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Exponential asymptotic stability via Krein-Rutman theorem for singularly perturbed parabolic periodic Dirichlet problems

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Abstract

We consider singularly perturbed semilinear parabolic periodic problems and assume the existence of a family of solutions. We present an approach to establish the exponential asymptotic stability of these solutions by means of a special class of lower and upper solutions. The proof is based on a corollary of the Krein-Rutman theorem.

1 Introduction

This paper concerns abstract singularly perturbed semilinear parabolic periodic problems of the type

$$\left\{ \begin{array}{rcl} \varepsilon\,(u'(t) + Au(t)) & = & F(t, u(t), \varepsilon), \\ u(t) & = & u(t+T), \end{array} \right. \tag{1}$$

as well as periodic Dirichlet problems for singularly perturbed reaction-diffusion-advection equations of the type

$$\left\{ \begin{array}{rcl} \varepsilon\,(\partial_t u - \partial_x^2 u) & = & g(t, x, u, \varepsilon)\partial_x u + h(t, x, u, \varepsilon) \text{ for } 0 < x < 1, \\ u(t, x) & = & u(t+T, x), \\ u(t, \pm 1) & = & 0. \end{array} \right. \tag{2}$$

Here, $\varepsilon > 0$ is the small singular perturbation parameter and $T > 0$ the given period.

Our goal is to establish a criterion for exponential asymptotic stability of a given family u_ε of solutions by means of families a_ε and b_ε of lower and upper solutions of asymptotic order q (see Definition 4.1) and which are close to each other in some topology with asymptotic order p, where $p > q$ (see Theorems 2.3, 4.2 and 4.4). The main tool for deriving such a criterion is the Krein-Rutman theorem.

2 Abstract singularly perturbed semilinear parabolic periodic problems

In this section we consider problem (1). We start by formulating the hypotheses under which we consider this problem.

Let X be a real Banach space with norm $\|.\|_X$. Concerning the operator A we suppose

(A_1) $A : D(A) \subseteq X \to X$ is linear, closed and densely defined.

(A_2) The resolvent set $\varrho(A)$ of A satisfies $\varrho(A) \subseteq \{\lambda \in \mathbb{C} : Re\ \lambda \leq 0\}$.

(A_3) There is some positive constant κ such that

$$||(A - \lambda I)^{-1}|| \leq \frac{\kappa}{|\lambda|} \quad \text{for all } \lambda \in \mathbb{C} \text{ with } Re\ \lambda \leq 0, \lambda \neq 0.$$

(A_4) A has a compact resolvent.

Assumptions $(A_1)-(A_3)$ imply that A is a sectorial operator (see, e.g., [1]). As usual, we denote by A^α the fractional power of A for $0 \leq \alpha \leq 1$ and by X^α its domain of definition. The space X^α equipped with the graph norm $\|A^\alpha u\|_X$ is a Banach space. Moreover, assumption (A_4) implies that the embeddings $D(A) \hookrightarrow X^\alpha \hookrightarrow X^\beta \hookrightarrow X$ are compact for $0 < \beta < \alpha < 1$. Concerning the nonlinearity F we suppose

(F_1) $F \in C^2(\mathbb{R} \times V \times (0,1); W)$, where V and W are a real Banach spaces with norms $\|.\|_V$ and $\|.\|_W$, respectively, such that

$$D(A) \hookrightarrow V \hookrightarrow W \hookrightarrow X, \tag{1}$$

and F is periodic in t with period $T > 0$, i.e.

$$F(t, u, \varepsilon) = F(t + T, u, \varepsilon) \quad \text{for all} \quad t \in \mathbb{R}, u \in V, \varepsilon \in (0,1).$$

To given $\varepsilon > 0$, a solution u to (1) is, by definition, a function $u \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; D(A))$ which satisfies (1) pointwise.

