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Waveform iteration and one–sided Lipschitz conditions

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Introduction

Since the end of the 1980's waveform-iteration(relaxation)-based algorithms have become an industrial standard for circuit simulation with the wider use of parallel computers [LRSV82, VC85, WSV82, JZ91]. Often they are implemented without further mathematical investigations for the correctness of the results. Experience has shown that for a wide class of VLSI circuits the waveform-iteration-based simulation gives a good performance and correct results. However, there are also examples for which they fail. For an application of waveform iteration to a wider class of problems we need further investigations in order to improve the efficiency and robustness of the algorithm.

One central question which arises is, how to control the step-size in such a way that the iteration process converges. Another question concerns the number of iterations which are necessary to get the correct waveform

The influence of the discretization step-size on the speed of convergence has been discussed in several publications for fixed step-sizes and linear systems of ODE's (or DAE's) [MN85, Nev87, MN87].

Results for nonlinear variable step-size and window technique based on Lipschitz condition are derived in [Bre93].

According to these investigations it is clear that the step-size influences not only the discretization error in each iteration but also the convergence behavior.

In [Bre93] Lipschitz conditions and exponentially weighted norms are used to prove the convergence of waveform iteration methods in both the continuous and the discrete cases. We also get estimates for step-size control with respect to the convergence behavior of the iteration scheme.

However, by using Lipschitz conditions these estimates for the upper bound of the discretization step-size are derived for the worst case of an exponential growth of the solutions.

In this way, small step-sizes are suggested in case of large Lipschitz constants even if the solution tends very fast to the steady state (in case of large negative eigenvalues of the linearization). The reason behind it is that Lipschitz conditions do not reflect the sign of eigenvalues of the Jacobian.

There are several problems (for example the dependence from initial values) where one-sided Lipschitz conditions are used to improve estimates [Deu90, GR92]. Our results extend the results in [Bre93] by using one-sided Lipschitz conditions. They allow a better adapted step-size control for exponentially decaying solutions.

1 Preliminaries

The problem we want to solve or, more precisely, to approximate by waveform iteration is the following initial value problem

$$\begin{array}{rcl} \dot{x} & = & g(t,x) \\ x(t_0) & = & x_0, \end{array} \tag{1}$$

where $x \in \mathbb{R}^n, t \in [t_0, t_e] \subset \mathbb{R}$ and g is continuous with respect to t and x. We consider (1) as a system of r interacting subsystems. Let $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$, $x = (x_1, \ldots, x_r)$, $x_i \in \mathbb{R}^{n_i}$, $g = (g_1, \ldots, g_r)$, $g_i : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n_i}$ be the corresponding decomposition.
With this notation we get the following iteration scheme as the simplest case of waveform iteration:

$$\dot{x}_i^k = g_i(t, u^{i,k})$$

with $u^{i,k} = (u_1^{i,k}, \ldots, u_n^{i,k})$ and $u_j^{i,k} = \left\{ \begin{array}{ll} x_j^k & j \leq i \\ x_j^{k-1} & j > i \end{array} \right.$ in case of Gauss-Seidel-Iteration and $u_j^{i,k} = x_j^{k-1}$ in case of Gauss-Jacobi-Iteration.
Similar to the iteration scheme we use the notation

$$g(t,x) = f(t,x,x)$$

and

$$\begin{array}{rcl} \dot{x}^k & = & f(t, x^k, x^{k-1}) \\ x^k(t_0) & = & x_0. \end{array} \tag{2}$$

Let us assume "classical" global Lipschitz conditions for the R.H.S. of (2)

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq L_x|x - \bar{x}| + L_y|y - \bar{y}| \tag{3}$$

for a suitable norm $|\cdot|$ in $\mathbb{R}^n$. In a Banach space B of continuous $\mathbb{R}^n$-valued functions over $[t_0, t_e]$ with exponentially weighted norms and the weighting parameter $\alpha > L_x + L_y$ we will define a contractive operator which maps y into x by solving the equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), y(s), s) ds. \tag{4}$$

The Lipschitz constant of this map is $L_y/(\alpha - L_x)$, the norm $\|\cdot\|_\alpha$ in B is defined by

$$\|x\|_\alpha := \sup_{t \in [t_0, t_e]} \left(e^{-\alpha(t-t_0)} |x(t)| \right).$$

For references see [Bre93, LRSV82, Nev87].

