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Global Weak Solutions of the Navier–Stokes–Vlasov–Poisson System

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Abstract

We consider the Navier-Stokes-Vlasov-Poisson system of partial differential equations, describing the motion of a viscous incompressible fluid with small solid charged particles therein. We prove the existence of a weak global solution of the initial boundary value problem for this system.

1 Introduction

The increasing interest in studying the motion of small solid particles in liquids and gases is stimulated by numerous applications of these processes in a wide range of engineering problems as well as by ecological needs. For example, we refer here to the problem of transport of fine–dispersed suspensions by aerial or liquid flows, the work of hydraulic or pneumatic transport devices, dust–collecting units, etc. The mathematical modeling of the motion of such matter (suspensions) is considered in a great number of papers (see, e.g., [18], [8] and the bibliography there). One of the models often used in the simulation of such processes is the two phase flow model. The main feature of this model is that the system of small solid particles is considered as a continuous matter. Then, the motion of a liquid with particles suspended therein is described as a motion of two inter-penetrating continuous phases — the carrying liquid and the "liquid of particles". However, this model is applicable only in the case when the size and the specific density of the particles are identical or slightly dispersed.

Another model – the Navier–Stokes–Liouville system – describe the motion of the mixture of a liquid and small solid particles, taking into account the high dispersion of their size. In the framework of this model the solid phase of the mixture is assumed to be a system of spherical particles of high specific density described by the distribution function of the particles depending on their coordinates, velocities and radii. This model is based on the homogenized Navier-Stokes system of equations describing the perturbation of the liquid by the motion of solid particles (see [13], [14]). This system involve the unknown distribution function of the particles $f(x, v, r, t)$. The distribution function satisfies the Liouville equation with account of Stokes forces. Combining this equation and the perturbated Navier–Stokes system we obtain a closed system of equations — the Navier–Stokes–Liouville system. The existence of a global weak solution of the initial boundary value problem for this system as well as the existence and uniqueness of a smooth solution in a small time interval was proved in [2] and [3].

In the present paper we consider a similar model which describes the motion of small solid charged particles with high dispersion of radii in a viscous incompressible and non-conducting fluid. We assume that the charges of all particles are of the same sign and proportional to their electric capacities. This means that the charge of a particle of radius r' is equal to $q\,r'$. In this case our model is described by the following system of equations:

$$\frac{\partial u}{\partial t} + (u\nabla)u - \nu\Delta u + \alpha \iint\limits_{a\mathbb{R}^3}^{b} r(u(x,t) - v) f(x,v,r,t)\,dvdr - \nabla p = g, \tag{1.1}$$

$$\mathrm{div} u = 0, \tag{1.2}$$

$$-\Delta\varphi = q \iint\limits_{a\mathbb{R}^3}^{b} r f(x,v,r,t)\,dvdr, \tag{1.3}$$

$$\frac{\partial f}{\partial t} + (v\nabla)f + \mathrm{div}_v[G(u, v, \nabla\varphi, r)f] = 0. \tag{1.4}$$

$$G = \beta r^{-2}[u(x,t) - v] - \gamma r^{-2}\nabla\varphi + g. \tag{1.5}$$

Here $u = u(x,t)$ and $p = p(x,t)$ are the velocity and the pressure of the liquid, respectively; $\varphi = \varphi(x,t)$ is the potential of the electric field, generated by the charged particles; $f(x, v, r, t)$ is a reduced distribution function associated to the real distribution function of the particles $f_\varepsilon(x, v, r, t)$ with respect to the space variable $x = (x_1, x_2, x_3)$, the velocities $v = (v_1, v_2, v_3)$ and radii $r' = \varepsilon r$ $(0 < a \le r \le b < \infty)$ by the formula:

$$f_\varepsilon(x, v, r', t) = \frac{1}{\varepsilon} f(x, v, \frac{r'}{\varepsilon}, t).$$

Here ε is the mean radius of particles (small parameter); α, β and γ are constants defined by:

$$\alpha = 6\pi\nu; \quad \beta = \frac{9\rho_f \nu}{2\rho_p \varepsilon^2}; \quad \gamma = \frac{3q}{4\pi\rho_p\varepsilon^2};$$

ν is the kinematic viscosity of the liquid; ρ_f and ρ_p are the specific densities of the liquid and the particles, respectively; $g = g(x)$ is the gravity.

We consider system (1.1)–(1.5) in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We assume the following boundary conditions on $\partial\Omega$:

$$u(x,t) = 0 \quad \text{on } S_T \equiv \partial\Omega \times [0, T], \tag{1.6}$$

$$\varphi(x,t) = 0 \quad \text{on } S_T, \tag{1.7}$$

$$f(x, v, r, t)(v, n) \ge 0 \quad \text{on } \partial\Omega \times \mathbb{R}^3 \times [a, b] \times [0, T], \tag{1.8}$$

where $n = n(x)$ is the outer normal vector to $\partial\Omega$ at the point x; $(\,\cdot\,,\,\cdot\,)$ in (1.8) denotes the scalar product in $\mathbb{R}^3$.

Condition (1.6) corresponds to the adhesion of the liquid to the fixed boundary $\partial\Omega$. Condition (1.7) means that the boundary is a perfectly conducting one. Finally, condition (1.8) means that a particle, reaching the boundary $\partial\Omega$ sticks there and comes to rest.

We complete system (1.1) – (1.5) by the following initial conditions:

$$u(x, 0) = u_0(x) \quad \text{in } \Omega; \tag{1.9}$$

$$f(x, v, r, 0) = f_0(x, v, r) \quad \text{in } \Omega \times \mathbb{R}^3 \times [a, b]. \tag{1.10}$$

Henceforth, we call system (1.1)–(1.5) the Navier–Stokes–Vlasov–Poisson system. It is a combination of the Navier–Stokes and the Vlasov–Poisson systems. Existence and uniqueness results for both of these systems were studied separately by many authors and by various methods (see, e.g., [11] – [1]).

Our goal is to prove the existence of a global weak solution of problem (1.1) – (1.10). The approach which is used in the present paper is a generalization of the methods developed in [4], [5].

The outline of the paper is the following. In Section 2 we introduce the notation of the weak solution of problem (1.1)–(1.10) and formulate the main result. In Section 3 we introduce a regularization of problem (1.1)–(1.10) and define a weak solution (u, f, φ) for the regularized problem. Then, we construct finite–dimensional approximations (u^n, f^n, φ^n) of the solution. To this end we use the modification of Galerkin's method developed in [4]. Following [5], we use an explicit construction for the solution of the Liouville equation (1.4). The compactness of the approximations (u^n, f^n, φ^n) is proved in Section 4. Finally, in Section 5 we pass to the limit as $n \to \infty$ in the integral identities which define the weak solution of the regularized problem and obtain the corresponding identities for the weak solution of the original problem.

2 Definition of the weak solution and formulation of the main result

Let Ω be a bounded convex domain in $\mathbb{R}^3$ with a sufficient smooth boundary. We introduce the following notation:

$$\begin{aligned} \Omega_T &= \Omega \times [0, T], \\ Q &= \Omega \times \mathbb{R}^3 \times [a, b],\ b > a > 0, \\ Q_T &= Q \times [0, T], \\ \mathbb{R}^6 &= \mathbb{R}^3 \times \mathbb{R}^3, \\ \mathbb{R}^6_T &= \mathbb{R}^6 \times [0, T]; \end{aligned}$$

$L_2(\Omega)$ and $L_2(\mathbb{R}^6)$ are Hilbert spaces with the scalar products

$$\begin{aligned} (f, g)_{2,\Omega} &= \int\limits_{\Omega} \sum_{i=1}^{3} f_i(x) g_i(x) dx, \\ (F, G)_{2,\mathbb{R}^6} &= \int\limits_{\mathbb{R}^6} F(x, v, r) G(x, v, r) dx dv; \end{aligned}$$

$J(\Omega)$ and $J^1(\Omega)$ are the closures of divergent-free $C^\infty(\bar{\Omega})$ functions with compact support in $L_2(\Omega)$ and $\overset{\circ}{W}{}^1_2(\Omega)$, respectively;

P_0 is an extension operator from $L_2(\Omega)$ to $L_2(\mathbb{R}^3)$ such that for any $u \in L_2(\Omega)$, $P_0 u = u$ in Ω and $P_0 u = 0$ in $\mathbb{R}^3 \setminus \Omega$;

S is a restriction operator from $L_2(\mathbb{R}^3)$ to $L_2(\Omega)$ such that for any $u \in L_2(\mathbb{R}^3)$ $Su = \chi_\Omega u$, where χ_Ω is the characteristic function of Ω.

We assume that the initial functions $u_0(x)$ and $f_0(x, v, r)$ in (1.9), (1.10) satisfy the following conditions

$$\mathrm{div} u_0 = 0, x \in \Omega, \quad u_0(x) = 0, x \in \partial\Omega,$$

$$0 \le f_0(x,v,r) \le A_1 < \infty, (x,v,r) \in Q, \tag{2.1}$$

$$\int\limits_Q f_0(x,v,r)dxdvdr = A_2 < \infty, \quad \int\limits_Q v^2 f_0(x,v,r)dxdvdr = A_3 < \infty.$$

We consider the vector function $(u(x,t), \varphi(x,t), f(x,v,r,t))$, where

$$u \in L_\infty(0,T;J(\Omega)) \cap L_2(0,T;J^1(\Omega)), \tag{2.2a}$$

$u(x,t)$ is a continuous function in t in the weak topology of $L_2(\Omega)$

$$\varphi \in L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega)) \tag{2.2b}$$

$$f(x,v,r,t) = S\tilde{f}(x,v,r,t), \tag{2.2c}$$

where $\tilde{f} \in L_\infty(\mathbb{R}^6_T \times [a,b])$, $\tilde{f} \in L_1(\mathbb{R}^6 \times [a,b])$ uniformly in $t \in [0,T]$ and $\tilde{f}$ is continuous in t in the weak topology of $L_1(\mathbb{R}^6 \times [a,b])$ at $t=0$.

Definition 1 *The vector $(u(x,t), \varphi(x,t), f(x,v,r,t))$ is a weak solution of problem (1.1) – (1.10) if the following integral identities hold*

$$(u_0, \zeta(0))_{2,\Omega} + \int\limits_0^T \Big\{ (u, \zeta_t + (u\nabla_x)\zeta)_{2,\Omega} - \nu(u,\zeta)_{J^1(\Omega)} -$$
$$-\alpha \left(\iint\limits_a^b r(u(x,t)-v)S\tilde{f}dvdr, \zeta \right)_{2,\Omega} + (g,\zeta)_{2,\Omega} \Big\} dt = 0 \tag{2.3}$$

$$\int\limits_0^T \left\{ (\nabla\varphi, \nabla\Phi)_{2,\Omega} - q\left(\iint\limits_a^b rS\tilde{f}dvdr, \Phi \right)_{2,\Omega} \right\} dt = 0 \tag{2.4}$$

$$\int\limits_0^T\int\limits_a^b (\tilde{f}, \Psi_t + (v\nabla_x)\Psi + (P_0 G\nabla_v)\Psi)_{2,\mathbb{R}^6} drdt + \int\limits_a^b (P_0 f_0, \Psi(0))_{2,\mathbb{R}^6} dr = 0 \tag{2.5}$$

for any vector functions ζ and functions Φ and Ψ which satisfy the following conditions

$$\zeta \in L_\infty(0,T;J(\Omega)) \cap L_4(0,T;J^1(\Omega)), \quad \zeta_t \in L_2(\Omega_T), \quad \zeta(x,t) = 0; \tag{2.6a}$$

$$\Phi \in L_2(0,T;W_2^1(\Omega)). \tag{2.6b}$$

$\Psi(x,v,r,t)$ is a function with compact support in $\mathbb{R}^6_T \times [a,b]$ on x and v,

$$\nabla_x\Psi \in L_1(\mathbb{R}^6_T \times [a,b]), \quad \nabla_v\Psi \in L_\infty(\mathbb{R}^6_T \times [a,b]), \tag{2.6c}$$

$$\Psi_T \in L_1(\mathbb{R}^6_T \times [a,b]), \quad \Psi(x,v,r,T) = 0.$$

Remark 1 *We introduced the operators P_0 and S for the following reason: At first we construct the solution of (1.4) on the whole domain $\mathbb{R}^6_T \times [a,b]$ and then restrict it to the set Q_T. Moreover, the convexity of the domain Ω implies condition (1.8).*

Theorem 1 *Let $g \in L_\infty(0,T;C^1(\Omega))$, $u_0 \in J(\Omega)$ and $f_0(x,v,r)$ satisfies (2.1). Then, there exists a weak solution of problem (1.1) – (1.10), such that*

$$\max_{0\le t\le T} \|u\|_{2,\Omega} + \max_{0\le t\le T} \int\limits_Q v^2 f dxdvdr + \int\limits_0^T \|u(t)\|^2_{J^1(\Omega)} dt + \max_{0\le t\le T} \|\nabla\varphi(t)\|^2_{2,\Omega} <$$

$$< C\left[\|u_0\|_{2,\Omega} + \int\limits_Q (1+v^2) f_0(x,v,r) dxdvdr + \|g\|^2_{L_\infty(0,T;C^1(\Omega))} \right].$$

This Theorem is proved in the sections 3–5.

3 Approximation of the solution

3.1 Regularization of the problem and definition of the solution

We consider the problem

$$\frac{\partial u}{\partial t} + (u\nabla_x)u - \nu\Delta u + \alpha \iint\limits_a^b r\theta_R((u-v)^2(u(x,t)-v) f dvdr - \nabla p = g, \tag{3.1}$$

$$\mathrm{div} u = 0, \tag{3.2}$$

$$\varepsilon\Delta^2\varphi - \Delta\varphi = q \iint\limits_a^b rf(x,v,r,t)dvdr, \tag{3.3}$$

$$\frac{\partial f}{\partial t} + (v\nabla_x)f + \mathrm{div}_v[G_{R,\varepsilon}(u,v,\nabla\varphi,g)f] = 0, \tag{3.4}$$

$$G_{R,\varepsilon}(u,v,\nabla\varphi,g) = \frac{\beta}{r^2}\theta_R((u-v)^2)[u-v] - \frac{\gamma}{r^2}\nabla\varphi + \chi_\varepsilon(x)\cdot g. \tag{3.5}$$

We complete this problem by the conditions (1.6) – (1.8) along with the following one

$$\frac{\partial\varphi}{\partial n} = 0, \quad (x,t) \in S_T. \tag{3.6}$$

Here $\varepsilon > 0$ is a sufficiently small parameter; $\theta_R \in C^\infty(\mathbb{R})$, such that $0 \le \theta_R(z) \le 1$ if $|z| \le R$, $\theta_R(z) = 0$ if $|z| > 2R$ and $\theta'_R \le 0$ if $z \ge 0$.