Let u_ε be a family of solutions to (1). It is well-known (see, e.g., [1, Theorem 8.1.1]) that the stability of the solution u_ε to (1) can be determined by means of the linearized initial value problem

$$\left\{ \begin{array}{rcl} \varepsilon\,(v'(t) + Av(t)) & = & \partial_u F(t, u_\varepsilon(t), \varepsilon)\ v(t), \quad t > t_0, \\ v(t_0) & = & v_0. \end{array} \right. \tag{2}$$

According to the assumptions $(A_1)-(A_3)$ and (F_1), there exists a family of operators $U_\varepsilon(t, t_0) \in \mathcal{L}(X)$ with $U_\varepsilon(t_0, t_0,) = I$ such that, for any given initial value $v_0 \in X$, the element $U_\varepsilon(t,t_0)\, v_0$ is the solution to (2) (see [1], Theorem 7.1.3). In particular, we have

$$U_\varepsilon(t,t_0)\, v_0 \in D(A) \text{ for all } v_0 \in X. \tag{3}$$

The operator $U_\varepsilon(T,0)$ is usually called the monodromy operator, and its spectral radius can be used to verify stability of u_ε by means of the following theorem (see, e.g., [1, Theorem 8.1.1]):

Theorem 2.1 *Suppose the assumptions $(A_1) - (A_3)$ and (F_1) are satisfied, and assume the spectral radius of $U_\varepsilon(T,0)$ to be less than one. Then u_ε is exponentially asymptotically stable.*

In the sequel we will use a corollary of the Krein-Rutman-Theorem in order to verify if the spectral radius of $U_\varepsilon(T,0)$ is less than 1. For this purpose we introduce some notation.

An order cone Q in a Banach space Y is a closed and convex cone with vertex at zero such that $Q \cap (-Q) = 0$. As usual, we write $u \le v$ iff $v - u \in P$, and $u < v$ iff $v - u \in P \setminus \{0\}$. We denote by $\mathrm{int}(Q)$ the set of interior points of Q. The next theorem is a crucial consequence of the Krein-Rutman Theorem (see, e.g., [6, Corollary 7.27] and [2, Theorem 7.3]).

Theorem 2.2 *Let Q be an order cone in a Banach space Y with int$(Q) \neq \emptyset$, and let $K : Y \to Y$ be a linear compact operator such that $Ku \in \mathrm{int}(Q)$ for all $u > 0$. Suppose that there exists $u_0 \in Q$ such that $u_0 > Ku_0$. Then the spectral radius of K is less than one.*

Now we are going to show how to verify the assumptions of Theorem 2.2 with $K = U_\varepsilon(T,0)$. We assume:

(P_1) There is an order cone P in X such that $U_\varepsilon(T,s)v \in P$ for all $v \in P$ and $0 \le s \le T$.

(P_2) There is a Banach space U with the properties: (i). There is some $0 < \alpha < 1$ such that $X^\alpha \hookrightarrow U \hookrightarrow X$. (ii). $\mathrm{int}(P_U) \neq \emptyset$, where $P_U := P \cap U$, and $\mathrm{int}(P_U)$ denotes the set of interior (in U) points of P_U. (iii). $U_\varepsilon(T,s)v \in \mathrm{int}(P_U)$ for all $v \in P \setminus \{0\}$ and $0 < s < T$.

(P_3) $\mathrm{int}(P_W) \neq \emptyset$, where $P_W := P \cap W$, and $\mathrm{int}(P_W)$ denotes the set of interior (in W) points of P_W.

The main result of this section which provides a criterion for exponential stability of the family of solutions u_ε to (1) under some appropriate properties of lower and upper solutions (see section 4) is the following one.