If we write the discretized algorithm as

$$\dot{x} = \Pi(f(\cdot, x, y)), \tag{5}$$

where Π is the discretization operator we get a contractive map with Lipschitz constant $\|\Pi\|_\alpha L_y/(\alpha - \|\Pi\|_\alpha L_x)$.
In the case of backward Euler discretization for $n = 1$ we have, e.g.

$$(\Pi(u))(t) := u(t_i), \forall t \in (t_{i-1}, t_i]$$

where $t_i := t_{i_1} + h_i$ and h_i is the step-size of the i-th discretization step. Here $\|\Pi\|_\alpha = e^{\alpha h}$, $h = \max h_i$. For $n > 1$ we have similar expressions [Bre93].
Because (in relevant cases) $\|\Pi\|_\alpha$ depends on αh we get a strict limitation for the maximal step-size which has a great disadvantage for large negative eigenvalues of the linearization of f.
With the use of one-sided Lipschitz condition with respect to the first argument of f, however, we get nearly the same contraction constants except that L_x may be negative. If we suppose $L_x < 0$ and if we assume that the partition of the initial value problem results in a block diagonal-dominant linearization, i.e. $0 \leq L_y < -L_x$, then there are no restrictions for (positive) α or for h with respect to convergence of the waveform iteration.

2 Convergence of waveform iteration with one-sided Lipschitz condition

Now let us consider one-sided Lipschitz conditions for f with respect to x

$$(f(t, x, y) - f(t, \bar{x}, y), x - \bar{x}) \leq \bar{L}_x(x - \bar{x}, x - \bar{x}) \tag{6}$$

We define by $|\cdot| := (\cdot, \cdot)^{1/2}$ a norm in $\mathbb{R}^n$

Theorem 2.1 *Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous, satisfy the Lipschitz condition (3) and the one-sided Lipschitz condition (6). Let $x, \bar{x}, y, \bar{y} \in C^1([t_0, t_e], \mathbb{R}^n)$ such that*

$$\begin{aligned} \dot{x} &= f(t, x, y), \\ \dot{\bar{x}} &= f(t, \bar{x}, \bar{y}), \\ x(t_0) &= \bar{x}(t_0) = x_0. \end{aligned}$$

Then for the exponentially weighted norm we have

$$\|x - \bar{x}\|_\alpha \leq \frac{L_y}{\alpha - \bar{L}_x} \|y - \bar{y}\|_\alpha. \tag{7}$$

For the proof of this theorem we need the following lemma.

Lemma 2.1 *Consider the initial value problems* $\dot{x} = g(t,x) + u(t), \dot{\bar{x}} = g(t,\bar{x}) + \bar{u}$, $x(t_0) = \bar{x}(t_0) = x_0$. *If we assume that* g *satisfies a one-sided Lipschitz condition with respect to* x *with Lipschitz constant* L *and* $\alpha > l$, *then for the corresponding solution we have*

$$\|x - \bar{x}\|_\alpha \leq \frac{1}{\alpha - L}\|u - \bar{u}\|_\alpha. \tag{8}$$

Proof. We denote

$$\begin{aligned} \Delta x &:= x - \bar{x}, \\ \Delta u &:= u - \bar{u} \end{aligned}$$

and

$$\mu(t) := \frac{1}{2}(\Delta x(t), \Delta x(t))e^{-2L(t-t_0)}$$
$$(\mu(t_0) = 0).$$

By differentiating and applying the one-sided Lipschitz condition for g we get

$$\begin{aligned} \dot{\mu}(t) &= ((\Delta\dot{x}(t), \Delta x(t)) - L(\Delta x(t), \Delta x(t)))\, e^{-2L(t-t_0)} \\ &= ((g(t,x(t)) - g(t,\bar{x}(t)), \Delta x(t)) - L(\Delta x(t), \Delta x(t)) + \\ &\qquad (u(t) - \bar{u}(t), \Delta x(t)))\, e^{-2L(t-t_0)} \\ &\leq (\Delta u(t), \Delta x(t))e^{-2L(t-t_0)}, \\ \mu(t) &\leq \int_{t_0}^{t} (\Delta u, \Delta x)e^{-2L(s-t_0)}ds. \end{aligned}$$

