Waveform iteration and one-sided Lipschitz conditions

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Introduction

From the end of the 1980's waveform-iteration(relaxation)-based algorithms become an industrial standard for circuit simulation by use of parallel computers [LRSV82, VC85, WSV82, JZ91]. Often they are implemented without further mathematical investigations for the correctness of the results. The experience has shown that for a wide class of VLSI circuits the waveform-iteration-based simulation gives a good performance and correct results but there are also examples for which they fail. To get a wider class of applications for the waveform iteration we need further investigations to improve the efficiency and robustness of the algorithm.

One question is, how to control the stepsize, such that the iteration process converges. Another question is, how many iterations are necessary to get the correct waveform.

The influence of the discretization stepsize on the speed of convergence has been discussed in several publications for fixed stepsizes and linear systems of ODE's (or DAE's) [MN85, Nev87, MN87].

Results for nonlinear variable stepsize and window technique based on Lipschitz condition are derives in [Bre93].

According to these investigations it is clear that the stepsize influences not only the error of the discretization in one iteration but also the convergence behavior.

In [Bre93] Lipschitz conditions and exponentially weighted norms are used to prove the convergence of waveform iteration methods in both the continuous and the discrete cases. We also get estimates for stepsize control with respect to to the convergence behavior of the iteration scheme.

However, by using Lipschitz conditions these estimates for the upper bound of the discretization stepsize are derived for the worst case, the case of an exponential growth of the solutions.

By this way, small stepsizes are suggested in case of large Lipschitz constants even if the solution tends very fast to the steady state (in case of large negative eigenvalues of the linearization). The the reason for that is that Lipschitz conditions don't reflect the sign of eigenvalues of the Jacobian.
There are several problems (for example the dependence from initial values) where one-sided Lipschitz conditions are used to improve estimates [Deu90, GR92]. Our results extend the results in [Bre93] by using one-sided Lipschitz conditions. They permit a better adapted stepsize control for exponentially decaying solutions.

1 Preliminaries

The problem we want to solve or better to approximate by waveform iteration is the following initial value problem

$$\begin{aligned} \dot{x} &= g(t,x) \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, t \in [t_0, t_e] \subset \mathbb{R}$ and g is continuous with respect to t and x.
We consider (1) as a system of r interacting subsystems. Let $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$, $x = (x_1, \ldots, x_r)$, $x_i \in \mathbb{R}^{n_i}$, $g = (g_1, \ldots, g_r)$, $g_i : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n_i}$ be the corresponding decomposition.
With this notation we get the following iteration scheme as the simplest case of waveform iteration:

$$\dot{x}_i^k = g_i(t, u^{i,k})$$

with $u^{i,k} = (u_1^{i,k}, \ldots, u_n^{i,k})$ and $u_j^{i,k} = \begin{cases} x_j^k & j \leq i \\ x_j^{k-1} & j > i \end{cases}$ in case of Gauss-Seidel-Iteration and $u_j^{i,k} = x_j^{k-1}$ in case of Gauss-Jacobi-Iteration.
Similar to the iteration scheme we use the notation

$$g(t,x) = f(t,x,x)$$

and

$$\begin{aligned} \dot{x}^k &= f(t, x^k, x^{k-1}) \\ x^k(t_0) &= x_0. \end{aligned} \tag{2}$$

Let us assume "classical" global Lipschitz conditions for the R.H.S. of (2)

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq L_x|x - \bar{x}| + L_y|y - \bar{y}| \tag{3}$$

for a suitable norm $|\cdot|$ in $\mathbb{R}^n$. In a Banach space B of continuous $\mathbb{R}^n$-valued functions over $[t_0, t_e]$ with exponentially weighted norms and the weighting parameter $\alpha > L_x + L_y$ we will define a contractive operator which maps y into x by solving the equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), y(s), s)ds. \tag{4}$$

The Lipschitz constant of this map is $L_y/(\alpha - L_x)$, the norm $\|\cdot\|_\alpha$ in B is defined by

$$\|x\|_\alpha := \sup_{t\in[t_0,t_e]} \left(e^{-\alpha(t-t_0)}|x(t)|\right).$$

For references see [Bre93, LRSV82, Nev87].

