as a result of the fact that the $L^\infty$-norm of the function $v$ is bounded above by its weighted $L^1$-norm. We now show that an $L^p$-*Gronwall estimate* still holds if the $L^\infty$-norm on the left-hand side is replaced with an $L^p$-norm for $p > 1$.

**Lemma 5.2** *Let $p > 1$ and its conjugate exponent $p' = p/(p-1)$ be fixed, and let $\alpha \in L^\infty(0,T)$, $\gamma_1 \in L^1(0,T)$, and $\gamma_2 \in L^{p'}(0,T)$ be given. Let nonnegative functions $v \in L^\infty(0,T)$, $s \in L^p(0,T)$ satisfy for a. e. $t \in (0,T)$ the inequality*

$$v(t) + \int_0^t s^p(\tau)\,\mathrm{d}\tau \;\le\; \alpha(t) + \int_0^t \gamma_1(\tau)\,v(\tau)\,\mathrm{d}\tau + \left(\int_0^t \gamma_2(\tau)\,s(\tau)\,\mathrm{d}\tau\right)^p .$$

*Then there exists a constant $M$ such that for a. e. $t \in (0,T)$ we have*

$$v(t) + \int_0^t s^p(\tau)\,\mathrm{d}\tau \le M \operatorname*{sup\,ess}_{0<\tau<t} \alpha(\tau)\,. \tag{5.15}$$

*Proof*. Set $G_2 = \left(\int_0^T \gamma_2^{p'}(\tau)\,\mathrm{d}\tau\right)^{1/p'}$. We fix $\delta$ such that for every $t \in [0,T]$ we have

$$\left(\int_{(t-\delta)^+}^t \gamma_2^{p'}(\tau)\,\mathrm{d}\tau\right)^{1/p'} \;\le\; \frac{1}{2}\,,$$

and consider first $t \in [0,\delta]$. By Hölder's inequality we have

$$\left(\int_0^t \gamma_2(\tau)\,s(\tau)\,\mathrm{d}\tau\right)^p \;\le\; \left(\int_0^t \gamma_2^{p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} \int_0^t s^p(\tau)\,\mathrm{d}\tau \;\le\; 2^{-p}\int_0^t s^p(\tau)\,\mathrm{d}\tau\,,$$

and the assertion follows from Lemma 5.1. Assume now that inequality (5.15) is proved for $t \in [0,k\delta]$ with a constant $M = M_k$, and consider $t \in (k\delta,(k+1)\delta]$. We have

$$\begin{aligned}
\left(\int_0^t \gamma_2(\tau)\,s(\tau)\,\mathrm{d}\tau\right)^p \;&=\; \left(\int_0^{t-\delta} \gamma_2(\tau)\,s(\tau)\,\mathrm{d}\tau + \int_{t-\delta}^t \gamma_2(\tau)\,s(\tau)\,\mathrm{d}\tau\right)^p\\
&\le\; 2^{p-1}\left(\left(\int_0^{t-\delta} \gamma_2(\tau)\,s(\tau)\,\mathrm{d}\tau\right)^p + \left(\int_{t-\delta}^t \gamma_2(\tau)\,s(\tau)\,\mathrm{d}\tau\right)^p\right)\\
&\le\; 2^{p-1}\,G_2^p \int_0^{t-\delta} s^p(\tau)\,\mathrm{d}\tau + \frac{1}{2}\int_{t-\delta}^t s^p(\tau)\,\mathrm{d}\tau\\
&\le\; 2^{p-1}\,G_2^p\,M_k \sup_{0\le\tau\le k\delta} \alpha(\tau) + \frac{1}{2}\int_0^t s^p(\tau)\,\mathrm{d}\tau\,.
\end{aligned}$$

Using Lemma 5.1 again, we complete the proof by induction over $k$. ■

We are able now to finish the proof of Theorem 2.5. Indeed, inequality (5.13) has the form as in Lemma 5.2, with $p = 2$, $\alpha$ replaced by $C\alpha$, $\gamma_1 = C\beta^2$, and $\gamma_2 = C^{1/p}\beta$, with $v$ and $s$ given by (5.14). The assertion of Theorem 2.5 therefore follows from inequality (5.13) and Lemma 5.2. ■

# A Appendix: An anisotropic embedding theorem

We prove here an embedding theorem for anisotropic Sobolev spaces that is needed in Section 4. For a vector $\mathbf{p} = (p_1, \ldots, p_N)$ we define the space $L^{\mathbf{p}}(\mathbb{R}^N)$ as the subspace of $L^1(\mathbb{R}^N)$ of functions $u$ such that the norm

$$\|u\|_{\mathbf{p}} = \left( \int_{\mathbb{R}} \left( \ldots \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |u(x)|^{p_1} \,\mathrm{d}x_1 \right)^{p_2/p_1} \mathrm{d}x_2 \ \ldots \right)^{p_N/p_{N-1}} \mathrm{d}x_N \right)^{1/p_N} \tag{A.1}$$

is finite. For a matrix $\mathbf{P} = (P_{ij})_{i,j=1}^N$, $P_{ij} = 1/p_{ij}$, we define the anisotropic Sobolev space

$$W^{1,\mathbf{P}}(\mathbb{R}^N) = \left\{ u \in L^1(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^{\mathbf{p}_i}(\mathbb{R}^N)\,, i = 1, \ldots, N \right\}, \tag{A.2}$$

where $\mathbf{p}_i = (p_{i1}, \ldots, p_{iN})$.