Substituting $\mu(t)$ and multiplying with $e^{-2L(t-t_0)}$ yields

$$(\Delta x(t), \Delta x(t)) \leq 2e^{2Lt}\int_{t_0}^{t} (\Delta u, \Delta x)e^{-2L(s-t_0)}ds,$$

$$|\Delta x(t)|^2 \leq 2e^{2L(t-t_0)}\int_{t_0}^{t} e^{2(\alpha-L)(s-t_0)}\|\Delta u\|_\alpha\|\Delta x\|_\alpha ds. \tag{9}$$

By integrating and multiplying with $e^{-2\alpha(t-t_0)}$ we finally get (8). □

Proof of the theorem. Setting $g(t,x) := f(t,x,y(t))$, $u(t) := 0$ and $\bar{u}(t) := f(t,\bar{x}(t),\bar{y}(t)) - f(t,\bar{x}(t),y(t))$, we get from Lemma 2.1

$$\|x - \bar{x}\|_\alpha \leq \frac{1}{\alpha - \bar{L}_x}\|\bar{u}\|_\alpha.$$

Using the Lipschitz condition on f the assertion of the theorem follows. □

For the discrete version we have the following theorem.

Theorem 2.2 *Let f be as in Theorem 2.1 Let $x, \bar{x}, y, \bar{y} \in C([t_0, t_e], \mathbb{R}^n)$ such that*

$$\dot{x} = \Pi(f(\cdot, x(\cdot), y(\cdot))), \dot{\bar{x}} = \Pi(f(\cdot, \bar{x}(\cdot), \bar{y}(\cdot))), x(t_0) = \bar{x}(t_0) = x_0.$$

Then for the exponential weighted norm we have

$$\|x - \bar{x}\|_\alpha \leq \frac{\|\Pi\|_\alpha L_y}{\alpha - \bar{L}_x - \|\Pi - I\|_\alpha L_x} \|y - \bar{y}\|_\alpha. \tag{10}$$

Proof. Let $g(t, x) := f(t, x, y(t))$, $u(t) := (\Pi f(\cdot, x, y))(t) - f(t, x(t), y(t))$, and $\bar{u}(t) := (\Pi f(\cdot, \bar{x}, \bar{y})) - f(t, \bar{x}(t), y(t))$. From Lemma 2.1 we get

$$\begin{aligned} \|x - \bar{x}\|_\alpha \quad \leq \quad & \frac{1}{\alpha - \bar{L}_x} \|(\Pi - I)(f(\cdot, x, y) - f(\cdot, \bar{x}, y)) + \\ & \qquad \Pi(f(\cdot, \bar{x}, y) - f(\cdot, \bar{x}, \bar{y}))\|. \end{aligned}$$

Thus

$$\|x - \bar{x}\|_\alpha \leq \frac{1}{\alpha - \bar{L}_x} (\|(\Pi - I)\|_\alpha L_x \|x - \bar{x}\|_\alpha + \|\Pi\|_\alpha L_y \|y - \bar{y}\|_\alpha)$$

which implies the assertion of the theorem. □

In the numerical practice of the waveform iteration method (for instance on parallel computers with distributed memory) the information of the whole system is not automatically known at the local node at which a subsystem is solved. So the conditions for the whole system have to be changed to a local one. In practice we relax the condition and merely require a one sided L-condition for the corresponding subsystem. Thus we have to extend the above results beginning with Lemma 2.1. For the definition of the corresponding Banach space we use the maximum of exponential weighted norms of the subsystems. (Compare [Bre93].)
With the notation of Lemma 2.1 let $g_i(t, x_i, x)$ the i-th component of g by a suitable ordering of the depending variables.