Theorem 2.3 *Suppose $(A_1)-(A_4)$, (F_1) and (P_1)– (P_3). Assume that there exist two families of T-periodic functions $a_\varepsilon, b_\varepsilon \in C^1(\mathbb{R};X)\cap C(\mathbb{R};D(A))$ with $b_\varepsilon(0)-a_\varepsilon(0)>0$, numbers $p>q>0$ and $c>0$ and elements $\varphi_l,\varphi_u \in int(P_W)$ which do not depend on ε and t such that for all sufficiently small ε and all t we have*

$$\varepsilon(a'_\varepsilon(t)+Aa_\varepsilon(t))-F(t,a_\varepsilon(t),\varepsilon)\le -\varepsilon^q\varphi_l, \tag{4}$$

$$\varepsilon(b'_\varepsilon(t)+Ab_\varepsilon(t))-F(t,b_\varepsilon(t),\varepsilon)\ge \ \varepsilon^q\varphi_u, \tag{5}$$

$$\|F(t,b_\varepsilon(t),\varepsilon)-F(t,a_\varepsilon(t),\varepsilon)-\partial_uF(t,u_\varepsilon(t),\varepsilon)(b_\varepsilon(t)-a_\varepsilon(t))\|_W\le c\varepsilon^p. \tag{6}$$

Then, for sufficiently small $\varepsilon>0$, the spectral radius of $U_\varepsilon(T,0)$ is less than one.

Proof. Introducing the function ψ_ε by

$$F(t,b_\varepsilon(t),\varepsilon)-F(t,a_\varepsilon(t),\varepsilon)-\partial_uF(t,u_\varepsilon(t),\varepsilon)\,(b_\varepsilon(t)-a_\varepsilon(t))=\varepsilon^p\psi_\varepsilon(t),$$

then we have by (6)

$$||\psi_\varepsilon(t)||_W\le c \quad \text{for} \quad t\in\mathbb{R}, \quad 0<\varepsilon\ll 1. \tag{7}$$

Hence, assumptions (4) and (5) imply

$$\begin{aligned}
&\varepsilon\left(b'_\varepsilon(t)-a'_\varepsilon(t)-(A+\partial_uF(t,u_\varepsilon(t),\varepsilon))\,(b_\varepsilon(t)-a_\varepsilon(t))\right)=\\
&\quad=\varepsilon\left(b'_\varepsilon(t)-a'_\varepsilon(t)-A(b_\varepsilon(t)-a_\varepsilon(t))\right)-F(t,b_\varepsilon(t),\varepsilon)+F(t,a_\varepsilon(t),\varepsilon)-\varepsilon^p\psi_\varepsilon(t)\ge\\
&\quad\ge\varepsilon^q\left(\varphi_l+\varphi_u+\varepsilon^{p-q}\psi_\varepsilon(t)\right).
\end{aligned}$$

Therefore, assumption (P_1) and the variation of constants formula yield

$$(I-U_\varepsilon(T,0))\,(b_\varepsilon(0)-a_\varepsilon(0))\ge\varepsilon^q\int_0^T U_\varepsilon(T,s)\left(\varphi_l+\varphi_u+\varepsilon^{p-q}\psi_\varepsilon(s)\right)ds.$$

Because of $\varphi_l,\varphi_u\in\mathrm{int}(P_W)$, $p>q$ and taking into account (7) we have $\varphi_l+\varphi_u+\varepsilon^{p-q}\psi_\varepsilon(s)\in P\setminus\{0\}$ for sufficiently small ε and any s. According to assumption (P_2) we get

$$\int_0^T U_\varepsilon(T,s)\left(\varphi_l+\varphi_u+\varepsilon^{p-q}\psi_\varepsilon(s)\right)ds\in\mathrm{int}(P_U).$$

Thus, for sufficiently small ε it holds

$$b_\varepsilon(0)-a_\varepsilon(0)-U_\varepsilon(T,0)(b_\varepsilon(0)-a_\varepsilon(0))>0.$$

Hence, the element $u_0:=b_\varepsilon(0)-a_\varepsilon(0)$ belongs to P_U and satisfies $u_0>Ku_0$ with $K:=U_\varepsilon(T,0)$.