We denote by $\mathbf{I}$ the identity $N \times N$ matrix, and by $\mathbf{1}$ the vector $\mathbf{1} = (1, 1, \ldots, 1)$. The spectral radius $\varrho(\mathbf{P})$ of $\mathbf{P}$ is defined as

$$\varrho(\mathbf{P}) \ = \ \max\{|\lambda| : \lambda \in \mathbb{C},\ \det(\mathbf{P} - \lambda\mathbf{I}) = 0\} \ = \ \limsup_{n\to\infty} |\mathbf{P}^n|^{1/n}\,. \tag{A.3}$$

**Theorem A.1** *Let $\varrho(\mathbf{P}) < 1$, and let*

$$(\mathbf{I} - \mathbf{P})^{-1}\mathbf{1} \ = \ \mathbf{b} \ = \ (b_1, \ldots, b_N)\,. \tag{A.4}$$

*Then $W^{1,\mathbf{P}}(\mathbb{R}^N)$ is embedded in $L^\infty(\mathbb{R}^N)$, and there exists a constant $C > 0$ such that each $u \in W^{1,\mathbf{P}}(\mathbb{R}^N)$ has for all $x, z \in \mathbb{R}^N$ the Hölder property*

$$|u(z) - u(x)| \ \le \ C\, \|u\|_{W^{1,\mathbf{P}}(\mathbb{R}^N)} \sum_{i=1}^N |z_i - x_i|^{1/b_i}\,. \tag{A.5}$$

The identity (A.4) can be written as

$$\mathbf{b} \ = \ (\mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \ldots)\,\mathbf{1}\,.$$

Since all entries of $\mathbf{P}$ are positive, we obtain $b_i > 1$ for all $i$, so that the right hand side of (A.5) is meaningful. Note also that in the isotropic case $p_{ij} = p$, Theorem A.1 gives the well known embedding condition $p > N$ with Hölder exponent $1/b = 1 - (N/p)$.

*Proof.* Following [3], we fix a smooth function $\Phi$ with compact support in $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N} \Phi(x)\,\mathrm{d}x = 1$, and for $\sigma > 0$ and $u \in W^{1,\mathbf{P}}(\mathbb{R}^N)$ set

$$u^\sigma(x) \ = \ \sigma^{-|\mathbf{b}|} \int_{\mathbb{R}^N} \Phi\Big(\frac{x - y}{\sigma^{\mathbf{b}}}\Big) u(y)\,\mathrm{d}y\,, \tag{A.6}$$

where $|\mathbf{b}| = \sum_{i=1}^N b_i$ and

$$\frac{x - y}{\sigma^{\mathbf{b}}} = \left( \frac{x_1 - y_1}{\sigma^{b_1}}, \ldots, \frac{x_N - y_N}{\sigma^{b_N}} \right)\,.$$

By substitution, we have the identity

$$u^\sigma(x) \;=\; \int_{\mathbb{R}^N} \Phi(z) u(x - \sigma^{\mathbf{b}} z)\,\mathrm{d}z\,, \tag{A.7}$$

which implies that

$$\lim_{\sigma\to 0} |u^\sigma - u|_1 = 0\,. \tag{A.8}$$

We differentiate $u^\sigma$ with respect to $\sigma$, integrate by parts with respect to $y$, and obtain

$$\frac{\partial u^\sigma(x)}{\partial\sigma} \;=\; -\sum_{i=1}^N \sigma^{-|\mathbf{b}|-1+b_i} \int_{\mathbb{R}^N} \Psi_i\Big(\frac{x-y}{\sigma^{\mathbf{b}}}\Big)\frac{\partial u(y)}{\partial y_i}\,\mathrm{d}y\,, \tag{A.9}$$

where

$$\Psi_i(z) \;=\; b_i z_i\,\Phi(z) \quad \text{ for } \; z\in\mathbb{R}^N\,.$$

By the anisotropic Hölder inequality we have

$$\left|\frac{\partial u^\sigma(x)}{\partial\sigma}\right| \;\le\; \sum_{i=1}^N \sigma^{-|\mathbf{b}|-1+b_i} \left\|\Psi_i\Big(\frac{\cdot}{\sigma^{\mathbf{b}}}\Big)\right\|_{\mathbf{p}_i'} \left\|\frac{\partial u}{\partial y_i}\right\|_{\mathbf{p}_i}\,, \tag{A.10}$$