We denote by Ω_ε a subdomain of Ω such that $\mathrm{dist}(\partial\Omega_\varepsilon, \partial\Omega) = \varepsilon$, and by $\chi_\varepsilon \in C_0^2(\Omega)$ a function such that $\chi_\varepsilon(x) = 1$ if $x \in \Omega_\varepsilon$ and $\chi_\varepsilon = 0$ if $x \in \partial\Omega$ and set $g_\varepsilon(x,t) = g(x,t)\chi_\varepsilon(x)$.

We consider the system $(u(x,t), \varphi(x,t), f(x,v,r,t))$ that satisfies conditions (2.2a), (2.2c), (1.7), (3.6) with

$$\varphi \in L_2(0,T; \overset{\circ}{W}{}_2^2(\Omega)). \tag{3.7}$$

We call the system $(u(x,t), \varphi(x,t), f(x,v,r,t))$ a weak solution of problem (3.1) – (3.5), (1.6) – (1.8) if the following integral identities hold

$$\begin{aligned}&\int_0^T \Big\{ (u, \zeta_t + (u\nabla_x)\zeta)_{2,\Omega} - \nu(u,\zeta)_{J^1(\Omega)} + (g,\zeta)_{2,\Omega} - \\ &- \alpha\left(\iint_a^b r\theta_R((u-v)^2)(u-v)S\tilde{f}dvdr, \zeta\right)_{2,\Omega}\Big\}dt + (u_0, \zeta(0))_{2,\Omega} = 0,\end{aligned} \tag{3.8}$$

$$\int_0^T \left\{ \varepsilon(\Delta\varphi, \Delta\Phi)_{2,\Omega} + (\nabla\varphi, \nabla\Phi)_{2,\Omega} - \left(q\int_a^b\int rS\tilde{f}dvdr, \Phi\right)_{2,\Omega}\right\} dt = 0, \tag{3.9}$$

$$\int_0^T\int_a^b (\tilde{f}, \Psi_t + (v\nabla_x)\Psi + P_0G_{R,\varepsilon}\nabla_v)\Psi)_{2,\mathbb{R}^6}drdt + \int_a^b (P_0f_0, \Psi(0))_{2,\mathbb{R}^6}dr = 0, \tag{3.10}$$

for any vector function ζ, any function Ψ satisfying the conditions (2.6a), (2.6c) and function $\Phi \in L_2(0,T; \overset{\circ}{W}{}_2^2(\Omega))$.

3.2 Construction of approximations

To construct the approximations of the regularized problem we make use of the following

Lemma 1 *Suppose that the function $f_0(x,v,r)$ satisfies condition (2.1). Then, there exists a sequence of non-negative functions $f_0^n(x,v,r)$ defined in Q such that for any fixed $n \in \mathbb{N}$ and $r \in [a,b]$, $f_0^n(x,v,r)$ is infinitely-differentiable in x, and in v, $f_0^n(x,v,r)$ has compact support in $\Omega \times \mathbb{R}^3 \times [a,b]$, $f_0^n(x,v,r)$ is bounded $\sup_Q f_0^n \le A_1$ and $f_0^n(x,v,r)$ satisfies the inequalities*

$$\int_Q f_0^n(x,v,r)dxdvdr \le A_2, \int_Q v^2 f_0^n(x,v,r)dxdvdr \le A_3 + 3A_2.$$

Moreover, $f_0^n \to f_0$ in $L_2(Q)$ as $n \to \infty$.

Proof. To construct the sequence $\{f_0^n\}$ we smooth the function f_0. Let $\omega(|\xi|)$, $\xi \in \mathbb{R}^3$ be a non-negative function such that: $\omega \in C^\infty(\mathbb{R}^3)$; $\omega(|\xi|) = 0$ if $|\xi| \geq 1$ and $\int\limits_{|\xi|\leq 1} \omega(|\xi|)d\xi = 1$.

We define the sequence

$$f_0^n(x, v, r) = n^6 \int\limits_{\Omega_n} \int\limits_{|v'|<n} \omega(n|x - x'|\omega(n|v - v'|) f_0(x', v', r) dx'dv',$$

where $\Omega_n \subset \Omega$ and $\operatorname{dist}(\partial\Omega_n, \partial\Omega) = 1/n$.

It is easy to see that the sequence $f_0^n(x, v, r)$ satisfies all assertions of the lemma. $\square$

Now, we construct an approximation by the method, developed in [4], which is a modification of Galerkin's method. We are looking for the approximations of (3.1), (3.2) in the form

$$u^n(x, t) = \sum_{l=1}^{n} C_{nl}(t)\Psi^l(x), \tag{3.11}$$

where $C_{nl} \in C^1(0, T)$ are unknown coefficients and $\Psi^l(x)$ $(l = 1, 2, \dots)$ is the orthonormal basis in $L_2(\Omega)$ consisting of the eigenfunctions of the problem

$$\Delta\Psi^l(x) - \nabla g^l = \mu_l \Psi^l(x), \quad \operatorname{div}\Psi^l(x) = 0, \quad x \in \Omega, \quad \Psi^l(x) = 0, \quad x \in \partial\Omega.$$

The corresponding approximations $\varphi^n(x, t)$, $\tilde{f}^n(x, v, r, t)$ for solutions of the equations (3.3) and (3.4) turn out as solutions of the following problem

$$\begin{aligned}
\varepsilon\Delta^2\varphi^n - \Delta\varphi^n &= q \int_a^b \int rS\tilde{f}^n(x, v, r, t)dvdr, && (3.12)\\
\varphi^n(x, t) &= \frac{\partial\varphi^n}{\partial n} = 0, \quad (x, t) \in S_T, && (3.13)\\
\frac{\partial\tilde{f}^n}{\partial t} + (v\nabla_x)\tilde{f}^n &+ \operatorname{div}_v \left\{ \left[\frac{\beta}{r^2}\theta_R((P_0u^n - v)^2)(P_0u^n - v) \right. \right. - \\
&- \left. \left. \frac{\gamma}{r^2}P_0\nabla\varphi^n + P_0 g_\varepsilon \right] \tilde{f}^n \right\} = 0, && (3.14)\\
\tilde{f}^n|_{t=0} &= P_0 f_0^n, && (3.15)
\end{aligned}$$

where the initial functions f_0^n are given in Lemma 1.

We define functions $X^n(x,v,r,t,\tau)$ and $V^n(x,v,r,t,\tau)$ as solutions of the following system of equations

$$\left.\begin{array}{rcl} \dfrac{dX^n}{d\tau} &=& V^n, \\ \dfrac{dV^n}{d\tau} &=& \dfrac{\beta}{r^2}\theta_R((P_0u^n(X^n,\tau)-V^n)^2)(P_0u^n(X^n,\tau)-V^n)- \\ &-& \dfrac{\gamma}{r^2}P_0\nabla\varphi^n(X^n,\tau)+P_0g_\varepsilon(X^n,\tau), \\ X^n|_{\tau=t} &=& x, \quad V^n|_{\tau=t}=v, \quad 0\le\tau\le t, \quad t\in[0,T]. \end{array}\right\} \tag{3.16}$$

The properties of the function Ψ^i (see [11]) imply

$$\sup_{\Omega_T}|\nabla u^n(x,t)|<\infty, \quad u^n|_{S_T}=0.$$

If for any $t\in[0,T]$ the function $\varphi^n(x,t)$ belongs to $C^2(\Omega)$ and the condition (3.13) is valid, then the right-hand side of system (3.16) satisfies the Lipschitz condition in X^n and V^n. So, we obtain the local solvability of (3.16). For any $\tau\in[0,t]$ the solutions X^n and V^n are bounded (see Lemma 2). Thus, we are able to extend them to $\tau=0$.

It is easy to verify that the solution of problem (3.14), (3.15) is given by the formula:

$$\begin{array}{rcl} \tilde{f}^n(x,v,r,t) &=& \exp\left\{\dfrac{\beta}{r^2}\displaystyle\int_0^t \big[3\theta_R((P_0u^n(X^n,\tau)-V^n)^2)+\right. \\ &+& \left. 2\theta_R'((P_0u^n(X^n,\tau)-V^n)^2)(P_0u^n(X^n,\tau)-V^n)^2\big]d\tau\right\}\times \\ &\times& P_0f_0^n(X^n(x,v,r,t,0),V^n(x,v,r,t,0),r). \end{array} \tag{3.17}$$

Lemma 2 *If $P_0f_0(x,v,r)$ has compact support with respect to x and v in $\mathbb{R}^6$, then the solution of problem (3.14), (3.15) also has compact support for any t.*

Proof. Suppose that $\mathrm{supp}P_0f_0^n\subset\Omega\times K_{R_0}(0)\times[a,b]$, where $K_{R_0}(0)=\{v\in\mathbb{R}^3 : |v|\le R_0\}$. We show that for any $x\in\mathbb{R}^3$, $r\in[a,b]$ $t\in[0,T]$ and any $\tau\in[0,T]$, the inequality

$$|v|>R_0+\frac{\beta}{a^2}T\sqrt{2R}+\frac{\gamma}{a^2}\|\varphi^n\|_{L_2(0,T;C^2(\Omega))}\sqrt{T}+\|g\|_{L_\infty(0,T;C^1(\Omega))}T=R_{\tilde{f}^n} \tag{3.18}$$

implies inequality $|V^n| > R_0$. To this end, we consider the following system of integral equations, equivalent to (3.16)

$$X^n(x,v,r,t,\tau) - x = \int_t^\tau V^n(x,v,r,t,s)ds, \tag{3.19}$$

$$\begin{aligned} V^n(x,v,r,t,\tau) - v &= \frac{\beta}{r^2}\int\limits_t^\tau \left[\theta_R((P_0u^n(X^n,s)-v)^2)(P_0u^n(X^n,s)-V^n)\right] ds - \\ &- \frac{\gamma}{r^2}\int\limits_t^\tau P_0\nabla\varphi^n(X^n,s)ds + \int\limits_t^\tau P_0 g_\varepsilon(X^n,s)ds. \end{aligned} \tag{3.20}$$

From (3.20) and (3.18) we obtain

$$|V^n(x,v,r,t,\tau)| \geq |v| - \frac{\beta}{r^2}\left|\int\limits_t^\tau \theta_R((P_0u^n(X^n,s)-v)^2)(P_0u^n(X^n,s)-V^n)ds\right| -$$

$$-\frac{\gamma}{r^2}\left|\int\limits_t^\tau P_0\nabla\varphi^n(X^n,s)ds\right| - \left|\int\limits_t^\tau P_0 g_\varepsilon(X^n,s)ds\right| > R_0.$$

On the other hand, from (3.20) follows

$$\sup_\tau |V^n| \leq |v| + \frac{\beta}{a^2}T\sqrt{2R} + \frac{\gamma}{a^2}\|\varphi^n\|_{L_2(0,T;C^2(\Omega))}\sqrt{T} + \|g\|_{L_\infty(0,T;C^1(\Omega))}T.$$

From this estimate and (3.19) we conclude

$$|X^n - x| \leq T\left(|v| + \frac{\beta}{a^2}T\sqrt{2R} + \frac{\gamma}{a^2}\|\varphi^n\|_{L_2(0,T;C^2(\Omega))}\sqrt{T} + \|g\|_{L_\infty(0,T;C^1(\Omega))}T\right).$$

Hence, (3.17) imply $\mathrm{supp}\tilde{f}^n \subset \Omega \times K_{R_{\tilde{f}^n}}(0)$ for any $t \in [0,T]$, $r \in [a,b]$. $\square$

Now, we show that the convexity of Ω implies the following boundary condition on the function $f^n(x,v,r,t) = S\tilde{f}^n(x,v,r,t)$:

$$f^n(x,v,r,t)(v,n(x)) \geq 0, \quad x \in \partial\Omega. \tag{3.21}$$

Since the function $f^n(x,v,r,t)$ is non-negative, condition (3.21) is equivalent to the following statement: If there exists a point $x_0 \in \partial\Omega$ such that $(v,n(x_0)) < 0$, then $f^n(x_0,v,r,t) = 0$. Thus, the convexity of Ω implies that for $\tau < t$ the particle is out of Ω and its motion is described by the equations (uniform linear motion):

$$\frac{dX^n}{d\tau} = V^n, \quad \frac{dV^n}{d\tau} = -\frac{\beta}{r^2}\theta_R((V^n)^2)V^n,$$

$$X^n|_{\tau=t} = x, \quad V^n|_{\tau=t} = v, \quad 0 \leq \tau \leq t.$$

Thus, the trajectory of the particle is a straight line if $\tau \in [0,t]$. Therefore, $P_0 f_0(X^n(x_0,v,t,0), V^n(x_0,v,t,0), r) = 0$ and, due to (3.17), the desired boundary condition (3.21) holds.

To find the coefficients $C_{nl}(t)$ in (3.11) we assume that identity (3.8) with respect to u^n and $\tilde{f}^n$ holds for all vector functions $\zeta(x,t) = H(t)\Psi^j(x)$, $j = 1,2,\ldots,n$. Here $H \in C^1(0,T)$, $H(T) = 0$. This assumption implies the following relations

$$\begin{aligned}\left(\frac{\partial u^n}{\partial t} + (u^n\nabla_x)u^n \right. &+ \left. \alpha \int\limits_a^b\!\!\int r\theta_R((u^n - v)^2)(u^n - v)S\tilde{f}^n dvdr, \Psi^k\right)_{2,\Omega} + \\ &+ \nu(u^n,\psi^k)_{J^1(\Omega)} = (g,\Psi^k)_{2,\Omega}, \quad k = 1,2,\ldots,n. \qquad (3.22)\end{aligned}$$

It is possible to represent these relations as a system of differential functional equations

$$\begin{aligned}\frac{dC_{nk}}{dt} &+ \sum_{l,m=1}^{n} \beta^k_{lm} C_{nl}(t)C_{nm}(t) + \sum_{l=1}^{n} \varepsilon^k_l C_{nl}(t) + \\ &+ \alpha\left(\int\limits_a^b\!\!\int r\theta_R\left(\left(\sum_{l=1}^{n} C_{nl}(t)\Psi^l - v\right)^2\right)\left(\sum_{l=1}^{n} C_{nl}(t)\Psi^l - v\right) S\tilde{f}^n dvdr, \Psi^k\right)_{2,\Omega} = \\ &= g^k, \quad k = 1,2,\ldots,n, \qquad (3.23)\end{aligned}$$

where

$$\beta^k_{lm} = ((\Psi^l\nabla)\Psi^m, \Psi^k)_{2,\Omega}, \quad \varepsilon^k_l = \nu(\Psi^l,\Psi^k)_{J^1(\Omega)}, \quad g^k = (g,\Psi^k)_{2,\Omega}.$$

Expanding the initial function $u_0(x)$ into a series with respect to the basis $\Psi^k(x)$

$$u_0(x) = \sum_{k=1}^{\infty} C_k\Psi^k(x)$$

we obtain the initial conditions

$$C_{nk}(0) = C_k, \quad k = 1,2,\ldots,n. \qquad (3.24)$$

3.3 A priori estimates of the approximations

Lemma 3 *The following estimates hold uniformly in n*

$$\sup_{\mathbb{R}^6_T\times[a,b]} \tilde{f}^n \le A, \qquad (3.25a)$$

$$\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} \tilde{f}^n(x,v,r,t)dxdvdr \le \int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} f_0(x,v,r)dxdvdr, \qquad (3.25b)$$

$$\max_{0\le t\le T}\|u^n(t)\|^2_{2,\Omega} + \max_{0\le t\le T}\int_a^b\int v^2\tilde f^n dxdvdr + \int_0^T \|u^n(t)\|^2_{J^1(\Omega)}dt +$$
$$+ \int_0^T\int_a^b\int \theta_R((P_0u^n-v)^2)(P_0u^n-v)^2\tilde f^n dxdvdrdt +$$
$$+ \varepsilon\max_{0\le t\le T}\|\Delta\varphi^n(t)\|^2_{2,\Omega} + \max_{0\le t\le T}\|\nabla\varphi^n(t)\|^2_{2,\Omega}\le A. \qquad (3.25c)$$

The constant A depends only on u_0, f_0, g, α, β, ν, and T.