In order to apply Theorem 2.2 it remains to show that K maps U compactly into U and that $Ku\in\mathrm{int}(P_U)$ for all $u\in P\setminus\{0\}$. The first property follows from (3) and assumption (P_2) and from the compactness of the embedding $D(A)\hookrightarrow X^\alpha$. The second property follows from assumption (P_2).

3 Sufficient conditions for (6)

Assumptions (4), (5) and (6) of Theorem 2.3 seem to be antagonistic: On the one hand, (6) is satisfied if a_ε and b_ε are sufficiently close asymptotically as $\varepsilon \to 0$. On the other hand, (4), (5) are not satisfied if a_ε and b_ε are too close asymptotically as $\varepsilon \to 0$. In this section we derive sufficient conditions for (6) in terms of asymptotical closeness of a_ε and u_ε and of b_ε and u_ε that allow (4) and (5) to be satisfied.

First we reformulate the expression on the left hand side of (6). A simple calculation (using the main theorem of differential and integral calculus) yields

$$\begin{aligned} & F(t, b_\varepsilon(t), \varepsilon) - F(t, a_\varepsilon(t), \varepsilon) - \partial_u F(t, u_\varepsilon(t), \varepsilon)\,(b_\varepsilon(t) - a_\varepsilon(t)) \\ & = \int_0^1 r \int_0^1 (\partial_u^2 F(t, u_\varepsilon(t) + rs(b_\varepsilon(t) - u_\varepsilon(t)), \varepsilon)(b_\varepsilon(t) - u_\varepsilon(t), b_\varepsilon(t) - u_\varepsilon(t)) \\ & -\partial_u^2 F(t, u_\varepsilon(t) + rs(a_\varepsilon(t) - u_\varepsilon(t)), \varepsilon)(a_\varepsilon(t) - u_\varepsilon(t), a_\varepsilon(t) - u_\varepsilon(t)))drds. \end{aligned} \tag{8}$$

3.1 A Setting for Reaction-Diffusion Equations

In this subsection we consider a setting where the domain of definition V of the nonlinearity F is “large”. This is typical for applications to reaction-diffusion equations, where V can be chosen as the Banach space of continuous functions with the usual maximum norm. Because the space V is “large”, its norm is “weak” and, hence, in many applications there exists a constant $c_1 > 0$ such that for all small $\varepsilon > 0$ and all t we have

$$\|u_\varepsilon(t)\|_V + \|a_\varepsilon(t)\|_V + \|b_\varepsilon(t)\|_V \le c_1. \tag{9}$$

From (8) follows that (6) is satisfied if, for example, there are positive constants c_2 and c_3 such that

$$\|a_\varepsilon(t) - u_\varepsilon(t)\|_W^2 + \|b_\varepsilon(t) - u_\varepsilon(t)\|_W^2 \le c_2 \varepsilon^p \tag{10}$$

and

$$\|\partial_u^2 F(t, u, \varepsilon)\| \le c_3 \text{ for all } u \in V \text{ with } \|u\|_V \le c_1, \tag{11}$$

where $\|.\|$ is the operator norm in the space of all bounded bilinear operators from $V \times V$ into W. So we get

Lemma 3.1 *Suppose (10) and (11). Then relation* (6) *holds.*

3.2 A Setting for Reaction-Diffusion-Advection Equations

In this subsection we consider a setting where the domain of definition V of the nonlinearity F is “small”. This is typical for applications to reaction-diffusion-advection equations, where V must be chosen, for example, as the Banach space of continuously differentiable functions with its usual norm. Because the space V is “small”, now its norm is "strong" and, hence, in most of the applications assumption (9) is not satisfied. For example, in case of singularly perturbed PDEs the functions $u_\varepsilon(t)$, $a_\varepsilon(t)$ and $b_\varepsilon(t)$ have large spatial gradients close to internal or boundary layers for small ε. Therefore, we assume concerning the larger space W that there is a constant $c_1 > 0$ such that

$$\|u_\varepsilon(t)\|_W + \|a_\varepsilon(t)\|_W + \|b_\varepsilon(t)\|_W \leq c_1. \tag{12}$$