Lemma 2.2 *Suppose that the i-th subsystem satisfies a one-sided Lipschitz condition with respect to x_i and a Lipschitz condition with respect to y, i.e.:*

$$\begin{aligned} (g_i(t, x_i, y) - g_i(t, \bar{x}_i, y), x_i - \bar{x}_i) \quad &\leq \quad \bar{L}_i(x_i - \bar{x}_i, x_i - \bar{x}_i), \\ |g_i(t, x_i, y) - g_i(t, x_i, \bar{y})| \quad &\leq \quad L_i|y - \bar{y}|. \end{aligned}$$

Then

$$\|x - \bar{x}\|_\alpha \leq \frac{1}{\alpha - L - \bar{L}} \|u - \bar{u}\|_\alpha \tag{11}$$

holds, with $\bar{L} := \max_i L_i, L := \max_i L_i$

Proof. For fixed i we use Lemma 2.1 where we replace g, u and $\bar{u}$ by $g_i(t, x_i, x(t))$, $u_i(t)$ and $\bar{u}_i(t)+g_i(t, \bar{x}_i(t), \bar{x}(t))-g_i(t, \bar{x}_i(t), x(t))$ respectively. Then (8) reads

$$\|\Delta x_i\|_\alpha \leq \frac{1}{\alpha - \bar{L}_i}(L_i\|\Delta x\|_\alpha + \|\Delta u_i\|_\alpha).$$

By taking the maximum of the right and left hand side we get

$$\|\Delta x\|_\alpha \leq \frac{1}{\alpha - \bar{L}}(L\|\Delta x\|_\alpha + \|\Delta u\|_\alpha)$$

□

Let us denote the i-th component of $f(t, x, y)$ by $f_i(t, x_i, x, y)$. We suppose one-sided Lipschitz conditions for the subsystems of (2) of the following form

$$(f_i(t, x_i, y, z) - f_i(t, \bar{x}_i, y, z), x_i - \bar{x}_i) \leq \bar{L}_{x,i}(x_i - \bar{x}_i, x_i - \bar{x}_i). \tag{12}$$

Theorem 2.3 *Under conditions of Theorem 2.1,the one-sided Lipschitz condition (12) and* $\bar{L}_x := \max_i L_{x,i}$ *we get*

$$\|x - \bar{x}\|_\alpha \leq \frac{L_y}{\alpha - \bar{L}_x - L_x}\|y - \bar{y}\|_\alpha.$$

The proof is similar to the proof of Theorem 2.1 by using Lemma 2.2 instead of Lemma 2.1.
In the same way we get a result corresponding to Theorem 2.2.

Theorem 2.4 *With the conditions of Theorem 2.2, (12) and* $\bar{L}_x$ *from Theorem 2.3 we have*

$$\|x - \bar{x}\|_\alpha \leq \frac{\|\Pi\|_\alpha L_y}{\alpha - \bar{L}_x - (1 + \|\Pi - I\|_\alpha)L_x}\|y - \bar{y}\|_\alpha.$$

Remark 1. Instead of using the Lipschitz constant L_x we can improve our result by using the Lipschitz constant with respect to the off-diagonal components of x. That means, splitting the dependent variables on the right hand side of (1) into three parts, say $\bar{f}(t, x, x, y)$ and supposing one-sided Lipschitz conditions with respect to the second variable based on scalar products on subsystems, and usual Lipschitz conditions with respect to the second, third and fourth variable, say $L_{x,d}, L_x, L_y$, then we have no change in Theorem 2.3 but Theorem 2.4 reads

$$\|x - \bar{x}\|_\alpha \leq \frac{\|\Pi\|_\alpha L_y}{\alpha - \bar{L}_x - L_x - (\|\Pi - I\|_\alpha)(L_{x,d} + L_x)}\|y - \bar{y}\|_\alpha.$$

In case of block Jacobi like iteration L_x is zero, so we have the same inequalities for Theorems 2.3 and 2.4 as in Theorems 2.1 and 2.2.

Remark 2. Our results show how to replace the standard Lipschitz condition by one-sided Lipschitz condition in the estimates needed for step-size control to guarantee convergence of the discrete waveform iteration scheme. The conditions are a mixture of local and global estimates. But it is easy to generalize our results in such a way that almost nothing but local estimates are used. We can do this first by defining an exponentially weighted norm for some parameter α_i for the i-th component of system (1) and then taking the maximum to get a norm in the function space. Then we have to correct the corresponding inequalities by factors of the form $e^{(\alpha_i-\alpha_j)(t_e-t_0)}$, so we obtain local upper bounds for the discretization step-size to ensure convergence.