where $\mathbf{p}_i'$ is the componentwise conjugate of $\mathbf{p}_i$. By substitution, we have

$$\left\|\Psi_i\Big(\frac{\cdot}{\sigma^{\mathbf{b}}}\Big)\right\|_{\mathbf{p}_i'} \;=\; \sigma^{\sum_{j=1}^N b_j/p_{ij}'}\|\Psi_i\|_{\mathbf{p}_i'} \;=\; \sigma^{|\mathbf{b}|-(\mathbf{P}\mathbf{b})_i}\|\Psi_i\|_{\mathbf{p}_i'}\,. \tag{A.11}$$

This and (A.4) yield the following estimate independent of $\sigma$ and $x$:

$$\left|\frac{\partial u^\sigma(x)}{\partial\sigma}\right| \;\le\; \sum_{i=1}^N \sigma^{b_i-(\mathbf{P}\mathbf{b})_i-1}\|\Psi_i\|_{\mathbf{p}_i'} \left\|\frac{\partial u}{\partial y_i}\right\|_{\mathbf{p}_i} \;=\; \sum_{i=1}^N \|\Psi_i\|_{\mathbf{p}_i'} \left\|\frac{\partial u}{\partial y_i}\right\|_{\mathbf{p}_i} \;=:\; U\,. \tag{A.12}$$

For $\sigma > \tilde\sigma > 0$ we have

$$|u^\sigma(x) - u^{\tilde\sigma}(x)| \;\le\; (\sigma-\tilde\sigma)\,U\,,$$

hence $u^\sigma$ converge uniformly in $L^\infty(\mathbb{R}^N)$ as $\sigma\to 0$. In view of (A.8), its limit is $u$, which thus belongs to $L^\infty(\mathbb{R}^N)\cap C(\mathbb{R}^N)$, and we have for all $\sigma>0$ the embedding inequality

$$|u(x)| \;\le\; |u^\sigma(x)| + \sigma U \le \sigma^{-|\mathbf{b}|}|u|_1 + \sigma U\,. \tag{A.13}$$

To prove the Hölder estimate, we replace $u(x)$ in (A.13) by $u(x+he_i)-u(x)$, where $e_i$ is the $i$-th unit coordinate vector and $h>0$ is arbitrary. We obtain

$$|u(x+he_i) - u(x)| \;\le\; |u^\sigma(x+he_i) - u^\sigma(x)| + 2\sigma U\,, \tag{A.14}$$

where

$$\begin{aligned} u^\sigma(x+he_i) - u^\sigma(x) \;&=\; \sigma^{-|\mathbf{b}|}\int_{\mathbb{R}^N} \Phi\Big(\frac{x-y}{\sigma^{\mathbf{b}}}\Big)(u(y+he_i)-u(y))\,\mathrm{d}y \\ &=\; -\sigma^{-|\mathbf{b}|}\int_0^h\int_{\mathbb{R}^N} \Phi\Big(\frac{x-y}{\sigma^{\mathbf{b}}}\Big)\frac{\partial u}{\partial y_i}(y-se_j)\,\mathrm{d}y\,\mathrm{d}s\,. \end{aligned} \tag{A.15}$$

This and (A.11) entail

$$|u^\sigma(x+he_i) - u^\sigma(x)| \;\le\; h\sigma^{-|\mathbf{b}|}\left\|\Psi_i\Big(\frac{\cdot}{\sigma^{\mathbf{b}}}\Big)\right\|_{\mathbf{p}_i'}\left\|\frac{\partial u}{\partial y_i}\right\|_{\mathbf{p}_i} \;\le\; h\sigma^{-(\mathbf{Pb})_i}\|\Psi_i\|_{\mathbf{p}_i'}\left\|\frac{\partial u}{\partial y_i}\right\|_{\mathbf{p}_i}. \tag{A.16}$$

We thus conclude from (A.14) that there exists a constant $C > 0$ such that for all $u \in W^{1,\mathbf{P}}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$, $\sigma > 0$, and $h > 0$ we have

$$|u(x+he_i) - u(x)| \;\le\; C\left(h\sigma^{-(\mathbf{Pb})_i} + \sigma\right)\sum_{j=1}^N \left\|\frac{\partial u}{\partial y_j}\right\|_{\mathbf{p}_j}. \tag{A.17}$$