The following lemma shows how to verify relation (6) in cases, when some more structure of the nonlinearity F is known. More precisely, we suppose that F has the representation

$$F(t,u,\varepsilon) = G(t,u,\varepsilon)u + H(t,u,\varepsilon) \tag{13}$$

with

$$G \in C^2\left(\mathbb{R} \times W \times [0,1], \mathcal{L}(V;W)\right), \ H \in C^2\left(\mathbb{R} \times W \times [0,1], W\right) \tag{14}$$

and that there is a constant c_4 such that

$$\left\{ \begin{array}{rcl} \|(\partial_u G(t,u,\varepsilon)w_1)v\|_W & \leq & c_4\|w_1\|_W\|v\|_V, \\ \|(\partial_u^2 G(t,u,\varepsilon)(w_1,w_2))v\|_W & \leq & c_4\|w_1\|_W\|w_2\|_W\|v\|_V, \\ \|\partial_u^2 H(t,u,\varepsilon)(w_1,w_2)\|_W & \leq & c_4\|w_1\|_W\|w_2\|_W \end{array} \right. \tag{15}$$

for all $v \in V$, $w_1, w_2 \in W$ and $u \in W$ with $\|u\|_W \leq c_1$. Moreover, we assume that there is a constant c_5 such that for all small $\varepsilon > 0$ and all t we have

$$\|a_\varepsilon(t) - u_\varepsilon(t)\|_W^2 + \|b_\varepsilon(t) - u_\varepsilon(t)\|_W^2 \leq c_5\varepsilon^{p+1}, \tag{16}$$

and

$$\|a_\varepsilon(t) - u_\varepsilon(t)\|_V^2 + \|b_\varepsilon(t) - u_\varepsilon(t)\|_V^2 \leq c_5\varepsilon^{p-1}. \tag{17}$$

Lemma 3.2 *Suppose (13)–(17). Then relation* (6) *holds.*

Proof. Because of (8) it suffices to show that for $0 \leq r, s \leq 1$

$$\begin{aligned} &\|\partial_u^2 F(t, u_\varepsilon(t) + rs(b_\varepsilon(t) - u_\varepsilon(t)), \varepsilon)(b_\varepsilon(t) - u_\varepsilon(t), b_\varepsilon(t) - u_\varepsilon(t)) \\ &\quad -\partial_u^2 F(t, u_\varepsilon(t) + rs(a_\varepsilon(t) - u_\varepsilon(t)), \varepsilon)(a_\varepsilon(t) - u_\varepsilon(t), a_\varepsilon(t) - u_\varepsilon(t))\|_W \\ &\quad = O(\varepsilon^p). \end{aligned} \tag{18}$$

From (13) we get

$$\partial_u^2 F(t,u,\varepsilon)(v,v) = (\partial_u^2 G(t,u,\varepsilon)(v,v))u + 2(\partial_u G(t,u,\varepsilon)v)v + \partial_u^2 H(t,u,\varepsilon)(v,v).$$

According to (15) we have

$$\begin{aligned} &\|(\partial_u^2 G(t, u_\varepsilon(t) + rs(b_\varepsilon(t) - u_\varepsilon(t)), \varepsilon)(b_\varepsilon(t) - u_\varepsilon(t), b_\varepsilon(t) - u_\varepsilon(t))u_\varepsilon(t)\|_W \leq \\ &\qquad \leq c_4\|b_\varepsilon(t) - u_\varepsilon(t)\|_W^2\|u_\varepsilon(t)\|_V \end{aligned}$$

and

$$\|2(\partial_u G(t, u_\varepsilon(t), \varepsilon)(b_\varepsilon(t) - u_\varepsilon(t))(b_\varepsilon(t) - u_\varepsilon(t))\|_W \leq 2c_4||b_\varepsilon(t) - u_\varepsilon(t)||_V||b_\varepsilon(t) - u_\varepsilon(t)\|_W$$

and

$$\|(\partial_u^2 H(t,u_\varepsilon(t),\varepsilon)(b_\varepsilon(t)-u_\varepsilon(t),b_\varepsilon(t)-u_\varepsilon(t))\|_W \le c_4\|b_\varepsilon(t)-u_\varepsilon(t)\|_W^2.$$