Remark 3. Suppose $\alpha > L$. A closer look at (9) shows that the inequality (8) can be improved to

$$\|x-\bar{x}\|_\alpha \leq \frac{1-e^{(L-\alpha)(t_e-t_0)}}{\alpha - L}\|u-\bar{u}\|_\alpha \tag{13}$$

which leads to

$$\|x-\bar{x}\|_\alpha \leq \frac{\left(1-e^{(L-\alpha)(t_e-t_0)}\right)L_y}{\alpha - \bar{L}_x}\|y-\bar{y}\|_\alpha \tag{14}$$

and

$$\|x-\bar{x}\|_\alpha \leq \frac{\|\Pi\|_\alpha\left(1-e^{(L-\alpha)(t_e-t_0)}\right)L_y}{\alpha - \bar{L}_x - \|\Pi - I\|_\alpha\left(1-e^{(L-\alpha)(t_e-t_0)}\right)L_x}\|y-\bar{y}\|_\alpha \tag{15}$$

instead of (7) and (10). In this notation we can read off how the length of the interval $[t_0, t_e]$ (window length) influences to the speed of convergence in the Banach space B. A smaller window length yields better convergence or fewer restrictions for the step-size of discretization.

In [Bre93] we have proved convergence results for a variable window technique. The proof is based on the following inequality for dependence on the initial values in terms of the norm of the Banach space. If $g(t,x)$ is Lipschitz continuous with respect to x and with constant L, $\dot{x} = g(t,x), x(t_0) = x_0$, $\dot{\bar{x}} = g(t,\bar{x}), \bar{x}(t_0) = \bar{x}_0$ we have [1]

$$\|x-\bar{x}\| \leq \frac{\alpha}{\alpha - \left(1-e^{(L-\alpha)(t_e-t_0)}\right)L}|x_0-\bar{x}_0|$$

By a well known result for one-sided Lipschitz conditions we have

$$|\Delta x(t)| \leq e^{Lt}|\Delta x(0)|.$$

[1] In [Bre93] we use $\|x\| := |x(t_0)| + \frac{1-e^{-\alpha()}}{\alpha}\|\dot{x}\|_\alpha$ to get results in C^1. But from $\|x\|_\alpha \leq \|x\|$ it is clear that one can derive similar results in C by using $\|\cdot\|_\alpha$.

The proof is the same as in the above lemma by setting $\Delta u = 0$ and by taking into account that $\mu(0) \neq 0$. ([GR92])
For our norm in Banach space this leads to

$$\|x - \bar{x}\|_\alpha \leq |x_0 - \bar{x}_0|$$

which is a better result as in [Bre93]. From this it is clear that with the same arguments as in [Bre93] Chapter 6.2 the convergence of waveform iteration with variable windows for a one-sided Lipschitz condition can be proven.

3 Conclusions

To make the connection between our results and step-size bounds for an implementation of the discretized algorithm we use the following inequality

$$\varepsilon_k \leq (\kappa\|\tilde{x}^k - \tilde{x}^{k-1}\| + \delta)/(1-\kappa),$$

where δ is an upper bound for the norm of the global discretization error for all iterations, $\tilde{x}^k$ denotes the solution of the k-th iteration of the discretized algorithm, κ is an upper bound for the (linear) convergence rate and finally ε_k denotes the norm of the difference between $\tilde{x}^k$ and the exact solution of (1). For κ we have $\frac{L_y}{\alpha - \bar{L}_x}$ by Theorem 2.1 or $\frac{L_y}{\alpha - \bar{L}_x - L_x}$ by Theorem 2.3.
If we have an estimate for the Lipschitz constants then we can choose α to get $\kappa < 1$. From Theorem 2.2 or 2.4 we get also upper bounds for the discretization step-size if we have estimates for $\|\Pi\|_\alpha$ and $\|I - \Pi\|_\alpha$ to ensure convergence of the discretized algorithm. Finally, by the above inequality we can choose the upper bound for the discretization error and the number of iterations we need to reach a given tolerance.
In practice we will do the estimates not for the whole interval but for smaller windows. According to Remark 3 we get smaller κ for smaller windows, so we can choose smaller α or larger step-size if we take the window length into account.

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