Proof. Using the boundedness of the functions $f_0^n(x,v,r)$ and the definition of $\theta_R(z)$, one obtains inequality (3.25a) from (3.17).

To prove (3.25b), we integrate equation (3.14) over the domain $\mathbb{R}^6\times[a,b]$. Since $\tilde f^n$ has compact support on $(x,v)\in\mathbb{R}^6$ we get

$$\frac{d}{dt}\int_a^b\int_{\mathbb{R}^6}\tilde f^n dxdvdr = 0.$$

Hence, inequality (3.25b) is proved.

Multiplying the k-th equation of system (3.22) by $C_{nk}(t)$ and adding over k from 1 to n we get

$$\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2_{2,\Omega} + \nu\|u^n\|^2_{J^1(\Omega)} +$$
$$+\alpha\left(\int_a^b\int r\theta_R((u^n-v)^2)(u^n-v)S\tilde f^n dvdr, u^n\right)_{2,\Omega} = (g,u^n)_{2,\Omega}.$$

Extending the vector functions u^n and g by zero to full $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}|P_0u^n\|^2_{2,\mathbb{R}^3} + \nu\|P_0u^n\|^2_{J^1(\mathbb{R}^3)} +$$
$$+ \alpha\left(\int_a^b\int r\theta_R((P_0u^n-v)^2)(P_0u^n-v)\tilde f^n dvdr, P_0u^n\right)_{2,\mathbb{R}^3} =$$
$$= (P_0g, P_0u^n)_{2,\mathbb{R}^3}. \qquad (3.26)$$

Multiplying (3.14) by v^2r^3 and integrating over $\mathbb{R}^6\times[a,b]$ we get

$$\frac{d}{dt}\Big\{\int_a^b\int\int r^3v^2\tilde f^n dxdvdr - 2\int_a^b\int\int\Big[\beta r\theta_R((P_0u^n-v)^2))(P_0u^n-v,v)-$$
$$-\gamma r(\nabla P_0\varphi^n, v) + r^3(v,P_0g_\varepsilon)\Big]\tilde f^n dxdvdr\Big\} = 0.$$

Taking the sum of the previous equation multiplied by $\frac{\alpha}{2\beta}$ and (3.26), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|P_0u^n\|^2_{2,\mathbb{R}^3} &+ \nu\|P_0u^n\|^2_{J^1(\mathbb{R}^3)} + \alpha\iint\limits_a^b r\theta_R((P_0u^n - v)^2)(P_0u^n - v)^2\tilde{f}^n dxdvdr + \\ &+ \frac{\alpha}{2\beta}\frac{d}{dt}\iint\limits_a^b r^3v^2\tilde{f}^n dxdvdr + \frac{\alpha\gamma}{\beta}\iint\limits_a^b r(\nabla P_0\varphi^n, v)\tilde{f}^n dxdvdr - \\ &- \frac{\alpha}{\beta}\iint\limits_a^b r^3\tilde{f}^n(v, P_0g_\varepsilon)dxdvdr = (P_0g, P_0u^n)_{2,\mathbb{R}^3}. \qquad (3.27)\end{aligned}$$

Differentiating (3.12) by t, multiplying it by $\varphi^n(x,t)$ and integrating the result over Ω, we obtain

$$\varepsilon\int\limits_\Omega \Delta^2\varphi^n_t\varphi^n dx - \int\limits_\Omega \Delta\varphi^n_t\varphi^n dx = q\iint\limits_{a\,Q}^b r\varphi^n(x,t)S\frac{\partial\tilde{f}^n(x,v,r,t)}{\partial t}dxdvdr.$$

Taking into account the boundary conditions (3.13), we get:

$$\frac{\varepsilon}{2}\frac{d}{dt}\int\limits_\Omega(\Delta\varphi^n)^2dx + \frac{1}{2}\frac{d}{dt}\int\limits_\Omega|\nabla\varphi^n|^2dx = q\iint\limits_{a\,Q}^b r\varphi^n(x,t)S\frac{\partial\tilde{f}^n}{\partial t}dxdvdr.$$

We extend the function $\varphi^n(x,t)$ by zero outside of Ω. Then,

$$\frac{\varepsilon}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^3}(\Delta P_0\varphi^n)^2dx + \frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^3}|\nabla P_0\varphi^n|^2dx = q\iint\limits_{a\,Q}^b rP_0\varphi^n(x,t)\frac{\partial\tilde{f}^n}{\partial t}dxdvdr.$$

Due to (3.14), the right-hand side of this equation can be rewritten as follows:

$$\varepsilon\frac{d}{dt}\int\limits_{\mathbb{R}^3}(\Delta P_0\varphi^n)^2dx + \frac{d}{dt}\int\limits_{\mathbb{R}^3}|\nabla P_0\varphi^n|^2dx = q\iint\limits_a^b r(\nabla P_0\varphi^n, v)\tilde{f}^n dxdvdr.$$

We multiply this equation by $\frac{\alpha\gamma}{\beta q}$ and plug it into (3.27), getting

$$\frac{1}{2}\frac{d}{dt}\|P_0u^n\|^2_{2,\mathbb{R}^3} + \nu\|P_0u^n\|^2_{J^1(\mathbb{R}^3)} + \alpha\iint\limits_a^b r\theta_R((P_0u^n - v)^2)(P_0u^n - v)^2\tilde{f}^n dxdvdr +$$

$$+\frac{\alpha}{2\beta}\frac{d}{dt}\iint\limits_a^b r^3v^2\tilde{f}^n dxdvdr + \frac{\varepsilon\alpha\gamma}{2\beta q}\frac{d}{dt}\int\limits_{\mathbb{R}^3}(\Delta P_0\varphi^n)^2dx +$$

$$+\frac{\alpha\gamma}{2\beta q}\frac{d}{dt}\int\limits_{\mathbb{R}^3}|\nabla P_0\varphi^n|^2dx = (P_0g, P_0u^n)_{2,\mathbb{R}^3} + \frac{\alpha}{\beta}\iint\limits_a^b r^3(v, P_0g_\varepsilon)\tilde{f}^n dxdvdr.$$

Using firstly Cauchy's inequality and afterwards Young's inequality with $p = q = 2$ and $\delta > 0$, we estimate the right-hand side:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|P_0u^n\|^2_{2,\mathbb{R}^3} &+ \nu\|P_0u^n\|^2_{J^1(\mathbb{R}^3)} + \\
&+ \alpha\iint\limits_a^b r\theta_R((P_0u^n - v)^2)(P_0u^n - v)^2\tilde{f}^n dxdvdr + \\
&+ \frac{\alpha}{2\beta}\frac{d}{dt}\iint\limits_a^b r^3v^2\tilde{f}^n dxdvdr + \frac{\varepsilon\alpha\gamma}{2\beta q}\frac{d}{dt}\int\limits_{\mathbb{R}^3}(\Delta P_0\varphi^n)^2dx + \\
&+ \frac{\alpha\gamma}{2\beta q}\frac{d}{dt}\int\limits_{\mathbb{R}^3}|\nabla P_0\varphi^n|^2dx \le \\
&\le \frac{\delta}{2}\|P_0u^n\|^2_{2,\mathbb{R}^3} + \frac{1}{2\delta}\|g\|^2_{2,\Omega} + \frac{\delta\alpha}{2\beta}\iint\limits_a^b r^3v^2\tilde{f}^n dxdvdr + \\
&+ \frac{\alpha}{2\delta\beta}\iint\limits_a^b r^3\tilde{f}^n(P_0g_\varepsilon)^2dxdvdr.
\end{aligned}$$

Integrating this equation with respect to the time variables, we get

$$
\begin{aligned}
\frac{1}{2}\|P_0u^n(t)\|^2_{2,\mathbb{R}^3} &+ \nu\int_0^t \|P_0u^n(\tau)\|^2_{J^1(\mathbb{R}^3)}d\tau + \\
&+ \alpha\int_0^t \iint_a^b r\theta_R((P_0u^n-v)^2)(P_0u^n-v)^2\tilde{f}^n dxdvdrd\tau + \\
&+ \frac{\alpha}{2\beta}\iint_a^b r^3v^2\tilde{f}^n(x,v,r,t)dxdvdr + \frac{\varepsilon\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}(\Delta P_0\varphi^n(x,t))^2dx + \\
&+ \frac{\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}|\nabla P_0\varphi^n(x,t)|^2dx \leq \\
&\leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + \frac{\delta}{2}\int_0^t \|P_0u^n(x,\tau)\|^2_{2,\mathbb{R}^3}d\tau + \frac{1}{2\delta}\int_0^t \|g\|^2_{2,\Omega}d\tau + \\
&+ \frac{\delta\alpha}{2\beta}\int_0^t\int_a^b\int_{\mathbb{R}^6} r^3v^2\tilde{f}^n(x,v,r,\tau)dxdvdrd\tau + \\
&+ \frac{\alpha}{2\delta\beta}\int_0^t\int_a^b\int_{\mathbb{R}^6} r^3\tilde{f}^n(x,v,r,\tau)(P_0g_\varepsilon)^2dxdvdrd\tau + \\
&+ \frac{\alpha}{2\beta}\iint_a^b r^3v^2P_0f_0^n(x,v,r)dxdvdr + \\
&+ \frac{\varepsilon\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}(\Delta P_0\varphi^n(x,0))^2dx + \frac{\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}|\nabla P_0\varphi^n(x,0)|^2dx = \\
&= \sum_{i=1}^{8} I_i. \qquad (3.28)
\end{aligned}
$$

defining 8 integrals. I_5, I_6 and I_7+I_8 we estimate successively.

To estimate I_7+I_8, we consider equation (3.12) for $t=0$:

$$
\varepsilon\Delta^2\varphi^n(x,0) - \Delta\varphi^n(x,0) = q\iint_a^b rf_0^n(x,v,r)dvdr. \qquad (3.29)
$$

It follows from Lemma 1 that the right-hand side of (3.29) belongs to the space $L_p(\Omega)$ with $p\in(\frac{3}{2},\frac{5}{3})$. In fact

$$
\int_\Omega\left(\iint_a^b rf_0^n(x,v,r)dvdr\right)^p dx \leq b^p\int_\Omega\left(\iint_a^b \frac{1}{(1+v^2)^{1/p}}(1+v^2)^{1/p}f_0^n(x,v,r)dvdr\right)^p dx.
$$

From Hölder's inequality, we conclude

$$\begin{aligned}
\int\limits_{\Omega}\left(\iint\limits_{a}^{b} r f_0^n(x,v,r)dvdr\right)^{p} dx \;&\leq\; b^p \int\limits_{\Omega}\left(\iint\limits_{a}^{b} \frac{dvdr}{(1+v^2)^{p/q}}\right)^{q/p} \times \\
&\times \;\left(\iint\limits_{a}^{b}(1+v^2)[f_0^n(x,v,r)]^p dvdr\right) dx = \\
&= \; b^p \left(\iint\limits_{a}^{b} \frac{dvdr}{(1+v^2)^{q/p}}\right)^{p/q} \times \\
&\times \; \int\limits_{Q}(1+v^2)[f_0^n(x,v,r)]^p dxdvdr < C_1.
\end{aligned}$$

The term $\displaystyle\iint\limits_{a}^{b} \frac{dvdr}{(1+v^2)^{p/q}}$ is bounded for given p. Hence, C_1 is a constant not depending on n.

We multiply (3.29) by $\varphi^n(x,0)$ and integrate the resulting equation over Ω. We have

$$\varepsilon \int\limits_{\Omega}(\Delta\varphi^n(x,0))^2 dx + \int\limits_{\Omega}|\nabla\varphi^n(x,0)|^2 dx = q\int\limits_{\Omega}\int\limits_{a}^{b} r f_0^n(x,v,r)\varphi^n(x,0)dvdrdx.$$

From Hölder's inequality, we get:

$$q\int\limits_{\Omega}\iint\limits_{a}^{b} r f_0^n(x,v,r)\varphi^n(x,0)dvdrdx \leq \left\|\iint\limits_{a}^{b} r f_0^n(x,v,r)dvdr\right\|_{L_p(\Omega)} \|\varphi^n(x,0)\|_{L_q(\Omega)}.$$

As it was shown above, the first term on the right-hand side of this inequality is bounded uniformly in n. To estimate the second term we make use of the embedding theorem and Friedrich's inequality:

$$\begin{aligned}
\varepsilon \int\limits_{\Omega}(\Delta\varphi^n(x,0))^2 dx \;&+\; \int\limits_{\Omega}|\nabla\varphi^n(x,0)|^2 dx \leq C_1^{1/p}\|\varphi^n(x,0)\|_{L_q(\Omega)} \leq \\
&\leq\; C_2\|\varphi^n(x,0)|_{W_2^1(\Omega)} \leq C_3\|\nabla\varphi^n(x,0)\|_{L_2(\Omega)}.
\end{aligned}$$

Thus, $\|\nabla\varphi^n(x,0)\|^2_{L_2(\Omega)} \leq C_3\|\nabla\varphi^n(x,0)\|_{L_2(\Omega)}$ or $\|\nabla\varphi^n(x,0)\|_{L_2(\Omega)} \leq C_3$, and we have

$$I_7 + I_8 \leq \frac{\alpha\gamma}{2\beta}C_3^2 \equiv \hat{C},$$

where $\hat{C}$ is a constant not depending on n.

According to Lemma 1, I_6 is uniformly bounded in n by the constant $\tilde{C}$.