Hence

$$\begin{aligned}&\|\partial_u^2 F(t,u_\varepsilon(t)+rs(b_\varepsilon(t)-u_\varepsilon(t)),\varepsilon)(b_\varepsilon(t)-u_\varepsilon(t),b_\varepsilon(t)-u_\varepsilon(t))\|_W\\ &\le c_4\Big(\|b_\varepsilon(t)-u_\varepsilon(t)\|_W^2\|u_\varepsilon(t)\|_V+2\|b_\varepsilon(t)-u_\varepsilon(t)\|_V\|b_\varepsilon(t)-u_\varepsilon(t)\|_W+\|b_\varepsilon(t)-u_\varepsilon(t)\|_W^2\Big).\end{aligned}$$

Taking into account (1), (16) and (17), we get

$$\|\partial_u^2 F(t,u_\varepsilon(t)+rs(b_\varepsilon(t)-u_\varepsilon(t)),\varepsilon)(b_\varepsilon(t)-u_\varepsilon(t),b_\varepsilon(t)-u_\varepsilon(t))\|_W = O(\varepsilon^p).$$

Analogously we obtain

$$\|\partial_u^2 F(t,u_\varepsilon(t)+rs(a_\varepsilon(t)-u_\varepsilon(t)),\varepsilon)(a_\varepsilon(t)-u_\varepsilon(t),a_\varepsilon(t)-u_\varepsilon(t))\|_W = O(\varepsilon^p).$$

Thus, relation (18) has been established.

4 Applications to Parabolic Periodic Dirichlet Problems

In this section we consider the periodic Dirichlet problem (2). Concerning the nonlinearities g and h in (2) we suppose

$$g,h\in C^2(\mathbb{R}\times[-1,1]\times\mathbb{R}\times[0,1],\mathbb{R}) \tag{1}$$

and

$$g(t,x,u,\varepsilon)=g(t+T,x,u,\varepsilon),\ h(t,x,u,\varepsilon)=h(t+T,x,u,\varepsilon). \tag{2}$$

We will show that, with appropriately chosen function spaces X, U, V and W, problem (2) can be written in the abstract form (1) such that the assumptions $(A_1)-(A_4),(F_1)$ and $(P_1)-(P_3)$ are fulfilled and that also the linearized initial boundary value problem

$$\left\{\begin{array}{rcl}\varepsilon\,(\partial_t v-\partial_x^2 v) &=& g(t,x,u,\varepsilon)\partial_x v+\partial_u g(t,x,u,\varepsilon)v\partial_x u+\partial_u h(t,x,u,\varepsilon)v,\\ v(0,x)&=&v_0(x),\\ v(t,\pm1)&=&0\end{array}\right. \tag{3}$$

can be represented in the form (2). For this end we set

$$X=L^2(-1,1),\ D(A)=H^2(-1,1)\cap H_0^1(-1,1),\ A=-\frac{d^2}{dx^2}.$$

It is well-known that in this setting the assumptions $(A_1)-(A_4)$ are valid, and it holds (see [1, Chapter 1.4]

$$X^\alpha=H^{2\alpha}(-1,1)\cap H_0^\alpha(-1,1).$$

Here, $H^{2\alpha}(-1,1)$ and $H_0^\alpha(-1,1)$ are the usual Sobolev spaces. Further, we take

$$U = \{u \in C^1([-1,1]) : \ u(\pm 1) = 0\}, \ \ \|u\|_U = \max_{-1 \le x \le 1} |u(x)| + \max_{-1 \le x \le 1} |u'(x)|.$$