In particular, for $\sigma = h^{1/b_i}$ we obtain, by virtue of (A.4), the formula

$$|u(x+he_i) - u(x)| \;\le\; C\,h^{1/b_i}\,\|u\|_{W^{1,\mathbf{P}}(\mathbb{R}^N)}, \tag{A.18}$$

and (A.5) follows from the triangle inequality. ■

**Corollary A.2** *The space $Y$ defined in (4.40) satisfies the condition in Theorem A.1 if and only if*

$$\frac{p_0'}{p_1 q_0} + \frac{1}{q_1} < \frac{1}{N-1}. \tag{A.19}$$

*Proof.* The matrix $\mathbf{P} - \lambda\mathbf{I}$ has the form

$$\mathbf{P} - \lambda\mathbf{I} = \begin{pmatrix} 1/q_1 - \lambda & 1/q_1 & \dots & 1/q_1 & 1/p_1 \\ 1/q_1 & 1/q_1 - \lambda & \dots & 1/q_1 & 1/p_1 \\ & & \dots & & \\ 1/q_1 & 1/q_1 & \dots & 1/q_1 - \lambda & 1/p_1 \\ 1/q_0 & 1/q_0 & \dots & 1/q_0 & 1/p_0 - \lambda \end{pmatrix},$$

and its determinant is

$$\det(\mathbf{P} - \lambda\mathbf{I}) = (-\lambda)^{N-2}\left(\left(\frac{N-1}{q_1} - \lambda\right)\left(\frac{1}{p_0} - \lambda\right) - \frac{N-1}{q_0 p_1}\right).$$

We easily check that all roots of the equation $\det(\mathbf{P} - \lambda\mathbf{I}) = 0$ are in absolute value smaller than 1 if and only if condition (A.19) holds. ■

**Remark A.3** The embedding formula (4.38) in $\mathbb{R}^3$ can be derived in a straightforward way from (A.6), where we set $b_1 = b_2 = \frac{1}{2}$, $b_3 = 1$. Put $u(y', y_N) = \xi(y', y_N)$ for $y_N > 0$, $u(y', y_N) = \xi(y', -y_N)$ for $y_N < 0$. Assuming that $\Phi(z) = \Phi(-z)$, we may set $\hat\Psi_1(z) = \int_{-\infty}^{z_1} \Psi_1(s, z_2, z_3)\mathrm{d}s$, $\hat\Psi_2(z) = \int_{-\infty}^{z_2} \Psi_2(z_1, s, z_3)\mathrm{d}s$. Then $\hat\Psi_1$, $\hat\Psi_2$ have compact support and we may integrate by parts in (A.9) to obtain

$$\begin{aligned} \frac{\partial u^\sigma}{\partial\sigma}(x) \;&=\; -\sum_{i=1}^2 \sigma^{-|\mathbf{b}|-1+2b_i}\int_{\mathbb{R}^3}\hat\Psi_i\Big(\frac{x-y}{\sigma^{\mathbf{b}}}\Big)\frac{\partial^2 u(y)}{\partial y_i^2}\,\mathrm{d}y \\ &\quad - \sigma^{-|\mathbf{b}|-1+b_3}\int_{\mathbb{R}^3}\Psi_3\Big(\frac{x-y}{\sigma^{\mathbf{b}}}\Big)\frac{\partial u(y)}{\partial y_3}\,\mathrm{d}y\,. \end{aligned}$$

Integrals of the form $\int_{\mathbb{R}^3} \Psi_*\Big(\frac{x-y}{\sigma^{\mathbf{b}}}\Big)u_*(y)\,\mathrm{d}y$ with $u_* \in L^2(\mathbb{R}^3)$ can be estimated in $L^4(\mathbb{R}^3)$ using the Young inequality for convolutions as

$$\Big|\int_{\mathbb{R}^3} \Psi_*\Big(\frac{\cdot - y}{\sigma^{\mathbf{b}}}\Big)u_*(y)\,\mathrm{d}y\Big|_4 \le \sigma^{(3/4)|\mathbf{b}|}|\Psi_*|_{4/3}|u_*|_2. \tag{A.20}$$

Hence, by virtue of the choice of $\mathbf{b}$, we have

$$\Big|\frac{\partial u^\sigma}{\partial \sigma}\Big|_4 \le C\sigma^{-1/2}\Big(\Big|\frac{\partial^2 u}{\partial y_1^2}\Big|_2 + \Big|\frac{\partial^2 u}{\partial y_2^2}\Big|_2 + \Big|\frac{\partial u}{\partial y_3}\Big|_2\Big), \qquad |u^\sigma|_4 \le C\sigma^{-1/2}|u|_2\,, \tag{A.21}$$

and (4.37) follows from the inequality

$$|u|_4 \le |u^\sigma|_4 + \Big|\int_0^\sigma \frac{\partial u^{\sigma'}}{\partial \sigma'}\mathrm{d}\sigma'\Big|_4. \tag{A.22}$$

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