It remains to estimate I_5:

$$I_5 \equiv \frac{\alpha}{2\delta\beta}\int_0^t\int_a^b\int_{\mathbb{R}^6} r^3\tilde{f}^n(x,v,r,t)(P_0 g_\varepsilon)^2 dxdvdrd\tau.$$

From (3.25b) and the definition of $g_\varepsilon(x,t)$ we get

$$I_5 \leq C\|g\|^2_{L_\infty(0,T;C^1(\Omega))}.$$

Thus, from (3.28), for any $t\in[0,T]$, we conclude

$$\begin{aligned}
\frac{1}{2}\|P_0u^n(t)\|^2_{2,\mathbb{R}^3} &+ \nu\int_0^t \|P_0u^n(\tau)\|^2_{J^1(\mathbb{R}^3}d\tau + \\
&+ \alpha\int_0^t\int_a^b\!\!\int r\theta_R((P_0u^n-v)^2)(P_0u^n-v)^2\tilde{f}^n dxdvdrd\tau + \\
&+ \frac{\alpha}{2\beta}\int_a^b\!\!\int r^3v^2\tilde{f}^n(x,v,r,t)dxdvdr + \frac{\varepsilon\alpha\gamma}{\beta q}\int_{\mathbb{R}^3}(\Delta P_0\varphi^n(x,t))^2dx + \\
&+ \frac{\alpha\gamma}{\beta q}\int_{\mathbb{R}^3}|\nabla P_0\varphi^n(x,t)|^2dx \leq \\
&\leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + \frac{\delta T}{2}\max_{0\leq t\leq T}\|P_0u^n(t)\|^2_{2,\mathbb{R}^3} + \frac{T}{2\delta}\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \\
&+ \frac{\alpha b^3\delta T}{2\beta}\max_{0\leq t\leq T}\int_a^b\int v^2\tilde{f}^n(x,v,r,t)dxdvdr + C\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \hat{C}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\max_{0\le t\le T}\|P_0u^n(t)\|^2_{2,\mathbb{R}^3} &+ \nu\int\limits_0^T \|P_0u^n\|^2_{J^1(\mathbb{R}^3)}dt + \\
&+ \alpha a\int\limits_0^T\int\limits_a^b\int \theta_R((P_0u^n-v)^2)(P_0u^n-v)^2\tilde{f}^n dxdvdrdt + \\
&+ \frac{a^3\alpha}{2\beta}\max_{0\le t\le T}\iint\limits_a^b v^2\tilde{f}^n(x,v,r,t)dxdvdr + \\
&+ \frac{\varepsilon\alpha\gamma}{\beta q}\max_{0\le t\le T}\int\limits_{\mathbb{R}^3}(\Delta P_0\varphi^n(x,t))^2dx + \\
&+ \frac{\alpha\gamma}{\beta q}\max_{0\le t\le T}\int\limits_{\mathbb{R}^3}|\nabla P_0\varphi^n(x,t)|^2dx \le \\
&\le \frac{1}{2}\|u_0\|^2_{2,\Omega} + \frac{\delta T}{2}\max_{0\le t\le T}\|P_0u^n(t)\|^2_{2,\mathbb{R}^3} + \frac{T}{2\delta}\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \\
&+ \frac{\alpha b^3\delta T}{2\beta}\max_{0\le t\le T}\iint\limits_a^b v^2\tilde{f}^n(x,v,r,t)dxdvdr + \\
&+ C\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \hat{C}.
\end{aligned}
$$

Choosing from this inequality the parameter δ in such a way that $\frac{\delta T}{2} < \frac{1}{4}$ and $\frac{\alpha b^3\delta T}{2\beta} < \frac{\alpha a^3}{4\beta}$, we obtain finally (3.25c). Lemma 3 is proved. $\square$

3.4 The existence of the approximations (u^n, φ^n, f^n)

Lemma 4 *For any $n = 1, 2, \ldots$ and any $R > 0$, $\varepsilon > 0$ there exists a unique solution $(u^n, \varphi^n, \tilde{f}^n)$ of problem (3.11) – (3.15), (3.23), (3.24).*

Proof. We denote by $C(0,T)$ the space of continuous vector functions $e(t) = (e_1(t), \ldots, e_n(t))$. This space is equipped with a norm $|e| = \max\limits_{0\le t\le T}\left[\sum\limits_{i=1}^n e_i^2(t)\right]^{1/2}$. We take $\varphi \in L_2(0,T;C^2(\Omega))$ and denote by $w = (e_1(t), \ldots, e_n(t), \varphi(x,t))$ an element of the space $C(0,T)\oplus L_2(0,T;C^2(\Omega))$ with the norm $|w| = |e| + \left(\int\limits_0^T \|\varphi(t)\|^2_{C^2(\Omega)}dt\right)^{1/2}$.

Here $\|\varphi\|_{C^2(\Omega)} = \max\limits_{x\in\Omega}\sum\limits_{|\alpha|=0}^2 |D^\alpha\varphi(x)|$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a nonnegative vector of integers, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$.

Let K be a bounded closed convex set in $C(0,T) \oplus L_2(0,T;C^2(\Omega))$:

$$K = \{w : |w| \le C_{R,\varepsilon}, e_i(0) = C_i, i = 1,2,\dots,n; \varphi(x,t) = \frac{\partial \varphi(x,t)}{\partial n} = 0, (x,t) \in S_T\}.$$

The constant $C_{R,\varepsilon}$ will be specified later. C_i are the coefficients defined in (3.24). Let $w^0 = (e_1^0(t), e_2^0(t), \dots, e_n^0(t), \varphi^0(x,t))$ be an arbitrary element of K.

We set

$$q^0(x,t) = \sum_{i=1}^{n} e_i^0 \Psi^i,$$

and consider the problem

$$\begin{aligned}
\frac{\partial \tilde f}{\partial t} &+ (v\nabla_x)\tilde f + \operatorname{div}_v \left\{ \left[\frac{\beta}{r^2}\theta_R((P_0q^0 - v)^2)(P_0q^0 - v) \right.\right. - \\
&- \left.\left. \frac{\gamma}{r^2}\nabla(P_0\varphi^0) + P_0 g_\varepsilon(x,t) \right] \tilde f \right\} = 0, \\
\tilde f|_{t=0} &= P_0 f_0^n(x,v,r).
\end{aligned}$$

Existence and uniqueness of the solution $\tilde f$ of this problem follows from the regular properties of the functions q^0, φ^0 and g_ε. More precisely, $q^0 \in C(0,T;C^1(\Omega))$, $\varphi^0 \in C^2(\Omega) \cap C_0^1(\Omega)$, $g_\varepsilon \in L_\infty(0,T;C^1(\Omega))$.

We are looking for a vector $q^1 = \sum_{i=1}^{n} e_i^1 \Psi^i$ as a solution of the system of ordinary differential equations

$$\begin{aligned}
\left(\frac{\partial q^1}{\partial t} + (q^0\nabla)q^1 \right. &+ \left. \alpha \iint_a^b r\theta_R((q^0 - v)^2)(q^0 - v)S\tilde f dvdr, \Psi^k \right)_{2,\Omega} + \\
&+ \nu(q^1, \Psi^k)_{J^1(\Omega)} = (g, \Psi^k)_{2,\Omega}, \quad k = 1,2,\dots,n,
\end{aligned} \tag{3.30}$$

This system is a linearization of system (3.23) and can be rewritten as:

$$\begin{aligned}
\frac{de_k^1}{dt} &+ \sum_{j,l=1}^{n} \beta_{jl}^k e_j^0 e_l^1 + \sum_{l=1}^{n} \varepsilon_l^k e_l^1 = \\
&= g^k - \alpha \left(\iint_a^b r\theta_R \left(\left(\sum_{l=1}^{n} e_l^0 \psi^l - v \right)^2 \right) \times \right. \\
&\times \left. \left(\sum_{l=1}^{n} e_l^0 \Psi^l - v \right) S\tilde f dvdr, \Psi^k \right)_{2,\Omega}, \quad k = 1,2,\dots,n.
\end{aligned} \tag{3.31}$$

The initial data are as follows:

$$w_k^1(0) = C_k = (u_0, \Psi^k)_{2,\Omega}, \quad k = 1,2,\dots,n. \tag{3.32}$$

The linear problem (3.31), (3.32) has a unique solution $\{e_k^1(t), k = 1, \dots, n\}$.

We are looking for $\varphi^1(x,t)$ as a solution of the problem

$$\varepsilon\Delta^2\varphi^1 - \Delta\varphi^1 = q \iint\limits_a^b rS\tilde{f}(x,v,r,t)dvdr, \tag{3.33}$$

$$\varphi^1(x,t) = \frac{\partial\varphi^1(x,t)}{\partial n} = 0, \quad (x,t) \in S_T. \tag{3.34}$$

Similar to the case of equation (3.29), one can conclude that the right-hand side of equation (3.33) belongs to the space $L_p(\Omega)$ with $p \in (\frac{3}{2}; \frac{5}{3})$, uniformly on $t \in [0,T]$. Thus, there exists a unique generalized solution of problem (3.33), (3.34) (see [9]), satisfying the inequality:

$$\|\varphi^1\|_{W_p^4(\Omega)} \le C \max_{0\le t\le T} \left\| q \int\limits_a^b \int rS\tilde{f}(x,v,r,t)dvdr \right\|_{L_p(\Omega)} \le C_1. \tag{3.35}$$

Thus, the vector $w^1 = (q^1, \varphi^1)$ is the image of $w^0 \in K$ of some operator Λ: $K \to C(0,T) \oplus L_2(0,T; C^2(\Omega))$. The fixed points of this operator together with the corresponding functions $\tilde{f}$ give the solution of the problem (3.11) – (3.15), (3.23), (3.24).

Now, we prove that the operator Λ maps the set K into itself. For this purpose we have to prove $|w^1| \le C_{R,\varepsilon}$ or

$$\max_{0\le t\le T} \|q^1\|_{J(\Omega)} + \left(\int\limits_0^T \|\varphi^1(t)\|^2_{C^2(\Omega)} dt \right)^{1/2} \le C_{R,\varepsilon}. \tag{3.36}$$

To this end, multiplying the k-th equation in (3.30) by $e_k^1(t)$ and adding over $k = \overline{1,n}$, we get

$$\frac{1}{2}\frac{d}{dt}\|q^1\|^2_{2,\Omega} + \nu\|q^1\|^2_{J^1(\Omega)} = (g, q^1)_{2,\Omega} - \alpha \left(\iint\limits_a^b r\theta_R((q^0-v)^2)(q^0-v)S\tilde{f}dvdr, q^1 \right)_{2,\Omega}.$$

To estimate the second term in the right hand side of this equation we make use of inequality (3.35), the definition of $\theta_R(z)$ and the embedding of the space $L_s(\Omega)$ for

$s \in [2,6]$ into $J^1(\Omega)$. We have

$$\left| \alpha \left(\int\limits_a^b \int r\theta_R((q^0-v)^2)(q^0-v)S\tilde{f}dvdr, q^1 \right)_{2,\Omega} \right|$$

$$\begin{aligned}
&\leq \alpha\sqrt{2R}b \int\limits_\Omega |q^1(x,t)| \iint\limits_a^b S\tilde{f}(x,v,r,t)dvdrdx \leq \\
&\leq \alpha b\sqrt{2R} \left\{ \int\limits_\Omega \left[\iint\limits_a^b S\tilde{f}(x,v,r,t)dvdr \right]^p dx \right\}^{1/p} \left\{ \int\limits_\Omega |q^1(x,t)|^s dx \right\}^{1/s} \leq \\
&\leq \hat{C}\|q^1\|_{J^1(\Omega)},
\end{aligned}$$

where $p \in (\frac{3}{2}, \frac{5}{3})$, $s \in (\frac{5}{2}, 3)$, and $\frac{1}{p} + \frac{1}{s} = 1$.

Similar to the case of equation (3.25c), one can obtain

$$\begin{aligned}
\frac{1}{4}\max_{0\leq t\leq T}\|q^1(t)\|^2_{2,\Omega} + \frac{\nu}{2}\|q^1\|^2_{L_2(0,T;J^1(\Omega))} &\leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + \\
+C\left[T^3\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \hat{C}^2T^2\right] &\equiv \frac{1}{4}[C^{(1)}_{R,\varepsilon}]^2.
\end{aligned}$$

Hence, it follows

$$\max_{0\leq t\leq T}\|q^1(t)\|_{2,\Omega} \leq C^{(1)}_{R,\varepsilon}.$$

An estimate for the second term in (3.36) follows from the embedding theorem (see [9], [17]). In fact,

$$\|\varphi^1(t)\|_{C^2(\Omega)} \leq C\|\varphi^1(t)\|_{W^4_p(\Omega)} \leq C^{(2)}_{R.\varepsilon}.$$

Thus,

$$\left(\int\limits_0^T \|\varphi^1(t)\|^2_{C^2(\Omega)}dt \right)^{1/2} \leq \sqrt{C^{(2)}_{R,\varepsilon}T},$$

and $|w^1| \leq C^{(1)}_{R,\varepsilon} + \sqrt{C^{(2)}_{R,\varepsilon}T} \equiv C_{R,\varepsilon}$.

Now, we show that the map Λ: $K \to K$ is compact. To this end we estimate the derivative $\dfrac{dw^1}{dt}$. Multiplying the k-th equation in (3.30) by $\dfrac{de^1_k}{dt}$ and summarizing over $k = \overline{1,n}$, we get:

$$\|q^1_t\|^2_{2,\Omega} + ((q^0\nabla)q^1, q^1_t)_{2,\Omega} + \frac{\nu}{2}\frac{d}{dt}\|q^1\|^2_{J^1(\Omega)} +$$

$$+\alpha\left(\iint\limits_a^b r\theta_R((q^0-v)^2)(q^0-v)S\tilde{f}dvdr, q^1_t \right)_{2,\Omega} = (g, q^1_t)_{2,\Omega}.$$

Then, we obtain

$$\|q_t^1\|_{2,\Omega}^2 + \frac{\nu}{2}\frac{d}{dt}\|q^1\|_{J^1(\Omega)}^2 \le$$
$$\le \|q_t^1\|_{2,\Omega}\left[\|q^0\|_{C(\Omega)}\|q^1\|_{J^1(\Omega)} + \|g\|_{L_\infty(0,T;C^1(\Omega))}(\mathrm{mes}\Omega)^{1/2} + C\right],$$

where $C \equiv \alpha b\sqrt{2R}A(b-a)\frac{4}{3}\pi R_{\tilde{f}}^3(\mathrm{mes}\Omega)^{1/2}$, A is the constant defined in (3.25a), and $R_{\tilde{f}}$ is defined in Lemma 2.

Since the functions Ψ^k are smooth and $\|q^0\|_{C(\Omega)} \le C_n$, applying Young's inequality and integrating with respect to t we get:

$$\int_0^T \|q_t^1\|_{2,\Omega}^2 dt \le C_n.$$

Thus, $\|e^1\|_{W_2^1(0,T)}^2 \le C_n$. Therefore, the function $e^1(t)$ belongs to the space $W_2^1(0,T)$, which is compactly embedded in $C(0,T)$ [17].

To complete the proof of the compactness of the operator Λ we make use of the following Lemma, proved in [12].