According to the embedding theorem of Sobolev we have $X^\alpha \hookrightarrow U$ for $\alpha > 3/4$. Finally, we take

$$W = C([-1,1]), \ \|u\|_W = \max_{-1 \le x \le 1} |u(x)|.$$

and

$$P = \{u \in L^2(-1,1) : \ u(x) \ge 0 \text{ for almost all } x\}.$$

Then, obviously,

$$\begin{array}{rcl} \text{int } P_U & = & \{u \in U : \ u(x) > 0 \text{ for all } x \in (0,1), \ u'(-1) > 0, \ u'(1) < 0\} \neq \emptyset, \\ \text{int } P_W & = & \{u \in W : \ u(x) > 0 \text{ for all } x \in [0,1]\} \neq \emptyset, \end{array}$$

i.e. (P_3) is satisfied. Moreover, assumptions (P_1) and (P_2) are satisfied, because the solution to (3) with non-negative initial function is non-negative for all $t > 0$ and $x \in [-1,1]$, and the solution to (2) with an initial function, which is non-negative and not identically zero, is positive for all $x \in (-1,1)$ and satisfies $\partial_x u(-1,t) > 0$ and $\partial_x u(1,t) < 0$ for all $t > 0$. This follows from the parabolic maximum principle (see, e.g., [4, Theorem 2.1.4 and Lemma 2.2.1]).

The spaces U and W introduced above are the same in the following subsections, the space V will be chosen according to the considered problem.

It is well-known that, under assumption (1), any solution to (2) in the sense of Section 2 is a classical solution, i.e., all derivatives in the differential equation exist and are continuous. Correspondingly we define the notions of upper and lower solutions to (2):

Definition 4.1 *To any fixed $\varepsilon > 0$, the functions $b_\varepsilon, a_\varepsilon : \mathbb{R} \times [-1,1] \to \mathbb{R}$ are called upper and lower solutions to (2) of asymptotic order $q > 0$, respectively, if they are sufficiently smooth, satisfy the periodicity and boundary conditions in (2) and the inequalities*

$$\left\{ \begin{array}{lcl} \varepsilon\left(\partial_t b_\varepsilon - \partial_x^2 b_\varepsilon\right) - g(t,x,b_\varepsilon,\varepsilon)\partial_x b_\varepsilon - h(t,x,b_\varepsilon,\varepsilon) & \ge & \varphi\varepsilon^q, \\ \varepsilon\left(\partial_t a_\varepsilon - \partial_x^2 a_\varepsilon\right) - g(t,x,a_\varepsilon,\varepsilon)\partial_x a_\varepsilon - h(t,x,a_\varepsilon,\varepsilon) & \le & -\varphi\varepsilon^q \end{array} \right.$$

for all $t \in \mathbb{R}$ and $x \in [-1,1]$, where $\varphi > 0$ is some constant.

Note that for the existence of solutions to (2) it is sufficient to construct ordered upper and lower solutions which satisfy weaker conditions than those used in Definition 4.1 (see, e.g., [2]). In order to prove the exponential asymptotic stability, the introduced notion in Definition 4.1 seems to be appropriate.

4.1 Reaction-Diffusion Equations

In this subsection we consider problems of the type (2) with $g = 0$. Assumption (F_1) can be easily verified, where the abstract function is defined by

$$F(t, u, \varepsilon)(x) := h(t, x, u(x), \varepsilon),$$

and the function space V is chosen as $V = W$.

Now, the following theorem is a direct consequence of Theorem 2.3.