If we write the discretized algorithm as

$$\dot{x} = \Pi(f(\cdot, x, y)), \tag{5}$$

where Π is the discretization operator we get a contractive map with Lipschitz constant $\|\Pi\| L_y/(\alpha - \|\Pi\| L_x)$.
In the case of backward Euler discretization and $n = 1$ we have eg.

$$(\Pi(u))(t) := u(t_i), \forall t \in (t_{i-1}, t_i]$$

where $t_i := t_{i_1} + h_i$ and h_i is the stepsize of the i-th discretization step. Here $\|\Pi\|_\alpha = e^{\alpha h}$, $h = \max h_i$. For $n > 1$ we have similar expressions [Bre93].
Because (in relevant cases) $\|\Pi\|$ depends on αh we get a strict limitation for the maximal stepsize which has a great disadvantage for large negative eigenvalues of the linearization of f.
With the use of one-sided Lipschitz condition with respect to the first argument of f however, we get nearly the same contraction constants except that L_x may be negative. If we suppose $L_x < 0$ and if we assume that the partition of the initial value problem results in a block diagonal-dominant linearization, i.e. $0 \leq L_y < -L_x$, then there are no restrictions for (positive) α or for h with respect to convergence of the waveform iteration.

2 Convergence of waveform iteration with one-sided Lipschitz condition

Now let us consider one-sided Lipschitz conditions for f with respect to x

$$(f(t, x, y) - f(t, \bar{x}, y), x - \bar{x}) \leq \bar{L}_x(x - \bar{x}, x - \bar{x}) \tag{6}$$

We denote $|\cdot| := (\cdot, \cdot)^{1/2}$ for the norm in $\mathbb{R}^n$

Theorem 2.1 *Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous with Lipschitz condition (3) and one-sided Lipschitz condition (6). Let $x, \bar{x}, y, \bar{y} \in C^1([t_0, t_e], \mathbb{R}^n)$ such that*

$$\dot{x} = f(t, x, y), \dot{\bar{x}} = f(t, \bar{x}, \bar{y}), x(t_0) = \bar{x}(t_0) = x_0.$$

Then for the exponential weighted norm we have

$$\|x - \bar{x}\|_\alpha \leq \frac{L_y}{\alpha - \bar{L}_x} \|y - \bar{y}\|_\alpha. \tag{7}$$

Lemma 2.1 *Supposing $\dot{x} = g(t, x) + u(t), \dot{\bar{x}} = g(t, \bar{x}) + \bar{u}$, $x(t_0) = \bar{x}(t_0) = x_0$ and a one-sided Lipschitz constant L for g with respect to x, $\alpha > L$. Then*

$$\|x - \bar{x}\|_\alpha \leq \frac{1}{\alpha - L} \|u - \bar{u}\|_\alpha \tag{8}$$

holds.

Proof. We denote

$$\Delta x := x - \bar{x}, \Delta u := u - \bar{u}$$

and

$$\mu(t) := \frac{1}{2}(\Delta x(t), \Delta x(t)) e^{-2L(t-t_0)}$$
$$(\mu(t_0) = 0).$$

By differentiating and applying the one-sided Lipschitz condition for g we get.

$$\begin{aligned}
\dot{\mu}(t) &= ((\Delta \dot{x}(t), \Delta x(t)) - L(\Delta x(t), \Delta x(t)))\, e^{-2L(t-t_0)} \\
&= ((g(t, x(t)) - g(t, \bar{x}(t)), \Delta x(t)) - L(\Delta x(t), \Delta x(t)) + \\
&\qquad (u(t) - \bar{u}(t), \Delta x(t)))\, e^{-2L(t-t_0)} \\
&\leq (\Delta u(t), \Delta x(t)) e^{-2L(t-t_0)} \\
\mu(t) &\leq \int_{t_0}^{t} (\Delta u, \Delta x) e^{-2L(s-t_0)} ds
\end{aligned}$$

Substituting $\mu(t)$ and multiplying with $e^{-2L(t-t_0)}$ yields

$$(\Delta x(t), \Delta x(t)) \leq 2e^{2Lt} \int_{t_0}^{t} (\Delta u, \Delta x) e^{-2L(s-t_0)} ds,$$

$$|\Delta x(t)|^2 \leq 2e^{2L(t-t_0)} \int_{t_0}^{t} e^{2(\alpha - L)(s-t_0)} \|\Delta u\|_\alpha \|\Delta x\|_\alpha ds. \tag{9}$$

By integrating and multiplying with $e^{-2\alpha(t-t_0)}$ we finally get (8). □

Proof of the theorem. In the above lemma let $g(t,x) := f(t,x,y(t))$ and $u(t) := 0, \bar{u}(t) := f(t,\bar{x}(t),\bar{y}(t)) - f(t,\bar{x}(t),y(t))$. From the lemma we have

$$\|x-\bar{x}\| \le \frac{1}{\alpha - \bar{L}_x}\|\bar{u}\|.$$

Using the Lipschitz condition on f the assertion of the theorem follows. $\square$

For the discrete version we have the following theorem.