Lemma *Let B_0, B and B_1 be Banach spaces such that $B_0 \subset B \subset B_1$. They are reflexive and the embedding B_0 in B_1 is compact. Consider the Banach space*

$$W = \left\{v : v \in L_{p_0}(0,T;B_0), v' = \frac{dv}{dt} \in L_{p_1}(0,T;B_1)\right\},$$

where $0 < T < +\infty$ is fixed and $1 < p_i < \infty$, $i = 0,1$.

The norm in the space W is defined by

$$\|v\|_{L_{p_0}(0,T;B_0)} + \|v'\|_{L_{p_1}(0,T;B_1)}.$$

Then, the embedding of W in $L_{p_0}(0,T;B_0)$ is compact.

This lemma implies that the Banach space

$$W = \{\varphi(x,t) : \varphi(x,t) \in L_2(0,T;W_p^4(\Omega)), \varphi_t' \in L_2(0,T;L_2(\Omega))\},$$

with norm $\|\varphi\|_{L_2(0,T;W_p^4(\Omega))} + \|\varphi_t'\|_{L_2(0,T;L_2(\Omega))}$ is compactly embedded in $L_2(0,T;C^2(\Omega))$.

Therefore, it remains to prove that $\dfrac{\partial\varphi^1}{\partial t} \in L_2(0,T;L_2(\Omega))$.

Differentiating equations (3.33) and (3.34) with respect to t, we obtain the following problem for φ_t^1:

$$\varepsilon\Delta^2\varphi_t^1 - \Delta\varphi_t^1 = q\int_a^b\int rS\frac{\partial\tilde{f}(x,v,r,t)}{\partial t}dvdr, \quad (x,t) \in \Omega_T, \tag{3.37}$$

$$\varphi_t^1 = \frac{\partial\varphi_t^1}{\partial n} = 0, \quad (x,t) \in S_T. \tag{3.38}$$

From equation (3.14) for the function $\tilde{f}$ we have

$$q\iint\limits_a^b rS\frac{\partial\tilde{f}(x,v,r,t)}{\partial t}dvdr = -q\iint\limits_a^b r(v\nabla_x)S\tilde{f}dvdr-$$

$$-q\iint\limits_a^b r\mathrm{div}_v\left\{\left[\frac{\beta}{r^2}\theta_R((q^0-v)^2)(q^0-v)-\frac{\gamma}{r^2}\nabla\varphi^0+g_\varepsilon(x,t)\right]S\tilde{f}\right\}dvdr, x\in\Omega_T.$$

Since the second term on the right hand side of this equation equals zero, equation (3.37) has the form

$$\varepsilon\Delta^2\varphi_t^1-\Delta\varphi_t^1=-q\iint\limits_a^b r(v\nabla_x)S\tilde{f}dvdr. \tag{3.39}$$

Multiplying (3.39) by φ_t^1 and integrating over Ω, we get:

$$\varepsilon\int\limits_\Omega(\Delta\varphi_t^1)^2dx+\int\limits_\Omega|\nabla\varphi_t^1|^2dx=-q\int\limits_\Omega\iint\limits_a^b r(v\nabla_x)S\tilde{f}\varphi_t^1dvdrdx.$$

Taking into account the conditions (3.38), we obtain the following equation

$$\varepsilon\int\limits_\Omega(\Delta\varphi_t^1)^2dx+\int\limits_\Omega|\nabla\varphi_t^1|^2dx=q\int\limits_\Omega\int\limits_a^b\int r(v,\nabla\varphi_t^1)S\tilde{f}dvdrdx.$$

The right hand side of the last equation we estimate as follows

$$q\int\limits_\Omega\iint\limits_a^b r(v,\nabla\varphi_t^1)S\tilde{f}dvdrdx=q\int\limits_\Omega\int\limits_a^b\int r(\sqrt{S\tilde{f}}v,\sqrt{S\tilde{f}}\nabla\varphi_t^1)dvdrdx\le$$

$$\le qb\left\{\int\limits_\Omega\iint\limits_a^b v^2S\tilde{f}dvdrdx\right\}^{1/2}\left\{\int\limits_\Omega\iint\limits_a^b S\tilde{f}|\nabla\varphi_t^1|^2dvdrdx\right\}^{1/2}\le$$

$$\le qb\sqrt{A}\left\{\int\limits_\Omega|\nabla\varphi_t^1|^2\iint\limits_a^b S\tilde{f}dvdrdx\right\}^{1/2}\le$$

$$\le qb\sqrt{A}\left(A(b-a)\frac{4}{3}\pi R_{\tilde{f}}^3\right)^{1/2}\left\{\int\limits_\Omega|\nabla\varphi_t^1|^2dx\right\}^{1/2}=$$

$$=\bar{C}_{R,\varepsilon}\left\{\int\limits_\Omega|\nabla\varphi_t^1|^2dx\right\}^{1/2}\le\frac{1}{2}\int\limits_\Omega|\nabla\varphi_t^1|^2dx+\frac{1}{2}\bar{C}_{R,\varepsilon}^2.$$

Then

$$\varepsilon\int\limits_{\Omega}(\Delta\varphi_t^1)^2dx+\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx\le\frac{1}{2}\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx+\frac{1}{2}\bar{C}^2_{R,\varepsilon},$$

and, therefore

$$\max_{0\le t\le T}\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx\le\bar{C}^2_{R,\varepsilon}.$$

Taking into account (3.38), we get $\varphi_t^1\in L_2(0,T;W_2^1(\Omega))$. Thus, it is proved that the image $\Lambda[K]$ of the set K, is a compact set in $C(0,T)\oplus L_2(0,T;C^2(\Omega))$. The continuity of the operator Λ follows from: the continuous dependence of the solutions of (3.31) on the initial data, the coefficients and the right hand sides; the continuous dependence of the solution of (3.14) on the coefficients that follows from (3.16) and (3.17); the a priori estimate of the right hand side of (3.12) and the embedding theorem of $W_p^4(\Omega))$ $(p\in(\frac{3}{2};\frac{5}{3}))$ in $C^2(\Omega))$.

From Schauder's theorem follows that the operator Λ has a fixed point in K. We denote it by $w=(e_1(t),\dots,e_n(t),\varphi_n(x,t))$.

The proof of the uniqueness of the solution of (3.11) – (3.15), (3.23), (3.24) is carried out in a standard way, considering the equation for the difference of two assumed solutions of this problem. Proving an estimates, similar to (3.25b) and (3.25c) we can conclude that the difference is equal to zero. Lemma 4 is proved completely. □

The mentioned procedure, along with formulae (3.11), (3.16) and (3.17) defines the finite approximations (u^n,φ^n,f^n) for the solutions of the regularized problem.

4 Compactness of the approximations (u^n,φ^n,f^n)

Due to the a priori estimates (3.25a), (3.25c) one can extract subsequences $\{u^n\}$, $\{\varphi^n\}$, and $\{\tilde{f}^n\}$, such that

$u^n\to u$ $*$-weakly in $L_\infty(0,T;J(\Omega))$ and weakly in $L_2(0,T;J^1(\Omega))$;

$\tilde{f}^n\to\tilde{f}$ $*$-weakly in $L_\infty(\mathbb{R}^6_T\times[a,b])$;

$\varphi^n\to\varphi$ $*$-weakly in $L_\infty(0,T;W_2^1(\Omega))$.

Lemma 5 *There exists a subsequence $\{\tilde{f}^n\}$ that converges uniformly with respect to $t\in[0,T]$ in the weak topology of $L_2(\mathbb{R}^6\times[a,b])$.*

Proof. We denote by $\{g_i(x,v,r)\}$ an orthonormal total sequence of smooth functions in $L_2(\mathbb{R}^6\times[a,b])$ and consider the sequence

$$\alpha_{ni}(t)=\int\limits_a^b\int\limits_{\mathbb{R}^6}\tilde{f}^n(x,v,r,t)g_i(x,v,r)dxdvdr,\quad i=1,2,\dots$$

Due to the estimates (3.25a) and (3.25b) this sequence is bounded for any fixed i, uniformly in n. Moreover, it follows from (3.14), (3.25a) and (3.25b) that

$$\left|\frac{d\alpha_{ni}(t)}{dt}\right| = \left|\int_a^b\int_{\mathbb{R}^6} g_i(x,v,r)\frac{\partial \tilde{f}^n}{\partial t}dxdvdr\right| =$$

$$= \Bigg|\int_a^b\int_{\mathbb{R}^6} g_i(x,v,r)\Big[(v\nabla_x)\tilde{f}^n + \mathrm{div}_v\Big\{\Big[\frac{\beta}{r^2}\theta_R((P_0u^n-v)^2)(P_0u^n-v)-$$

$$-\frac{\gamma}{r^2}P_0\nabla\varphi^n + P_0g_\varepsilon(x,t)\Big]\tilde{f}^n\Big\}dxdvdr\Bigg| =$$

$$= \Bigg|\int_a^b\int_{\mathbb{R}^6}\Big\{\tilde{f}^n(v\nabla_x)g_i + \frac{\beta}{r^2}\Big(\Big[\theta_R((P_0u^n-v)^2)(P_0u^n-v)\tilde{f}^n\Big]\nabla_v\Big)g_i-$$

$$-\frac{\gamma}{r^2}\Big([P_0\nabla\varphi^n\tilde{f}^n]\nabla_v\Big)g_i(x,v,r) + \Big([P_0g_\varepsilon(x,t)\tilde{f}^n]\nabla_v\Big)g_i(x,v,r)\Big\}dxdvdr\Bigg| \le$$

$$\le AC_i + \frac{\beta}{a^2}\Bigg(\int_a^b\int_{\mathbb{R}^6}|P_0u^n|\tilde{f}^n|\nabla_v g_i|dxdvdr + \int_a^b\int_{\mathbb{R}^6}\tilde{f}^n|v||\nabla_v g_i|dxdvdr\Bigg)+$$

$$+\frac{\gamma}{a^2}\int_a^b\int_{\mathbb{R}^6}\left(|P_0\nabla\varphi^n| + |P_0g_\varepsilon|\right)\tilde{f}^n|\nabla_v g_i|dxdvdr \le$$

$$\le AC_i + \frac{\beta}{a^2}\left(\int_a^b\int_{\mathbb{R}^6}(\tilde{f}^n)^2dxdvdr\right)^{1/2}\left(\int_a^b\int_{\mathbb{R}^6}|P_0u^n|^2|\nabla_v g_i|^2dxdvdr\right)^{1/2} + AC_i+$$

$$+\frac{\gamma}{a^2}\left(\int_a^b\int_{\mathbb{R}^6}(\tilde{f}^n)^2dxdvdr\right)^{1/2}\left(\int_a^b\int_{\mathbb{R}^6}|P_0\nabla\varphi^n|^2|\nabla_v g_i|^2dxdvdr\right)^{1/2} +$$

$$+\frac{\gamma}{a^2}C_iA\|g\|_{L_\infty(0,T;C^1(\Omega))} \le$$

$$\le \tilde{C}_i\left(1 + \|u^n\|_{2,\Omega} + \|\nabla\varphi^n\|_{2,\Omega}\right).$$

This estimate, along with (3.25c) implies that the sequence $\{\alpha_{ni}(t)\}$ is equicontinuous for any i. So, one can extract a subsequence that converges uniformly in t for any fixed interval $(0,T]$ and for any i. We keep the same notation for this subsequence.

Let $b(x,v,r)$ be an arbitrary function from $L_2(\mathbb{R}^6\times[a,b])$ and β_i its Fourier coefficients with respect to $\{g_i(x,v,r)\}$. Then, we have

$$\sup_{0\le t\le T}\left|\int_a^b\int_{\mathbb{R}^6} b(x,v,r)[\tilde{f}^n(x,v,r,t) - \tilde{f}^m(x,v,r,t)]dxdvdr\right| =$$

$$= \sup_{0\le t\le T}\left|\int_a^b\int_{\mathbb{R}^6}\left\{\left(b(x,v,r)-\sum_{i=1}^N\beta_i g_i(x,v,r)\right)[\tilde f^n-\tilde f^m]+\right.\right.$$

$$\left.\left.+\sum_{i=1}^N\beta_i g_i(x,v,r)[\tilde f^n-\tilde f^m]\right\}dxdvdr\right|\le$$

$$\le \sup_{0\le t\le T}\left|\sum_{i=1}^N\beta_i\left(\int_a^b\int_{\mathbb{R}^6} g_i(x,v,r)\tilde f^n dxdvdr-\int_a^b\int_{\mathbb{R}^6} g_i(x,v,r)\tilde f^m dxdvdr\right)\right|+$$

$$+\sup_{0\le t\le T}\left\{\int_a^b\int_{\mathbb{R}^6}\left|b(x,v,r)-\sum_{i=1}^N\beta_i g_i(x,v,r)\right|^2 dxdvdr\right\}^{1/2}\times$$

$$\times\left\{\left(\int_a^b\int_{\mathbb{R}^6}(\tilde f^n)^2dxdvdr\right)^{1/2}+\left(\int_a^b\int_{\mathbb{R}^6}(\tilde f^m)^2dxdvdr\right)^{1/2}\right\}\le$$

$$\le \sup_{0\le t\le T}\left|\sum_{i=1}^N\beta_i(\alpha_{in}(t)-\alpha_{im}(t))\right|+2A\left\{\int_a^b\int_{\mathbb{R}^6}\left|b(x,v,r)-\sum_{i=1}^N\beta_i g_i(x,v,r)\right|^2 dxdvdr\right\}^{\frac12}.$$

For sufficiently large N, n and m, the right hand side of this inequality is arbitrarily small. This proves the lemma. ☐

Lemma 6 *The limit function $\tilde f(x,v,r,t)$ satisfies the following conditions:*

$$\tilde f(x,v,r,t)\ge 0 \text{ almost everywhere in } \mathbb{R}^6_T\times[a,b]; \tag{4.1}$$

$$\int_a^b\int_{\mathbb{R}^6}\tilde f(x,v,r,t)dxdvdr=\int_Q f_0(x,v,r)dxdvdr; \tag{4.2}$$

$$\sup_{0<t<T}\int_a^b\int_{\mathbb{R}^6} v^2\tilde f(x,v,r,t)dxdvdr<\infty. \tag{4.3}$$

Proof. We denote by B an arbitrary measurable set in $\mathbb{R}^6\times[a,b]$ with $\mathrm{mes}B<\infty$. According to Lemma 5, we have

$$\int_B\tilde f(x,v,r,t)dxdvdr=\lim_{n\to\infty}\int_B\tilde f^n(x,v,r,t)dxdvdr.$$

Due to (3.17), $\tilde f^n(x,v,r,t)\ge 0$. Thus, (4.1) is proved.

From inequality (3.25c) and the boundedness of the support of $\tilde{f}^n(x,v,r,t)$ follows

$$\int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} \tilde{f}^n(x,v,r,t)dxdvdr \leq \delta At + \hat{A}, \tag{4.4}$$

where, $\hat{A}$ is a constant depending only on f_0; A is the constant defined in Lemma 3, and $\delta \in (0,1)$ is arbitrary.