Theorem 4.2 *Suppose (1) and (2) with $g = 0$. Let u_ε be a family of solutions to (2) with $g = 0$, and let a_ε and b_ε be families of lower solutions and upper solutions of asymptotic order $q > 0$ to (2) with $g = 0$, respectively. Suppose that for all ε, t and x it holds*

$$|u_\varepsilon(t, x)| + |a_\varepsilon(t, x)| + |b_\varepsilon(t, x)| \leq \kappa_1,$$
$$|b_\varepsilon(t, x) - u_\varepsilon(t, x)| + |a_\varepsilon(t, x) - u_\varepsilon(t, x)| \leq \kappa_2 \varepsilon^{\frac{p}{2}},$$

where $\kappa_1 > 0$, $\kappa_2 > 0$ and $p > q$ are constants. Further, suppose that for all ε we have $b_\varepsilon(0, x) \geq a_\varepsilon(0, x)$ for all $x \in [-1, 1]$ *and $b_\varepsilon(0, x_0) > a_\varepsilon(0, x_0)$* for some $x_0 \in (-1, 1)$. *Then, for sufficiently small $\varepsilon > 0$, the solutions u_ε to (2) are exponentially asymptotically stable.*

Remark 4.3 *Upper and lower solutions which satisfy the assumptions of Theorem 4.2 have been constructed for problems with interior layers in [3, 5].*

4.2 Reaction-Diffusion-Advection Equations

In this subsection we consider problems of the general type (2) with (1) and (2). Then conditions (F_1) with (13)–(15) can be easily verified, where the abstract functions G and H are defined by

$$(G(t, u, \varepsilon)v)(x) := g(t, x, u(x), \varepsilon)v'(x),\ H(t, u, \varepsilon)(x) := h(t, x, u(x), \varepsilon),$$

and the space V is chosen as

$$V = C^1([-1, 1]),\ \|u\|_V = \max_{|x| \leq 1} |u(x)| + \max_{|x| \leq 1} |u'(x)|.$$

Now, the following theorem is a direct consequence of Theorem 2.3 and Lemma 3.2:

Theorem 4.4 *Suppose (1) and (2). Let u_ε be a family of solutions to (2), and let a_ε and b_ε be families lower solutions and upper solutions of order $q > 0$ to (2), respectively. Suppose that for all ε, t and x it holds*

$$|u_\varepsilon(t, x)| + |a_\varepsilon(t, x)| + |b_\varepsilon(t, x)| \leq \kappa_1,$$

$$|b_\varepsilon(t,x) - u_\varepsilon(t,x)| + |a_\varepsilon(t,x) - u_\varepsilon(t,x)| \le \kappa_2 \varepsilon^{\frac{p+1}{2}},$$
$$|\partial_x b_\varepsilon(t,x) - \partial_x u_\varepsilon(t,x)| + |\partial_x a_\varepsilon(t,x) - \partial_x u_\varepsilon(t,x)| \le c_2 \varepsilon^{\frac{p-1}{2}},$$

where κ_1*,* κ_2 *and* $p > q$ *are constants. Further, suppose that for all* ε *we have* $b_\varepsilon(0,x) \ge a_\varepsilon(0,x)$ for all $x \in [-1,1]$ *and* $b_\varepsilon(0,x_0) > a_\varepsilon(0,x_0)$ for some $x_0 \in (-1,1)$. *Then, for sufficiently small* $\varepsilon > 0$*, the solutions* u_ε *to (2) are exponentially asymptotically stable.*

Remark 4.5 *The upper and the lower solutions which satisfy the assumptions of Theorem 4.4 will be presented in our forthcoming paper.*

The approach described above seems to have a wide range of applicability in dealing with transition and boundary layers.

References

[1] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. **840**, Springer–Verlag, 1981

[2] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Pitman Research Notes in Math. Series **247**, Longman Scientific & Technical, 1991

[3] N. N. Nefedov, *An asymptotic method of differential inequalities for the investigation of periodic contrast structures: existence, asymptotics, and stability* , Differ. Uravn. **36**, 262-269 (2000)

[4] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, 1992

[5] V.T. Volkov, N. N. Nefedov, *Development of the asymptotic method of differential inequalities for investigation of periodic contrast structures in reaction-diffusion-advection equations*, Computational Mathematics and Mathematical Physics **46**, No 4, 585-593 (2006)

[6] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed Point Theorems*, Springer–Verlag, 1986