Theorem 2.2 *Let f like in theorem 2.1 Let $x,\bar{x},y,\bar{y} \in C([t_0,t_e],\mathbb{R}^n)$ such that*

$$\dot{x} = \Pi(f(\cdot,x(\cdot),y(\cdot))), \dot{\bar{x}} = \Pi(f(\cdot,\bar{x}(\cdot),\bar{y}(\cdot))), x(t_0) = \bar{x}(t_0) = x_0.$$

Then for the exponential weighted norm we have

$$\|x-\bar{x}\|_\alpha \le \frac{\|\Pi\|L_y}{\alpha - \bar{L}_x - \|\Pi - I\|L_x}\|y-\bar{y}\|_\alpha \tag{10}$$

Proof. Let $g(t,x) := f(t,x,y(t))$, $u(t) := (\Pi f(\cdot,x,y))(t) - f(t,x(t),y(t))$ and $\bar{u}(t) := (\Pi f(\cdot,\bar{x},\bar{y})) - f(t,\bar{x}(t),y(t))$. From lemma 2.1 we get

$$\begin{aligned}\|x-\bar{x}\| \;\; &\le \;\; \frac{1}{\alpha-\bar{L}_x}\|(\Pi - I)(f(\cdot,x,y) - f(\cdot,\bar{x},y)) + \\ &\qquad\qquad \Pi(f(\cdot,\bar{x},y) - f(\cdot,\bar{x},\bar{y}))\|.\end{aligned}$$

thus

$$\|x-\bar{x}\| \le \frac{1}{\alpha-\bar{L}_x}\left(\|(\Pi - I)\|L_x\|x-\bar{x}\| + \|\Pi\|L_y\|y-\bar{y}\|\right)$$

that means the assertion of the theorem is true. $\square$

In the numerical practice of waveform iteration method for instance on parallel computers with distributed memory the information of the whole system is not automatically known on the local node at which a subsystem is solved. So the conditions for the whole system have to be changed to a local one. In practice we relax the condition and merely require a one sided L-condition for the corresponding subsystem. Thus we have to extend the above results beginning with lemma 2.1. For the definition of Banach space we use the maximum of exponential weighted norms of the components corresponding to subsystems. (Compare [Bre93].)
With the notation of lemma 2.1 let $g_i(t,x_i,x)$ the i-th component of g by a corresponding ordering of the depending variables.

Lemma 2.2 *Suppose we have one-sided Lipschitz conditions in subsystems that is*

$$(g_i(t,x_i,y) - g_i(t,\bar{x}_i,y), x_i - \bar{x}_i) \le \bar{L}_i(x_i - \bar{x}_i, x_i - \bar{x}_i)$$

and Lipschitz conditions for each component of g with respect to y:

$$|g_i(t, x_i, y) - g_i(t, x_i, \bar{y})| \le L_i |y - \bar{y}|$$

then

$$\|x - \bar{x}\| \le \frac{1}{\alpha - L - \bar{L}} \|u - \bar{u}\| \tag{11}$$

holds with $\bar{L} := \max_i L_i$, $L := \max_i L_i$

Proof. For fixed i we use lemma 2.1 with $g_i(t, x_i, x(t))$ in place of g and $u_i(t)$, $\bar{u}_i(t) + g_i(t, \bar{x}_i(t), \bar{x}(t)) - g_i(t, \bar{x}_i(t), x(t))$ in place of $u(t)$, $\bar{u}(t)$ respectively. Then (8) reads

$$\|\Delta x_i\| \le \frac{1}{\alpha - \bar{L}_i} (L_i \|\Delta x\| + \|\Delta u_i\|).$$

By taking the maximum of the right hand side and left hand side we get

$$\|\Delta x\| \le \frac{1}{\alpha - \bar{L}} (L \|\Delta x\| + \|\Delta u\|)$$

□

Let us denote the i-th component of $f(t, x, y)$ by $f_i(t, x_i, x, y)$. We suppose one-sided Lipschitz conditions for the subsystems of (2) in the following form

$$(f_i(t, x_i, y, z) - f_i(t, \bar{x}_i, y, z), x - \bar{x}) \le \bar{L}_{x,i}(x_i - \bar{x}_i, x_i - \bar{x}_i). \tag{12}$$

Theorem 2.3 *With the conditions of theorem 2.1, (12) and* $\bar{L}_x := \max_i L_{x,i}$ *we get*

$$\|x - \bar{x}\|_\alpha \le \frac{L_y}{\alpha - \bar{L}_x - L_x} \|y - \bar{y}\|_\alpha.$$

The proof is similar to the proof of theorem 2.1 by using lemma 2.2 instead of lemma 2.1.