In fact, from equation (3.14), we have:

$$\left| \frac{d}{dt} \int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} \tilde{f}^n(x,v,r,t)dxdvdr \right| = \left| \int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} (v\nabla_x) \tilde{f}^n dxdvdr \right| =$$

$$= \left| \delta \int_a^b \int_{\mathbb{R}^6} (v,x)(1+x^2)^{\frac{\delta}{2}-1} \tilde{f}^n dxdvdr \right| \leq \delta \int_a^b \int_{\mathbb{R}^6} (1+v^2) \tilde{f}^n dxdvdr \leq \delta A.$$

(4.4) immediately follows from this inequality.

Now, we prove the following statement: For any $\varepsilon_1 > 0$ there exists $R_1(\varepsilon_1) < \infty$, such that, for any n and $t \in [0,T]$, the following inequality holds:

$$\int_a^b \int_{\mathbb{R}^3} \int_{|x|>R_1(\varepsilon_1)} \tilde{f}^n(x,v,r,t)dxdvdr + \int_a^b \int_{|v|>R_1(\varepsilon_1)} \int_{\mathbb{R}^3} \tilde{f}^n(x,v,r,t)dxdvdr < \varepsilon_1. \tag{4.5}$$

In fact, we have

$$\int_a^b \int_{\mathbb{R}^3} \int_{|x|>R} \tilde{f}^n dxdvdr + \int_a^b \int_{|v|>R} \int_{\mathbb{R}^3} \tilde{f}^n dxdvdr \leq$$

$$\leq \frac{1}{(1+R^2)^{\delta/2}} \int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} \tilde{f}^n dxdvdr + \frac{1}{1+R^2} \int_a^b \int_{\mathbb{R}^6} (1+v^2) \tilde{f}^n dxdvdr \leq$$

$$\leq \frac{\delta AT + A_1}{(1+R^2)^{\delta/2}} + \frac{A}{1+R^2}.$$

It follows from (4.5) and non-negativity of $\tilde{f}(x,v,r,t)$ that

$$\int_a^b \int_{\mathbb{R}^6} \tilde{f}(x,v,r,t)dxdvdr = \lim_{R\to\infty} \lim_{n\to\infty} \int_a^b \int_{|v|<R} \int_{|x|<R} \tilde{f}^n dxdvdr =$$

$$= \lim_{n\to\infty} \int_a^b \int_{\mathbb{R}^6} \tilde{f}^n(x,v,r,t)dxdvdr = \lim_{n\to\infty} \int_a^b \int_{\mathbb{R}^6} \tilde{f}^n(x,v,r,0)dxdvdr =$$

$$= \lim_{n\to\infty} \int_a^b \int_{\mathbb{R}^6} P_0 f_0^n(x,v,r)dxdvdr = \lim_{n\to\infty} \int_Q f_0^n(x,v,r)dxdvdr =$$

$$= \int_Q f_0(x,v,r)dxdvdr.$$

Hence, (4.2) is proved.

It remains to prove (4.3). Let $b_R(x,v)$ be a function of x and v in the space $\mathbb{R}^6$, such that $|b_R(x,v)| \le 1$ and $b_R(x,v) = 0$ if $|x| \ge R$ and $|v| \ge R$, where R is a positive parameter.

Thus

$$\int_a^b \int_{\mathbb{R}^6} v^2 b_R(x,v)\tilde{f}(x,v,r,t)dxdvdr =$$

$$= \int_a^b \int_{\mathbb{R}^6} v^2 b_R(x,v)(\tilde{f} - \tilde{f}^n)dxdvdr + \int_a^b \int_{\mathbb{R}^6} v^2 b_R(x,v)\tilde{f}^n dxdvdr.$$

According to Lemma 5, the first term on the right hand side tends to zero as $n \to \infty$ for any fixed R. From the definition of the function $b_R(x,v)$ and (3.25c) it follows that the second term is bounded uniformly in n and R by the constant A defined in Lemma 3. Lemma 6 is proved. $\square$

Lemma 7 *The sequence $\{\tilde{f}^n\}$ converges in the weak topology of $L_1(\mathbb{R}^6 \times [a,b])$ uniformly in $t \in [0,T]$.*

Proof. Let $g(x,v,r) \in L_\infty(\mathbb{R}^6 \times [a,b])$. Then, we have:

$$\left| \int_a^b \int_{\mathbb{R}^6} g(x,v,r)[\tilde{f}^n(x,v,r,t) - \tilde{f}^m(x,v,r,t)]dxdvdr \right| \le \tag{4.6}$$

$$\le \left| \int_a^b \int_{|v|\le R_1} \int_{|x|\le R_1} g(x,v,r)[\tilde{f}^n(x,v,r,t) - \tilde{f}^m(x,v,r,t)]dxdvdr \right| +$$

$$+ \left[\int_a^b \int \int_{|x|>R_1} (\tilde{f}^n + \tilde{f}^m)dxdvdr + \int_a^b \int_{|v|>R_1} \int (\tilde{f}^n + \tilde{f}^m)dxdvdr \right] \|g\|_{L_\infty(\mathbb{R}^6\times[a,b])}.$$

Due to inequality (4.5) and the choice of R_1 the second term in the right hand side of (4.6) is sufficiently small, uniformly in n, m and t. According to Lemma 5, the first term on the right hand side of (4.6) is smaller then any $\varepsilon_1 > 0$ for any fixed R_1 and $n, m > N(\varepsilon_1)$. Thus, the sequence $\{\tilde{f}^n\}$ is weakly fundamental in $L_1(\mathbb{R}^6 \times [a,b])$, uniformly in t. Therefore, it is weakly convergent in $L_1(\mathbb{R}^6 \times [a,b])$, uniformly in t.

Corollary 1 *The limit function $\tilde{f}(x,v,r,t)$ is continuous in $t \in [0,T]$ in the weak topology of $L_1(\mathbb{R}^6 \times [a,b])$.*

Lemma 8 *The vector function $u(x,t)$ is weakly continuous in t in the norm of $L_2(\Omega)$.*

Proof. First, we show that for any fixed k and $n \geq k$, the functions $C_{nk}(t)$ in (3.11) represent a uniformly bounded and equicontinuous set of functions on the interval $[0,T]$.

The uniform boundedness of $C_{nk}(t)$ follows from the a priori estimate (3.25c). Since $C_{nk}(t) = (u^n(t), \Psi^k)_{2,\Omega}$, the equicontinuity follows from (3.22). Indeed, integrating (3.22) with respect to τ from t to $t+\Delta t$, estimating the right hand side and using Cauchy's inequality, we have

$$|C_{nk}(t+\Delta t) - C_{nk}(t)| \leq \nu \int\limits_t^{t+\Delta t} \|u^n(\tau)\|_{J^1(\Omega)} \|\Psi^k\|_{J^1(\Omega)} d\tau + \int\limits_t^{t+\Delta t} \|g(\tau)\|_{2,\Omega} d\tau +$$

$$+ \max_{x\in\Omega} |\Psi^k(x)| \int\limits_t^{t+\Delta t} \|u^n(\tau)\|_{2,\Omega} \|u^n(\tau)\|_{J^1(\Omega)} d\tau +$$

$$+\alpha b \int\limits_t^{t+\Delta t} \int\limits_\Omega \int\limits_a^b \int \theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v||\Psi^k(x)| S\tilde{f}^n(x,v,r,\tau) dv dr dx d\tau \leq$$

$$\leq \nu \|\Psi^k\|_{J^1(\Omega)} \|u^n\|_{L_2(0,T;J^1(\Omega))} \sqrt{\Delta t} + \max_{x\in\Omega} |\Psi^k(x)| \|u^n\|_{L_\infty(0,T;J(\Omega))} \sqrt{\Delta t} \|u^n\|_{L_2(0,T;J^1(\Omega))} +$$

$$+ \int\limits_t^{t+\Delta t} \|g(\tau)\|_{2,\Omega} d\tau + \alpha b \max_{x\in\Omega} |\Psi^k(x)| \left\{ \int\limits_t^{t+\Delta t} \int\limits_\Omega \int\limits_a^b \int S\tilde{f}^n(x,v,r,\tau) dv dr dx d\tau \right\}^{1/2} \times$$

$$\times \left\{ \int\limits_0^T \int\limits_\Omega \int\limits_a^b \int \theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v|^2 S\tilde{f}^n(x,v,r,\tau) dv dr dx d\tau \right\}^{1/2}.$$

From (3.25b), (3.25c) and the properties of the functions $\Psi^k(x)$, we get:

$$|C_{nk}(t+\Delta t) - C_{nk}(t)| \leq A(k) \left(\sqrt{\Delta t} + \int\limits_t^{t+\Delta t} \|g(\tau)\|_{2,\Omega} d\tau \right).$$

It is clear that for any fixed k and $n \geq k$ the right hand side of this inequality tends to zero, uniformly in t, as $\Delta t \to 0$. By the usual diagonal process we extract a subsequence n_l. For any fixed k, the functions $C_{n_l k}(t)$ converge uniformly to a continuous function $C_k(t)$, as $l \to \infty$. For this subsequence we keep the same notation $C_{nk}(t)$.

Now, we prove that the sequence of functions $u^n(x,t)$ converges in the weak topology of $L_2(\omega)$, uniformly with respect to $t \in [0,T]$.

We denote by $g(x)$ an arbitrary vector function from $L_2(\Omega)$ and by g_k the Fourier coefficients of this function with respect to the system $\{\Psi^k(x)\}$. Then,

$$\sup_{[0,T]} \left| \int_\Omega (u^n(x,t) - u^m(x,t), g(x))dx \right| =$$

$$= \sup_{[0,T]} \left| \int_\Omega \left[\left(g(x) - \sum_{k=1}^N g_k \psi^k(x), u^n(x,t) - u^m(x,t) \right) + \right.\right.$$

$$\left.\left. + \sum_{k=1}^N g_k(\psi^k(x), u^n(x,t) - u^m(x,t)) \right] dx \right| \leq$$

$$\leq \sup_{[0,T]} \left\{ \int_\Omega \left| g(x) - \sum_{k=1}^N g_k \Psi^k(x) \right|^2 dx \right\}^{1/2} \times$$

$$\times \left\{ \left[\int_\Omega |u^n(x,t)|^2 dx \right]^{1/2} + \left[\int_\Omega |u^m(x,t)|^2 dx \right]^{1/2} \right\} +$$

$$+ \sup_{[0,T]} \left| \sum_{k=1}^N g_k (C_{nk}(t) - C_{mk}(t)) \right| \leq$$

$$\leq \left(\|u^n\|_{L_\infty(0,T;J(\Omega))} + \|u^m\|_{L_\infty(0,T;J(\Omega))} \right) \left\{ \int_\Omega \left| g(x) - \sum_{k=1}^N g_k \Psi^k(x) \right|^2 dx \right\}^{1/2} +$$

$$+ \sup_{[0,T]} \left| \sum_{k=1}^N g_k (C_{nk}(t) - C_{mk}(t)) \right|.$$

The right hand side of this inequality is arbitrarily small, if we choose at first a sufficiently large N and afterwards some sufficiently large n and m.

It follows from the convergence of the sequence $\{u^n(x,t)\}$ to the function $u(x,t)$ that this limit is continuous in t in the weak topology of $L_2(\Omega)$.

Lemma 9 *Galerkin's approximations $\{u^n\}$ satisfy the following inequality:*

$$\int_0^{T-\rho} \|u^n(t+\rho) - u^n(t)\|^2_{2,\Omega} dt < C\rho^{1/2},$$

where ρ is an arbitrary constant $0 < \rho < T$ and C is a constant not depending on n and ρ.

Proof. It follows from (3.22) for fixed ρ $(0 < \rho < T)$, t $(0 \le t \le T-\rho)$ and $\tau \in [t, t+\rho]$ that

$$\left(\frac{\partial u^n}{\partial \tau} + (u^n \nabla_x) u^n + \alpha \int\limits_a^b \int r\theta_R((u^n - v)^2)(u^n - v) S\tilde{f}^n dv dr, \Phi \right)_{2,\Omega} + \tag{4.7}$$

$$+\nu(u^n, \Phi)_{J^1(\Omega)} = (g, \Phi)_{2,\Omega},$$

where Φ is an arbitrary function from $J^1(\Omega)$ such that $\Phi = \sum_{k=1}^n d_k \Psi^k$ with constant d_k. We set $\Phi = u^n(x, t+\rho) - u^n(x,t)$. Integrating (4.7) with respect to τ in the interval $[t, t+\rho]$, we get

$$\|u^n(t+\rho) - u^n(t)\|^2_{2,\Omega} = \int\limits_t^{t+\rho} \Big\{ (u^n(\tau), (u^n(\tau)\nabla)[u^n(t+\rho) - u^n(t)])_{2,\Omega} -$$

$$-\nu(u^n(\tau), u^n(t+\rho) - u^n(t))_{J^1(\Omega)} + (g(\tau), u^n(t+\rho) - u^n(t))_{2,\Omega} -$$

$$-\alpha \left(\int\limits_a^b \int r\theta_R((u^n(\tau) - v)^2)(u^n(\tau) - v) S\tilde{f}^n(v, a, \tau) dv dr, u^n(t+\rho) - u^n(t) \right)_{2,\Omega} \Big\} d\tau.$$

Therefore,

$$\begin{aligned}
\|u^n(t+\rho) - u^n(t)\|^2_{2,\Omega} \ &\le \ \int\limits_t^{t+\rho} \int\limits_\Omega \Big\{ (|u^n(x,\tau)|^2 + \nu |Du^n(x,\tau)|) \times \\
&\times \ (|Du^n(x,t+\rho)| + |Du^n(x,t)|) + \\
&+ \ |g(x,\tau)(|u^n(x,t+\rho)| + u^n(x,t)|) + \\
&+ \ \alpha \int\limits_a^b \int S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau) - v)^2) \times \\
&\times \ |u^n(x,\tau) - v| \times \\
&\times \ (|u^n(x,t+\rho)| + |u^n(x,t)|) dv dr \Big\} dx d\tau = \\
&= \ \sum_{k=1}^{8} I_k(t),
\end{aligned} \tag{4.8}$$

where

$$|Du^n| = \left(\sum_{i,j=1}^{3} \left(\frac{\partial u_i^n}{\partial x_j} \right)^2 \right)^{1/2}.$$

The terms I_k are defined further in the text.

Integrating inequality (4.8) with respect to t in the interval $[0, T-\rho]$ and estimating the terms $I_k(t)$, $(k = 1, 2, \dots, 8)$ on the right hand side, we show that

$$\int_0^{T-\rho} I_k(t)dt \leq \eta_k \rho^{1/2}, \quad k = 1, 2, \dots, 8, \tag{4.9}$$

where η_k are constants not depending on n and ρ. For

$$I_1(t) \equiv \int_t^{t+\rho} \int_\Omega |u^n(x,\tau)|^2 |Du^n(x, t+\rho)| dx d\tau,$$

using Cauchy's inequality and the embedding theorem of $J^1(\Omega)$ in $L_4(\Omega)$, we get

$$\begin{aligned} \int_0^{T-\rho} I_1(t)dt \quad &\leq \quad \int_0^{T-\rho} \int_t^{t+\rho} \left\{ \int_\Omega |u^n(x,\tau)|^4 dx \right\}^{1/2} \left\{ \int_\Omega |Du^n(x,t+\rho)|^2 dx \right\}^{1/2} d\tau dt \leq \\ &\leq \quad C \int_0^{T-\rho} \int_t^{t+\rho} \|u^n(\tau)\|^2_{J^1(\Omega)} \|u^n(t+\rho)\|_{J^1(\Omega)} d\tau dt. \end{aligned} \tag{4.10}$$

where C denotes the constant coming from the embedding theorem.

We change the order of integration supposing that $u^n(x,t) = 0$ for $t > T$ and $t < 0$. It follows

$$\int_0^{T-\rho} I_1(t)dt \leq C \int_0^T \|u^n(\tau)\|^2_{J^1(\Omega)} \int_{\tau-\rho}^{\tau} \|u^n(t+\rho)\|_{J^1(\Omega)} dt d\tau.$$

We estimate the inner integral in t by Cauchy's inequality. Changing the interval of integration from $[\tau - \rho, \tau]$ to $[0, T]$, we obtain that the right hand side of (4.10), due to estimate (3.25c), is not grater then

$$C\sqrt{\rho} \left(\int_0^T \|u^n(t)\|^2_{J^1(\Omega)} dt \right)^{3/2} \equiv \eta_1 \sqrt{\rho}.$$

In the same way, one can obtain the estimate for

$$I_2(t) \equiv \int_t^{t+\rho} \int_\Omega |u^n(x,\tau)|^2 |Du^n(x,t)| dx d\tau.$$

Now, we consider

$$I_3 + I_4 \equiv \nu \int_t^{t+\rho} \int_\Omega |Du^n(x,\tau)| (|Du^n(x,t+\rho)| + |Du^n(x,t)|) dx d\tau.$$

Cauchy's inequality and estimate (3.25c) imply

$$\begin{aligned}\int_0^{T-\rho} I_3(t)dt &\leq \nu \int_0^{T-\rho}\int_t^{t+\rho} \|u^n(\tau)\|_{J^1(\Omega)}\|u^n(t+\rho)\|_{J^1(\Omega)}d\tau dt \leq \\ &\leq \nu \int_0^{T-\rho} \|u^n(t+\rho)\|_{J^1(\Omega)} \left\{\rho \int_t^{t+\rho} \|u^n(\tau)\|^2_{J^1(\Omega)}d\tau\right\}^{1/2} dt \leq \\ &\leq \nu \left\{\rho \int_0^T \|u^n(\tau)\|^2_{J^1(\Omega)}d\tau\right\}^{1/2} \int_0^T \|u^n(t)\|_{J^1(\Omega)}dt \leq \\ &\leq \nu(A\rho)^{1/2}\left(T\int_0^T \|u^n(t)\|^2_{J^1(\Omega)}dt\right)^{1/2} \leq \nu AT\rho^{1/2} \leq \eta_3\sqrt{\rho}.\end{aligned}$$

A similar bound can be easily proved for I_4.

Now, we consider the terms I_5 and I_6 from (4.8)

$$I_5 + I_6 \equiv \int_t^{t+\rho}\int_\Omega |g(x,\tau)|(|u^n(x,t+\rho)| + |u^n(x,t)|)dxd\tau.$$

From Cauchy's inequality, we have

$$I_k(t) \leq \max_{0\leq t\leq T} \|u^n(t)\|_{J(\Omega)} \int_t^{t+\rho} \|g(\tau\|_{2,\Omega}d\tau, \quad k = 5, 6.$$

Changing the order of integration, we obtain:

$$\int_0^{T-\rho}\int_t^{t+\rho} \|g(\tau)\|_{2,\Omega}d\tau dt \leq \rho \int_0^T \|g(\tau)\|_{2,\Omega}dt.$$

From this inequality we obtain (4.9) for I_5 and I_6.

Now, we consider the last two terms in (4.8), i.e., I_7 and I_8. We have

$$I_7(t) \equiv \alpha \int_t^{t+\rho}\int_\Omega\int_a^b\int S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v||u^n(x,t+\rho)|dvdrdxd\tau.$$

Using Cauchy's inequality, we get

$$I_7(t) \leq \alpha \int_t^{t+\rho}\int_\Omega |u^n(x,t+\rho)| \left\{\int_a^b\int S\tilde{f}^n(x,v,r,\tau)dvdr\right\}^{1/2} \times$$

$$\times \left\{ \int_a^b \int S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v|^2 dvdr \right\}^{1/2} dxd\tau.$$

To estimate the integral over Ω we make use of Hölder's inequality.

$$I_7(t) \leq \alpha \int_t^{t+\rho} \left\{ \int_\Omega |u^n(x,t+\rho)|^6 dx \right\}^{1/6} \left\{ \int_\Omega \left(\int_a^b \int S\tilde{f}^n(x,v,r,\tau) dvdr \right)^{3/2} dx \right\}^{1/3} \times$$

$$\times \left\{ \int_\Omega \int_a^b \int S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v|^2 dvdrdx \right\}^{1/2} d\tau. \quad (4.11)$$

Considering the second term on the right hand side of (4.11)

$$\tilde{I}_7(\tau) \equiv \left\{ \int_\Omega \left(\int_a^b \int S\tilde{f}^n(x,v,r,\tau) dvdr \right)^{3/2} dx \right\}^{1/3},$$

it is easy to see that

$$\begin{aligned} \tilde{I}_7(\tau) &\leq \int_\Omega \left(\int_a^b \int (1+v^2)[S\tilde{f}^n(x,v,r,\tau)]^{3/2} dvdr \right) \left(\int_a^b \int \frac{dvdr}{(1+v^2)^2} \right)^{1/2} dx \leq \\ &\leq C_1 \int_\Omega \int_a^b \int (1+v^2) S\tilde{f}^n(x,v,r,\tau) dvdrdx \leq C_2. \end{aligned}$$

Here, we use the a priori estimates (3.25a) and (3.25c) and the bound

$$\int_a^b \int \frac{dvdr}{(1+v^2)^2} < \infty.$$

Thus, from (4.11) follows

$$I_7(t) \leq C_3 \int_t^{t+\rho} \left\{ \int_\Omega |u^n(x,t+\rho)|^6 dx \right\}^{1/6} \times$$

$$\times \left\{ \int_\Omega \int_a^b \int S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau)-v)^2)|u^n x,\tau)-v|^2 dvdrdx \right\} d\tau.$$

Changing the order of integration and using Cauchy's inequality, we get

$$\int_0^{T-\rho} I_7(t)dt \ \le \ C_3\rho \left[\int_0^T \left\{ \int_\Omega |u^n(x,t)|^6 dx \right\}^{1/3} dt \right]^{1/2} \times$$

$$\times \left\{ \int_0^T \int_Q S\tilde{f}^n(x,v,r,t)\theta_R((u^n(x,t)-v)^2)(u^n(x,t)-v)^2 dvdrdxdt \right\}^{1/2}.$$

Due to (3.25c) and the embedding of $J^1(\Omega)$ in $L_6(\Omega)$, we have

$$\int_0^{T-\rho} I_7(t)dt \le \eta_7\rho.$$

The term $I_8(t)$ can be estimated in a similar way.

Thus, we proved inequality (4.9) and consequently Lemma 9. $\square$

From Lemma 9 and estimates (3.25a) - (3.25c) we obtain that the sequence $\{u^n\}$ is compact in $L_2(\Omega_T)$.

5 The limit $n \to \infty$

Using the lemmas 3, 5 – 9 and assuming that $R = n$, $\varepsilon = \frac{1}{n}$, we pass to the limit as $n \to \infty$ in the integral identities (3.8) – (3.10) .

5.1 The identity (2.3)

We multiply (3.22) by $H_i(t)$ and summarize over j. Then, integrating by parts, we obtain (3.8) for u^n and $\tilde{f}^n$, where the test functions ζ are defined by

$$\zeta(x,t) = \sum_{j=1}^n H_j(t)\Psi^j(x), \quad H_j(t) \in C^1(0,T) \quad H_i(T) = 0. \tag{5.1}$$

Now, we show that the limits of the subsequences $\{u^n\}$ and $\{\tilde{f}^n\}$ satisfy (2.3). First, we prove that

$$\lim_{n\to\infty} (u^n, (u^n\nabla_x)\zeta)_{2,\Omega_T} = (u, (u\nabla_x)\zeta)_{2,\Omega_T}.$$

To this end, we use the formula

$$(u^n, (u^n\nabla_x)\zeta)_{2,\Omega_T} - (u, (u\nabla_x)\zeta)_{2,\Omega_T} = (u^n - u, (u\nabla_x)\zeta)_{2,\Omega_T} + (u^n, ((u^n-u)\nabla_x)\zeta)_{2,\Omega_T}.$$

Due to the strong convergence of u^n to u in $L_2(\Omega_T)$ and the uniform boundedness of the norm of u^n in $L_2(\Omega_T)$ and (2.6a), the right hand side of the last equation tends to zero as $n \to \infty$. (Notice that the set of functions ζ from (5.1) is dense in the set of functions satisfying (2.6a)).

Now, we pass to the limit as $n \to \infty$ in the term of (3.8) containing $\tilde{f}^n$. Notice that for any $\varepsilon > 0$, there exists $R_1(\varepsilon) > 0$ that

$$I \equiv \int\limits_{Q_T \cap \{v:|v|\geq R_1\}} r\theta_n((u^n - v)^2) S\tilde{f}^n(x,v,r,t)|u^n - v||\zeta(x,t)|dxdvdrdt < \varepsilon, \quad (5.2)$$

uniformly on n. Taking into account that $\zeta \in L_4(0,T;J^1(\Omega))$ and the results of Lemma 3 we obtain, similar to the proof of the bound for I_7 in Lemma 9,

$$I \leq \sqrt{A} \left\{ \int\limits_a^b \int\limits_{|v|\geq R_1} \frac{dvda}{(1+v^2)^2} \right\}^{1/6}.$$

Inequality (5.2) immediately follows from this estimate.

Now, we prove that for any $R_1 > 0$,

$$\lim_{n\to\infty} \int\limits_{Q_T \cap \{v:|v|\leq R_1\}} r\theta_n((u^n - v)^2) S\tilde{f}^n(x,v,r,t)(u^n - v, \zeta)dxdvdrdt = \quad (5.3)$$

$$= \int\limits_{Q_T \cap \{v:|v|\leq R_1\}} rS\tilde{f}(x,v,r,t)(u(x,t) - v, \zeta(x,t))dxdvdrdt.$$

To this end, we consider the integrals

$$\begin{aligned}\sum_{j=1}^{4} I_j \equiv & \int\limits_{Q_T \cap \{v:|v|\leq R_1\}} \Big\{ rS(\tilde{f}^n - \tilde{f})(u - v, \zeta) + \\ & + rS\tilde{f}^n[\theta_n((u^n - v)^2) - \theta_n((u - v)^2)](u^n - v, \zeta) + \\ & + rS\tilde{f}^n[\theta_n((u - v)^2) - 1](u^n - v, \zeta) + rS\tilde{f}^n(u^n - u, \zeta)\Big\} dxdvdrdt \quad (5.4)\end{aligned}$$

Since $\tilde{f}^n \to \tilde{f}$ converges $*$-weakly in $L_\infty(\mathbb{R}^6_T \times [a,b])$ and $(u - v, \zeta) \in L_1(Q_T \cap \{v : |v| \leq R_1|\}$, I_1 tends to zero as $n \to \infty$. Indeed, we have

$$\int\limits_{Q_T \cap \{v:|v|\leq R_1\}} |u - v||\zeta(x,t)|dxdvdrdt \leq (b - a)\Bigg[\int\limits_0^T \int\limits_{|v|\leq R_1} \int\limits_\Omega |u(x,t)||\zeta(x,t)|dxdvdt +$$

$$+ \int\limits_0^T \int\limits_{|v|\leq R_1} \int\limits_\Omega |v||\zeta(x,t)|dxdvdrdt \leq \frac{4}{3}\pi R_1^3 \left\{ \int\limits_0^T\int\limits_\Omega |u|^2 dxdt \right\}^{\frac{1}{2}} \left\{ \int\limits_0^T\int\limits_\Omega |\zeta(x,t)|^2 dxdt \right\}^{\frac{1}{2}} +$$

$$+\frac{4}{3}\pi R_1^4 \int_0^T\int_\Omega |\zeta(x,t)|dxdt \leq C\|\zeta\|_{L_\infty(0,T;J(\Omega))}(1+\|u\|_{L_\infty(0,T;J(\Omega))}).$$

Estimating the integral I_2 in (5.4), we show firstly that

$$\lim_{n\to\infty}\int_0^T\int_{|v|\leq R_1}\int_\Omega |\theta_n((u^n-v)^2)-\theta_n((u-v)^2)|^3 dxdvdt = 0. \tag{5.5}$$

Since

$$\begin{aligned} |\theta_n((u^n-v)^2)-\theta_n((u-v)^2)| &\leq C|(u^n-v)^2-(u-v)^2| \leq \\ &\leq C(|u^n-u||u^n|+|u||u^n-u|+2|v||u^n-u|), \end{aligned}$$

it follows

$$\int_0^T\int_{|v|\leq R_1}\int_\Omega |\theta_n((u^n-v)^2)-\theta_n((u-v)^2)|dxdvdt \leq$$

$$\leq \hat{C}\|u^n-u\|_{2,Q_T}\left(\|u^n\|_{L_\infty(0,T;J(\Omega))}+\|u\|_{L_\infty(0,T;J(\Omega))}+1\right).$$

Due to the convergence of u^n to u in $L_2(Q_T)$ as $n\to\infty$ and

$$|\theta_n((u^n-v)^2)-\theta_n((u-v)^2)| \leq 2,$$

this inequality implies (5.5).