On the same way we get a (corresponding) result compared to theorem 2.2

Theorem 2.4 *With the conditions of theorem 2.2, (12) and* $\bar{L}_x$ *from theorem 2.3 we have*

$$\|x - \bar{x}\|_\alpha \le \frac{\|\Pi\| L_y}{\alpha - \bar{L}_x - (1 + \|\Pi - I\|) L_x} \|y - \bar{y}\|_\alpha.$$

Remark. Instead of using the Lipschitz constant L_x we can improve our result by using the Lipschitz constant with respect to the components of x "outside of the diagonal" Properly speaking, splitting the depending variables on the right hand side into three parts, say $\bar{f}(t, x, x, y)$ and supposing one-sided Lipschitz conditions with respect to the first x entry based on scalar products on subsystems, and (normal) Lipschitz conditions with respect to both x entries and y, say $L_{x,d}, L_x, L_y$. Then in theorem 2.3 is no change but theorem 2.4 reads

$$\|x-\bar{x}\|_\alpha \leq \frac{\|\Pi\| L_y}{\alpha - \bar{L}_x - L_x - (\|\Pi - I\|)(L_{x,d} + L_x)} \|y - \bar{y}\|_\alpha.$$

In case of block Jacobi like iteration L_x is zero, so we have the same inequalities for theorems 2.3 and 2.4 as in theorems 2.1 and 2.2.

3 Conclusions

Suppose, we have $\alpha > L$. A closer look to (9) shows that the inequality (8) can be improved to

$$\|x-\bar{x}\| \leq \frac{1-e^{(L-\alpha)(t_e-t_0)}}{\alpha - L} \|u-\bar{u}\| \tag{13}$$

which leads to

$$\|x-\bar{x}\|_\alpha \leq \frac{\left(1-e^{(L-\alpha)(t_e-t_0)}\right) L_y}{\alpha - \bar{L}_x} \|y-\bar{y}\|_\alpha \tag{14}$$

and

$$\|x-\bar{x}\|_\alpha \leq \frac{\|\Pi\| \left(1-e^{(L-\alpha)(t_e-t_0)}\right) L_y}{\alpha - \bar{L}_x - \|\Pi - I\| \left(1-e^{(L-\alpha)(t_e-t_0)}\right) L_x} \|y-\bar{y}\|_\alpha \tag{15}$$

in place of (7) and (10). In this notation we see the influence of the length of the interval $[t_0, t_e]$ (window length) to the speed of convergence in Banach space. The smaller window length yields better convergence or fewer restrictions for the stepsize of discretization.
In [Bre93] we have proved convergence results for a variable window technique. The proof is based on the following inequality for dependence from initial values in terms of the norm of the Banach space. If $g(t,x)$ is Lipschitz continuous with respect to x and with constant L, $\dot{x} = g(t,x), x(t_0) = x_0$, $\dot{\bar{x}} = g(t,\bar{x}), \bar{x}(t_0) = \bar{x}_0$ we have [1]

$$\|x-\bar{x}\| \leq \frac{\alpha}{\alpha - \left(1-e^{(L-\alpha)(t_e-t_0)}\right) L} |x_0 - \bar{x}_0|$$

By a well known result for one-sided Lipschitz conditions we have

$$|\Delta x(t)| \leq e^{Lt} |\Delta x(0)|$$

[1] In [Bre93] we use $\|x\| := |x(t_0)| + \frac{1-e^{-\alpha()}}{\alpha} \|\dot{x}\|_\alpha$ to get results in C^1. But from $\|x\|_\alpha \leq \|x\|$ it is clear that one can derive similar results in C by using $\|\cdot\|_\alpha$.

The proof is the same as in the above lemma with setting $\Delta u = 0$ and taking in account that $\mu(0) \neq 0$. ([GR92])
For our norm in Banach space this leads to

$$\|x - \bar{x}\| \leq |x_0 - \bar{x}_0|$$

which is a better result as in [Bre93]. From this it is clear that with the same arguments as in [Bre93] chapter 6.2 the convergence of waveform iteration with variable windows for a one-sided Lipschitz condition can be proven.

Our results shows how to replace Lipschitz condition with one-sided Lipschitz condition in the estimates needed for stepsize control to guarantee convergence of the discrete waveform iteration scheme. They are still in a mixture of local and global estimates. But is simple to generalize our results to use almost only local estimates. We can do this by first defining an exponential weighted norm with parameter α_i for each component i of System (1) an then taking the maximum to get a norm in the overall function space. Then we have to correct the corresponding inequalities by factors of the form $e^{(\alpha_i - \alpha_j)(t_e - t_0)}$, but we get local upper boundaries for the discretization stepsize with respect to needs of convergence.

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