For

$$I_2 \equiv \int_{Q_T\cap\{v:|v|\leq R_1\}} rS\tilde{f}^n[\theta_n((u^n-v)^2)-\theta_n((u-v)^2)](u^n-v,\zeta)dxdvdrdt$$

follows from (3.25a)

$$|I_2| \leq A(b-a)\int_0^T\int_{|v|\leq R_1}\int_\Omega |u^n-v||\theta_n((u-v)^2)-\theta_n((u^n-v)^2)||\zeta|dxdvdt.$$

Using Hölder's inequality, we get

$$
\begin{aligned}
|I_2| \;&\leq\; A(b-a)\int\limits_0^T \left[\left\{\int\limits_{|v|\leq R_1}\int\limits_{\Omega} |u^n - v|^2 dxdv\right\}^{1/2} \times\right.\\
&\times \left\{\int\limits_{|v|\leq R_1}\int\limits_{\Omega} |\theta_n((u^n - v)^2) - \theta_n((u-v)^2)|^3 dxdv\right\}^{1/3} \times\\
&\times \left.\left\{\int\limits_{|v|\leq R_1}\int\limits_{\Omega} |\zeta(x,t)|^6 dxdv\right\}^{1/6}\right] dt \leq\\
&\leq \tilde{A}(1 + \|u^n\|_{L_\infty(0,T;J(\Omega))}) \times\\
&\times \int\limits_0^T \left\{\int\limits_{|v|\leq R_1}\int\limits_{\Omega} |\theta_n((u^n - v)^2) - \theta_n((u-v)^2)|^3 dxdv\right\}^{1/3} \times\\
&\times \left\{\int\limits_{|v|\leq R_1}\int\limits_{\Omega} |\zeta(x,t)|^6 dxdv\right\}^{1/6} dt.
\end{aligned}
$$

Using Hölder's inequality once again, we get

$$
\begin{aligned}
|I_2| \;&\leq\; \hat{A}\left\{\int\limits_0^T \int\limits_{|v|\leq R_1}\int\limits_{\Omega} |\theta_n((u^n - v)^2) - \theta_n((u-v)^2)|^3 dxdvdt\right\}^{1/3} \times\\
&\times \left\{\int\limits_0^T \left[\int\limits_{|v|\leq R_1}\int\limits_{\Omega} |\zeta(x,t)|^6 dxdv\right]^{1/4} dt\right\}^{2/3}.
\end{aligned}
$$

Using (5.5) and the embedding of $J^1(\Omega)$ in $L_6(\Omega)$, we get

$$\lim_{n\to\infty} I_2 = 0.$$

Now, we consider

$$I_3 \equiv \int\limits_{Q_T \cap \{v:|v|\leq R_1\}} rS\tilde{f}[\theta_n((u-v)^2) - 1](u^n - v, \zeta)dxdvdrdt.$$

We divide domain Ω_T in two parts:

$$\Omega_T^1 = \{(x,t) \in \Omega_T : |u(x,t)| \leq B\}, \quad \Omega_T^2 = \Omega_T \setminus \Omega_T^1,$$

where B is a positive constant.

Since $u(x,t) \in L_2(\Omega_T)$, for any $\delta > 0$ there exists such B that $\mathrm{mes}\Omega_T^2 < \delta$. Hence,

$$
\begin{aligned}
|I_3| \ \leq \ & A\Big(\int\limits_a^b \int\limits_{|v|\leq R_1} \int\limits_{\Omega_T^1} |\theta_n((u-v)^2) - 1||(u^n - v, \zeta)| dx dv dr dt + \\
& + \int\limits_a^b \int\limits_{|v|\leq R_1} \int\limits_{\Omega_T^2} |\theta_n((u-v)^2) - 1||(u^n - v, \zeta)| dx dv dr dt \Big).
\end{aligned}
$$

The argument of the function θ_n is bounded on the set $\Omega_T^1 \times \{v : |v| \leq R_1\}$. The sequence $\{\theta_n\}$ uniformly converges to 1 as $n \to \infty$ on any compact set. Therefore, since $(u^n - v, \zeta) \in L_1(Q_T \cap \{v : |v| \leq R_1\})$, for sufficiently large n, the integral over $\Omega_T^1 \times \{v : |v| \leq R_1 \times [a,b]\}$ is arbitrary small. The second integral is arbitrary small, due to the choice of δ.

Now, we consider the last term in (5.4),

$$
I_4 \equiv \int\limits_{Q_T \cap \{v:|v|\leq R_1\}} rS\tilde{f}^n(u^n - u, \zeta) dx dv dr dt.
$$

For this integral we have

$$
\begin{aligned}
|I_4| \ \leq \ & A\frac{4}{3}\pi R_1^3 \int\limits_0^T \int\limits_\Omega |u^n - u||\zeta| dx dt \leq \\
\leq \ & A\frac{4}{3}\pi R_1^3 \left\{ \int\limits_{\Omega_T} |u^n - u|^2 dx dt \right\}^{1/2} \left\{ \int\limits_0^T \int\limits_\Omega |\zeta|^2 dx dt \right\}^{1/2} \leq \\
\leq \ & \tilde{A} \|u^n - u\|_{2,\Omega_T} \|\zeta\|_{L_\infty(0,T;J(\Omega))}.
\end{aligned}
$$

Thus, $\lim\limits_{n\to\infty} I_4 = 0$ and (5.3) is proved.

From (5.2) and (5.3) follows

$$
\int\limits_{Q_T} rS\tilde{f}|u(x,t) - v|\zeta(x,t) dx dv dr dt < \infty.
$$

This estimate along with (5.2), (5.3), allows us to pass to the limit as $n \to \infty$ in the fourth terms of (3.8). The remaining terms in (3.8) are linear with respect to the u^n and we pass to the limit without any difficulties. Identity (2.3) is proved.

5.2 The identity (2.4)

Multiplying equation (3.12) by $\Phi(x,t)$ and integrating by parts we obtain the identity (3.10) for φ^n and $\tilde{f}^n$ with $\varepsilon = \frac{1}{n}$.

The first term in (3.9) tends to zero as $n \to \infty$. Indeed,

$$\left| \frac{1}{n} \int_0^T (\Delta\varphi^n, \Delta\Phi)_{2,\Omega} dt \right| \le \frac{1}{\sqrt{n}} \int_0^T \frac{1}{\sqrt{n}} \|\Delta\varphi^n\|_{2,\Omega} \|\Delta\Phi\|_{2,\Omega} dt \le \frac{1}{\sqrt{n}} \sqrt{A} \int_0^T \|\Delta\Phi\|_{2,\Omega} dt,$$

where A is the constant defined in (3.25c).

It was shown in the beginning of Section 4 that from the sequence $\{\varphi^n\}$ one can extract a subsequence, converging $*$-weakly in $L_\infty(0,T;W_2^1(\Omega))$. Taking into account the properties of the functions $\Phi(x,t)$, we can pass to the limit as $n \to \infty$ in the second term of (3.9).

Lemma 7 and Lebesgue's theorem allow us to pass to the limit as $n \to \infty$ in the third term of (3.9) for functions $\Phi(x,t) \in L_\infty(0,T;\overset{\circ}{W}{}_2^2(\Omega))$ being dense in $L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega))$. Thus, identity (2.4) is proved.

5.3 The identity (2.5)

Setting $R = n$ and $\varepsilon = \frac{1}{n}$ and taking into account (3.5) we get from identity (3.10) for the approximations $\tilde{f}^n(x,v,r,t)$

$$\int_0^T \int_a^b \left(\tilde{f}^n, \Psi_t + (v\nabla_x)\Psi + (g_n(x,t)\nabla_v)\Psi \right)_{2,\mathbb{R}^6} drdt + \int_a^b (P_0 f_0^n, \Psi(0))_{2,\mathbb{R}^6} dr + \tag{5.6}$$

$$+\beta \int_0^T \int_a^b \frac{1}{r^2} \left(\tilde{f}^n N, (P_0\theta_n((u^n - v)^2)[u^n(x,t) - v]\nabla_v)\Psi \right)_{2,\mathbb{R}^6} drdt -$$

$$-\gamma \int_0^T \int_a^b \frac{1}{r^2} \left(\tilde{f}^n, (P_0 \nabla_x \varphi^n(x,t)\nabla_v)\Psi \right)_{2,\mathbb{R}^6} drdt = 0,$$

where $\Psi(x,v,r,t)$ is an arbitrary function satisfying the conditions (2.6c).

Due to the $*$-weak convergence of the sequence $f^n(x,v,r,t)$ in $L_\infty(\mathbb{R}_T^6 \times [a,b])$ (see Lemma 3) and the properties of the functions $g_n(x,t)$ and $\Psi(x,v,r,t)$ we can pass to the limit as $n \to \infty$ in the first integral of (5.6). According to Lemma 1 and the properties of the functions $f_0^n(x,v,r)$ we can pass to the limit as $n \to \infty$ in the second integral of (5.6). In the third integral of (5.6) we pass to the limit in an analogous manner as we did in the term containing $\tilde{f}^n$ in identity (3.8) (see Sec. 4.1).

Considering the fourth integral in (5.6), we denote by $\mathcal{G}_n(x,y)$ Green's function of problem (3.12), (3.13) with $\varepsilon = \frac{1}{n}$ and by $\mathcal{G}(x,y)$ Green's function of problem (1.3), (1.7). We introduce the following notations

$$
\begin{aligned}
F_n(x,t) &= q\int_a^b\int r\tilde{f}^n(x,v,r,t)dvdr, \\
F(x,t) &= q\int_a^b\int r\tilde{f}(x,v,r,t)dvdr, \\
\varphi^n(x,t) &= \int_\Omega \mathcal{G}_n(x,y)F_n(y,t)dy, &\quad (5.7)\\
\varphi(x,t) &= \int_\Omega \mathcal{G}(x,y)F(y,t)dy, \\
\tilde{\varphi}^n(x,t) &= \int_\Omega \mathcal{G}(x,y)F_n(y,t)dy, &\quad (5.8)
\end{aligned}
$$

where $\tilde{f}(x,v,r,t)$ is the $*$-weak limit of the sequence $\{\tilde{f}^n(x,v,r,t)\}$ in $L_\infty(\mathbb{R}^6_T\times[a,b])$.
The functions $F_n(x,t)$ belong to $L_p(\Omega)$ ($p\in(\frac{3}{2};\frac{5}{3})$) uniformly on n and $t\in[0,T]$ (see Lemma 4). According to Lemma 7, the sequence $\{F_n(x,t)\}$ to $F(x,t)$ converges in the weak topology of $L_1(\Omega)$, uniformly in $t\in[0,T]$. Thus, $F(x,t)\in L_p(\Omega)$ ($p\in(\frac{3}{2};\frac{5}{3})$).
The operator with integral kernel $\nabla\mathcal{G}(x,t)$ acting from $L_1(\Omega)$ to $L_1(\Omega)$ is completely continuous (see [18]). Therefore, the sequence $\{\nabla\tilde{\varphi}^n(x,t)\}$ converges to $\nabla\varphi(x,t)$ in $L_1(\Omega)$ for any $t\in[0,T]$. From Lebesgue's theorem the strong convergence in $L_1(\Omega_T)$ follows. It is evident that the functions $\varphi^n(x,t)$ and $\varphi(x,t)$ are the solutions of the problems (3.12), (3.13) and (1.3), (1.7) correspondingly.

Thus, we can rewrite the fourth integral in (5.6) in the form

$$
\gamma\int_0^T\int_a^b \frac{1}{r^2}\left(\tilde{f}^n,(P_0\nabla_x\varphi^n(x,t)\nabla_v)\Psi\right)_{2,\mathbb{R}^6} drdt =
$$

$$
=\gamma\int_0^T\int_a^b\int_\Omega\int \frac{1}{r^2}S\tilde{f}^n(x,v,r,t)(\nabla_x\varphi(x,t)\nabla_v)\Psi(x,v,r,t)dvdxdrdt+
$$

$$
+\gamma\int_0^T\int_a^b\int_\Omega\int \frac{1}{r^2}S\tilde{f}^n(x,v,r,t)\left([\nabla_x\tilde{\varphi}^n(x,t)-\nabla_x\varphi(x,t)]\nabla_v\right)\Psi dvdxdrdt+
$$

$$
+\gamma\int_0^T\int_a^b\int_\Omega\int \frac{1}{r^2}S\tilde{f}^n(x,v,r,t)\left([\nabla_x\varphi^n(x,t)-\nabla_x\tilde{\varphi}^n(x,t)]\nabla_v\right)\Psi dvdxdrdt=
$$

$$
=I_1^{(n)}+I_2^{(n)}+I_3^{(n)}.
$$

Since $\Psi(x,v,r,t)$ satisfies conditions (2.6c) and $\nabla_x\varphi(x,t) \in L_1(\Omega_T)$, from the $*$-weak convergence of the sequence $\{\tilde{f}^n(x,v,r,t)\}$ to $\tilde{f}(x,v,r,t)$ in $L_\infty(\mathbb{R}^6_T \times [a,b])$ follows

$$\lim_{n\to\infty} I_1^{(n)} = \int\limits_0^T \int\limits_a^b \frac{\gamma}{r^2}\left(\tilde{f}, (P_0\nabla_x\varphi(x,t)\nabla_v)\Psi\right)_{2,\mathbb{R}^6} drdt.$$

Using the convergence of sequence $\{\nabla_x\tilde{\varphi}(x,t)\}$ to $\nabla_x\varphi(x,t)$ in $L_1(\Omega_T)$ and the uniform boundedness of the approximations $\tilde{f}^n(x,v,r,t)$ (see (3.25a)), we get

$$\lim_{n\to\infty} I_2^{(n)} = 0.$$

Considering $I_3^{(n)}$, we take into account (3.25a) and obtain

$$\begin{aligned}
|I_3^{(n)}| &\le \frac{\gamma}{a^2}\int\limits_0^T\int\limits_a^b\int\limits_\Omega\int\limits_\Omega\int S\tilde{f}^n(x,v,r,t)|\nabla_x\mathcal{G}_n(x,y)-\nabla_x\mathcal{G}(x,y)| \times \\
&\times |F_n(y,t)||\nabla_v\Psi(x,v,r,t)|dvdxdydrdt \le \\
&\le C\int\limits_0^T\int\limits_\Omega\int\limits_\Omega |\nabla_x\mathcal{G}_n(x,y)-\nabla_x\mathcal{G}(x,y)||F_n(y,t)|dxdydt \le \\
&\le C\int\limits_0^T \left\{\int\limits_\Omega\left[\int\limits_\Omega |\nabla_x\mathcal{G}_n(x,y)-\nabla_x\mathcal{G}(x,y)|dx\right]^\beta dy\right\}^{\frac{1}{\beta}}\left\{\int\limits_\Omega |F_n(y,t)|^p dy\right\}^{\frac{1}{p}} dt,
\end{aligned}$$

where $p^{-1}+\beta^{-1}=1$, $\frac{3}{2}<p<\frac{5}{3}$, $\frac{5}{2}<\beta<3$.

To estimate I_3^n we use the following equality

$$\lim_{n\to\infty}\int\limits_\Omega\left(\int\limits_\Omega |\nabla_x\mathcal{G}_n(x,y)-\nabla_x\mathcal{G}(x,y)|dx\right)^\beta dy = 0, \tag{5.9}$$

where $0<\beta\le 3$. This equality can be proved by the method of Lyusternic-Vischik (see [19]) with the help of the explicit form of the fundamental solutions of the corresponding equations (see [9]).

Since the functions $F_n(y,t)$ lie in $L_p(\Omega)$ ($p\in(\frac{3}{2},\frac{5}{3})$) uniformly in n and $t\in[0,T]$, due to (5.9), we get

$$\lim_{n\to\infty} I_3^{(n)} = 0.$$

Thus, equality (2.5) is proved and the proof of the Theorem is complete.

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